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Partial Differential Equations — *Stability for parabolic quasi minimizers in metric measure spaces*, by Yohei Fujishima and Jens Habermann, communicated on January 12, 2018.

Abstract. — We are concerned with the stability property for parabolic quasi minimizers in metric measure spaces. More precisely we consider a doubling metric measure space $\mathscr{X}$ which supports a weak Poincaré inequality and a parabolic domain $\Omega_T = \Omega \times (0, T)$ on the product space $\mathscr{X} \times \mathbb{R}$, where $\Omega \subset \mathscr{X}$ is a domain whose boundary $\partial\Omega$ is regular in the sense that its complement satisfies a uniform capacity density condition. We then show that a parabolic $\mathscr{Q}$ quasi minimizer of the p energy, $p \geq 2$, with fixed initial boundary data on the parabolic boundary of Ω_T is stable with respect to the variation of $\mathscr{Q}$ and p. The manuscript at hand is an extension of the result [7] to the setting of metric measure spaces.

Key words: Parabolic quasi minimizer, metric measure space, stability, higher integrability

Mathematics Subject Classification: 35K55, 35B65, 49N60, 30L99

1. Introduction

Let $(\mathscr{X}, \mathrm{d}, \mu)$ be a metric measure space with a doubling measure μ and which supports a weak $(1, p)$-Poincaré inequality. For a parabolic domain $\Omega_T := \Omega \times (0, T)$, with $T > 0$ and $\Omega \subset \mathscr{X}$ being an open bounded set, we consider parabolic $\mathscr{Q}$-quasiminimizers of the Dirichlet p-energy. For the special case $\mathscr{X} \equiv \mathbb{R}^n$, $n \geq 2$ and μ being the Lebesgue measure, parabolic $\mathscr{Q}$-quasiminimizers of the Dirichlet p-energy are functions $u \in L^p(0, T; W^{1,p}(\Omega))$ which satisfy the inequality

$$-\int_{\Omega_T} u\partial_t\Phi \,\mathrm{d}z + \frac{1}{p}\int_{\operatorname{spt}\Phi} |Du|^p \,\mathrm{d}z \leq \frac{\mathscr{Q}}{p}\int_{\operatorname{spt}\Phi} |Du - D\Phi|^p \,\mathrm{d}z,$$

for all test functions $\Phi \in C_c^\infty(\Omega_T)$, where $\mathscr{Q} \geq 1$ and $p > 1$ are fixed. In the case $\mathscr{Q} = 1$, parabolic 1-minimizers for $p > \frac{2n}{n+2}$ are weak solutions of the parabolic p-Laplace equation

$$\partial_t u - \operatorname{div}(|Du|^{p-2}Du) = 0 \quad \text{on } \Omega_T.$$

Parabolic quasiminimizers have first been introduced in the Euclidean setting in [34] and then been defined in more general metric measure spaces in [18]. In the past five years the investigation of parabolic quasiminimizers in metric measure spaces has gained increasing interest, see for example [18, 19, 29, 30, 28, 31, 9].

In the case of a general metric measure space $(\mathscr{X}, \mathrm{d}, \mu)$ we are considering here, there is no sense in speaking of a parabolic equation and the notion of parabolic $\mathscr{Q}$-quasiminimizers involves so called p-weak upper gradients instead of weak derivatives, see Definition 2.1 for the exact notion. In this general setting, quasiminimizers have been introduced in the time-independent (elliptic) case in [21] and then been studied by many authors, for example in [2, 27, 32, 26].

We are interested in stability issues for parabolic quasiminimizers. More precisely, we investigate the stability of global $\mathscr{Q}$-quasiminimizers of the p-energy with fixed initial-boundary-data $\eta : \partial_{\mathrm{par}}\Omega_T \to \mathbb{R}$, under a variation of the parameters $\mathscr{Q}$ and p. Problems of this type have been studied first in the elliptic case for systems of partial differential equations on $\mathbb{R}^n$ in [24, 23] and for parabolic systems in [20]. In the case of parabolic quasiminimizers the discussion of stability questions is more involved since one cannot use the structure of an underlying PDE. For time independent problems, results have first been proven in [26], where the authors treated the Euclidean case and the more general setting of metric measure spaces as well. Parabolic problems in the Euclidean setting have been recently studied in [7] and [8]. To the best of our knowledge the manuscript at hand is the first stability result for time dependent problems on metric measure spaces.

A fundamental ingredient for the stability proof with varying exponent p is a higher integrability property for parabolic quasiminimizers, since it provides uniform energy bounds for quasiminimizers and then allows to conclude convergence at least for subsequences by compactness properties of the underlying function spaces. Hence the proof of our main theorem is based on the global (up to the boundary) higher integrability estimates for quasiminimizers, recently shown in [6] (see also [10]), which is true only for exponents p which satisfy $p > \frac{2\mathrm{n}}{\mathrm{n}+2}$, where n denotes the 'dimension' of the metric measure space, related to the doubling constant. This is the reason, why we do not get stability in the case that $1 < p \le \frac{2\mathrm{n}}{\mathrm{n}+2}$ with our methods. Higher integrability for quasiminimizers (but also for solutions of equations) can in general not be achieved for this range of exponents. Moreover, we have to assume a weak regularity property for the boundary $\partial\Omega$ of the domain Ω under consideration. In fact, counterexamples, already for elliptic and parabolic systems show (see for example in [24, 20]) that stability cannot hold in case of an irregular boundary $\partial\Omega$. The boundary regularity we are assuming, is formulated in terms of a capacity density condition for the complement $\mathscr{X}\backslash\Omega$ – called uniform p thickness – and a deep self improving result for uniform p thickness, which goes back to [22] and was carried over to the context of metric measure spaces in [4].

Let us make some remarks on the proof and give some technical aspects. To prove Theorem 2.2 we follow basically the strategy of [7]: We first prove a Caccioppoli type estimate (Lemma 4.5) for parabolic quasiminimizers and use then the global higher integrability results of [6] to obtain uniform energy bounds. Starting then with a sequence $\{u_i\}$ of parabolic $\mathscr{Q}_i$-quasiminimizers of the p_i-energy, we get by compactness results for parabolic spaces $L^p(0, T; \mathscr{N}^{1,p}(\Omega))$ in terms of Lemma 3.10 – using the Rellich Kondrachov theorem in metric measure spaces (Lemma 3.11) and the existence of a weak time derivative $\partial_t u_i$ (Lemma 4.2) – the convergence $u_i \to u$ in $L^q(\Omega_T)$ for a $q > p$ of a subsequence and also the weak convergence $g_{u_i} \rightharpoonup g$ in $L^q(\Omega_T)$ of the sequence of minimal p-weak upper gradients to a limit function g. Lemma 4.1 allows to identify the limit g of this sequence as a p-weak upper gradient of the limit function u. Once having the limit function u at hand, we show, that u satisfies the initial-boundary conditions. This is done by a characterization of Newtonian functions with zero boundary values with the help of Hardy type estimates (Lemmas 3.8 and 3.9). In a last step we show that the limit function is indeed a parabolic $\mathcal{Q}$-minimizer of the p-energy. The proofs in the metric measure space setting is at many stages technically more involved than the proof in the Euclidean case. A reason for this is – in contrast to weak derivatives – the nonlinear behavior of p-weak upper gradients, which affect the proofs in many ways.

2. Statements and theorems

In this manuscript, let $(\mathscr{X}, \mu, \mathrm{d})$ be a locally linearly convex metric measure space which satisfies the doubling property with doubling constant $c_d \geq 1$ and which supports a weak $(1, p)$-Poincaré inequality. We say that $(\mathscr{X}, \mu, \mathrm{d})$ is called a *locally linearly convex metric measure space* if there exist positive constants $C > 0$ and $r_* > 0$ such that, for all balls $B_r(x) \subset \mathscr{X}$ with $r \in (0, r_*]$, every two points $y_1, y_2 \in B_{2r}(x) \setminus \overline{B_r(x)}$ can be connected by a curve lying in the annulus $B_{2Cr}(x) \setminus B_{C^{-1}r}(x)$, where $B_r(x) := \{y \in \mathscr{X} : \mathrm{d}(y, x) < r\}$ denotes the open ball of radius r and center x with respect to the metric d and $\overline{B_r(x)}$ is the closure of $B_r(x)$ with respect to the metric d. See [4] and [14]. Furthermore, the doubling property of the measure μ means the following: There exists a constant $c \geq 1$ such that

$$0 < \mu(B_{2r}(x)) \leq c \cdot \mu(B_r(x)) < +\infty, \tag{2.1}$$

for all radii $r > 0$ and all $x \in \mathscr{X}$. We define the doubling constant

$$c_d := \inf\{c \in (1, \infty) : (2.1) \text{ holds}\}. \tag{2.2}$$

We denote with $\mathrm{n} := \log_2 c_d$ the dimension from below of the metric measure space. Following the concept of Cheeger [5], Heinonen and Koskela [14], a Borel-function $g : \mathscr{X} \to [0, \infty]$ is called an 'upper gradient' for an extended real-valued function $u : \mathscr{X} \to [-\infty, +\infty]$, if for all rectifiable curves $\gamma : [0, \ell_\gamma] \to \mathscr{X}$ there holds

$$|u(\gamma(0)) - u(\gamma(\ell_\gamma))| \leq \int_\gamma g \, \mathrm{d}s, \tag{2.3}$$

and g is called a p-weak upper gradient of u if (2.3) holds for p-almost every path, which means that it fails only for a path family Γ on $\mathscr{X}$ satisfying

$$\mathrm{Mod}_p(\Gamma) := \inf\left\{\int_X \varrho^p \, \mathrm{d}\mu : \varrho \geq 0, \int_\gamma \varrho \, \mathrm{d}s \geq 1 \text{ for all } \gamma \in \Gamma\right\} = 0.$$

We define for $1 \le p < \infty$ and for a fixed open subset $\Omega \subset \mathscr{X}$ the vector space

$$\tilde{\mathscr{N}}^{1,p}(\Omega) := \{u \in L^p(\Omega) : \text{there exists a } p\text{-integrable } p\text{-weak upper gradient of } u\}.$$

This space can be endowed with a norm

$$\|u\|_{\tilde{\mathscr{N}}^{1,p}(\Omega)} := \|u\|_{L^p(\Omega)} + \inf_g \|g\|_{L^p(\Omega)} = \|u\|_{L^p(\Omega)} + \|g_u\|_{L^p(\Omega)},$$

where the infimum is taken over all p-integrable p-weak upper gradients of u, and g_u denotes the minimal p-weak upper gradient of the function u. Introducing the equivalence relation

$$u \sim v :\Leftrightarrow \|u - v\|_{\tilde{N}^{1,p}(\Omega)} = 0, \tag{2.4}$$

we define the *Newtonian space* $N^{1,p}(\Omega)$ as the quotient space

$$\mathscr{N}^{1,p}(\Omega) := \tilde{\mathscr{N}}^{1,p}(\Omega)/\sim. \tag{2.5}$$

In a standard way one may also define the spaces $\mathscr{N}_o^{1,p}(\Omega)$ and $\mathscr{N}_{\mathrm{loc}}^{1,p}(\Omega)$ and we refer the reader to [3] for more details on Newtonian spaces.

Poincaré inequality and Sobolev embedding

We demand that the metric measure space $(\mathscr{X}, \mathrm{d}, \mu)$ supports a weak $(1, p)$-Poincaré inequality in the sense that there exist constants $c_P > 0$ and $\Gamma > 1$ such that for all open balls $B_\varrho(x_o) \subset B_{\Gamma\varrho}(x_o) \subset \mathscr{X}$, for all p-integrable functions u on $\mathscr{X}$ and all upper gradients g of u there holds

$$⨍_{B_\varrho(x_o)} |u - u_{\varrho,x_o}| \,\mathrm{d}\mu \le c_P \varrho \left[⨍_{B_{\Gamma\varrho}(x_o)} g^p \,\mathrm{d}\mu \right]^{\frac{1}{p}}, \tag{2.6}$$

where the symbol

$$u_{\varrho,x_o} := ⨍_{B_\varrho(x_o)} u \,\mathrm{d}\mu := \frac{1}{\mu(B_\varrho(x_o))} \int_{B_\varrho(x_o)} u \,\mathrm{d}\mu$$

denotes the mean value integral of the function u on the ball $B_\varrho(x_o)$ with respect to the measure μ. The distance in the space $\mathscr{X}$ is denoted by d and in the usual way we define the distance between a point x and a set $Y \subset \mathscr{X}$ as well as between two sets $Y_1, Y_2 \subset \mathscr{X}$.

By Hölder's inequality it directly follows that if a metric space supports a weak $(1, p)$-Poincaré inequality, then it supports a weak $(1, q)$-Poincaré inequality for all $q \ge p$. On the other hand it was shown in [16] that if a complete metric space is endowed with a doubling measure and supports a weak $(1, p)$-Poincaré inequality, then it supports also a weak $(1, p - \varepsilon)$-Poincaré inequality for some $\varepsilon \equiv \varepsilon(c_P, \Gamma, c_d, p) > 0$, and therefore also a weak $(1, q)$-Poincaré inequality for all $q \in [p-\varepsilon, p]$. Moreover, from [12] we know that if we assume a weak $(1,p)$-Poincaré inequality, then the Sobolev embedding theorem holds and hence a weak (q,p)-Poincaré inequality holds for all $q \le p^*$, with

$$p^* := \begin{cases} \dfrac{p\mathrm{n}}{\mathrm{n}-p}, & p < \mathrm{n}, \\ +\infty, & p \ge \mathrm{n}. \end{cases} \tag{2.7}$$

On the other hand it was shown in [21], see also [14, 11, 12], that in this case for every $u \in N^{1,p}(B_{2\Gamma\varrho}(x_o))$ with $B_{2\Gamma\varrho}(x_o) \subset \mathscr{X}$ the following Sobolev-type inequality holds:

$$\left[\fint_{B_\varrho(x_o)} |u - u_{\varrho,x_o}|^q \,\mathrm{d}\mu \right]^{\frac{1}{q}} \le c_* \varrho \left[\fint_{B_{2\Gamma\varrho}(x_o)} g_u^p \,\mathrm{d}\mu \right]^{\frac{1}{p}}, \quad \text{for all } 1 \le q \le p^*. \tag{2.8}$$

The constant c_* in the above inequality depends only on c_d and on the constant c_P in the weak $(1,p)$-Poincaré inequality.

Poincaré and Sobolev inequalities hold also on more general domains. More precisely, the Poincaré inequality holds on bounded measurable subsets E of the metric space $\mathscr{X}$ such that the p-capacity of the complement $\mathscr{X} \backslash E$ does not vanish. In detail we have

$$\int_E |u|^p \,\mathrm{d}\mu \le C_E \int_E g_u^p \,\mathrm{d}\mu,$$

for every function $u \in N_o^{1,p}(E)$ and for every bounded measurable set $E \subset \mathscr{X}$ with $\mathrm{cap}_p(\mathscr{X} \backslash E) > 0$. See (2.13) for the definition of capacity. The constant C_E depends on c_P, c_d, p and E.

Parabolic Newtonian spaces

Since we are dealing with time-dependent problems, we have to introduce the parabolic Newtonian space $L^p(0,T;\mathscr{N}^{1,p}(\Omega))$ or its local version, respectively. We consider the product space $\mathscr{X} \times \mathbb{R}$, which we endow with the 'parabolic distance'

$$\mathrm{d}_{\mathrm{par}}(z_1, z_2) := \max\{\mathrm{d}(x_1, x_2), \sqrt{|t_1 - t_2|}\},$$

for points $z_i = (x_i, t_i) \in \mathscr{X} \times \mathbb{R}$. We will consider functions on a parabolic domain $\Omega_T := \Omega \times (0,T) \subset \mathscr{X} \times \mathbb{R}$ with an open bounded set $\Omega \subset \mathscr{X}$ (with respect to the metric d) and $T > 0$. For such a set we denote the parabolic boundary of Ω_T by

$$\partial_{\mathrm{par}}\Omega_T := (\Omega \times \{t = 0\}) \cup (\partial\Omega \times (0,T)).$$

The parabolic Newtonian space

$$L^p(0,T;\mathscr{N}^{1,p}(\Omega))$$

consists in all functions $u : \Omega \times (0, T) \to \mathbb{R}$ for which $u(\cdot, t) \in \mathcal{N}^{1,p}(\Omega)$ for almost all $t \in (0, T)$ and moreover

$$\int_0^T \|u(\cdot, t)\|^p_{\mathcal{N}^{1,p}(\Omega)} \, \mathrm{d}t < \infty.$$

The minimal p-weak upper gradient of a function $u \in L^p(0, T; \mathcal{N}^{1,p}(\Omega))$ has to be understood in the sense that

$$g_u(x, t) = g_{u(\cdot, t)}(x). \tag{2.9}$$

We equip the space $L^p(0, T; \mathcal{N}^{1,p}(\Omega))$ with the norm

$$\|u\|_{L^p(0,T,\mathcal{N}^{1,p}(\Omega))} := \|u\|_{L^p(\Omega_T)} + \|g_u\|_{L^p(\Omega_T)},$$

where g_u denotes the minimal p-weak upper gradient of the function u.

Parabolic quasiminimizers and statement of the main theorems

For the whole manuscript, let $(\mathcal{X}, \mathrm{d}, \mu)$ be a metric measure space with all the properties mentioned above. We first introduce the concept of parabolic $\mathcal{Q}$-quasiminimizers:

Definition 2.1 (Global parabolic $\mathcal{Q}$-minimizer). Let $\mathcal{Q} \geq 1$, $p > 1$ and $\Omega \subset \mathcal{X}$ be a bounded open set and $T > 0$. We say that $u \in L^p(0, T; \mathcal{N}^{1,p}(\Omega))$ is a global parabolic $\mathcal{Q}$-minimizer of the p-energy with initial-boundary data $\eta \in L^p(0, T; \mathcal{N}^{1,p}(\Omega))$ if

$$-\iint_{\Omega_T} u \partial_t \Phi \, \mathrm{d}\mu \, \mathrm{d}t + \frac{1}{p} \iint_{\operatorname{spt}\Phi} g_u^p \, \mathrm{d}\mu \, \mathrm{d}t \leq \frac{\mathcal{Q}}{p} \iint_{\operatorname{spt}\Phi} g_{u-\Phi}^p \, \mathrm{d}\mu \, \mathrm{d}t, \tag{2.10}$$

for all test functions $\Phi \in \operatorname{Lip}_c(\Omega_T)$ and moreover

$$\begin{cases} u(\cdot, t) - \eta(\cdot, t) \in \mathcal{N}_o^{1,p}(\Omega) & \text{for almost every } t \in (0, T), \\ \lim\limits_{h \to 0} \dfrac{1}{h} \displaystyle\int_0^h \int_\Omega |u - \eta|^2 \, \mathrm{d}\mu \, \mathrm{d}t = 0. \end{cases} \tag{2.11}$$

Before stating our main theorems, we define the *uniform p-thickness* of a domain. The uniform p-thickness of the complement $\mathcal{X} \backslash \Omega$ is a major assumption in our theorem. It means that there exist positive constants μ and r_o such that

$$\operatorname{cap}_p((\mathcal{X} \backslash \Omega) \cap \overline{B_r(x)}, B_{2r}(x)) \geq \mu \operatorname{cap}_p(\overline{B_r(x)}, B_{2r}(x)), \tag{2.12}$$

for all $x \in \mathcal{X} \backslash \Omega$ and $r \in (0, r_o)$. Here cap_p denotes the variational p-capacity, which is defined for an open set $\mathcal{O} \subset \mathcal{X}$ and a subset $E \subset \mathcal{O}$ as follows:

$$\operatorname{cap}_p(E, \mathcal{O}) := \inf_{\substack{f \geq 1 \text{ on } E \\ f \in \mathcal{N}_o^{1,p}(\mathcal{O})}} \int_{\mathcal{O}} g_f^p \, \mathrm{d}\mu, \tag{2.13}$$

where g_f denotes the minimal p-weak upper gradient of f. We refer the reader to [3] for more details on the capacity in metric spaces. The uniform p-thickness is a weak regularity property: Domains satisfying (2.12), do not have thin external cusps. A deep self-improving property of the uniform p-thickness has been proved in [22, 4] and says that for every set $E \subset \mathscr{X}$ which is uniformly p-thick there exists $q < p$ such that E is also uniformly q-thick. This property plays an essential role in stability estimates since we only assume for the domain Ω that $\mathscr{X} \setminus \Omega$ is uniformly p-thick, there p is the limit exponent of the sequence $\{p_i\}$. However note that for Lipschitz domains this property is always satisfied.

Our main theorem reads as follows:

Theorem 2.2. *Let $T > 0$, $p > \frac{2\mathrm{n}}{\mathrm{n}+2}$, where $\mathrm{n} = \log_2 c_d$. Let $\Omega \subset \mathscr{X}$ be such that $\mathscr{X} \setminus \Omega$ is uniformly p-thick. Moreover, let $\{p_i\}_{i \in \mathbb{N}}$ and $\{\mathscr{Q}_i\}_{i \in \mathbb{N}}$ be two sequences with $p_i > \frac{2\mathrm{n}}{\mathrm{n}+2}$ and $\mathscr{Q}_i \geq 1$ such that*

$$\begin{cases} p_i \to p \\ \mathscr{Q}_i \to \mathscr{Q} \end{cases} \quad \text{as } i \to \infty.$$

Let $u_i \in L^{p_i}(0, T; \mathscr{N}^{1,p_i}(\Omega))$ be a parabolic $\mathscr{Q}_i$-minimizer of the p_i-energy in the sense that

$$-\iint_{\Omega_T} u_i \partial_t \Phi \, \mathrm{d}\mu \, \mathrm{d}t + \frac{1}{p_i} \iint_{\operatorname{spt}\Phi} g_{u_i}^{p_i} \, \mathrm{d}\mu \, \mathrm{d}t \leq \frac{\mathscr{Q}_i}{p_i} \iint_{\operatorname{spt}\Phi} g_{u_i - \Phi}^{p_i} \, \mathrm{d}\mu \, \mathrm{d}t,$$

for all test functions $\Phi \in \mathrm{Lip}_c(\Omega_T)$, with initial-boundary data $\eta \in \mathrm{Lip}(\overline{\Omega_T})$ in the sense of (2.11)*. Suppose that there exists a strongly measurable function u such that*

$$\lim_{i \to \infty} u_i(x, t) = u(x, t) \quad \text{for almost every } (x, t) \in \Omega_T.$$

Then $u \in L^p(0, T; \mathscr{N}^{1,p}(\Omega))$ and moreover it is a global parabolic $\mathscr{Q}$-minimizer of the p-energy with initial boundary data η in the sense of Definition 2.1.

Remark 2.3. The proof of Theorem 2.2 shows that for the sequence $\{u_i\}_i$ of $\mathscr{Q}_i$ quasiminimizers holds that

$$u_i \to u \quad \text{strongly in } L^p(\Omega_T), \quad \text{and} \quad g_{u_i} \rightharpoonup g \quad \text{weakly in } L^p(\Omega_T),$$

as $i \to \infty$, where g denotes a p-weak upper gradient of u, not necessarily the minimal one.

In the case that $\mathscr{Q} = 1$, the limit function in the above Theorem 2.2 is a parabolic minimizer of the p energy. For this case we get the following stronger result:

Theorem 2.4. *Under the assumptions of Theorem 2.2 we get: If $\mathcal{Q} = 1$, then additionally to the assertions of Theorem 2.2 there holds*

$$g_{u_i} \to g_u \quad \textit{strongly in } L^p(\Omega_T)$$

as $i \to \infty$, where g_u is the minimal p-weak upper gradient of the function u.

Remark 2.5. The strong L^p convergence in Theorem 2.4 means that

$$\iint_{\Omega_T} |g_{u_i} - g_u|^p \, \mathrm{d}\mu \, \mathrm{d}t \to 0$$

as $i \to \infty$. In particular, we do not get the strong convergenge $g_{u_i - u} \to 0$ in $L^p(\Omega_T)$ and hence we cannot conclude that $u_i \to u$ in $L^p(0, T; \mathcal{N}^{1,p}(\Omega))$.

3. Preliminaries

We start with a number of properties of parabolic Newtonian spaces, which are essential for the stability proofs.

Definition 3.1 (Parabolic Newtonian space). We denote by $L^p(0, T; \mathcal{N}^{1,p}(\Omega))$ the space of all functions $u : \Omega \times (0, T) \to \mathbb{R}$ such that $(0, T) \ni t \mapsto u(\cdot, t) \in \mathcal{N}^{1,p}(\Omega)$ is strongly measurable and the functions $(0, T) \ni t \mapsto \|u(\cdot, t)\|_{\mathcal{N}^{1,p}(\Omega)}$ are contained in $L^p(0, T)$.

Remark 3.2. *Strongly measurable* means that there exists a sequence of simple functions $u_k : (0, T) \to \mathcal{N}^{1,p}(\Omega)$ such that

$$\|u(\cdot, t) - u_k(\cdot, t)\|_{\mathcal{N}^{1,p}(\Omega)} \to 0, \quad \text{as } k \to \infty.$$

Definition 3.3 (Dual space). The dual space $[L^p(0, T; \mathcal{N}_o^{1,p}(\Omega))]^*$ of the parabolic Newtonian space $L^p(0, T; \mathcal{N}_o^{1,p}(\Omega))$ is defined as the space of continuous linear functionals on $L^p(0, T; \mathcal{N}_o^{1,p}(\Omega))$.

Remark 3.4. We do not have characterizations of dual spaces in terms of integration by parts formulas, so we cannot obtain an identification like

$$[W_o^{1,p}(\Omega)]^* \cong W^{-1,q}(\Omega), \quad \frac{1}{p} + \frac{1}{q} = 1,$$

as we have it for Sobolev spaces, however since the space $\mathcal{N}^{1,p}(\Omega)$ is reflexive for any $p > 1$, there holds

$$[L^p(0, T; \mathcal{N}_o^{1,p}(\Omega))]^* \cong L^q(0, T; [\mathcal{N}_o^{1,p}(\Omega)]^*), \quad \frac{1}{p} + \frac{1}{q} = 1,$$

where the dual pairing between $L^q(0,T;[\mathcal{N}_o^{1,p}(\Omega)]^*)$ and $L^p(0,T;\mathcal{N}_o^{1,p}(\Omega))$ is

$$\langle u,v\rangle = \int_0^T \langle u(\cdot,t),v(\cdot,t)\rangle_* \, \mathrm{d}t.$$

Here $\langle\cdot,\cdot\rangle_*$ denotes the dual pairing between $[\mathcal{N}_o^{1,p}(\Omega)]^*$ and $\mathcal{N}_o^{1,p}(\Omega)$.

Definition 3.5 (Weak time derivative). We call a function $v \in [L^p(0,T;\mathcal{N}_o^{1,p}(\Omega))]^*$ weak time derivative of $u \in L^p(0,T;\mathcal{N}^{1,p}(\Omega))$ – and write $v = \partial_t u$ in the weak sense – if

$$\langle v,\varphi\rangle = -\iint_{\Omega_T} u(\partial_t\varphi)\,\mathrm{d}\mu\,\mathrm{d}t,$$

for all $\varphi \in \mathrm{Lip}_c(\Omega_T)$.

3.1. *Smoothing in time*

We consider regularizations of parabolic quasi minimizers in the time variable. For $\varepsilon > 0$ we denote by $\sigma_\varepsilon(s) := \varepsilon^{-1}\sigma(s/\varepsilon)$ with a standard smoothing kernel $\sigma \in C_c^\infty$ with $\operatorname{spt}\sigma \subset (-1,1)$. For $\Phi : \Omega\times(0,T)\to\mathbb{R}$ we denote the smoothed function

$$[\Phi]_\varepsilon(x,t) := \int_{\mathbb{R}} \Phi(x,t-s)\sigma_\varepsilon(s)\,\mathrm{d}s.$$

Note that if $\Phi \in L^p(0,T;\mathcal{N}^{1,p}(\Omega))$, we have that $[\Phi]_\varepsilon(x,\cdot) \in C^\infty(0,T)$ for almost all $x \in \Omega$ and

$$\|\Phi(x,\cdot) - [\Phi]_\varepsilon(x,\cdot)\|_{L^p(0,T)} \to 0 \quad \text{as } \varepsilon \to 0, \text{ a.e. on } \Omega.$$

Testing (2.10) with $[\Phi]_\varepsilon$ instead of Φ and using an integration by parts in the time variable, we obtain

$$\iint_{\Omega_T} \partial_t[u]_\varepsilon \Phi\,\mathrm{d}\mu\,\mathrm{d}t + \frac{1}{p}\iint_{\operatorname{spt}[\Phi]_\varepsilon} g_u^p\,\mathrm{d}\mu\,\mathrm{d}t \le \frac{\mathcal{Q}}{p}\iint_{\operatorname{spt}[\Phi]_\varepsilon} g_{u-[\Phi]_\varepsilon}^p\,\mathrm{d}\mu\,\mathrm{d}t, \tag{3.1}$$

for all $\Phi \in \mathrm{Lip}_c(\Omega_T)$ and $\varepsilon > 0$ small enough. Here $[u]_\varepsilon$ denotes the smoothing of u with respect to the time variable. On the other hand, from [29, Lemma 2, Corollary 1] we know that for any function $\psi \in L_c^p(0,T;\mathcal{N}_o^{1,p}(\Omega))$ and every $\varepsilon > 0$ there exists a function $\phi \in \mathrm{Lip}_c(\Omega_T)$ such that

$$\|\psi-\phi\|_{L^p(0,T;\mathcal{N}^{1,p}(\Omega))} < \varepsilon, \quad \|\psi-\phi\|_{L^2(\Omega_T)} < \varepsilon, \quad |\operatorname{spt}\phi\backslash\operatorname{spt}\psi| < \varepsilon.$$

Using this we can easily conclude that (3.1) holds also for all test functions $\Phi \in L_c^p(0,T;\mathcal{N}_o^{1,p}(\Omega))$.

The following approximation Lemma is proved in [31]:

LEMMA 3.6 (Smoothing). *Let $u \in L^p_{\mathrm{loc}}(0,T;\mathcal{N}^{1,p}_{\mathrm{loc}}(\Omega))$ and $[u]_\varepsilon$ be the mollification with respect to the time variable. Then $g_{u-[u]_\varepsilon} \to 0$ in $L^p_{\mathrm{loc}}(\Omega_T)$ and pointwise $(\mu \times \mathscr{L}^1)$-almost everywhere on Ω_T.*

3.2. Netwonian spaces with zero boundary values

In order to prove that the limit function attains the boundary values η, we use the following characterization of Newtonian spaces with zero boundary values, which has been shown for Sobolev spaces in [13] and for Newtonian spaces in [26].

LEMMA 3.7. *Let $(\mathscr{X},\mu,\mathrm{d})$ be a doubling metric measure space which supports a weak $(1,q)$-Poincaré inequality for some $1 \le q < \infty$. Let $p > q$ and let $\Omega \subset \mathscr{X}$ be an open bounded set such that $\mathscr{X}\backslash\Omega$ is uniformly p-thick. Then*

$$\mathcal{N}^{1,p}_o(\Omega) = \mathcal{N}^{1,p}(\Omega) \cap \bigcap_{s<p} \mathcal{N}^{1,s}_o(\Omega).$$

An important characterization of Newtonian spaces with zero boundary values uses a Hardy type inequality. It has been proved for $\mathscr{X} = \mathbb{R}^N$ in [1, 22] and on metric measure spaces in [4, 17].

LEMMA 3.8. *Let $\Omega \subset \mathscr{X}$ be a bounded open set and assume that $\mathscr{X}\backslash\Omega$ is uniformly p-thick for some $p > 1$. Then there exists a constant $c \equiv c(\Omega,p) > 0$ such that*

$$\int_\Omega \Big(\frac{|u(x)|}{\mathrm{dist}(x,\mathscr{X}\backslash\Omega)}\Big)^p \,\mathrm{d}\mu \le c\|u\|^p_{\mathcal{N}^{1,p}(\Omega)},$$

for all $u \in \mathcal{N}^{1,p}_o(\Omega)$.

LEMMA 3.9. *Let $\Omega \subset \mathscr{X}$ be an open set. If $u \in \mathcal{N}^{1,p}(\Omega)$ satisfies*

$$\int_\Omega \Big(\frac{|u(x)|}{\mathrm{dist}(x,\mathscr{X}\backslash\Omega)}\Big)^p \,\mathrm{d}\mu < \infty,$$

then $u \in \mathcal{N}^{1,p}_o(\Omega)$.

3.3. Compactness properties

We have to apply Simon's compactness result on parabolic spaces to our setting of Newtonian spaces. Therefore let us make some remarks. The statement of Simon, [33, Corollary 8] goes as follows:

LEMMA 3.10. *Let X, B, Y be Banach spaces with $X \subset B \subset Y$, the embedding $X \to B$ being compact, the embedding $B \to Y$ being continuous and there exist constants $\theta \in (0,1)$ and C such that*

$$\|v\|_B \le C\|v\|^{1-\theta}_X \|v\|^\theta_Y, \quad \textit{for all } v \in X. \tag{3.2}$$

Moreover, let $1 \leq p_o \leq \infty$, $1 \leq r_1 \leq \infty$ *and let* $\{u_i\}_i$ *be a bounded sequence in* $L^{p_o}(0, T; X)$ *such that* $\{\partial_t u_i\}_i$ *is bounded in* $L^{r_1}(0, T; Y)$, *and* $\theta(1 - 1/r_1) \leq (1 - \theta)/p_o$. *Then the sequence* $\{u_i\}_i$ *is relatively compact in the space* $L^p(0, T; B)$ *for all* $p < p_*$, *where* $1/p_* = (1 - \theta)/p_o - \theta(1 - 1/r_1)$.

To apply the above Lemma in the parabolic setting, we also have to use the following compactness result of Rellich–Kondrachov type in the metric space setting, which has ben shown in [12]:

Lemma 3.11. *Let* $(\mathscr{X}, \mathrm{d}, \mu)$ *be a metric measure space with a doubling measure* μ *and* $\mathrm{n} := \log_2 c_d$, *which supports a weak* $(1, p)$*-Poincaré inequality with dilation constant* $\Gamma \geq 1$. *Then the following holds true: Let* B *be a fixed ball and* $\{u_i, g_i\}$ *be a sequence of functions* u_i *with* p*-weak upper gradients* g_i *such that* $\|u_i\|_{L^1(B)} + \|g_i\|_{L^p(5\Gamma B)}$ *is uniformly bounded. Then there exists a subsequence of* $\{u_i\}_i$ *that converges strongly in* $L^q(B)$ *for each* $1 \leq q \leq \mathrm{n}p/(\mathrm{n} - p)$ *when* $p < \mathrm{n}$ *and for each* $q \geq 1$ *when* $p \geq \mathrm{n}$.

3.4. Higher integrability up to the boundary

To establish uniform global energy bounds for the sequence of parabolic quasi minimizers, a fundamental ingredient is the following global higher integrability result, which was proved in [6].

Lemma 3.12. *Let* $\Omega \subset \mathscr{X}$ *be a bounded open set such that* $\mathscr{X} \setminus \Omega$ *is uniformly* p*-thick. For fixed* $\mathscr{Q} \geq 1$, $p > \frac{2\mathrm{n}}{\mathrm{n}+2}$ *and given initial-boundary-data* $\eta \in \mathrm{Lip}(\overline{\Omega_T})$ *let* $u \in L^p(0, T; \mathscr{N}^{1,p}(\Omega))$ *be a parabolic* $\mathscr{Q}$*-minimizer of the* p*-energy, satisfying the conditions* (2.11). *Then there exists a constant* $\delta \equiv \delta(\mathrm{n}, p, \mathscr{Q}, \Omega) > 0$ *such that*

$$u \in L^{p+\delta}(0, T; \mathscr{N}^{1,p+\delta}(\Omega))$$

and furthermore

$$\iint_{\Omega_T} g_u^{p+\delta} \,\mathrm{d}\mu \,\mathrm{d}t \leq C,$$

for a constant C *which depends only on* n, p, $\mathscr{Q}$, Ω, δ, η *and* $\|g_u\|_{L^p(\Omega_T)}$.

Remark 3.13. The constant in the above Lemma depends also on $\mathscr{Q}$. However, a close look at the proof of the statement in [6] shows that it is stable with respect to the variation of $\mathscr{Q}$ in a compact interval $[1, \mathscr{Q}_o]$ and therefore can be also replaced by a constant which depends only on $\mathscr{Q}_o$. Since we are interested in stability estimates for $\mathscr{Q}$ varying in such a compact interval (in particular we do not consider the case $\mathscr{Q} \to \infty$), the dependence upon $\mathscr{Q}$ is not crucial for our purpose.

4. Proof of the stability theorem

We start our proof with a convergence result for p-weak upper gradients, which is well known in the elliptic setting (see for example [15, Lemma 3.1]). However, we could not find this result for the parabolic case and this is why we state this result as a first step.

Lemma 4.1. *Let $\{u_i\}_{i \in \mathbb{N}}$ be a sequence of functions on the parabolic domain Ω_T with $u_i \in L^p(\Omega_T)$ for all $i \in \mathbb{N}$. Let $\{g_{u_i}\}_{i \in \mathbb{N}}$ be a sequence such that $g_{u_i} \in L^p(\Omega_T)$ is a p-weak upper gradient of u_i for every $i \in \mathbb{N}$. Moreover let*

$$u_i \rightharpoonup u \quad \text{and} \quad g_{u_i} \rightharpoonup g \quad \text{both weakly in } L^p(\Omega_T).$$

Then $g \in L^p(\Omega_T)$ is a p-weak upper gradient of u.

Proof. We use (twice) Mazur's Lemma (see for example [3, Lemma 6.2]) on the Banach space $L^p(\Omega_T)$ as follows: There exist convex combinations

$$g_j := \sum_{i=j}^{N_j} a_{j,i} g_{u_i}, \quad \tilde{u}_j := \sum_{i=j}^{N_j} a_{j,i} u_i,$$

with $N_j \in \mathbb{N}$ for every j, $a_{j,i} \geq 0$ and $\sum_{i=j}^{N_j} a_{j,i} = 1$ such that

$$\tilde{u}_j \to u, \quad g_j \to g \quad \text{both strongly in } L^p(\Omega_T), \text{ as } j \to \infty,$$

and moreover by the basic calculus rules for p-weak upper gradients we have that g_j is a p-weak upper gradient of $\tilde{u}_j$ for every j. Passing to a subsequence we therefore get that

$$\tilde{u}_j(\cdot, t) \to u(\cdot, t), \quad g_j(\cdot, t) \to g(\cdot, t), \quad \text{strongly in } L^p(\Omega),$$

for almost all $t \in (0, T)$. We apply now [3, Proposition 2.3] to conclude that $g(\cdot, t)$ is a p-weak upper gradient of $u(\cdot, t)$ for almost all $t \in (0, T)$ and this implies (by definition of parabolic p-weak upper gradients) that g is a p-weak upper gradient of the function u. □

We next show that parabolic quasi-minimizers possess a time derivative in a weak sense.

Lemma 4.2. *Let $\Omega \subset \mathscr{X}$ be bounded and open, $p > 1$, $T > 0$ and $\mathscr{Q} \geq 1$. Let $u \in L^p(0, T; \mathscr{N}^{1,p}(\Omega))$ be a parabolic $\mathscr{Q}$-minimizer in the sense of* (2.10). *Then*

$$\partial_t u \in [L^p(0, T; \mathscr{N}_o^{1,p}(\Omega))]^*$$

and

$$|\langle \partial_t u, \varphi \rangle| \leq \frac{2^p \mathscr{Q}}{p} \|u\|_{L^p(0,T;\mathscr{N}^{1,p}(\Omega))}^{p-1} \|g_\varphi\|_{L^p(\Omega_T)},$$

for all $\varphi \in L^p(0, T; \mathscr{N}_o^{1,p}(\Omega))$.

PROOF. We define $v = \partial_t u$ as the continuous linear functional on $\mathrm{Lip}_c(\Omega_T)$ such that

$$v[\varphi] := \langle v, \varphi \rangle = -\iint_{\Omega_T} u \partial_t \varphi \, \mathrm{d}\mu \, \mathrm{d}t.$$

In order to extend this functional on the whole space $L^p(0, T; \mathcal{N}^{1,p}(\Omega))$ we first prove that

$$\left| \int_0^T \int_\Omega u \partial_t \varphi \, \mathrm{d}\mu \, \mathrm{d}t \right| \le c \|u\|_{L^p(0,T;\mathcal{N}^{1,p}(\Omega))}^{p-1}, \tag{4.1}$$

for all $\varphi \in \mathrm{Lip}_c(\Omega_T)$. We may assume that $\|u\|_{L^p(0,T;\mathcal{N}^{1,p}(\Omega))} \neq 0$ because otherwise there is nothing to prove. Let $\varphi \in \mathrm{Lip}_c(\Omega_T)$ be a test function satisfying $\|D\varphi\|_{L^p(\Omega_T)} = 1$ and set $\Phi := \|u\|_{L^p(0,T;\mathcal{N}^{1,p}(\Omega))} \varphi$. By the quasi minimizing property (2.10) of u we get

$$\begin{aligned}
&\|u\|_{L^p(0,T;\mathcal{N}^{1,p}(\Omega))} \int_0^T \int_\Omega u \partial_t \varphi \, \mathrm{d}\mu \, \mathrm{d}t \\
&= \int_0^T \int_\Omega u \partial_t \Phi \, \mathrm{d}\mu \, \mathrm{d}t \\
&\ge -\frac{\mathcal{Q}}{p} \iint_{\mathrm{spt}\,\varphi} g_{u-\Phi}^p \, \mathrm{d}\mu \, \mathrm{d}t \\
&\ge -\frac{2^{p-1}\mathcal{Q}}{p} \iint_{\mathrm{spt}\,\varphi} (g_u^p + g_\Phi^p) \, \mathrm{d}\mu \, \mathrm{d}t \\
&\ge -\frac{2^{p-1}\mathcal{Q}}{p} \iint_{\mathrm{spt}\,\varphi} (g_u^p + \|u\|_{L^p(0,T;\mathcal{N}^{1,p}(\Omega))}^p |D\varphi|^p) \, \mathrm{d}\mu \, \mathrm{d}t \\
&\ge -\frac{2^p \mathcal{Q}}{p} \|u\|_{L^p(0,T;\mathcal{N}^{1,p}(\Omega))}.
\end{aligned}$$

Here we have used that $g_\Phi^p \le \|u\|_{L^p(0,T;\mathcal{N}^{1,p}(\Omega))}^p g_\varphi^p$, moreover $g_\varphi = |D\varphi|$ since $\varphi \in \mathrm{Lip}(\Omega_T)$ and in the very last step that $\|D\varphi\|_{L^p(\Omega_T)} = 1$. Hence we obtain

$$\int_0^T \int_\Omega u \partial_t \varphi \, \mathrm{d}\mu \, \mathrm{d}t \ge -\frac{2^p \mathcal{Q}}{p} \|u\|_{L^p(0,T;\mathcal{N}^{1,p}(\Omega))}^{p-1}.$$

On the other hand, replacing Φ by $-\Phi$ in the above argument, we also get

$$\int_0^T \int_\Omega u \partial_t \varphi \, \mathrm{d}\mu \, \mathrm{d}t \le \frac{2^p \mathcal{Q}}{p} \|u\|_{L^p(0,T;\mathcal{N}^{1,p}(\Omega))}^{p-1},$$

and therefore we finally conclude (4.1) with a constant $c \equiv \frac{2^p \mathcal{Q}}{p}$. Since $\mathrm{Lip}_c(\Omega_T)$ is dense in the space $L^p(0,T;\mathcal{N}_o^{1,p}(\Omega))$ (see Thm. 5.45 in [3]), we can now extend the functional by (4.1) to a continuous linear functional on the space $L^p(0,T;\mathcal{N}_o^{1,p}(\Omega))$ and therefore we get that $v = \partial_t u \in [L^p(0,T;\mathcal{N}_o^{1,p}(\Omega))]^*$ together with the desired estimate. □

Remark 4.3. We may repeat the argument above with the mollified function $[u]_\varepsilon$ for $\varepsilon > 0$ to conclude the following: Being $u \in L^p(0,T;\mathcal{N}^{1,p}(\Omega))$ the parabolic $\mathcal{Q}$-minimizer of Lemma 4.2, there exists $\varepsilon \equiv \varepsilon(p, \|u\|_{L^p(0,T;\mathcal{N}^{1,p}(\Omega))})$ such that

$$|\langle \partial_t [u]_\varepsilon, \varphi\rangle| \le \frac{3^p \mathcal{Q}}{p} \|u\|^{p-1}_{L^p(0,T;\mathcal{N}^{1,p}(\Omega))} \|g_\varphi\|_{L^p(\Omega_T)},$$

for all $\varphi \in L^p(0,T;\mathcal{N}_o^{1,p}(\Omega))$. To see this we argue as follows: We may again assume that $\|u\|_{L^p(0,T;\mathcal{N}^{1,p}(\Omega))} > 0$, because otherwise by definition of the mollification there would be again nothing to prove. Since u is a parabolic $\mathcal{Q}$-minimizer, the mollified function $[u]_\varepsilon$ fulfills the inequality (3.1). Using this instead of (2.10) and testing with $\Phi := \|u\|_{L^p(0,T;\mathcal{N}^{1,p}(\Omega))}\varphi$, where $\varphi \in \mathrm{Lip}_c(\Omega_T)$ such that $\|D\varphi\|_{L^p(\Omega_T)} = 1$, we get with a similar argument

$$\begin{aligned}
&\|[u]_\varepsilon\|_{L^p(0,T;\mathcal{N}^{1,p}(\Omega))} \int_0^T \int_\Omega [u]_\varepsilon \partial_t \varphi \, \mathrm{d}\mu \, \mathrm{d}t \\
&\quad \ge -\frac{2^{p-1}\mathcal{Q}}{p} (\|u\|^p_{L^p(0,T;\mathcal{N}^{1,p}(\Omega))} + \|[u]_\varepsilon\|^p_{L^p(0,T;\mathcal{N}^{1,p}(\Omega))}).
\end{aligned}$$

Repeating the same argument with $-\varphi$ instead of φ we therefore obtain

$$\left|\int_0^T \int_\Omega [u]_\varepsilon \partial_t \varphi \, \mathrm{d}\mu \, \mathrm{d}t\right| \le \frac{2^{p-1}\mathcal{Q}}{p} \left(\|u\|^{p-1}_{L^p(0,T;\mathcal{N}^{1,p}(\Omega))} + \frac{\|[u]_\varepsilon\|^p_{L^p(0,T;\mathcal{N}^{1,p}(\Omega))}}{\|u\|_{L^p(0,T;\mathcal{N}^{1,p}(\Omega))}} \right).$$

By the definition of the $L^p - \mathcal{N}^{1,p}$-norm we have

$$\begin{aligned}
&|\|u\|_{L^p(0,T;\mathcal{N}^{1,p}(\Omega))} - \|[u]_\varepsilon\|_{L^p(0,T;\mathcal{N}^{1,p}(\Omega))}| \\
&\quad \le \|u - [u]_\varepsilon\|_{L^p(\Omega_T)} + \|g_{u-[u]_\varepsilon}\|_{L^p(\Omega_T)}.
\end{aligned}$$

Choosing now $\varepsilon > 0$ small enough, by Lemma 3.6 we may achieve that

$$\|u - [u]_\varepsilon\|_{L^p(\Omega_T)} + \|g_{u-[u]_\varepsilon}\|_{L^p(\Omega_T)} \le \frac{1}{2} \|u\|_{L^p(0,T;\mathcal{N}^{1,p}(\Omega))},$$

and therefore

$$\frac{1}{2}\|u\|_{L^p(0,T;\mathcal{N}^{1,p}(\Omega))} \le \|[u]_\varepsilon\|_{L^p(0,T;\mathcal{N}^{1,p}(\Omega))} \le \frac{3}{2}\|u\|_{L^p(0,T;\mathcal{N}^{1,p}(\Omega))}. \tag{4.2}$$

Hence $\varepsilon \equiv \varepsilon(p, \|u\|_{L^p(0,T;\mathscr{N}^{1,p}(\Omega))})$. Plugging this into the previous expression on the right hand side we then obtain the desired estimate.

LEMMA 4.4. *Let $\xi \in \mathrm{Lip}_c(0,T;L^\infty(\Omega))$. Assume that $u \in L^p(0,T;\mathscr{N}^{1,p}(\Omega))$ and $\partial_t u \in [L^p(0,T;\mathscr{N}_o^{1,p}(\Omega))]^*$ with $p > \frac{2n}{n+2}$. Then $\partial_t(\xi u) \in [L^p(0,T;\mathscr{N}_o^{1,p}(\Omega))]^*$ and there exists a constant $c > 0$ such that*

$$
\begin{aligned}
(4.3)\qquad |\langle \partial_t(\xi u), \varphi\rangle| \le{}& \|\xi\|_{L^\infty(\Omega_T)} \|\partial_t u\|_{[L^p(0,T;\mathscr{N}_o^{1,p}(\Omega))]^*} \|\varphi\|_{L^p(0,T;\mathscr{N}^{1,p}(\Omega))} \\
&+ c\|\partial_t\xi\|_{L^\infty(\Omega_T)} \|u\|_{L^p(0,T;\mathscr{N}^{1,p}(\Omega))} \|\varphi\|_{L^p(0,T;\mathscr{N}^{1,p}(\Omega))}
\end{aligned}
$$

for every function $\varphi \in L^p(0,T;\mathscr{N}_o^{1,p}(\Omega))$.

PROOF. Since $\Omega_T \subset \mathscr{X} \times \mathbb{R}$ is bounded we have that $\varphi\xi \in \mathrm{Lip}_c(\Omega_T)$ for every $\varphi \in \mathrm{Lip}_c(\Omega_T)$. Then we have

$$
\langle \partial_t u, \xi\varphi\rangle = -\iint_{\Omega_T} u \cdot \partial_t(\xi\varphi)\, d\mu\, dt = -\iint_{\Omega_T} (u\varphi\partial_t\xi + \xi u \partial_t\varphi)\, d\mu\, dt.
$$

Since the Sobolev embedding theorem and $p > \frac{2n}{n+2}$ shows that $L^p(0,T;\mathscr{N}_o^{1,p}(\Omega)) \subset L^2(\Omega_T)$ is continuous, by the Hölder inequality we see that

$$
\begin{aligned}
\left| \iint_{\Omega_T} u\varphi\partial_t\xi\, d\mu\, dt \right| &\le \|\partial_t\xi\|_{L^\infty(\Omega_T)} \|u\|_{L^2(\Omega_T)} \|\varphi\|_{L^2(\Omega_T)} \\
&\le c\|\partial_t\xi\|_{L^\infty(\Omega_T)} \|u\|_{L^p(0,T;\mathscr{N}^{1,p}(\Omega))} \|\varphi\|_{L^p(0,T;\mathscr{N}^{1,p}(\Omega))},
\end{aligned}
$$

where $c > 0$ is a constant independent of u and φ. On the other hand, we have

$$
|\langle \partial_t u, \xi\varphi\rangle| \le \|\xi\|_{L^\infty(\Omega_T)} \|\partial_t u\|_{[L^p(0,T;\mathscr{N}_o^{1,p}(\Omega))]^*} \|\varphi\|_{L^p(0,T;\mathscr{N}^{1,p}(\Omega))},
$$

therefore we obtain

$$
\begin{aligned}
\left| \iint_{\Omega_T} \xi u \partial_t\varphi\, d\mu\, dt \right| \le{}& \|\xi\|_{L^\infty(\Omega_T)} \|\partial_t u\|_{[L^p(0,T;\mathscr{N}_o^{1,p}(\Omega))]^*} \|\varphi\|_{L^p(0,T;\mathscr{N}^{1,p}(\Omega))} \\
&+ c\|\partial_t\xi\|_{L^\infty(\Omega_T)} \|u\|_{L^p(0,T;\mathscr{N}^{1,p}(\Omega))} \|\varphi\|_{L^p(0,T;\mathscr{N}^{1,p}(\Omega))}
\end{aligned}
$$

for all $\varphi \in \mathrm{Lip}_c(\Omega_T)$. This implies that (4.3) holds for all $\varphi \in \mathrm{Lip}_c(\Omega_T)$. Then we approximate $\varphi \in L^p(0,T;\mathscr{N}^{1,p}(\Omega))$ by a sequence of $\{\varphi_n\} \subset \mathrm{Lip}_c(\Omega_T)$ and obtain (4.3) actually holds for all $\varphi \in L^p(0,T;\mathscr{N}^{1,p}(\Omega))$. This completes the proof of Lemma 4.4. □

In the next step we prove a global Caccioppoli type inequality for quasi minimizers.

LEMMA 4.5. *For any $\bar\delta > 0$ there exists a constant $c \equiv c(\bar\delta)$ such that for all $i \in \mathbb{N}$ there holds*

$$\sup_{t\in(0,T)}\int_\Omega |u_i(\cdot,t)-\eta(\cdot,t)|^2\,\mathrm{d}\mu + \iint_{\Omega_T} g_{u_i}^{p_i}\,\mathrm{d}\mu\,\mathrm{d}t$$

$$\leq c\iint_{\Omega_T}|\partial_t\eta|^{\frac{p_i}{p_i-1}}\,\mathrm{d}\mu\,\mathrm{d}t + \overline{\mathcal{Q}}\iint_{\Omega_T} g_\eta^{p_i}\,\mathrm{d}\mu\,\mathrm{d}t + \bar\delta\iint_{\Omega_T}|u_i-\eta|^{p_i}\,\mathrm{d}\mu\,\mathrm{d}t.$$

Here $\overline{\mathcal{Q}}$ denotes the upper bound of $\{\mathcal{Q}_i\}$.

Proof. The proof is similar to the one in the Euclidean case, see [6, Lemma 4.1]. Therefore we are focussing only on the differences to the Euclidean case. We test the quasi minimality with the function

$$\Phi_\varepsilon^h(x,t) := \chi_s^{h,h}(t)([u_i]_\varepsilon - [\eta]_\varepsilon) \in \mathrm{Lip}_c(0,T;\mathcal{N}_o^{1,p_i}(\Omega)),$$

for $0<s<T$ fixed and $h \ll 1$, where we denote for this proof and and also for later proofs the piecewise affine function in time

$$\chi_s^{h,k}(t) := \begin{cases} 0 & 0\leq t\leq h,\\ \frac{t-h}{h} & h\leq t\leq 2h,\\ 1 & 2h\leq t\leq s-k,\\ \frac{s-t}{k} & s-k\leq t\leq s,\\ 0 & s\leq t\leq T.\end{cases} \tag{4.4}$$

(3.1) then gives

$$-\iint_{\Omega_T}[u_i]_\varepsilon\partial_t\Phi_\varepsilon^h\,\mathrm{d}\mu\,\mathrm{d}t + \frac{1}{p_i}\iint_{\mathrm{spt}[\Phi_\varepsilon^h]_\varepsilon} g_{u_i}^{p_i}\,\mathrm{d}\mu\,\mathrm{d}t \leq \frac{\mathcal{Q}_i}{p_i}\iint_{\mathrm{spt}[\Phi_\varepsilon^h]_\varepsilon} g_{u_i-[\Phi_\varepsilon^h]_\varepsilon}^{p_i}\,\mathrm{d}\mu\,\mathrm{d}t.$$

The first term on the left hand side is estimated exactly as in the Euclidean case and we obtain for every $\bar\delta>0$

$$\lim_{\varepsilon,h\to 0}\iint_{\Omega_T}[u_i]_\varepsilon\partial_t\Phi_\varepsilon^h\,\mathrm{d}\mu\,\mathrm{d}t \geq \int_\Omega |u_i(\cdot,s)-\eta(\cdot,s)|^2\,\mathrm{d}\mu$$

$$-\bar\delta\iint_{\Omega_T}|u_i-\eta|^{p_i}\,\mathrm{d}\mu\,\mathrm{d}t - c\iint_\Omega|\partial_t\eta|^{\frac{p_i}{p_i-1}}\,\mathrm{d}\mu\,\mathrm{d}t.$$

To estimate the term on the right hand side we have to use properties of upper gradients to get in a first step

$$g_{u_i-[\Phi_\varepsilon^h]_\varepsilon} \leq g_{u_i-[[u_i]_\varepsilon]_\varepsilon} + g_{[[u_i]_\varepsilon]_\varepsilon-[\Phi_\varepsilon^h]_\varepsilon}.$$

The first term on the right hand side goes to zero as $\varepsilon\to 0$, since $g_{u_i-[[u_i]_\varepsilon]_\varepsilon}\to 0$ in L^p. To treat the second term we note that $[[u_i]_\varepsilon]_\varepsilon - [\Phi_\varepsilon^h]_\varepsilon = [[\eta]_\varepsilon]_\varepsilon$ on $[2h+\varepsilon, s-2h-\varepsilon]$ and therefore we get

$$\iint_{\operatorname{spt}[\Phi_\varepsilon^h]_\varepsilon} g^{p_i}_{[[u_i]_\varepsilon]_\varepsilon - [\Phi_\varepsilon^h]_\varepsilon} \, d\mu \, dt \le \iint_{\operatorname{spt}[\Phi_\varepsilon^h]_\varepsilon \cap [2h+\varepsilon, s-2h-\varepsilon]} g^{p_i}_{[[\eta]_\varepsilon]_\varepsilon} \, d\mu \, dt$$
$$+ \iint_{\operatorname{spt}[\Phi_\varepsilon^h]_\varepsilon \cap ([h-\varepsilon, 2h+\varepsilon] \cup [s-2h-\varepsilon, s-h+\varepsilon])} g^{p_i}_{[[u_i]_\varepsilon]_\varepsilon - [\Phi_\varepsilon^h]_\varepsilon} \, d\mu \, dt$$

The second term on the right hand side converges to 0 as $\varepsilon \to 0$ and $h \to 0$ (by dominated convergence), whereas for the first term holds

$$\lim_{\varepsilon, h \to 0} \iint_{\operatorname{spt}[\Phi_\varepsilon^h]_\varepsilon \cap [2h+\varepsilon, s-2h-\varepsilon]} g^{p_i}_{[[\eta]_\varepsilon]_\varepsilon} \, d\mu \, dt \le \iint_{\operatorname{spt}(u_i - \eta)} g^{p_i}_\eta \, d\mu \, dt,$$

and hence we get

$$\lim_{\varepsilon, h \to 0} \iint_{\operatorname{spt}[\Phi_\varepsilon^h]_\varepsilon} g^{p_i}_{u_i - [\Phi_\varepsilon^h]_\varepsilon} \, d\mu \, dt \le \iint_{\Omega_T} g^{p_i}_\eta \, d\mu \, dt.$$

Now combining these estimates, the proof follows exactly as in the Euclidean case. □

Direct consequences of the Caccioppoli type estimate and the higher integrability properties of Lemma 3.12 are:

COROLLARY 4.6. *For the sequence of Theorem* 2.2 *there holds*

$$\sup_{i \in \mathbb{N}} \left[\sup_{t \in (0,T)} \|u_i(\cdot, t)\|^2_{L^2(\Omega)} + \|u_i\|^{p_i}_{L^{p_i}(\Omega_T)} + \|g_{u_i}\|^{p_i}_{L^{p_i}(\Omega_T)} \right] < \infty,$$

and

$$\|g_{u_i}\|^{p-q}_{L^{p-q}(\Omega_T)} \le c[\|u_i\|^{p-q}_{L^{p_i}(\Omega_T)} + \|\partial_t \eta\|^{\frac{p-q}{p_i - 1}}_{L^{p_i/(p_i-1)}(\Omega_T)} + \|\eta\|^{p-q}_{L^{p_i}(0,T;\mathcal{N}^{1,p_i}(\Omega))}],$$

for every $q \in (0, p-1)$ *and with a constant* $c \equiv c(\mathrm{n}, \sup_i p_i, \Omega_T)$, *and moreover, there exists a constant* $\delta > 0$ *such that*

$$M := \sup_{i \in \mathbb{N}} (\|u_i\|^{p+\delta}_{L^{p+\delta}(\Omega_T)} + \|g_{u_i}\|^{p+\delta}_{L^{p+\delta}(\Omega_T)}) < \infty. \tag{4.5}$$

PROOF. The estimates are direct consequences of the Caccioppoli type inequality in Lemma 4.5, higher integrability in terms of Lemma 3.12 and Hölder's inequality. □

We will now conclude suitable convergence properties of the sequence $\{u_i\}_i$.

LEMMA 4.7. *For the sequence* $\{u_i\}_i$ *of Theorem* 2.2 *there exists a subsequence, which we denote again by* $\{u_i\}_i$ *and a weak upper gradient* g *of* u *such that*

$$u_i \to u \text{ in } L^{p+\delta}(\Omega_T) \cap L^2(\Omega_T), \quad g_{u_i} \rightharpoonup g \text{ weakly in } L^{p+\delta}(\Omega_T),$$

as $i \to \infty$. Moreover we have that $\partial_t u \in [L^{p+\delta}(0,T;\mathcal{N}_o^{1,p+\delta}(\Omega))]^*$ and

$$\partial_t u_i \overset{*}{\rightharpoonup} \partial_t u \quad \text{in the weak-}{*}\text{ topology on } [L^{p+\delta}(0,T;\mathcal{N}_o^{1,p+\delta}(\Omega))]^*.$$

Remark 4.8. By lower semi continuity we then directly obtain for the minimal weak upper gradient of u:

$$\|g_u\|_{L^q(\Omega_T)} \le \liminf_{i\to\infty} \|g_{u_i}\|_{L^q(\Omega_T)},$$

for all $q < p + \delta$.

Proof. First we note that by Lemma 4.2 and the uniform bound (4.5) we obtain that

$$\sup_{i\in\mathbb{N}} \|\partial_t u_i\|_{[L^{p+\delta}(0,T;\mathcal{N}_o^{1,p+\delta}(\Omega))]^*} < \infty. \tag{4.6}$$

Next we will apply Lemma 3.10 for the both cases $p \ge 2$ and $\frac{2n}{n+2} < p < 2$ as follows: In the case $p \ge 2$ we choose

$$X = \mathcal{N}^{1,p+\delta}(\Omega), \quad Y = [\mathcal{N}_o^{1,p+\delta}(\Omega)]^*, \quad B = L^{p+\delta}(\Omega).$$

The inclusion $\mathcal{N}^{1,p+\delta}(\Omega) \subset L^{p+\delta}(\Omega)$ is compact by the Rellich–Kondrachov theorem in the metric version in terms of Lemma 3.11. The second inclusion $L^{p+\delta}(\Omega) \subset [\mathcal{N}_o^{1,p+\delta}(\Omega)]^*$ is continuous, as one can see by Hölder's inequality (note that $p+\delta > 2$). By [25, Theorem 4.1] and [25, Corollary 3.1] (3.2) holds for this choice of spaces. In the case $\frac{2n}{n+2} < p < 2$ we apply Lemma 3.10 with the spaces

$$X = \mathcal{N}^{1,p}(\Omega), \quad Y = [\mathcal{N}_o^{1,p}(\Omega)]^*, \quad B = L^2(\Omega).$$

The inclusion $\mathcal{N}^{1,p}(\Omega) \subset L^2(\Omega_T)$ is compact again by the Rellich–Kondrachov theorem in terms of Lemma 3.11. The second inclusion $L^2(\Omega_T) \subset [\mathcal{N}_o^{1,p}(\Omega)]^*$ is continuous with the following argument: Setting

$$u[\varphi] := \int_\Omega u \cdot \varphi \, d\mu,$$

and using Hölder's inequality and the Sobolev embedding we identify every $u \in L^2(\Omega)$ as an element of $[\mathcal{N}^{1,p}(\Omega)]^*$: For every $\varphi \in \mathcal{N}_o^{1,p}(\Omega)$ we have

$$|u[\varphi]| \le \|u\|_{L^2(\Omega)} \|\varphi\|_{L^2(\Omega)} \le c\|u\|_{L^2(\Omega)} \cdot \|\varphi\|_{\mathcal{N}^{1,p}(\Omega)}.$$

Now we use once again [25, Theorem 4.1] together with [25, Corollary 3.1] to conclude that (3.2) holds also for this choice of spaces.

All together we conclude that in any case $p > \frac{2n}{n+2}$ for a subsequence there holds

$$\begin{cases} u_i \to u & \text{in } L^{p+\delta}(\Omega_T) \cap L^2(\Omega_T), \\ g_{u_i} \rightharpoonup g & \text{weakly in } L^{p+\delta}(\Omega_T). \end{cases}$$

By Lemma 4.1 g is a p-weak upper gradient of u. In order to obtain the weak-$*$ convergence of the time derivatives, we first see that by (4.6), we have for a subsequence that

$$\partial_t u_i \overset{*}{\rightharpoonup} v \text{ in the weak-}* \text{ topology on } [L^{p+\delta}(0,T;\mathcal{N}_o^{1,p+\delta}(\Omega))]^*,$$

for a function $v \in [L^{p+\delta}(0,T;\mathcal{N}_o^{1,p+\delta}(\Omega))]^*$ and it remains to show that $v = \partial_t u$: By the strong convergence $u_i \to u$ in $L^p(\Omega_T)$ we see that for every function $\phi \in \mathrm{Lip}_c(\Omega_T)$ there holds

$$\langle \partial_t u_i, \phi \rangle = -\iint_{\Omega_T} u_i \partial_t \phi \, d\mu \, dt \to -\iint_{\Omega_T} u \partial_t \phi \, d\mu \, dt,$$

and on the other hand by the weak-$*$ convergence $\partial_t u_i \to v$ we get

$$\langle \partial_t u_i, \phi \rangle \to \langle v, \phi \rangle.$$

Hence we deduce

$$\langle v, \phi \rangle = \langle \partial_t u, \phi \rangle,$$

and moreover by the Poincaré inequality

$$\begin{aligned} |\langle \partial_t u, \phi \rangle| &\le \|v\|_{[L^{p+\delta}(0,T;\mathcal{N}_o^{1,p+\delta}(\Omega))]^*} \|\phi\|_{L^{p+\delta}(0,T;\mathcal{N}^{1,p+\delta}(\Omega_T))} \\ &= \|v\|_{[L^{p+\delta}(0,T;\mathcal{N}_o^{1,p+\delta}(\Omega))]^*} \left(\|\phi\|_{L^{p+\delta}(\Omega_T)} + \|g_\phi\|_{L^{p+\delta}(\Omega_T)} \right) \\ &\le c \|v\|_{[L^{p+\delta}(0,T;\mathcal{N}_o^{1,p+\delta}(\Omega))]^*} \|g_\phi\|_{L^{p+\delta}(\Omega_T)}. \end{aligned}$$

This shows that $\partial_t u \in [L^{p+\delta}(0,T;\mathcal{N}_o^{1,p+\delta}(\Omega))]^*$ and $v = \partial_t u$. □

PROOF (OF THEOREM 2.2). We prove the theorem in two steps: First we prove that the limit function u satisfies the initial-boundary conditions (2.11), and thereafter we show that u is in fact a $\mathcal{Q}$ minimizer of the p energy. To prove that $u(\cdot,t) - \eta(\cdot,t) \in \mathcal{N}_o^{1,p}(\Omega)$ for almost every $t \in (0,T)$ we first show that there exists $\varepsilon_o > 0$ such that

$$u(\cdot,t) - \eta(\cdot,t) \in \mathcal{N}_o^{1,p-\varepsilon}(\Omega),$$

for almost all $t \in (0,T)$ and for every $\varepsilon \in (0,\varepsilon_o)$. To this aim, we use the Hardy type characterization of Newtonian functions with zero boundary values in terms of Lemmas 3.8 and 3.9 as follows: For a given $\varepsilon > 0$ small enough, which will be fixed later, there exists $I \equiv I(\varepsilon) \in \mathbb{N}$ such that $p - \varepsilon < p_i < p + \varepsilon$ for all $i \ge I(\varepsilon)$.

In particular there holds $u_i(\cdot, t) - \eta(\cdot, t) \in \mathcal{N}_o^{1,p-\varepsilon}(\Omega)$ for almost all $t \in (0, T)$ and all $i \geq I(\varepsilon)$. We therefore get by the pointwise almost everywhere convergence $u_i(x, t) \to u(x, t)$, Fatou's Lemma, Lemma 3.8, Corollary 4.6 and Hölder's inequality

$$\begin{aligned}
&\iint_{\Omega_T} \Big(\frac{|u(x,t) - \eta(x,t)|}{\operatorname{dist}(x, \mathscr{X}\backslash\Omega)}\Big)^{p-\varepsilon} \, \mathrm{d}\mu \, \mathrm{d}t \\
&\quad \leq \liminf_{i\to\infty} \iint_{\Omega_T} \Big(\frac{|u_i(x,t) - \eta(x,t)|}{\operatorname{dist}(x, \mathscr{X}\backslash\Omega)}\Big)^{p-\varepsilon} \, \mathrm{d}\mu \, \mathrm{d}t \\
&\quad \leq c \sup_{i\in\mathbb{N}} \|u_i - \eta\|^{p-\varepsilon}_{L^{p-\varepsilon}(0,T,\mathcal{N}^{1,p-\varepsilon}(\Omega))} \\
&\quad \leq c \sup_{i\in\mathbb{N}} [\|u_i\|^{p-\varepsilon}_{L^{p-\varepsilon}(\Omega_T)} + \|g_{u_i}\|^{p-\varepsilon}_{L^{p-\varepsilon}(\Omega_T)} + \|\eta\|^{p-\varepsilon}_{L^{p-\varepsilon}(0,T;\mathcal{N}^{1,p-\varepsilon}(\Omega))}] \\
&\quad \leq c \sup_{i\in\mathbb{N}} [1 + \|u_i\|^{p-\varepsilon}_{L^{p+\varepsilon}(\Omega_T)} + \|\eta\|^{p-\varepsilon}_{L^{p+\varepsilon}(0,T;\mathcal{N}^{1,p+\varepsilon}(\Omega))} + \|\partial_t \eta\|^{p-\varepsilon}_{L^{\frac{p-\varepsilon}{p-\varepsilon-1}}(\Omega_T)}].
\end{aligned}$$

By the uniform energy bound (4.5) the right hand side of the preceding estimate is bounded independently of the index i. This estimate implies that

$$\int_\Omega \Big(\frac{|u(x,t) - \eta(x,t)|}{\operatorname{dist}(x, \mathscr{X}\backslash\Omega)}\Big)^{p-\varepsilon} \, \mathrm{d}\mu < \infty,$$

for almost every $t \in (0, T)$. Hence, by Lemma 3.9 we conclude that $u(\cdot, t) - \eta(\cdot, t) \in \mathcal{N}_o^{1,p-\varepsilon}(\Omega)$ for almost every t. Moreover the above argument holds uniformly for every $\varepsilon \in (0, \varepsilon_o)$ so that we have $u(\cdot, t) - \eta(\cdot, t) \in \mathcal{N}_o^{1,p-\varepsilon}(\Omega)$ for every $\varepsilon \in (0, \varepsilon_o)$. By Lemma 3.7 we finally conclude that $u(\cdot, t) - \eta(\cdot, t) \in \mathcal{N}_o^{1,p}(\Omega)$ for almost every $t \in (0, T)$, which is the statement $(2.11)_1$.

To prove $(2.11)_2$, we test (3.1) with the test function $\Phi_\varepsilon^h := \chi_\tau^{h,k}([u_i]_\varepsilon - [\eta]_\varepsilon)$, where $\chi_\tau^{h,k}$ denotes the piecewise affine function in the time variable, which we have defined in (4.4). An argument similar to the one in the proof of Lemma 4.5 together with the strong convergence $u_i \to u$ in $L^{p+\delta}(\Omega_T)$ and in $L^2(\Omega_T)$ directly gives us, letting first $h \to 0$ and using that u_i attains the initial data η, and letting then $i \to \infty$:

$$\frac{1}{k}\int_{\tau-k}^{\tau}\int_\Omega |u - \eta|^2 \, \mathrm{d}\mu \, \mathrm{d}t \leq c \sup_{i\in\mathbb{N}} \int_0^\tau \int_\Omega (g_{u_i}^{p_i} + |u_i - \eta|^{p_i} + |\partial_t \eta|^{\frac{p_i}{p_i-1}} + g_\eta^{p_i}) \, \mathrm{d}\mu \, \mathrm{d}t.$$

Since the term on the left-hand side of the preceding inequality tends to

$$\int_\Omega |u(x,\tau) - \eta(x,\tau)|^2 \, \mathrm{d}\mu,$$

as $k \to 0$, we conclude the assertion $(2.11)_2$ by the global higher integrability (4.5), since it implies that the right-hand side of the preceding inequality tends to zero as $\tau \to 0$.

Next, we show that the limit function u is a parabolic quasi minimizer of the p energy, which means that it satisfies (2.10). For this aim, let $\Phi \in \mathrm{Lip}_c(\Omega_T)$ be an arbitrary Lipschitz function and set $K := \operatorname{spt}\Phi$. Since $K \Subset \Omega_T = \Omega \times (0, T)$ we have that $\bar{\delta} := \mathrm{d}_{\mathrm{par}}(K, \mathrm{d}_p\Omega_T) > 0$. We consider for $s < \bar{\delta}$ the open sets

$$K^s := \{z \in \Omega_T : \mathrm{d}_{\mathrm{par}}(z, K) < s\} \subset \Omega_T.$$

For fixed $\alpha > 0$ we can now find $s \equiv s(\alpha) > 0$ so small that

$$\iint_{K^{2s}\setminus K} g_u^{p+\delta}\,\mathrm{d}\mu\,\mathrm{d}t < \alpha, \tag{4.7}$$

where $\delta > 0$ denotes the higher integrability exponent from Lemma 3.12. Moreover we can assume that $s < \bar{\delta}/4$ to get that

$$K \Subset K^s \Subset K^{2s} \Subset \Omega_T.$$

Now let $\xi \in \mathrm{Lip}_c(\Omega_T)$ be a Lipschitz cutoff function on Ω_T with the property that $0 \le \xi \le 1$, $\xi \equiv 1$ on $K^{s/2}$, $\xi \equiv 0$ on $\Omega_T \setminus K^{3s/4}$ with Lipschitz constant $L \equiv L(s) < \infty$. Then the function $t \mapsto \xi(x, t)$ is differentiable for almost every $t \in (0, T)$ and every fixed $x \in \Omega$ and $|\partial_t \xi| \le L$. Moreover there holds $g_\xi \le L$. We define the function

$$\Phi_{i,\varepsilon} := \Phi + \xi([u_i]_\varepsilon - [u]_\varepsilon),$$

and choose the constant $\varepsilon > 0$ small enough to have $\Phi_{i,\varepsilon} \in \mathrm{Lip}_c(\Omega_T)$ with $\operatorname{spt}\Phi_{i,\varepsilon} \Subset K^{3s/4}$ and $\operatorname{spt}[\Phi_{i,\varepsilon}]_\varepsilon \Subset K^s$. Then $\Phi_{i,\varepsilon}$ is an admissible function for the formulation (3.1) which gives

$$\begin{aligned} -\iint_{\Omega_T} [u_i]_\varepsilon \partial_t(\Phi_{i,\varepsilon})\,\mathrm{d}\mu\,\mathrm{d}t + \frac{1}{p_i}\iint_{\operatorname{spt}[\Phi_{i,\varepsilon}]_\varepsilon} g_{u_i}^{p_i}\,\mathrm{d}\mu\,\mathrm{d}t \\ \le \frac{\mathcal{Q}_i}{p_i}\iint_{\operatorname{spt}[\Phi_{i,\varepsilon}]_\varepsilon} g_{u_i - [\Phi_{i,\varepsilon}]_\varepsilon}^{p_i}\,\mathrm{d}\mu\,\mathrm{d}t. \end{aligned} \tag{4.8}$$

For fixed $\beta > 0$ we find $i_o \in \mathbb{N}$ such that $p - \beta \le p_i$ for all $i \ge i_o$ and hence by Hölder's inequality and the monotonicity of the product measure $\mu \times \mathcal{L}^1$ on $\mathcal{X} \times \mathbb{R}$ there holds

$$\iint_K g_{u_i}^{p-\beta}\,\mathrm{d}\mu\,\mathrm{d}t \le \left[\iint_K g_{u_i}^{p_i}\,\mathrm{d}\mu\,\mathrm{d}t\right]^{\frac{p-\beta}{p_i}} [(\mu \times \mathcal{L}^1)(\Omega_T)]^{1-\frac{p-\beta}{p_i}}.$$

Now, by the lower semicontinuity, Remark 4.8 and since $\beta > 0$ is arbitrary, we obtain, using also that $K \subset \operatorname{spt}[\Phi_{i,\varepsilon}]_\varepsilon$ for every $i \in \mathbb{N}$, that

$$\iint_K g_u^p\,\mathrm{d}\mu\,\mathrm{d}t \le \liminf_{i\to\infty} \iint_{\operatorname{spt}[\Phi_{i,\varepsilon}]_\varepsilon} g_{u_i}^{p_i}\,\mathrm{d}\mu\,\mathrm{d}t. \tag{4.9}$$

Now we have a look at the first integral on the left hand side of (4.8). We write in a first step

$$\iint_{\Omega_T} [u_i]_\varepsilon \partial_t(\Phi_{i,\varepsilon})\,\mathrm{d}\mu\,\mathrm{d}t = \iint_{\Omega_T} [u_i]_\varepsilon \partial_t(\Phi_{i,\varepsilon} - \Phi)\,\mathrm{d}\mu\,\mathrm{d}t + \iint_{\Omega_T} [u_i]_\varepsilon \partial_t \Phi\,\mathrm{d}\mu\,\mathrm{d}t =: I_1 + I_2,$$

with the obvious labeling of I_1 and I_2. By the strong convergenve $[u_i]_\varepsilon \to u_i$ in $L^2(\Omega_T)$ as $\varepsilon \to 0$ and $u_i \to u$ in $L^2(\Omega_T)$ as $i \to \infty$ and since $\Phi \in \mathrm{Lip}_c(\Omega_T)$, we immediately deduce

$$I_2 \to \iint_{\Omega_T} u \partial_t \Phi\,\mathrm{d}\mu\,\mathrm{d}t$$

as $\varepsilon \to 0$, $i \to \infty$. For the integral I_1 we write

$$I_1 = \iint_{\Omega_T} [u_i - u]_\varepsilon \partial_t(\xi [u_i - u]_\varepsilon)\,\mathrm{d}\mu\,\mathrm{d}t + \iint_{\Omega_T} [u]_\varepsilon \partial_t(\xi [u_i - u]_\varepsilon)\,\mathrm{d}\mu\,\mathrm{d}t = I_{11} + I_{12}.$$

For the integral I_{11} we get by integration by parts that

$$\begin{aligned} I_{11} &= \iint_{\Omega_T} \partial_t \xi |[u_i - u]_\varepsilon|^2\,\mathrm{d}\mu\,\mathrm{d}t + \iint_{\Omega_T} \xi [u_i - u]_\varepsilon \partial_t [u_i - u]_\varepsilon\,\mathrm{d}\mu\,\mathrm{d}t \\ &= \frac{1}{2} \iint_{\Omega_T} \partial_t \xi |[u_i - u]_\varepsilon|^2\,\mathrm{d}\mu\,\mathrm{d}t, \end{aligned}$$

and the last integral converges to 0 as $\varepsilon \to 0$, $i \to \infty$ by the strong convergence $[u_i - u]_\varepsilon \to u_i - u$, $u_i \to u$ in $L^2(\Omega_T)$ and the fact that $|\partial_t \xi| \le L$ on Ω_T. For the integral I_{12} we write

$$\begin{aligned} I_{12} &= \iint_{\Omega_T} \partial_t \xi [u]_\varepsilon [u_i - u]_\varepsilon\,\mathrm{d}\mu\,\mathrm{d}t + \iint_{\Omega_T} \xi([u]_\varepsilon - u) \partial_t [u_i - u]_\varepsilon\,\mathrm{d}\mu\,\mathrm{d}t \\ &\quad + \iint_{\Omega_T} \xi u \partial_t [u_i - u]_\varepsilon\,\mathrm{d}\mu\,\mathrm{d}t. \end{aligned}$$

The first integral tends to zero as $\varepsilon \to 0$, $i \to \infty$ since $|\partial_t \xi| \le L$ on Ω_T and $[u_i - u]_\varepsilon \to 0$ in $L^2(\Omega_T)$.

In order to estimate the second integral, we argue as follows: We first show

$$\begin{aligned} &|\langle \partial_t [u]_\varepsilon, \varphi \rangle| \le c(u) \|\varphi\|_{L^p(0,T;\mathcal{N}^{1,p}(\Omega))}, \\ &|\langle \partial_t [u_i]_\varepsilon, \varphi \rangle| \le c(u_i) \|\varphi\|_{L^{p_i}(0,T;\mathcal{N}^{1,p_i}(\Omega))}, \end{aligned} \tag{4.10}$$

for all $\varphi \in \mathrm{Lip}_c(\Omega_T)$, where $c(u)$, $c(u_i) > 0$ are constants independent of ε. We only show the first inequality. The second inequality can be shown in the same manner. Noting that $\partial_t u \in [L^p(0,T;\mathcal{N}_o^{1,p}(\Omega))]^*$, we have

$$|\langle \partial_t [u]_\varepsilon, \varphi\rangle| = \left|\iint_{\Omega_T} [u]_\varepsilon \partial_t \varphi \,\mathrm{d}\mu\,\mathrm{d}t\right| = \left|\iint_{\Omega_T} u\partial_t[\varphi]_\varepsilon \,\mathrm{d}\mu\,\mathrm{d}t\right| = |\langle \partial_t u, [\varphi]_\varepsilon\rangle|$$
$$\le c(u)\|[\varphi]_\varepsilon\|_{L^p(0,T;\mathcal{N}^{1,p}(\Omega))} \le \frac{3c(u)}{2}\|\varphi\|_{L^p(0,T;\mathcal{N}^{1,p}(\Omega))}$$

for all $\varphi \in \mathrm{Lip}_c(\Omega_T)$, where $c(u) > 0$ is a constant independent of ε. The last estimate can be derived from the argument as in (4.2). This implies the first inequality of (4.10).

By the density of $\mathrm{Lip}_c(\Omega_T)$ in $L^p(0,T;\mathcal{N}^{1,p}(\Omega))$ and $L^{p_i}(0,T;\mathcal{N}^{1,p_i}(\Omega))$ with their norms we can apply (4.10) with the choice $\varphi \equiv \xi(u-[u]_\varepsilon)$ to get for $i \gg 1$ and $\varepsilon \ll 1$:

$$\left|\iint_{\Omega_T} \xi([u]_\varepsilon - u)\partial_t[u_i - u]_\varepsilon \,\mathrm{d}\mu\,\mathrm{d}t\right|$$
$$\le |\langle \partial_t[u]_\varepsilon, \xi(u-[u]_\varepsilon)\rangle| + |\langle \partial_t[u_i]_\varepsilon, \xi(u-[u]_\varepsilon)\rangle|$$
$$\le c\|g_{\xi(u-[u]_\varepsilon)}\|_{L^p(\Omega_T)} + c\|g_{\xi(u-[u]_\varepsilon)}\|_{L^{p_i}(\Omega_T)}$$
$$\le c(p,\mathcal{Q},u,u_i)\|g_{\xi(u-[u]_\varepsilon)}\|_{L^{p+\delta}(\Omega_T)}.$$

Here we have used Hölder's inequality to replace the exponents p and p_i by $p+\delta$. Since by the calculus rules for upper gradients we have

$$|g_{\xi(u-[u]_\varepsilon)}| \le \xi g_{u-[u]_\varepsilon} + |u-[u]_\varepsilon| g_\xi,$$

we get

$$\left|\iint_{\Omega_T} \xi([u]_\varepsilon - u)\partial_t[u_i - u]_\varepsilon \,\mathrm{d}\mu\,\mathrm{d}t\right| \le c(p,\mathcal{Q},u,u_i,\xi)\|u-[u]_\varepsilon\|_{\mathcal{N}^{1,p+\delta}(\Omega)}.$$

Now applying Lemma 3.6 we conclude that the right hand side converges to zero as $\varepsilon \to 0$.

By Lemma 4.4 we can conclude that $\partial_t(\xi u) \in [L^p(0,T;\mathcal{N}_o^{1,p}(\Omega))]^*$ and

$$\iint_{\Omega_T} \xi u \partial_t[u_i - u]_\varepsilon \,\mathrm{d}\mu\,\mathrm{d}t = -\langle \partial_t(\xi u), [u_i-u]_\varepsilon\rangle$$
$$\xrightarrow{\varepsilon\to 0} -\langle \partial_t(\xi u), u_i - u\rangle = \langle \xi u, \partial_t(u_i-u)\rangle \xrightarrow{i\to\infty} 0.$$

The convergence to zero as $i \to \infty$ is a consequence of Lemma 4.7.

Combining the last estimates we deduce

$$\lim_{i\to\infty}\lim_{\varepsilon\to 0}\iint_{\Omega_T} [u_i]_\varepsilon \partial_t(\Phi_{i,\varepsilon})\,\mathrm{d}\mu\,\mathrm{d}t = \iint_{\Omega_T} u\partial_t\Phi\,\mathrm{d}\mu\,\mathrm{d}t.$$

Up to now we have therefore shown that

$$-\iint_{\Omega_T} u\partial_t\Phi \,\mathrm{d}\mu\,\mathrm{d}t + \frac{1}{p}\iint_K g_u^p \,\mathrm{d}\mu\,\mathrm{d}t \le \frac{\mathcal{Q}_i}{p_i}\lim_{\varepsilon\to 0}\iint_{\operatorname{spt}[\Phi_{i,\varepsilon}]_\varepsilon} g_{u_i-[\Phi_{i,\varepsilon}]_\varepsilon}^{p_i}\,\mathrm{d}\mu\,\mathrm{d}t + \mathcal{A}_i, \tag{4.11}$$

where $\lim_{i\to\infty}\mathcal{A}_i = 0$. In a next step we will focus on the limit $\varepsilon \to 0$ on the right hand side of the preceding estimate. For this aim we split the domain of integration according to $\operatorname{spt}[\Phi_{i,\varepsilon}]_\varepsilon \Subset K^s = K \cup (K^s\setminus K)$. On $K \subset K^{s/2}$ we have

$$u_i - [\Phi_{i,\varepsilon}]_\varepsilon = u - \Phi + \Phi - [\Phi]_\varepsilon + u_i - u - [[u_i - u]_\varepsilon]_\varepsilon,$$

hence we get

$$\iint_K g_{u_i-[\Phi_{i,\varepsilon}]_\varepsilon}^{p_i}\,\mathrm{d}\mu\,\mathrm{d}t \le \iint_K [g_{u-\Phi} + g_{\Phi-[\Phi]_\varepsilon} + g_{u-u_i-[[u-u_i]_\varepsilon]_\varepsilon}]^{p_i}\,\mathrm{d}\mu\,\mathrm{d}t.$$

Since $\Phi \in \operatorname{Lip}_c(\Omega_T)$ we have that $\Phi - [\Phi]_\varepsilon \to 0$ uniformly on K as $\varepsilon \to 0$ and therefore the second term in the preceding integral tends to zero as $\varepsilon \to 0$. Moreover, we have by the higher integrability, Lemma 3.12, that $g_{u-u_i} \in L^{p+\delta}$ for $i \in \mathbb{N}$ large enough and therefore we have by Lemma 3.6 that

$$g_{u_i-u-[[u_i-u]_\varepsilon]_\varepsilon} \to 0 \quad \text{as } \varepsilon \to 0 \tag{4.12}$$

pointwise almost everywhere on K and also in $L^{p+\delta}(K)$. Using now the elementary estimate

$$||\mu|^a - |\mu|^b| \le \left[\frac{1}{\gamma}|\mu|^{\max\{a,b\}+\gamma} + \frac{1}{\gamma}\left(\frac{1}{a}+\frac{1}{b}\right)\right]|b-a|, \tag{4.13}$$

which holds for all $\mu \in \mathbb{R}^k$, $a, b > 0$ and $\delta > 0$, see [20], and which we apply with $\mu = g_{u_i-u-[[u_i-u]_\varepsilon]_\varepsilon}$, $a = p_i$, $b = p$ and $\gamma = \delta/2$, we obtain by dominated convergence that

$$\lim_{\varepsilon\to 0}\iint_K g_{u_i-[\Phi_{i,\varepsilon}]_\varepsilon}^{p_i}\,\mathrm{d}\mu\,\mathrm{d}t \le \iint_K g_{u-\Phi}^p\,\mathrm{d}\mu\,\mathrm{d}t + c(\delta, p, \|g_u\|_{L^{p+\delta}(\Omega_T)})|p_i - p|.$$

Next, we estimate the integral on the set $K^s\setminus K$. We first note that $[\Phi]_\varepsilon \equiv 0$ on $\Omega_T\setminus K^{s/2}$ for $\varepsilon > 0$ small enough and hence we conclude that

$$u_i - [\Phi_{i,\varepsilon}]_\varepsilon = u_i - [\xi([u_i]_\varepsilon - [u]_\varepsilon)]_\varepsilon$$

on $K^s\setminus K^{3s/4}$. Consequently we can write

$$\iint_{K^s\setminus K} g_{u_i-[\Phi_{i,\varepsilon}]_\varepsilon}^{p_i}\,\mathrm{d}\mu\,\mathrm{d}t \le \iint_{K^s\setminus K}[g_{u_i-[\xi([u_i]_\varepsilon-[u]_\varepsilon)]_\varepsilon} + \chi_{K^{3s/4}}g_{[\Phi]_\varepsilon}]^{p_i}\,\mathrm{d}\mu\,\mathrm{d}t,$$

where $\chi_{K^{3s/4}}$ denotes the characteristic function of the set $K^{3s/4}$. Now, as $\varepsilon \to 0$, we deduce with Lemma 3.6 that

$$\lim_{\varepsilon \to 0} \iint_{K^s \setminus K} g_{u_i - [\Phi_{i,\varepsilon}]_\varepsilon}^{p_i} \, \mathrm{d}\mu \, \mathrm{d}t \le \iint_{K^s \setminus K} g_{u_i - \xi(u_i - u)}^{p_i} \, \mathrm{d}\mu \, \mathrm{d}t.$$

Now we consider the integral

$$I_3 := \iint_{K^s \setminus K} g_{u_i - \xi(u_i - u)}^{p_i} \, \mathrm{d}\mu \, \mathrm{d}t.$$

To estimate this integral, we first note that, since ξ is a Lipschitz cut-off function, we get by basic calculus rules for p weak upper gradients that

$$g_{u_i(1-\xi)+\xi u} \le (1-\xi) g_{u_i} + \xi g_u + |u_i - u| g_\xi.$$

Hence we get for the expression I_3 the estimate

$$I_3 \le c \iint_{K^s \setminus K} \left((1-\xi)^{p_i} g_{u_i}^{p_i} + g_\xi^{p_i} |u_i - u|^{p_i} + \xi^{p_i} g_u^{p_i} \right) \mathrm{d}\mu \, \mathrm{d}t, \tag{4.14}$$

and thus the right hand side divides into three integrals. For the second one we get by Hölder's inequality, $g_\xi \le L$ and by Lemma 4.7:

$$\iint_{K^s \setminus K} g_\xi^{p_i} |u_i - u|^{p_i} \, \mathrm{d}\mu \, \mathrm{d}t \le c \|u_i - u\|_{L^{p+\delta}(\Omega_T)}^{\frac{p}{p+\delta}} \to 0,$$

as $i \to \infty$. The third integral is estimated with the help of Hölder's inequality and (4.7) as follows:

$$\iint_{K^s \setminus K} \xi^{p_i} g_u^{p_i} \, \mathrm{d}\mu \, \mathrm{d}t \le \left[\iint_{K^s \setminus K} g_u^{p+\delta} \, \mathrm{d}\mu \, \mathrm{d}t \right]^{\frac{p_i}{p+\delta}} (\mu \times \mathscr{L}^1)(K^s \setminus K)^{1 - \frac{p_i}{p+\delta}},$$

and hence we deduce

$$\limsup_{i \to \infty} \iint_{K^s \setminus K} \xi^{p_i} g_u^{p_i} \, \mathrm{d}\mu \, \mathrm{d}t \le c \alpha^{\frac{p}{p+\delta}}.$$

It remains to estimate the first integral

$$\iint_{K^s \setminus K} (1-\xi)^{p_i} g_{u_i}^{p_i} \, \mathrm{d}\mu \, \mathrm{d}t. \tag{4.15}$$

To do so, we proceed basically as in [7]. We show that for $D \Subset \Omega_T$ being compact and for almost every $r \in (0, r_o)$, where $r_o := \mathrm{d}_{\mathrm{par}}(K, (\mathscr{X} \times \mathbb{R}_+) \setminus \Omega_T)$ there exists a constant $c \equiv c(\mathcal{Q}, p)$ such that

$$\limsup_{i\to\infty} \iint_{D(r)} g_{u_i}^{p_i} \,d\mu\,dt \le c \iint_{D(r)} g_u^p \,d\mu\,dt, \tag{4.16}$$

where

$$D(r) := \{z \in \Omega_T : \mathrm{d_{par}}(z, D) < r\}.$$

For this aim, let $0 < \varrho < r < r_o$ and $\xi \in \mathrm{Lip}_c(\Omega_T)$ be a cut-off function such that

$$0 \le \xi \le 1, \quad \xi \equiv 1 \quad \text{on } D(\varrho), \quad \xi \equiv 0 \quad \text{on } \Omega_T \backslash D(r).$$

We test the quasi minimality of u_i with the test function $\Phi_{i,\varepsilon} := \xi([u_i]_\varepsilon - [u]_\varepsilon)$, where $\varepsilon > 0$ is sufficiently small. By (3.1) we get

$$-\iint_{\Omega_T} [u_i]_\varepsilon \partial_t \Phi_{i,\varepsilon} \,d\mu\,dt + \frac{1}{p_i} \iint_{\mathrm{spt}[\Phi_{i,\varepsilon}]_\varepsilon} g_{u_i}^{p_i} \,d\mu\,dt \le \frac{\mathcal{Q}_i}{p_i} \iint_{\mathrm{spt}[\Phi_{i,\varepsilon}]_\varepsilon} g_{u_i - [\Phi_{i,\varepsilon}]_\varepsilon}^{p_i} \,d\mu\,dt.$$

Using the same arguments as after (4.8), we conclude that

$$\lim_{i\to\infty} I_i = \lim_{i\to\infty} \lim_{\varepsilon\to 0} \iint_{\Omega_T} [u_i]_\varepsilon \partial_t \Phi_{i,\varepsilon} \,d\mu\,dt = 0,$$

with the obvious notation for I_i. Letting $\varepsilon \to 0$ in the above inequality we arrive at

$$\iint_{D(\varrho)} g_{u_i}^{p_i} \,d\mu\,dt + I_i \le 2\mathcal{Q} \iint_{D(r)} g_{u_i - \xi(u_i - u)}^{p_i} \,d\mu\,dt.$$

This we see as follows: The integral on the left hand side comes up by exactly the same argument as in (4.9). For the integral on the right hand side we use that $\mathrm{spt}[\Phi_{i,\varepsilon}]_\varepsilon \subset D(r)$ for $\varepsilon > 0$ sufficiently small and then an argument similar to (4.12). Using now once again the calculus rules for upper gradients we get

$$g_{u_i - \xi(u_i - u)} = g_{(1-\xi)u_i + \xi u} \le (1 - \xi) g_{u_i} + \xi g_u + |u_i - u| g_\xi.$$

Plugging this into the right hand side of the preceding inequality and using that $1 - \xi \equiv 0$ on $D(\varrho)$ we arrive at

$$\begin{aligned}\iint_{D(\varrho)} g_{u_i}^{p_i} \,d\mu\,dt + I_i &\le c \iint_{D(r)\backslash D(\varrho)} g_{u_i}^{p_i} \,d\mu\,dt \\ &\quad + c \iint_{D(r)} (g_\xi^{p_i} |u_i - u|^{p_i} + \xi^{p_i} g_u^{p_i}) \,d\mu\,dt.\end{aligned}$$

Now we define for $r \in (0, r_o)$ the quantity

$$\Psi(r) := \limsup_{i\to\infty} \iint_{D(r)} g_{u_i}^{p_i} \,d\mu\,dt.$$

Since $D(r) \subset D(\tilde{r})$ for $r < \tilde{r}$, the function $r \mapsto \Psi(r)$ is nondecreasing and by the higher integrability of g_{u_i} it is also finite. Hence, the set of points of discontinuity of Ψ is at most countable. Now we add on both sides of the above inequality the quantity

$$\iint_{D(\varrho)} g_{u_i}^{p_i} \, d\mu \, dt,$$

and conclude that for every point of continuity $r \in (0, r_o)$ of Ψ that

$$(1+c)\Psi(\varrho) \le c\Psi(r) + \limsup_{i \to \infty} \iint_{D(r)} g_\xi^{p_i} |u_i - u|^{p_i} \, d\mu \, dt + c \iint_{D(r)} g_u^p \, d\mu \, dt.$$

Here we have used once again the elementary inequality (4.13) and the arguments after (4.13) so 'replace' in the integral on the right hand side the exponent p_i by the exponent p. By Hölder's inequality and the strong convergence $u_i \to u$ in the space $L^{p+\delta}$, we see that the second integral on the right hand side is zero and therefore we get

$$(1+c)\Psi(\varrho) \le \Psi(r) + c \iint_{D(r)} g_u^p \, d\mu \, dt,$$

where all constants c are the same constants and the inequality holds for all $\varrho \in (0, r)$. Now, since Ψ is continuous in r, we get as $\varrho \nearrow r$:

$$(1+c)\Psi(r) \le \Psi(r) + c \iint_{D(r)} g_u^p \, d\mu \, dt,$$

and therefore finally

$$\Psi(r) \le c \iint_{D(r)} g_u^p \, d\mu \, dt,$$

which is the desired estimate (4.16).

We use this estimate now as follows to estimate the integral (4.15): Since $\xi \equiv 1$ on $K^{s/2}$ we may achieve that

$$\iint_{K^s \setminus K} (1-\xi)^{p_i} g_{u_i}^{p_i} \, d\mu \, dt \le \iint_D g_{u_i}^{p_i} \, d\mu \, dt,$$

where $D = \overline{K^s} \setminus K^{s/2}$ is compact. Now we apply the preceding argument and the estimate (4.16) with the set D and therefore $D(r) \subset K^{2s} \setminus K$, if we choose r sufficiently small. Estimate (4.16) provides a constant which does not depend on i such that

$$\limsup_{i \to \infty} \iint_{D(r)} g_{u_i}^{p_i} \, d\mu \, dt \le c \iint_{D(r)} g_u^p \, d\mu \, dt.$$

Combining this with (4.7) we therefore get

$$\begin{aligned}\limsup_{i\to\infty}\iint_{K^s\setminus K}(1-\xi)^{p_i}g_{u_i}^{p_i}\,\mathrm{d}\mu\,\mathrm{d}t &\le \limsup_{i\to\infty}\iint_{D(r)} g_{u_i}^{p_i}\,\mathrm{d}\mu\,\mathrm{d}t\\ &\le c\iint_{D(r)} g_u^p\,\mathrm{d}\mu\,\mathrm{d}t\\ &\le c\Big(\iint_{K^{2s}\setminus K} g_u^{p+\delta}\,\mathrm{d}\mu\,\mathrm{d}t\Big)^{\frac{p}{p+\delta}}(\mu(\Omega)T)^{1-\frac{p}{p+\delta}}\\ &\le c(\mu(\Omega)T)^{1-\frac{p}{p+\delta}}\alpha^{\frac{p}{p+\delta}}.\end{aligned}$$

By (4.14) we finally obtain

$$\iint_{K^s\setminus K} g_{u_i-\xi(u_i-u)}^{p_i}\,\mathrm{d}\mu\,\mathrm{d}t \le c\alpha^{\frac{p}{p+\delta}} + c(\mu(\Omega)T)^{1-\frac{p}{p+\delta}}\alpha^{\frac{p}{p+\delta}},$$

where the constant c is not depending on α. Since $\alpha > 0$ was arbitrary, combining this with (4.11) and letting $i \to \infty$ we end up with the inequality

$$-\iint_{\Omega_T} u\partial_t\Phi\,\mathrm{d}\mu\,\mathrm{d}t + \frac{1}{p}\iint_K g_u^p\,\mathrm{d}\mu\,\mathrm{d}t \le \frac{\mathscr{Q}}{p}\iint_K g_{u-\Phi}^p\,\mathrm{d}\mu\,\mathrm{d}t,$$

which holds for all test functions $\Phi \in \mathrm{Lip}_c(\Omega_T)$ with $K = \operatorname{spt}\Phi$. This is the quasi-minimality of the limit function u and the proof of Theorem 2.2 is complete. □

Proof (of Theorem 2.4). Let u_i be a parabolic $\mathscr{Q}_i$-minimizer of the p_i-energy. By Theorem 2.2 we know that u is the parabolic 1-minimizer of the p-energy and $u_i \to u$ strongly in $L^p(\Omega)$, $g_u \rightharpoonup g$ weakly in $L^p(\Omega)$ and g is a p-weak upper gradient of u. Note that at this point we do not know if g is in fact the minimal p-weak upper gradient of u. To prove the assertion of Theorem 2.4, we first show that

$$\lim_{i\to\infty}\|g_{u_i}\|_{L^p(\Omega_T)} = \|g\|_{L^p(\Omega)}, \tag{4.17}$$

because since the space $L^p(\Omega)$ with the product measure $\mu \times \mathscr{L}^1$ is a uniformly convex space and hence the weak convergence $g_{u_i} \rightharpoonup g$ in $L^p(\Omega_T)$ together with the norm convergence (4.17) provides the strong convergence $g_{u_i} \to g$ in $L^p(\Omega_T)$. By lower semicontinuity of the L^p-norm we immediately get

$$\|g\|_{L^p(\Omega_T)} \le \liminf_{i\to\infty}\|g_{u_i}\|_{L^p(\Omega_T)} \tag{4.18}$$

and therefore it remains to show that

$$\limsup_{i\to\infty}\|g_{u_i}\|_{L^p(\Omega_T)} \le \|g\|_{L^p(\Omega_T)}. \tag{4.19}$$

In a first step we prove that

$$\limsup_{i\to\infty} \iint_{\Omega_T} g_{u_i}^{p_i}\, \mathrm{d}\mu\, \mathrm{d}t \le \iint_{\Omega_T} g_u^p\, \mathrm{d}\mu\, \mathrm{d}t.$$

For this aim, we test the quasi-minimality (3.1) of u_i with the test function

$$\Phi_\varepsilon^h(x,t) := \chi_{T-h}^{h,h}[u_i - u]_\varepsilon,$$

where $\chi_{T-h}^{h,h}(t) \in \mathrm{Lip}_c(0,T)$ is the affine function defined in (4.4), which is $\equiv 1$ on the interval $(2h, T-2h)$ and $\equiv 0$ on the intervals $(0,h)$ and $(T-h,T)$. This gives

$$-\iint_{\Omega_T} [u_i]_\varepsilon \partial_t \Phi_\varepsilon^h\, \mathrm{d}\mu\, \mathrm{d}t + \frac{1}{p_i} \iint_{\mathrm{spt}[\Phi_\varepsilon^h]_\varepsilon} g_{u_i}^{p_i}\, \mathrm{d}\mu\, \mathrm{d}t \le \frac{\mathcal{Q}_i}{p_i} \iint_{\mathrm{spt}[\Phi_\varepsilon^h]_\varepsilon} g_{u_i - [\Phi_\varepsilon^h]_\varepsilon}^{p_i}\, \mathrm{d}\mu\, \mathrm{d}t.$$

To estimate the first term we first write

$$\iint_{\Omega_T} [u_i]_\varepsilon \partial_t \Phi_\varepsilon^h\, \mathrm{d}\mu\, \mathrm{d}t = \iint_{\Omega_T} [u_i - u]_\varepsilon \partial_t \Phi_\varepsilon^h\, \mathrm{d}\mu\, \mathrm{d}t + \iint_{\Omega_T} [u]_\varepsilon \partial_t \Phi_\varepsilon^h\, \mathrm{d}\mu\, \mathrm{d}t =: I_1 + I_2.$$

By the definition of Φ_ε^h and an integration by parts we get for the integral I_1:

$$I_1 = \frac{1}{2} \iint_{\Omega_T} |[u_i - u]_\varepsilon|^2 \partial_t \chi_{T-h}^{h,h}\, \mathrm{d}\mu\, \mathrm{d}t.$$

Now using (2.11) for both u_i and u and $\partial_t \chi_{T-h}^{h,h} = \pm 1/h$ on $(h, 2h)$ and $(T-2h, T-h)$ respectively we obtain by the strong convergence $[u_i - u]_\varepsilon \to u_i - u$ in $L^2(\Omega_T)$ that

$$\lim_{h\to 0} \lim_{\varepsilon\to 0} I_1 = -\int_\Omega |u_i(\cdot, T) - u(\cdot, T)|^2\, \mathrm{d}\mu \le 0.$$

For the second integral we get by integration by parts

$$\begin{aligned} I_2 &= -\langle \partial_t [u]_\varepsilon, \chi_{T-h}^{h,h}[u_i - u]_\varepsilon \rangle \\ &= -\langle \partial_t([u]_\varepsilon - u), \chi_{T-h}^{h,h}[u_i - u]_\varepsilon \rangle - \langle \partial_t u, \chi_{T-h}^{h,h}([u_i - u]_\varepsilon - (u_i - u)) \rangle \\ &\quad - \langle \partial_t u, \chi_{T-h}^{h,h}(u_i - u) \rangle = I_{21} + I_{22} + I_{23}. \end{aligned}$$

For the first term, we perform again an integration by parts and obtain

$$\begin{aligned} I_{21} &= \langle \partial_t (\chi_{T-h}^{h,h}[u_i - u]_\varepsilon), [u]_\varepsilon - u \rangle \\ &= \langle \partial_t \chi_{T-h}^{h,h}[u_i - u]_\varepsilon, [u]_\varepsilon - u \rangle + \langle \chi_{T-h}^{h,h} \partial_t [u_i - u]_\varepsilon, [u]_\varepsilon - u \rangle. \end{aligned}$$

Since both u_i and u are parabolic quasiminimizers, the second term can be estimated with the help of Remark 4.3 (and $|\chi_{T-h}^{h,h}| \le 1$) as follows:

$$\begin{aligned}|\langle\chi^{h,h}_{T-h}\partial_t[u_i-u]_\varepsilon,[u]_\varepsilon-u\rangle| &\le |\langle\partial_t[u_i]_\varepsilon,[u]_\varepsilon-u\rangle|+|\langle\partial_t[u]_\varepsilon,[u]_\varepsilon-u\rangle|\\ &\le \frac{3^p\mathscr{Q}^p}{p}\|u\|^{p-1}_{L^p(0,T;\mathscr{N}^{1,p}(\Omega))}\|g_{[u]_\varepsilon-u}\|_{L^p(\Omega_T)}\\ &\quad+\frac{3^{p_i}\mathscr{Q}_i^{p_i}}{p_i}\|u_i\|^{p_i-1}_{L^{p_i}(0,T;\mathscr{N}^{1,p_i}(\Omega))}\|g_{[u]_\varepsilon-u}\|_{L^{p_i}(\Omega_T)}\\ &\le c(p,\mathscr{Q},M)\|g_{[u]_\varepsilon-u}\|_{L^{p+\delta}(\Omega_T)}\xrightarrow{\varepsilon\to0}0.\end{aligned}$$

In the last step we have once again used the argument of Remark 4.3, Hölder's inequality and the uniform energy bound (4.5) to replace the $L^p-\mathscr{N}^{1,p}$-norms by a constant $c\equiv c(\mathscr{Q},p,M)$. For the first term we obtain moreover

$$|\langle\partial_t\chi^{h,h}_{T-h}[u_i-u]_\varepsilon,[u]_\varepsilon-u\rangle|\le|\partial_t\chi^{h,h}_{T-h}|\langle[u_i-u]_\varepsilon,[u]_\varepsilon-u\rangle|\xrightarrow{\varepsilon\to0}0,$$

and hence we conclude that

$$\lim_{h\to0}\lim_{\varepsilon\to0}|I_{21}|=0.$$

Using once again the fact that u is a parabolic minimizer, we obtain by Lemma 4.2 that

$$|I_{22}|\le c(p,\mathscr{Q})\|u\|^{p-1}_{L^p(0,T;\mathscr{N}^{1,p}(\Omega))}\|g_{[u_i-u]_\varepsilon-(u_i-u)}\|_{L^p(\Omega_T)}\xrightarrow{\varepsilon\to0}0.$$

Combining the estimates for I_{21} and I_{22} we obtain

$$\lim_{\varepsilon\to0}I_2=-\langle\partial_tu,\chi^{h,h}_{T-h}(u_i-u)\rangle=-\langle\partial_tu,u_i-u\rangle+\langle\partial_tu,(1-\chi^{h,h}_{T-h})(u_i-u)\rangle.$$

For the second term on the right hand side we get, using Lebsgue's convergence theorem and once again Lemma 4.2:

$$\begin{aligned}&|\langle\partial_tu,(1-\chi^{h,h}_{T-h})(u_i-u)\rangle|\\ &\quad\le c(p,\mathscr{Q})\|u\|^{p-1}_{L^p(0,T;\mathscr{N}^{1,p}(\Omega))}\|g_{(1-\chi^{h,h}_{T-h})(u_i-u)}\|_{L^p(\Omega_T)}\\ &\quad\le c(p,\mathscr{Q})\|u\|^{p-1}_{L^p(0,T;\mathscr{N}^{1,p}(\Omega))}\|(1-\chi^{h,h}_{T-h})g_{u_i-u}\|_{L^p(\Omega_T)}\xrightarrow{h\to0}0,\end{aligned}$$

since $\chi^{h,h}_{T-h}(t)\to1$ as $h\to0$. Therefore we conclude that

$$\lim_{h\to0}\lim_{\varepsilon\to0}I_2=-\langle\partial_tu,u_i-u\rangle.$$

In a next step we want to see that the integral on the right hand side tends to zero as $i\to\infty$. For this aim we use again a time-mollification $[u]_\varepsilon$ for $\varepsilon>0$ small enough of the minimizer u, writing

$$\langle\partial_tu,u_i-u\rangle=\langle\partial_t[u]_\varepsilon,u_i-u\rangle+\langle\partial_t(u-[u]_\varepsilon),u_i-u\rangle.$$

Since $\partial_t[u]_\varepsilon \in L^2(\Omega_T)$ we can estimate the fist term by Hölder's inequality to get

$$|\langle \partial_t[u]_\varepsilon, u_i - u\rangle| \le \|\partial_t[u]_\varepsilon\|_{L^2(\Omega)}\|u_i - u\|_{L^2(\Omega)} \xrightarrow{i\to\infty} 0,$$

by the strong convergence $u_i \to u$ in $L^p(\Omega_T)$. For the second term we use perform first an integration by parts to move the time derivative to the right hand side and then use once again Lemma 4.2 to obtain

$$|\langle \partial_t(u - [u]_\varepsilon), u_i - u\rangle| = |\langle \partial_t(u_i - u), u - [u]_\varepsilon\rangle| \le c(p, \mathcal{Q}, M)\|g_{[u]_\varepsilon - u}\|_{L^p(\Omega_T)},$$

where the constant does not depend on i. Hence we have that for every $\varepsilon > 0$ small enough that

$$\limsup_{i\to\infty} |\langle \partial_t u, u_i - u\rangle| \le c(p, \mathcal{Q}, M)\|g_{[u]_\varepsilon - u}\|_{L^p(\Omega_T)}.$$

Since $\varepsilon > 0$ was arbitrary we conclude

$$\lim_{i\to\infty} |\langle \partial_t u, u_i - u\rangle| = 0.$$

Combining now all the estimates from before we arrive at

$$\limsup_{i\to\infty} \iint_{\Omega_T} g_{u_i}^{p_i}\,\mathrm{d}\mu\,\mathrm{d}t \le \limsup_{i\to\infty} \lim_{\varepsilon,h\to 0} \mathcal{Q}_i \iint_{\mathrm{spt}[\Phi_\varepsilon^h]_\varepsilon} g_{u_i - [\Phi_\varepsilon^h]_\varepsilon}^{p_i}\,\mathrm{d}\mu\,\mathrm{d}t.$$

For the right hand side of this inequality we proceed exactly as in the proof of Lemma 4.5. Note here that the test function in the proof of Lemma 4.5 differs from the one here just by the fact that u is replaced by η there. However, arguing exactly in the same way leads us to

$$\lim_{h\to 0}\lim_{\varepsilon\to 0} \iint_{\mathrm{spt}[\Phi_\varepsilon^h]_\varepsilon} g_{u_i - [\Phi_\varepsilon^h]_\varepsilon}^{p_i}\,\mathrm{d}\mu\,\mathrm{d}t \le \iint_{\Omega_T} g_u^{p_i}\,\mathrm{d}\mu\,\mathrm{d}t,$$

and applying once again the elementary inequality (4.13) to replace the exponent p_i in the above integral by the exponent p, we obtain

$$\limsup_{i\to\infty} \iint_{\Omega_T} g_{u_i}^{p_i}\,\mathrm{d}\mu\,\mathrm{d}t \le \limsup_{i\to\infty} \mathcal{Q}_i \iint_{\Omega_T} g_u^{p_i}\,\mathrm{d}\mu\,\mathrm{d}t = \iint_{\Omega_T} g_u^{p}\,\mathrm{d}\mu\,\mathrm{d}t. \tag{4.20}$$

Here we also used that $\mathcal{Q}_i \to 1$ as $i \to \infty$. In order to conclude the convergence (4.19) we have to replace the exponent p_i on the left hand side of the above estimate by the exponent p. This can be done by higher integrability in terms of the energy bound (4.5) as follows: We proceed analogous to the argument in [7, Proof of Theorem 2.2]: We fix $\gamma > 0$ and let $i \in \mathbb{N}$ be large enough to have $|p_i - p| < \gamma$. By Hölder's inequality we get

$$(4.21)\qquad \iint_{\Omega_T} g_{u_i}^p \,\mathrm{d}\mu\,\mathrm{d}t = \iint_{\Omega_T} g_{u_i}^{p-\gamma} g_{u_i}^{\gamma} \,\mathrm{d}\mu\,\mathrm{d}t \le \left[\iint_{\Omega_T} g_{u_i}^{p_i} \,\mathrm{d}\mu\,\mathrm{d}t\right]^{\frac{p-\gamma}{p_i}} \left[\iint_{\Omega_T} g_{u_i}^{\frac{\gamma p_i}{p_i-p+\gamma}} \,\mathrm{d}\mu\,\mathrm{d}t\right]^{\frac{p_i-p+\gamma}{p_i}}.$$

Being $\delta > 0$ the higher integrability exponent of Lemma 3.12, we choose $q_i := (p+\delta)\frac{p_i-p+\gamma}{\gamma p_i}$ to achieve that $\frac{\gamma p_i}{p_i-p+\gamma} q_i = p+\delta$. Note that $q_i > 1$ if $i \in \mathbb{N}$ is large enough. Then by Hölder's inequality and (4.5) we get

$$\left[\iint_{\Omega_T} g_{u_i}^{\frac{\gamma p_i}{p_i-p+\gamma}} \,\mathrm{d}\mu\,\mathrm{d}t\right]^{\frac{p_i-p+\gamma}{p_i}} \le \left[\iint_{\Omega_T} g_{u_i}^{p+\delta} \,\mathrm{d}\mu\,\mathrm{d}t\right]^{\frac{\gamma}{p+\delta}} (\mu(\Omega)T)^{\frac{p_i-p+\gamma}{p_i}-\frac{\gamma}{p+\delta}} \le c(M)^{\gamma} (\mu(\Omega)T)^{\frac{p_i-p+\gamma}{p_i}-\frac{\gamma}{p+\delta}}.$$

Combining this with (4.20) and (4.21) we get

$$\limsup_{i\to\infty} \iint_{\Omega_T} g_{u_i}^p \,\mathrm{d}\mu\,\mathrm{d}t \le c(M)^{\gamma} \left[\iint_{\Omega_T} g_u^p \,\mathrm{d}\mu\,\mathrm{d}t\right]^{\frac{p-\gamma}{p}} (\mu(\Omega)T)^{\gamma(1/p-1/(p+\delta))},$$

and since $\gamma > 0$ was arbitrary, we finally conclude that

$$\limsup_{i\to\infty} \iint_{\Omega_T} g_{u_i}^p \,\mathrm{d}\mu\,\mathrm{d}t \le \iint_{\Omega_T} g_u^p \,\mathrm{d}\mu\,\mathrm{d}t.$$

Note here that on the right hand side integral appears the minimal p-weak upper gradient g_u of u. By its minimality the inequality obviously holds also for the p-weak upper gradient g and therefore (4.19) is shown. Moreover we see, combining the last estimate with (4.18) we get that

$$\|g\|_{L^p(\Omega_T)} \le \liminf_{i\to\infty} \|g_{u_i}\|_{L^p(\Omega_T)} \le \limsup_{i\to\infty} \|g_{u_i}\|_{L^p(\Omega_T)} \le \|g_u\|_{L^p(\Omega_T)} \le \|g\|_{L^p(\Omega_T)},$$

where the last inequality holds since g_u is minimal. Hence, we have equality in the above estimate and therefore g is in fact the minimal p-weak upper gradient. This finishes the proof of Theorem 2.4. □

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