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Partial Differential Equations — *Regularizing effect of the lower order terms in some elliptic problems: old and new*, by Lucio Boccardo and Luigi Orsina, communicated on January 12, 2018.[1]

Abstract. — This paper is divided in two parts: the first one is a survey about recent results on the regularizing effect of lower order terms on solutions of nonlinear Dirichlet problems; in the second part we prove two existence theorems (related to the results of the first part) concerning elliptic systems.

Key words: Elliptic equations, Calderon–Zygmund theory, regularizing effects

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1. A survey on some recent results

In this paper Ω is a bounded, open subset of $\mathbb{R}^N$, with $N > 2$, $f(x)$ belongs to $L^m(\Omega)$, with $m \geq 1$, $M(x)$ is a measurable matrix such that

$$\alpha|\xi|^2 \leq M(x)\xi\xi, \quad |M(x)| \leq \beta, \tag{1.1}$$

for almost every x in Ω, and for every ξ in $\mathbb{R}^N$, with $0 < \alpha \leq \beta$. Furthermore $m^* = \frac{mN}{N-m}$ (if $1 < m < N$) and $m^{**} = \frac{mN}{N-2m}$ (if $1 < m < N/2$).

1.1. *The Stampacchia–Calderon–Zygmund theory for linear operators with discontinuous coefficients*

Let us consider the following boundary value problem

$$\begin{cases} -\mathrm{div}(M(x)\nabla u) = f, & \text{in } \Omega; \\ u = 0, & \text{on } \partial\Omega; \end{cases} \tag{1.2}$$

If $f \in L^m(\Omega)$, $m \geq \frac{2N}{N+2}$, existence and uniqueness of a weak solution $u \in W_0^{1,2}(\Omega)$ is a consequence of the Lax–Milgram theorem (finite energy solutions); therefore there exists a unique $u \in W_0^{1,2}(\Omega)$ such that

$$\int_\Omega M(x)\nabla u \nabla v = \int_\Omega f(x)v(x), \quad \forall v \in W_0^{1,2}(\Omega). \tag{1.3}$$

[1] This paper is related to a talk given at "XXVII Convegno Nazionale di Calcolo delle Variazioni" – Levico Terme (Trento) 6–10 February, 2017.

Moreover, G. Stampacchia (see [24]) proved the following summability theorem:

$$(1.4)\qquad \text{if } \begin{cases} 2N/(N+2) < m < N/2, & \text{then } u \in L^{m^{**}}(\Omega); \\ m = N/2, & \text{then } u \text{ has exponential summability;} \\ m > N/2, & \text{then } u \in L^{\infty}(\Omega). \end{cases}$$

Thus the summability $\frac{mN}{N-2m}$ of u is a strictly increasing function of m (recall that Ω is bounded).

However,

(1.5) the summability of ∇u is not a strictly increasing function of m (see [7]).

Note that, by a result by Meyers (see [22]), the gradient of finite energy solutions belongs to $L^p(\Omega)$ for some $p > 2$, independent on m, and dependent on the ellipticity constants α and β of M.

On the other hand, if $f \in L^m(\Omega)$, $1 \le m < \frac{2N}{N+2}$, we are outside of the finite energy solutions framework and the existence of a distributional solution u such that

$$(1.6)\qquad \int_{\Omega} M(x)\nabla u \nabla \varphi = \int_{\Omega} f(x)\varphi(x), \quad \forall \varphi \in \mathscr{D}(\Omega),$$

is proved in [8] and [9], where it is also proved that (infinite energy solutions)

$$(1.7)\qquad \text{if } \begin{cases} 1 < m < 2N/(N+2), & \text{then } u \in W_0^{1,m^*}(\Omega); \\ m = 1, & \text{then } u \in W_0^{1,q}(\Omega),\ q < \frac{N}{N-1}. \end{cases}$$

Note that solutions in distributional sense may not be unique (see for example [23]): in order to overcome this problem, the notion of *duality solution* has been introduced by Stampacchia in [24], who then proved that it is unique.

Note that here the summability of ∇u is a strictly increasing function of m, in contrast with (1.5).

1.2. The Stampacchia–Calderon–Zygmund theory for nonlinear operators with nonregular data

Now we consider nonlinear differential operators of the type

$$A(v) = -\mathrm{div}(a(x, v, \nabla v)).$$

A is a Leray–Lions operator defined from $W_0^{1,p}(\Omega)$, $1 < p < N$, into its dual and $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function (that is, measurable with respect to x in Ω for every (s, ξ) in $\mathbb{R} \times \mathbb{R}^N$, and continuous with respect to (s, ξ) in $\mathbb{R} \times \mathbb{R}^N$ for almost every x in Ω). We assume that there exist two real positive constants α and β, and a nonnegative function $k(x)$ in $L^{p'}(\Omega)$, such that for almost every x in Ω, for every s in $\mathbb{R}$, for every ξ and η in $\mathbb{R}^N$ $(\xi \neq \eta)$,

$$\alpha|\xi|^p \le a(x,s,\xi)\xi,$$
$$|a(x,s,\xi)| \le \beta(k(x) + |s|^{p-1} + |\xi|^{p-1}),$$
$$0 < [a(x,s,\xi) - a(x,s,\eta)][\xi - \eta].$$

Under these assumptions, A is pseudomonotone, and is hence surjective (see [21]).

The first example is the p-Laplace operator: $\mathrm{div}(\tilde{a}(x)|\nabla v|^{p-2}\nabla v)$.

If $f \in L^m(\Omega)$, $m \ge (p^*)'$, the existence of a weak solution (finite energy solutions) $u \in W_0^{1,p}(\Omega)$ is due to Leray and Lions (see [21])

$$(1.8) \qquad u \in W_0^{1,p}(\Omega) : \int_\Omega a(x,v,\nabla u)\nabla v = \int_\Omega f(x)v(x), \quad \forall v \in W_0^{1,p}(\Omega).$$

Moreover

- if $m > N/p$ (respectively $m > N/p$) the results of [24] says that u belongs to $L^\infty(\Omega)$ (respectively u has exponential summability);
- if $(p^*)' < m < N/p$, in [13] is proved that $u \in L^{[(p-1)m^*]^*}(\Omega)$.

If $f \in L^m(\Omega)$, $1 \le m < (p^*)'$, the Calderon–Zygmund theory for linear operators with nonregular data is studied in [8], [9], [11], where the existence of distributional solution is proved:

- if $\frac{N}{N(p-1)+1} < m < \frac{Np}{pN+p-N} = (p^*)'$, then there exists a distributional solution u of

$$(1.9) \qquad u \in W_0^{1,q}(\Omega) : \int_\Omega a(x,v,\nabla u)\nabla\varphi = \int_\Omega f(x)\varphi(x), \quad \forall\varphi \in \mathscr{D}(\Omega),$$

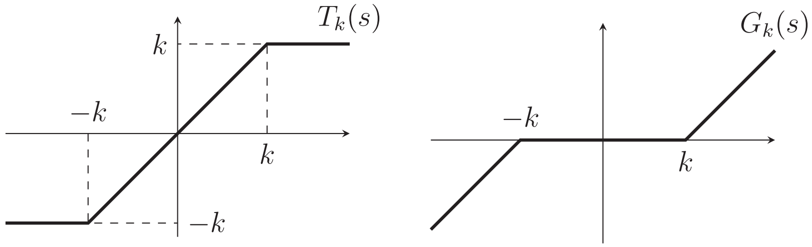
  where

$$(1.10) \qquad q = (p-1)m^*;$$

- if $m = \frac{N}{N(p-1)+1}$, then there exists a distributional solution u of (1.9), and $u \in W_0^{1,1}(\Omega)$; a radial example shows that $u \notin W_0^{1,q}(\Omega)$, for every $q > 1$;
- if $1 \le m < \frac{N}{N(p-1)+1}$, a definition of solution $u \notin W_0^{1,1}(\Omega)$, weaker than distributional solution and which uses the truncation, is needed: this part is studied in [3], where also a suitable definition of gradient is introduced.

Here we recall the definition of the truncation $T_k(s)$ and of $G_k(s)$:

$$T_k(s) = \max(-k, \min(s,k)), \quad G_k(s) = s - T_k(s) = (|s| - k)^+ \,\mathrm{sgn}(s).$$



1.3. The impact of some lower order terms

In this subsection we will recall how the presence of lower order terms may lead to an improvement of the summability of the solutions.

1.3.1. Semilinear problems. Here we consider the following boundary value problem, which can be seen as a perturbation of (1.2):

$$\begin{cases} -\mathrm{div}(M(x)\nabla u) + g(u) = f, & \text{in } \Omega; \\ u = 0, & \text{on } \partial\Omega; \end{cases} \tag{1.11}$$

where $g(s)$ is a continuous increasing (without growth conditions) real function, with $g(0) = 0$.

The existence of a solution $u \in W_0^{1,2}(\Omega)$, such that $ug(u) \in L^1(\Omega)$, is contained in the existence results of [18], if $f \in L^m(\Omega)$, $m > \frac{2N}{N+2}$.

Moreover, in [20] is proved (even in a more general setting) the regularizing effect of a polynomial lower order term $g(s) = s|s|^{r-1}$. Indeed for the boundary value problem

$$\begin{cases} -\mathrm{div}(M(x)\nabla u) + u|u|^{r-1} = f \in L^m(\Omega), & \text{in } \Omega; \\ u = 0, & \text{on } \partial\Omega; \end{cases} \tag{1.12}$$

it is possible to prove the existence of weak solutions, even beyond the natural duality pairing; that is: there exists a weak solution $u \in W_0^{1,2}(\Omega) \cap L^{(r-1)m}(\Omega)$, if

$$r' \le m < \frac{2N}{N+2}, \qquad r > 2^*. \tag{1.13}$$

Remark 1.1. Other regularizing effects are studied in [14], [10], [12].

1.3.2. Regularizing effect of the interplay between coefficients. In [1] the regularizing effect of the interaction between the coefficient of the zero order term and the datum in some nonlinear elliptic problems is studied.

The simplest example is the linear problem

$$\begin{cases} \displaystyle\int_\Omega M(x)\nabla u \nabla\varphi + \int_\Omega a(x)u\varphi = \int_\Omega f(x)\varphi, \\ \forall \varphi \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega), \end{cases}$$

where $0 \le a(x) \in L^1(\Omega)$. Even if $f(x)$ only belongs to $L^1(\Omega)$, in [1] it is proved that the assumption

$$\text{there exists } Q > 0 \text{ such that } |f(x)| \le Qa(x) \tag{1.14}$$

implies the existence of a weak solution u belonging to $W_0^{1,2}(\Omega)$ and such that $|u(x)| \le Q$:

$$\begin{cases} u \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega) : \\ \displaystyle\int_\Omega M(x)\nabla u \nabla\varphi + \int_\Omega a(x)u\varphi = \int_\Omega f(x)\varphi, \\ \forall\varphi \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega), \end{cases} \tag{1.15}$$

Outline of the proof. Formally: we use $G_Q(u)$ as test function and we deduce the inequality (we drop the first positive term and we use the assumption (1.14))

$$\int_\Omega a(x)|u|\,|G_Q(u)| \le \int_\Omega |f(x)|\,|G_Q(u)| \le Q\int_\Omega a(x)|G_Q(u)|,$$

that is

$$\int_{\{|u|>Q\}} a(x)[|u| - Q]^2 \le 0,$$

which implies

$$|u| \le Q. \qquad \square$$

Moreover a simple radial example shows the above boundedness result is sharp.

Moreover in [15] it is possible to find an example showing that the bounded solution u of (1.15) is not Hölder-continuous.

The case where Q in (1.14) is not a constant, but a function belonging to some Lebesgue space, is studied in [2].

2. New results concerning elliptic systems related to models for chemotaxis

2.1. A system with bounded solutions

In this section, we study the existence of solutions u, φ of the system (see also [6] and the references therein)

$$\begin{cases} -\mathrm{div}(A(x)\nabla u) + \rho(x)u = -\mathrm{div}(uM(x)\nabla\varphi) + f(x), & \text{in } \Omega; \\ -\mathrm{div}(M(x)\nabla\varphi) = g(x)u, & \text{in } \Omega; \\ u = 0 = \varphi, & \text{on } \partial\Omega; \end{cases}$$

where A and M are elliptic matrices satisfying (1.1) and f, ρ, g are functions belonging to $L^1(\Omega)$ and such that

$$|f(x)| \le Q\rho(x), \quad 0 \le g(x) \le R\rho(x), \tag{2.1}$$

where Q and R belong to $\mathbb{R}^+$ are such that

$$0 < QR < \frac{1}{4}. \tag{2.2}$$

THEOREM 2.1. *If* (2.1) *and* (2.2) *hold, there exists* $u \in W_0^{1,2}(\Omega)$ *and* $\varphi \in W_0^{1,q}(\Omega)$, $q < \frac{N}{N-1}$, *such that for every* v, w *smooth:*

$$\begin{cases} \int_\Omega A(x)\nabla u\nabla v + \int_\Omega \rho(x)uv = \int_\Omega uM(x)\nabla\varphi\nabla v + \int_\Omega f(x)v(x), \\ \int_\Omega M(x)\nabla\varphi\nabla w = \int_\Omega g(x)u(x)w(x). \end{cases} \tag{2.3}$$

PROOF. Our starting point is the following approximate system

$$\begin{cases} -\mathrm{div}(A(x)\nabla u_n) + \rho_n(x)u_n \\ \quad = -\mathrm{div}\Big(\dfrac{u_n}{1+\frac{1}{n}|u_n|}M(x)\nabla\varphi_n\Big) + f_n(x), & \text{in } \Omega; \\ -\mathrm{div}(M(x)\nabla\varphi_n) = g_n(x)\dfrac{u_n}{1+\frac{1}{n}|u_n|}, & \text{in } \Omega; \\ u_n = \varphi_n = 0, & \text{on } \partial\Omega; \end{cases} \tag{2.4}$$

where

$$\rho_n(x) = \frac{\rho(x)}{1+\frac{1}{n}|\rho|}, \quad f_n(x) = \frac{f(x)}{1+\frac{1}{nQ}|f|}, \quad g_n(x) = \frac{g(x)}{1+\frac{1}{nR}|g|}.$$

We prove the existence of solutions u_n and φ_n by means of the Schauder fixed point theorem. Let $n \in \mathbb{N}$ be fixed. For every $\gamma(x)$ in $W_0^{1,2}(\Omega)$, let Z be the unique solution of the linear Dirichlet problem

$$Z \in W_0^{1,2}(\Omega) : -\mathrm{div}(M(x)\nabla Z) = g_n(x)\frac{\gamma(x)}{1+\frac{1}{n}|\gamma|}.$$

Then let Γ be the unique solution of the linear Dirichlet problem

$$\Gamma \in W_0^{1,2}(\Omega) : -\mathrm{div}(A(x)\nabla\Gamma) + \rho_n(x)\Gamma = -\mathrm{div}\Big(\frac{\gamma(x)}{1+\frac{1}{n}|\gamma|}M(x)\nabla Z\Big) + f_n(x).$$

Choosing Z as test function in the first equation, one gets (since the right hand side is bounded by a constant times n^2)

$$\alpha \int_\Omega |\nabla Z|^2 \le n^2 \int_\Omega |Z| \le C_0 n^2 \Big(\int_\Omega |\nabla Z|^2\Big)^{\frac{1}{2}},$$

where in the last passage we have used Sobolev inequality. Therefore,

$$\int_\Omega |\nabla Z|^2 \le C_1 n^4. \tag{2.5}$$

Choosing Γ as test function in the second equation, one then obtains (dropping a positive term)

$$\begin{aligned}\alpha \int_\Omega |\nabla \Gamma|^2 &\le n \int_\Omega M(x)\nabla Z \nabla \Gamma + n \int_\Omega |\Gamma| \\ &\le \beta n \Big(\int_\Omega |\nabla Z|^2\Big)^{\frac{1}{2}} \Big(\int_\Omega |\nabla \Gamma|^2\Big)^{\frac{1}{2}} + C_2 n \Big(\int_\Omega |\nabla \Gamma|^2\Big)^{\frac{1}{2}},\end{aligned}$$

where once again we used the Sobolev inequality. Therefore, recalling (2.5), we obtain

$$\int_\Omega |\nabla \Gamma|^2 \le C_3 n^6, \tag{2.6}$$

which implies that $\|\Gamma\|_{W_0^{1,2}(\Omega)} \le C_4 n^3$, independently of γ; hence the ball of radius $C_4 n^3$ in $W_0^{1,2}(\Omega)$ is invariant for the map $\Sigma : \gamma \to \Gamma$. Moreover, the map Σ is completely continuous in $W_0^{1,2}(\Omega)$, since it is continuous and if there exists d in $\mathbb{R}$ such that $\|\gamma\|_{W_0^{1,2}(\Omega)} \le d$, then $\dfrac{\gamma(x)}{1+\frac{1}{n}|\gamma|}$ is compact in $W^{-1,2}(\Omega)$; hence the map $\gamma \to Z$ is compact in $W_0^{1,2}(\Omega)$, and so the map $Z \to \Gamma$ is compact in $W_0^{1,2}(\Omega)$.

Thus we proved that there exists (u_n, φ_n) in $W_0^{1,2}(\Omega) \times W_0^{1,2}(\Omega)$, solution of the system (2.4).

The first step is the proof of an a priori estimate on the sequence $\{u_n\}$ in $L^\infty(\Omega)$.

Let $h > 0$ and $j > 0$ be fixed. In (2.4), we use as test functions $D(u_n)$ (in the first equation) and $H_n(u_n)$ (in the second equation), where $D(s) = T_h(G_j(s))$ and

$$H_n(s) = \int_0^s \frac{\tau}{1+\frac{1}{n}|\tau|} D'(\tau)\, d\tau.$$

Note that

$$|H_n(s)| \le \int_0^s |\tau| D'(\tau)\, d\tau = \psi_{j,h}(t) = \begin{cases} 0 & \text{if } |t| \le j, \\ \frac{t^2 - j^2}{2} & \text{if } j \le |t| \le j+h, \\ hj + \frac{h^2}{2} & \text{if } |t| \ge j+h, \end{cases}$$

If we define $B_{j,h} = \{x \in \Omega : j \le |u_n| < j+h\}$ and $A_j = \{x \in \Omega : |u_n| \ge j\}$, we have

$$\int_{B_{j,h}} A(x)\nabla u_n \nabla u_n + \int_{A_j} \rho(x) u_n T_h(G_j(u_n)) = \int_{B_{j,h}} \frac{u_n}{1+\frac{1}{n}|u_n|} M(x)\nabla\varphi_n \nabla u_n + \int_{A_j} f_n T_h(G_j(u_n)). \tag{2.7}$$

and

$$\int_{B_{j,h}} M(x)\nabla\varphi_n \nabla u_n H_n'(u_n) = \int_{A_j} g_n(x) u_n H_n(u_n). \tag{2.8}$$

Substituting this identity in the first one, dropping the positive first term and dividing by h, we obtain

$$\int_{A_j} \rho_n(x) u_n \frac{T_h(G_j(u_n))}{h} \le \int_{A_j} \rho_n(x)|u_n| \frac{\psi_{j,h}(u_n)}{h} + \int_{A_j} f \frac{T_h(G_j(u_n))}{h} \le R\int_{A_j} \rho_n(x)|u_n| \frac{\psi_{j,h}(u_n)}{h} + Q\int_{A_j} \rho_n(x).$$

Letting h tend to zero, $\frac{T_h(s)}{h}$ tends to $\operatorname{sign}(s)$ and $\frac{\psi_{j,h}(s)}{h}$ tends to j. Therefore

$$\int_{A_j} \rho_n(x)|u_n| \le Rj\int_{A_j} \rho_n(x)|u_n| + Q\int_{A_j} \rho_n(x),$$

that is

$$\int_{A_j} \rho_n(x)[(1-Rj)|u_n| - Q] \le 0.$$

Let $0 < j < \frac{1}{R}$, so that

$$[(1-Rj)j - Q]\int_{A_j} \rho_n(x) \le 0. \tag{2.9}$$

Now we look for j such that $[(1-Rj)j - Q] > 0$, that is

$$\frac{1-\sqrt{1-4QR}}{2R} < j < \frac{1+\sqrt{1-4QR}}{2R}, \quad \text{with the condition } QR < \frac{1}{4}.$$

Observe that $0 < \frac{1-\sqrt{1-4QR}}{2R}$ and $\frac{1+\sqrt{1-4QR}}{2R} < \frac{1}{R}$. If we define $C_{Q,R} = \frac{1-4QR}{4R}$, we have that

$$\|u_n\|_{L^\infty(\Omega)} \le C_{Q,R}. \tag{2.10}$$

The second step is the proof of an a priori estimate on the sequence $\{u_n\}$ in $W_0^{1,2}(\Omega)$. In the first part of this proof, we choose as test functions in (2.4) u_n (in the first equation) and $H_n(u_n)$ (in the second equation) with now

$$H_n(s) = \int_0^s \frac{\tau}{1+\frac{1}{n}|\tau|}\, d\tau$$

we have

$$\int_\Omega A(x)\nabla u_n \nabla u_n + \int_\Omega \rho(x) u_n^2 = \int_\Omega \frac{u_n}{1+\frac{1}{n}|u_n|} M(x)\nabla\varphi_n \nabla u_n + \int_\Omega f_n u_n. \tag{2.11}$$

and, thanks to (2.10) and to the assumption (2.1),

$$\int_\Omega M(x)\nabla\varphi_n \nabla u_n H_n'(u_n) \le R\int_\Omega \rho(x) C_{Q,R} H_n(C_{Q,R}). \tag{2.12}$$

Substituting this inequality in (2.11), we have

$$\alpha\int_\Omega |\nabla u_n|^2 \le [R C_{Q,R} H_n(C_{Q,R}) + Q C_{Q,R}]\int_\Omega \rho(x).$$

Thus there exist $u \in W_0^{1,2}(\Omega)$ and a sequence still denoted by $\{u_n\}$ such that $\{u_n\}$ weakly converges in $W_0^{1,2}(\Omega)$ and a.e. to u.

Moreover the estimate (2.10) and the assumption (2.1) say that the right hand side of the second equation is bounded in $L^1(\Omega)$, since

$$\left| g_n(x)\frac{u_n}{1+\frac{1}{n}|u_n|}\right| \le |g(x)u_n| \le R\rho C_{Q,R},$$

so that the theory of Dirichlet problems in $L^1(\Omega)$ gives the boundedness of the sequence $\{\varphi_n\}$ in $W_0^{1,q}(\Omega)$, for $q < \frac{N}{N-1}$. Thus there exist $\varphi \in W_0^{1,q}(\Omega)$ and a sequence still denoted by $\{\varphi_n\}$ such that $\{\varphi_n\}$ weakly converges in $W_0^{1,q}(\Omega)$ and a.e. to φ.

Since the principal parts of the differential operators are linear, we can pass to the limit in the weak formulation of (2.4), even with the weak convergences of the sequences $\{u_n\}$ and $\{\varphi_n\}$, and we have the existence of u, φ solutions of (2.3). □

Remark 2.2. Since u belongs to $L^\infty(\Omega)$ the summability of φ depends only on the summability of ρ; in particular, $\varphi \in W_0^{1,2}(\Omega)$ if $\rho \in L^m(\Omega)$, with $m \ge \frac{2N}{N+2}$.

2.2. *A second system*

In this section we study the following system, which is similar to the previous one, but for the fact that the "bad" divergence term is now on the left hand side of the equation (i.e., it has the "good" sign):

$$\begin{cases} -\operatorname{div}(A(x)\nabla u) + u - \operatorname{div}(uM(x)\nabla\varphi) = f(x), & \text{in } \Omega; \\ -\operatorname{div}(A^*(x)\nabla\varphi) + \varphi = |u|^p, & \text{in } \Omega; \\ u = \varphi = 0, & \text{on } \partial\Omega; \end{cases}$$

where A and M are elliptic matrices satisfying (1.1), $0 \le f \in L^m(\Omega)$, $m > 2$ and $m \ge p+1$, $p > 0$.

THEOREM 2.3. *Under the above assumptions, there exist* $u \in W_0^{1,q}(\Omega)$, $q < \frac{2m}{m+2}$, *and* $\varphi \in W_0^{1,2}(\Omega)$, *such that for every* v, w *smooth:*

$$\begin{cases} \int_\Omega A(x)\nabla u\nabla v + \int_\Omega uv - \int_\Omega uM(x)\nabla\varphi\nabla v = \int_\Omega f(x)v(x), \\ \int_\Omega A^*(x)\nabla\varphi\nabla w + \int_\Omega \varphi w(x) = \int_\Omega u(x)^p w(x). \end{cases} \tag{2.13}$$

PROOF. Our starting point is the boundary value problem (with the same notations of the previous case)

$$\begin{cases} -\operatorname{div}(A(x)\nabla u_n) + u_n - \operatorname{div}\Big(\dfrac{u_n}{1+\frac{1}{n}|u_n|}M(x)\nabla\varphi_n\Big) = f_n(x), & \text{in } \Omega; \\ -\operatorname{div}(A^*(x)\nabla\varphi_n) + \varphi_n = |T_n(u_n)|^p, & \text{in } \Omega; \\ u_n = \varphi_n = 0, & \text{on } \partial\Omega; \end{cases}$$

The positivity of the function $f(x)$ (hence the positivity of $f_n(x)$) implies the positivity of $u_n(x)$ (see [6]); so that our starting point is the boundary value problem

$$\begin{cases} -\operatorname{div}(A(x)\nabla u_n) + u_n - \operatorname{div}\Big(\dfrac{u_n}{1+\frac{1}{n}|u_n|}M(x)\nabla\varphi_n\Big) = f_n(x), & \text{in } \Omega; \\ -\operatorname{div}(A^*(x)\nabla\varphi_n) + \varphi_n = T_n(u_n)^p, & \text{in } \Omega; \\ u_n = \varphi_n = 0, & \text{on } \partial\Omega; \end{cases}$$

We now choose φ_n as test function in the formulation for u_n, and u_n as test function in the formulation for φ_n. We get

$$\int_\Omega A(x)\nabla u_n\nabla\varphi_n + \int_\Omega u_n\varphi_n + \int_\Omega M(x)\nabla\varphi_n\nabla\varphi_n\frac{u_n}{1+\frac{1}{n}|u_n|} = \int_\Omega f_n\varphi_n,$$

and

$$\int_\Omega A^*(x)\nabla u_n\nabla\varphi_n + \int_\Omega u_n\varphi_n = \int_\Omega T_n(u_n)^p u_n.$$

Dropping a positive term (recall that $u_n \geq 0$), we then obtain the inequality

$$\int_\Omega T_n(u_n)^{p+1} \leq \int_\Omega T_n(u_n)^p u_n \leq \int_\Omega f_n \varphi_n \leq \int_\Omega f \varphi_n$$
$$\leq \|f\|_{L^m(\Omega)} \left[\int_\Omega (\varphi_n)^{m'}\right]^{\frac{1}{m'}} \leq \|f\|_{L^m(\Omega)} \left[\int_\Omega T_n(u_n)^{pm'}\right]^{\frac{1}{m'}}.$$

Here note that $pm' \leq p+1$, since $m \geq p+1$. Thus we have

$$\left[\int_\Omega T_n(u_n)^m\right]^{\frac{1}{m}} \leq \|f\|_{L^m(\Omega)}. \tag{2.14}$$

Since $m > 2 > \frac{2N}{N+2}$, the above estimate implies that the sequence

$$\{T_n(u_n)^p\} \text{ is compact in } L^{\frac{2N}{N+2}}(\Omega), \tag{2.15}$$

which implies that the sequence

$$\{\varphi_n\} \text{ is compact in } W_0^{1,2}(\Omega). \tag{2.16}$$

In [5] (see also [6]) is proved that since $\{\varphi_n\}$ is bounded in $W_0^{1,2}(\Omega)$, then

$$\int_\Omega \frac{|\nabla u_n|^2}{(1+u_n)^2} \leq C, \tag{2.17}$$

which then implies the a.e. convergence of $\{u_n(x)\}$ to a function $u(x)$.

Moreover, for every fixed $k > 0$, (2.14) implies that, for $n > k$,

$$k^m \operatorname{meas}(\{x : k < u_n\}) \leq \int_{k<u_n} T_n(u_n)^m = \int_{k<T_n(u_n)} T_n(u_n)^m$$
$$\leq \int_\Omega T_n(u_n)^m \leq \|f\|^m_{L^m(\Omega)},$$

so that the sequence $\{u_n\}$ is bounded in the Marcinkiewicz space $M^m(\Omega)$; therefore we have that

$$\{u_n\} \text{ strongly converges to } u \text{ in } L^r(\Omega), \quad 2 < r < m. \tag{2.18}$$

Then, (2.17) and (2.18) imply that there exists q such that

$$\{\nabla u_n\} \text{ is bounded in } W_0^{1,q}(\Omega), \quad 1 < q < \frac{2m}{m+2}. \tag{2.19}$$

Indeed, we use the Hölder inequality with exponents $2/q$ and $2/(2-q)$ and we have

$$\int_\Omega |\nabla u_n|^q = \int_\Omega \frac{|\nabla u_n|^q}{(1+|u_n|)^q}(1+|u_n|)^q \le C_R\left[\int_\Omega (1+|u_n|)^{\frac{2q}{2-q}}\right]^{\frac{2-q}{2}},$$

which is bounded because of (2.18).

Then (2.16) and (2.18) allow us to pass to the limit in L^1 in

$$\frac{u_n}{1+\frac{1}{n}u_n}M(x)\nabla\varphi_n$$

in order to prove the existence of solutions of the system (2.13). □

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