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Number Theory — *Three counterexamples concerning the Northcott property of fields*, by Arno Fehm, communicated on November 10, 2017.

Abstract. — We give three examples of fields concerning the Northcott property on elements of small height: The first one has the Northcott property but its Galois closure does not even satisfy the Bogomolov property. The second one has the Northcott property and is pseudo-algebraically closed, i.e. every variety has a dense set of rational points. The third one has bounded local degree at infinitely many rational primes but does not have the Northcott property.

Key words: Height, algebraic number, Northcott property, pseudo-algebraically closed field

Mathematics Subject Classification: 11G50, 12E30, 11R04, 12F05

1. Introduction

Northcott's theorem on the finiteness of elements of bounded height in number fields is of central importance in diophantine geometry, for example very classically in the proof of the Mordell–Weil theorem. Motivated by that, Bombieri and Zannier [BZ01] say that a field $K \subseteq \overline{\mathbb{Q}}$ has the *Northcott property* (N) if for each $T > 0$ the set

$$K_T := \{\alpha \in K^\times : h(\alpha) < T\}$$

is finite, where $h : \overline{\mathbb{Q}} \to \mathbb{R}$ denotes the absolute logarithmic Weil height. In the same paper, the authors introduce another closely related notion: A field K has the *Bogomolov property* (B) if there exists $T > 0$ such that K_T consists only of the roots of unity in K. Note that clearly (N) implies (B). These and related properties have since been studied by various authors, see e.g. [AZ00, DZ08, Wid11, CW13, Hab13, Pot15, GR17].

One theme in this area is whether properties like (N) and (B) are preserved under taking Galois closures. For example, [Wid11, Cor. 2] gives a field $K \subseteq \overline{\mathbb{Q}}$ with (N) whose Galois closure over $\mathbb{Q}$ does not have (N). Similarly, [Pot16, Example 3.1] gives a field $K \subseteq \overline{\mathbb{Q}}$ with (B) whose Galois closure over $\mathbb{Q}$ does not have (B), and states that "It would be interesting to know whether the Galois closure of a field with the Northcott property necessarily satisfies the Bogomolov property." Our first result is that the answer to this is negative:

Proposition 1.1. *There exists an algebraic extension $K/\mathbb{Q}$ such that K has the Northcott property but the Galois closure of $K/\mathbb{Q}$ does not have the Bogomolov property.*

The intuition being that varieties over fields with (B), or even more so, with (N), have 'few' point, Amoroso, David and Zannier [ADZ14, Problem 6.1] asked whether there exists a field K with (B) that is *pseudo-algebraically closed*, i.e. every geometrically irreducible variety V over K has a K-rational point[1], and they present "some evidences for a negative answer". However, Pottmeyer [Pot16, Example 3.2] showed that such fields do exist, and while this was seen as surprising, it was apparently expected that at least there should be no pseudo-algebraically closed fields with (N): Our second result is that such fields do in fact exist, and can even be chosen Galois over $\mathbb{Q}$ (which might be interesting in light of Proposition 1.1):

Proposition 1.2. *There exists a Galois extension $K/\mathbb{Q}$ such that K is pseudo-algebraically closed and has the Northcott property.*

As the Northcott property implies a variety of other well-studied properties of fields (see e.g. [CW13, Theorem 6.8]), for example on pre-periodic points of polynomial mappings, this proposition might also give surprising counter-examples to some of the questions there, but we will not discuss these implications here.

The construction of the example in Proposition 1.1 is completely elementary, while the construction of the example in Proposition 1.2 uses some (known) results on specializations of covers of curves. The Northcott property in both cases follows from a very general criterion of Widmer [Wid11], which we recall is Section 2.

Pottmeyer [Pot15, Question 4.8] asks whether the Northcott property is implied by the so-called *universal strong Bogomolov property* (*USB*). The following example answers this questions negatively:

Proposition 1.3. *There exists a Galois extension $K/\mathbb{Q}$ such that infinitely many prime numbers are totally split in K but K does not have the Northcott property.*

Namely, Pottmeyer [Pot15, Theorem 4.3] shows that every Galois extension of $\mathbb{Q}$ that has finite local degree at infinitely many prime numbers (in particular, any K as in Proposition 1.3) satisfies (USB). Moreover, since by [Pot15, Lemma 4.2], (USB) implies also the Narkiewicz property (R) (cf. [CW13, Definition 6.6]), this shows that (N) is not implied by (R). The construction of the example in Proposition 1.3 builds on a result of Bombieri and Zannier [BZ01].

2. Widmer's criterion

We start by quoting the criterion of Widmer [Wid11, Theorem 3] and state a special case that is sufficient for our constructions:

[1] This property first occurred in the work of Ax on the elementary theory of finite fields. The term *pseudo-algebraically closed* was coined by Frey.

THEOREM 2.1. *Let $K_0 \subseteq K_1 \subseteq \cdots$ be a tower of number fields with*

$$\inf_{K_{i-1} \subsetneq M \subseteq K_i} N_{K_{i-1}/\mathbb{Q}}(D_{M/K_{i-1}})^{([M:K_0][M:K_{i-1}])^{-1}} \to \infty \quad \text{as } i \to \infty,$$

where the infimum is taken over intermediate fields M, and D_{M/K_i} denotes the relative discriminant. Then $K := \bigcup_{i=0}^{\infty} K_i$ has (N).

COROLLARY 2.2. *Let $K_0 \subseteq K_1 \subseteq \cdots$ be a tower of number fields and let $k_i = [K_i : \mathbb{Q}]$. If for each intermediate field $K_{i-1} \subsetneq M \subseteq K_i$ there exists a prime number $p > i^{k_i^2}$ that is unramified in K_{i-1} but ramified in M, then $K := \bigcup_{i=0}^{\infty} K_i$ has* (N).

PROOF. Let $K_{i-1} \subsetneq M \subseteq K_i$. If p is ramified in M but not in K_{i-1}, there is a prime $\mathfrak{p}$ of K_{i-1} over p that ramifies in M. Then $\mathfrak{p} | D_{M/K_{i-1}}$, hence $p | N := N_{K_{i-1}/\mathbb{Q}}(D_{M/K_{i-1}})$. Thus $N \geq p > i^{k_i^2}$, hence $N^{([M:K_0][M:K_{i-1}])^{-1}} \geq i$, so Theorem 2.1 applies. □

3. PROOF OF PROPOSITION 1.1

Let K_0 be any proper finite extension of $\mathbb{Q}$ in $\overline{\mathbb{Q}}$. We fix an algebraic integer $0 \neq \alpha \in K_0$ and $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ such that $\beta := \sigma\alpha/\alpha$ is not a root of unity (this is always possible, but take for example $K_0 = \mathbb{Q}(i)$, $\alpha = 2 + i$ and complex conjugation as σ). Choose a sequence of prime numbers l_i with $l_i \to \infty$ and let $k_i = [K_0 : \mathbb{Q}] \cdot l_1 \cdots l_i$. We now construct a certain tower of number field $K_0 \subseteq K_1 \subseteq \cdots$ with $[K_i : \mathbb{Q}] = k_i$. Suppose we already constructed $K_0, \ldots, K_{i-1}$. Fix a prime number $p_i > i^{k_i^2}$ that in addition does not ramify in K_{i-1} and does not divide $N_{K_0/\mathbb{Q}}(\alpha)$, let γ_i be an l_i-th root of $p_i\alpha$ in $\overline{\mathbb{Q}}$ and define $K_i = K_{i-1}(\gamma_i)$.

We claim that $K := \bigcup_i K_i$ has the desired properties: For each i, p_i ramifies in K_i but not in K_{i-1}, and there are no other intermediate fields $K_{i-1} \subsetneq M \subseteq K_i$. Therefore, Corollary 2.2 applies and gives that K has (N). However, if $\hat{K}$ denotes the Galois closure of K over $\mathbb{Q}$, then for each i, $\hat{K}$ contains both γ_i and $\sigma\gamma_i$ and therefore also $\frac{\sigma\gamma_i}{\gamma_i}$, which satisfies

$$\left(\frac{\sigma\gamma_i}{\gamma_i}\right)^{l_i} = \frac{\sigma(p_i\alpha)}{p_i\alpha} = \frac{\sigma\alpha}{\alpha} = \beta.$$

So since β is not a root of unity, neither is $\frac{\sigma\gamma_i}{\gamma_i}$, and $h\left(\frac{\sigma\gamma_i}{\gamma_i}\right) = \frac{1}{l_i} h(\beta) \to 0$, hence $\hat{K}$ does not satisfy (B).

4. PROOF OF PROPOSITION 1.2

We want to construct a certain field $K \subseteq \overline{\mathbb{Q}}$ and prove that it is pseudo-algebraically closed. It is well-known that for this it suffices to show that every geometrically irreducible curve X over $\mathbb{Q}$ has a K-rational point, see [FJ08, Theorem 11.2.3]. Moreover, since every curve admits a finite cover which is itself a Galois cover of $\mathbb{P}^1$ (see [FJ08, Theorem 18.9.3]), it suffices to prove the statement for the latter curves. Therefore, let $X_1, X_2, \ldots$ be an enumeration of the geometrically irreducible curves over $\mathbb{Q}$ that admit a Galois morphism to $\mathbb{P}^1$.

For each i we will construct a suitable finite Galois extension N_i of $\mathbb{Q}$ of degree n_i such that $X_i(N_i) \neq \emptyset$, let K_i be the compositum of $N_1, \ldots, N_i$, and K the union of the K_i (i.e. the compositum of all N_i). Suppose we already constructed $N_1, \ldots, N_{i-1}$ of degrees $n_1, \ldots, n_{i-1}$. Fix a Galois morphism $\varphi_i : X_i \to \mathbb{P}^1$, which induces a Galois extension of function fields $F := \mathbb{Q}(\mathbb{P}^1) \subseteq \mathbb{Q}(X_i) =: E$. Let $n_i := \deg(\varphi_i) = [E : F]$ and $d_i := n_1 \cdots n_i$.

We now apply a version of Hilbert's irreducibility theorem that allows some control on the ramification. While there are several such results in the literature, we intend to use [Leg16, Corollary 3.3]. For this, list the intermediate fields[2] $F \subsetneqq M \subseteq E$ that are Galois over F as $M_1, \ldots, M_r$ and observe that each M_j/F ramifies in some branch point $\alpha_j \in \mathbb{A}^1(\overline{\mathbb{Q}}) = \overline{\mathbb{Q}}$ by the Riemann–Hurwitz formula. In particular, there is a corresponding inertia subgroup $I \subseteq \mathrm{Gal}(E/F)$ not contained in $\mathrm{Gal}(E/M_j)$. Pick $g \in I \backslash \mathrm{Gal}(E/M_j)$ and let $C_j := g^{\mathrm{Gal}(E/F)}$ be the conjugacy class of g. If $m_j \in \mathbb{Q}[X]$ denotes the minimal polynomial of α_j over $\mathbb{Q}$, by the Chebotarev density theorem there are infinitely many prime numbers p such that $m_j \in \mathbb{Z}_{(p)}[X]$ and m_j has a zero modulo p. We can therefore choose primes $p_1, \ldots, p_r$ that are

(1) pairwise distinct,
(2) greater than $i^{d_i^2}$,
(3) not among the finitely many *bad primes* of the cover φ_i (cf. [Leg16, Def. 2.6]),
(4) not among the finitely many prime numbers that ramify in K_{i-1},
(5) and such that m_j has a zero modulo p_j for $j = 1, \ldots, r$.

Now [Leg16, Corollary 3.3] gives $x \in \mathbb{P}^1(\mathbb{Q})$ such that the fiber $\varphi_i^{-1}(x)$ is irreducible with function field a Galois extension N_i of $\mathbb{Q}$ with $\mathrm{Gal}(N_i/\mathbb{Q}) \cong \mathrm{Gal}(E/F)$ and such that the inertia group at each p_j is generated by an element of C_j. In particular, $[N_i : \mathbb{Q}] = n_i$ and in each (not necessarily Galois) subextension $\mathbb{Q} \subsetneqq M \subseteq N_i$, one of the $p_1, \ldots, p_r$ ramifies.

Note that $K_i = N_1 \cdots N_i$ satisfies $k_i := [K_i : \mathbb{Q}] \leq d_i$, and let $K = \bigcup_i K_i$. By construction, $X_i(K) \supseteq X_i(N_i) \neq \emptyset$ for each i, so K is pseudo-algebraically closed. Moreover, K satisfies (N) as the conditions of Corollary 2.2 are met: Each $K_{i-1} \subsetneqq M \subseteq K_i = K_{i-1}N_i$ is of the form $M = K_{i-1}M_0$ for some $\mathbb{Q} \subsetneqq M_0 \subseteq N_i$, and by construction there is a prime $p > i^{d_i^2} \geq i^{k_i^2}$ that ramifies in M_0 (and therefore in M) but not in K_{i-1}.

REMARK 4.1. Lukas Pottmeyer pointed out to me that replacing $i^{d_i^2}$ in (2) by $(i+1)^{4d_i^2}$ will achieve that $K_{\log(2)} = \{0, \pm 1\}$, i.e. $\alpha = 2$ is of smallest positive height in K.

[2] In fact, it would suffice to work with the *minimal* such fields.

5. Proof of Proposition 1.3

For a prime number p we denote by $\mathbb{Q}^{\mathrm{t}p}$ the field of *totally p-adic numbers*, i.e. the maximal Galois extension of $\mathbb{Q}$ in which p is totally split. We first recall a result of Bombieri and Zannier [BZ01, Example 2]. They prove that for any finite set of prime numbers $p_1, \dots, p_n$, the intersection $L := \bigcap_{i=1}^{n} \mathbb{Q}^{\mathrm{t}p_i}$ does not have (N). More precisely, they show that

$$\liminf_{\alpha \in L} h(\alpha) \le \sum_{i=1}^{n} \frac{\log p_i}{p_i - 1}. \tag{1}$$

To start our construction, fix any $T > 0$ and choose a sequence $d_1, d_2, \dots$ such that $d_i > e$ for each i and $\sum_{i=1}^{\infty} \frac{\log d_i}{d_i - 1} < T$. We want to construct an infinite sequence of primes $p_1, p_2, \dots$ and pairwise distinct elements $x_1, x_2, \dots \in \bigcap_{i=1}^{\infty} \mathbb{Q}^{\mathrm{t}p_i}$ with $p_i > d_i$ and $h(x_i) < T$ for each i. Suppose we already constructed primes $p_1, \dots, p_{n-1}$ and $x_1, \dots, x_{n-1} \in \bigcap_{i=1}^{n-1} \mathbb{Q}^{\mathrm{t}p_i}$ with $p_i > d_i$ and $h(x_i) < T$ for $i = 1, \dots, n-1$. By the Chebotarev density theorem, there are infinitely many primes p such that p is totally split in the Galois closure of $\mathbb{Q}(x_1, \dots, x_{n-1})$, in other words, $x_1, \dots, x_{n-1} \in \mathbb{Q}^{\mathrm{t}p}$. Choose such a prime $p_n > d_n$ and note that $x_1, \dots, x_{n-1} \in \bigcap_{i=1}^{n} \mathbb{Q}^{\mathrm{t}p_i}$. Now by (1), there exists $x_n \in \bigcap_{i=1}^{n} \mathbb{Q}^{\mathrm{t}p_i} \setminus \{x_1, \dots, x_{n-1}\}$ with

$$h(x_n) \le \sum_{i=1}^{n} \frac{\log p_i}{p_i - 1} \le \sum_{i=1}^{n} \frac{\log d_i}{d_i - 1} < T.$$

Continuing this construction, we arrive at $K := \bigcap_{i=1}^{\infty} \mathbb{Q}^{\mathrm{t}p_i}$ with $K_T \supseteq \{x_1, x_2, \dots\}$ infinite, so K does not satisfy (N). As $d_i \to \infty$, the set $\{p_1, p_2, \dots\}$ is infinite.

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