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Functional Analysis — *On the Moser–Trudinger inequality in fractional Sobolev–Slobodeckij spaces*, by Enea Parini and Bernhard Ruf, communicated on December 15, 2017.[1]

Abstract. — We give a contribution to the problem of finding the optimal exponent in the Moser–Trudinger inequality in the fractional Sobolev–Slobodeckij space $\tilde{W}_0^{s,p}(\Omega)$, where $\Omega \subset \mathbb{R}^N$ is a bounded domain, $s \in (0,1)$, and $sp = N$. We exhibit an explicit exponent $\alpha^*_{s,N} > 0$, which does not depend on Ω, such that the Moser–Trudinger inequality does not hold true for $\alpha \in (\alpha^*_{s,N}, +\infty)$.

Key words: Fractional Moser–Trudinger inequality, fractional Sobolev space, optimal exponent

Mathematics Subject Classification: 46E35, 35A23

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded, open domain with Lipschitz boundary. A celebrated result by Trudinger [15] and Moser [9] (see also the contributions by Yudovich [16], Pohozaev [12] and Strichartz [13]) states that functions in $W_0^{1,N}(\Omega)$ enjoy summability of exponential type: more precisely, there exists an exponent $\alpha_N > 0$ such that

$$\sup\left\{\int_\Omega \exp(\alpha|u|^{\frac{N}{N-1}}) \,|\, u \in W_0^{1,N}(\Omega),\ \|\nabla u\|_{L^N} \le 1\right\} < +\infty \tag{1}$$

holds true for every $\alpha \in [0, \alpha_N]$, and fails for $\alpha \in (\alpha_N, +\infty)$. This optimal exponent is given by $\alpha_N = N(N\omega_N)^{\frac{1}{N-1}}$, where ω_N is the volume of the N-dimensional unit ball. Subsequently, Adams [1] was able to extend the results to higher order Sobolev spaces $W_0^{k,p}(\Omega)$ with $kp = N$. His proof is based on expressing functions belonging to the space as Riesz potentials of their gradients of order k. This approach can be extended to Bessel potential spaces of fractional order. Martinazzi considered in [8] the space $\tilde{H}^{s,p}(\Omega)$, which is defined for $s \in (0,1)$ and $p \in (1, +\infty)$ as

$$\tilde{H}^{s,p}(\Omega) := \{u \in L^p(\mathbb{R}^N) \,|\, (-\Delta)^{\frac{s}{2}} u \in L^p(\Omega),\ u \equiv 0 \text{ in } \mathbb{R}^N \backslash \Omega\}.$$

[1] This paper is related to a talk given at "XXVII Convegno Nazionale di Calcolo delle Variazioni" – Levico Terme (Trento) 6–10 February, 2017.

In [10] we investigated the case of the fractional Sobolev–Slobodeckij space $\tilde{W}_0^{s,p}(\Omega)$, defined as the completion of $C_c^\infty(\Omega)$ with respect to the norm

$$u \mapsto (\|u\|^p_{L^p(\Omega)} + [u]^p_{W^{s,p}(\mathbb{R}^N)})^{\frac{1}{p}},$$

where

$$[u]_{W^{s,p}(\mathbb{R}^N)} := \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,dx\,dy.$$

If Ω has a Lipschitz boundary, this space can also be equivalently defined as

$$\tilde{W}_0^{s,p}(\Omega) := \left\{ u \in L^p(\mathbb{R}^N) \,\middle|\, \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,dx\,dy < +\infty,\ u \equiv 0 \text{ in } \mathbb{R}^N\backslash\Omega \right\}.$$

It is important to observe that the Sobolev–Slobodeckij space $\tilde{W}_0^{s,p}(\Omega)$ is in general different from the Bessel potential spaces $\tilde{H}^{s,p}(\Omega)$, unless $p=2$. Here the precise statement of the main result.

THEOREM 1.1. *Let Ω be a bounded, open domain of $\mathbb{R}^N$ $(N \geq 2)$ with Lipschitz boundary, and let $s \in (0,1)$, $sp = N$. Then there exists $\alpha_* = \alpha_*(s,\Omega) > 0$ such that*

$$\sup\left\{ \int_\Omega \exp(\alpha|u|^{\frac{N}{N-s}}) \,|\, u \in \tilde{W}_0^{s,p}(\Omega),\ [u]_{W^{s,p}(\mathbb{R}^N)} \leq 1 \right\} < +\infty \quad \textit{for } \alpha \in [0,\alpha_*).$$

Moreover,

$$\sup\left\{ \int_\Omega \exp(\alpha|u|^{\frac{N}{N-s}}) \,|\, u \in \tilde{W}_0^{s,p}(\Omega),\ [u]_{W^{s,p}(\mathbb{R}^N)} \leq 1 \right\} = +\infty \quad \textit{for } \alpha \in (\alpha^*_{s,N}, +\infty),$$

where

$$\alpha^*_{s,N} := N\Big(\frac{2(N\omega_N)^2\Gamma(p+1)}{N!}\sum_{k=0}^{\infty}\frac{(N+k-1)!}{k!}\frac{1}{(N+2k)^p}\Big)^{\frac{s}{N-s}}.$$

The proof of the validity of the Moser–Trudinger inequality for some value of $\alpha > 0$ follows the approach by Trudinger and is essentially contained in [11].

In order to give an upper bound to the optimal exponent $\bar{\alpha}$ such that

$$\sup\left\{ \int_\Omega \exp(\alpha|u|^{\frac{N}{N-s}}) \,|\, u \in \tilde{W}_0^{s,p}(\Omega),\ [u]_{W^{s,p}(\mathbb{R}^N)} \leq 1 \right\} < +\infty \tag{2}$$

for $\alpha \in [0,\bar{\alpha})$, one shows that it is possible to restrict to the case where Ω is a ball. Moreover, by a simple scaling argument it is easy to see that the optimal exponent does not depend on the radius of the ball. The following formula for the Gagliardo seminorm of a radially symmetric function $u \in W^{s,p}(\mathbb{R}^N)$ is needed in our proofs, and can be of independent interest.

PROPOSITION 1.2. *Let $u \in W^{s,p}(\mathbb{R}^N)$ be a radially symmetric function. Suppose that $sp = N$. Then,*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,dx\,dy$$
$$= (N\omega_N)^2 \int_0^{+\infty}\int_0^{+\infty} |u(r)-u(t)|^p r^{N-1} t^{N-1} \frac{r^2+t^2}{|r^2-t^2|^{N+1}}\,dr\,dt.$$

Then, one considers the family of functions defined by

$$u_\varepsilon(x) = \begin{cases} |\ln\varepsilon|^{\frac{N-s}{N}} & \text{if } |x| \le \varepsilon \\ \dfrac{|\ln|x||}{|\ln\varepsilon|^{\frac{s}{N}}} & \text{if } \varepsilon < |x| < 1 \\ 0 & \text{if } |x| \ge 1 \end{cases} \tag{3}$$

whose restrictions to the unit ball B belong to $\tilde{W}_0^{s,p}(B)$. For $s = 1$, this is the Moser-sequence used in [9], which satisfies

$$\|\nabla u_\varepsilon\|_2^2 = N\omega_N \quad \text{for every } \varepsilon > 0.$$

For $s \in (0,1)$, we cannot expect that $[u_\varepsilon]_{W^{s,p}(\mathbb{R}^N)}$ is constant, therefore it is essential to perform some lengthy calculations in order to obtain the limit as $\varepsilon \to 0$ of the quantity

$$I(\varepsilon) := \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_\varepsilon(x)-u_\varepsilon(y)|^p}{|x-y|^{N+sp}}\,dx\,dy$$
$$= (N\omega_N)^2 \int_0^{+\infty}\int_0^{+\infty} |u_\varepsilon(r)-u_\varepsilon(t)|^p r^{N-1} t^{N-1} \frac{r^2+t^2}{|r^2-t^2|^{N+1}}\,dr\,dt.$$

The results obtained are consistent with the local case. For example, if $N = 2$ we have

$$\lim_{s\to 1^-} (1-s)\alpha_{s,2}^* = 2\pi^2$$

which coincides with the optimal exponent $\alpha_{1,2}^* = 4\pi$ (see [9]), up to the multiplicative constant

$$K(2,2) := \frac{1}{2}\int_{S^1} |\langle \sigma, \mathbf{e}\rangle|^2\, d\mathscr{H}^{N-1}(\sigma) = \frac{\pi}{2}$$

which appears in the asymptotic behaviour of Gagliardo seminorms in the limit $s \to 1^-$ (see [2]).

It is an open problem to determine whether the exponent $\alpha^*_{s,N}$ is optimal. If this is the case, does the Moser–Trudinger inequality hold true also for $\alpha = \alpha^*_{opt}$, as in the classical case? And is the supremum attained, similarly to the results of [3] and [7]? This question was addressed by a recent paper by Takahashi [14] in the case of the space $H^{\frac{1}{2},2}(\mathbb{R})$.

We mention that Iula extended our analysis to the case $N = 1$ in [5]. He was able to prove that, for $s = \frac{1}{2}$ and $p = 2$, the exponent $\alpha^*_{\frac{1}{2},1}$ is equal to $2\pi^2$ and it coincides with the optimal exponent $\alpha_2 = \pi$ determined in [6] for the space $\tilde{H}^{\frac{1}{2},2}(I)$, up to a normalization constant relating the seminorms in the two spaces (see [4, Proposition 3.6]).

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