Rend. Lincei Mat. Appl. 29 (2018), 321–342
DOI 10.4171/RLM/809

Partial Differential Equations — *Spectral gaps and non-Bragg resonances in a water channel*, by Valeria Chiadò Piat, Sergey A. Nazarov and Keijo M. Ruotsalainen, communicated on January 12, 2018.

This paper is dedicated to the memory of Professor Ennio De Giorgi.

Abstract. — In this paper the essential spectrum of the linear problem of water-waves on a 3d-channel with gently periodic bottom will be studied. We show that under a certain geometric condition on the bottom profile the essential spectrum has spectral gaps. In classical analysis of waveguides it is known that the Bragg resonances at the edges of the Brillouin zones create band gaps in the spectrum. Here we demonstrate that the band gaps can be opened also in the frequency range far from the Bragg resonances. The position and the length of the gaps are found out by applying an asymptotic analysis to the model problem in the periodicity cell.

Key words: Water waves, spectral problem, asymptotic analysis, Bragg resonances

Mathematics Subject Classification: 35P30, 80M35, 76B15

1. Introduction

During the last decades the propagation of waves through periodic structures has attracted considerable attention. This is partly due to the invention of photonic crystals, which exhibit extraordinary properties that are supposed to bring about a new technological revolution in optics, information transmission, and other areas. The main tasks have been in controlling the wave propagation either by guiding the wave in some preferred direction, or to prevent its propagation at certain frequencies.

From a historical point of view the first object in the study of wave propagation was the water waves. As can be seen, for example, from Euler's seminal work [7] or Lord Rayleigh's investigations [18]. Especially, the propagation of waves through periodic media has been in the focus of the research. This is also the intent of our paper. We study the surface waves on the channel over an undulating bottom.

The analysis of wave interaction with periodic structures has been an important and active field in hydrodynamics. The focus has been mainly on the scattering by the bottom topography or the propagation of trapping modes along the periodic topography [17, 2, 13, 14, 28, 15, 19, 21, 29, 20, 30, 8]. Even though the Bragg scattering/resonance is closely related to the question of band gaps in the spectrum of a periodic waveguide, there are very few papers which directly address this issue in hydrodynamics. We mention here only the articles by Chou [6] and Linton [16] where the band gap structure is explicitly mentioned and investigated. In our previous paper [5], we considered a two-dimensional problem and, using the asymptotic analysis, showed that periodic bottom always creates a big family of spectral gaps in the spectrum. There we have demonstrated that the Bragg resonances occur at the edges of the Brillouin zones resulting to the gap opening. However, other experimental works (see, e.g., [32]) hint the existence of non-Bragg resonances, which appear in the frequency range far from the edges of the Brillouin zone.

In this paper, we study the surface water waves in a three-dimensional rectangular duct with a corrugated bottom. Analogously to our previous work [5] (see also [23]) the opening of the Bragg and non-Bragg gaps may occur at the intersections of the folded dispersion curves of the unperturbed case, but unlike the Bragg gaps, the non-Bragg gaps arise away from the edges of the Brillouin zone. Moreover, we will present sufficient conditions for the width and height of the channel as well as the profile of the bottom undulation which lead to the band gaps in the frequency spectrum. In this way our results will provide new insight in the creation and control of band gaps in periodic waveguides.

From early on, it has been clear that the theoretical study of wave propagation is related to the spectral properties of self-adjoint elliptic operators in unbounded media. In other words, the spectral theory of elliptic operators became the focus of studies. Naturally, the spectral theory has a bottomless source of problems in the gargantuous jungle of phenomena related to the wave propagation. From a mathematical point of view the central question is the structure of the spectrum. Is it continuous? Does it contain gaps, i.e., intervals of frequencies on which the waves do not propagate through the media?

From the study of the spectral properties of the Neumann-Laplacian [23, 25, 1], or Dirichlet–Laplacian [4, 26, 25], it is known that the periodic perturbation of the cylindrical waveguide creates gaps in the spectrum of these operators. From these sources the questions of the present paper have emerged. The difference with previously mentioned articles is that the spectral parameter appears now in the boundary condition, making the analysis quite different.

The main tool of our study is the asymptotic analysis which entitles us to detect a gap in the spectrum when the periodic perturbation of the channel bottom is small enough. To fill in the theoretical analysis, we also present some numerical results in order to establish to which extent our asymptotic analysis is valid.

2. Formulation of the problem

2.1. *The corrugated channel*

We consider a three-dimensional channel

$$\Omega_\varepsilon = \left\{ (x, y, z) : |x| < \frac{l}{2},\ y \in \mathbb{R},\ z \in (-d + \varepsilon h(x, y), 0) \right\} \tag{2.1}$$

where $d, l > 0$ are fixed numbers, $\varepsilon > 0$ is a small parameter and h is a smooth function, 1-periodic with respect to $y \in \mathbb{R}$. Without loss of generality, we will assume that h has zero mean value. With this in mind d is the average depth of the duct and l is the width. The boundary $\partial\Omega_\varepsilon$ splits into the liquid surface Σ_o at level $z = 0$, the corrugated bottom $\Sigma_{d,\varepsilon}$ at level $z = -d + \varepsilon h(x, y)$, and the lateral boundary Σ_l.

Under the assumptions of incompressible, inviscid and irrotational fluid motion, linear water waves in the waveguide can be described by a velocity potential $\Phi^\varepsilon(x, y, z, t)$ [12]. For a harmonic mode with an angular frequency θ, the velocity potential may be sought in the form

$$\Phi^\varepsilon(x, y, z, t) = u^\varepsilon(x, y, z)e^{i\theta t}.$$

Assuming the linearized kinematic boundary condition at the free surface and the no-flow condition at the bottom and vertical walls, we obtain the Steklov spectral problem

$$-\Delta u^\varepsilon = 0 \quad \text{in } \Omega_\varepsilon, \tag{2.2}$$

$$\partial_z u^\varepsilon = \lambda^\varepsilon u^\varepsilon \quad \text{on } \Sigma_0, \tag{2.3}$$

$$\partial_n u^\varepsilon = 0 \quad \text{on } \Sigma_l \cup \Sigma_{d,\varepsilon}. \tag{2.4}$$

where ∂_n denotes the exterior normal derivative and $\lambda^\varepsilon = \theta^2/g$ the spectral parameter, where g is the acceleration due to gravity.

The main question, we investigate in this paper, is to understand which values of $\lambda^\varepsilon \in \mathbb{C}$ belong to the spectrum of the above problem (2.2)–(2.4).

2.2. *The problem in the periodicity cell*

The analysis of the wave propagation in an infinite periodic structure can be reduced to the analysis of the wave propagation in the bounded periodicity cell with $\varepsilon > 0$

$$\omega_\varepsilon = \{(x, y, z) : |x| < l/2, \ |y| < 1/2, \ z \in (-d + \varepsilon h(x, y), 0)\}.$$

This will be done using the Floquet–Bloch theory based on the Gelfand transform [9]

$$U(x, y, z; \eta) = (Gu)(x, y, z; \eta) = (2\pi)^{-1/2} \sum_{p \in \mathbb{Z}} e^{-i\eta p} u(x, y + p, z).$$

Applying the Gelfand transform to our spectral problem (2.2)–(2.4), we obtain for each $\eta \in [0, 2\pi)$ a spectral problem in a periodicity cell

$$-\Delta U^{\varepsilon} = 0 \quad \text{in } \omega_{\varepsilon}, \tag{2.5}$$

$$\partial_z U^{\varepsilon} = \Lambda^{\varepsilon}(\eta) U^{\varepsilon} \quad \text{on } \sigma_0, \tag{2.6}$$

$$\partial_n U^{\varepsilon} = 0 \quad \text{on } \{z = -d + \varepsilon h(x, y)\} \cup \{x = \pm l/2\}, \tag{2.7}$$

$$U^{\varepsilon}\Big(x, -\frac{1}{2}, z; \eta\Big) = e^{-i\eta} U^{\varepsilon}\Big(x, \frac{1}{2}, z; \eta\Big), \tag{2.8}$$

$$\partial_y U^{\varepsilon}\Big(x, -\frac{1}{2}, z; \eta\Big) = e^{-i\eta} \partial_y U^{\varepsilon}\Big(x, \frac{1}{2}, z; \eta\Big), \tag{2.9}$$

where we have denoted the free water surface by

$$\sigma_0 = \{(x, y, z) : |x| < l/2,\ |y| < 1/2,\ z = 0\}.$$

By the spectral theory of elliptic partial differential operators, for each fixed $\eta \in [0, 2\pi)$, this problem has a *monotone unbounded sequence* of real non-negative eigenvalues

$$0 \le \Lambda_1^{\varepsilon}(\eta) \le \Lambda_2^{\varepsilon}(\eta) \le \cdots \to +\infty$$

From the literature (see, e.g., [9, 22, 11, 27, 31]), we know that λ^{ε} belongs to the spectrum of our original problem in the unbounded channel Ω_{ε} if and only if λ^{ε} equals $\Lambda_j^{\varepsilon}(\eta)$ for some $j \in \mathbb{N}\backslash\{0\}$ and $\eta \in [0, 2\pi)$. The functions $\eta \mapsto \Lambda_j^{\varepsilon}(\eta)$ are continuous and 2π-periodic, hence the spectrum is a union of the closed segments Υ_j^{ε}, $j \in \mathbb{N}\backslash\{0\}$, where

$$\Upsilon_j^{\varepsilon} = \{\lambda \in \mathbb{R} : \lambda = \Lambda_j^{\varepsilon}(\eta),\ \eta \in [0, 2\pi)\}.$$

We obtain *a spectral gap* in the spectrum if there exist an open non-empty interval in the positive real semi-axis which does not intersect any of the closed segments above. However, when the segments overlap each other, no spectral gap opens. One aim is to show the existence of some gaps in the spectrum under appropriate sufficient conditions.

2.3. *The problem in the straight channel: $\varepsilon = 0$*

The same problem with $\varepsilon = 0$ in the channel Ω_0 with a flat bottom can be solved by separation of variables. Then for every $j \in \mathbb{Z}$ the pair (λ_j^K, u_j^K) defined by

$$\lambda_j^K = K \tanh(d \cdot K) =: D(K), \quad |K| \ge |k_j|, \tag{2.10}$$

$$u_j^K(x, y, z) = \cos\Big(k_j\Big(x + \frac{l}{2}\Big)\Big) e^{\pm iy\sqrt{K^2 - k_j^2}} (e^{zK} + e^{-(z+2d)K})$$

is a solution of the spectral problem. Here and in the sequel we denote by $k_j = \frac{j\pi}{l}$. The dispersion relation (2.10) is a functional relationship between the temporal frequency ω and the wave number $|K|$. The longitudinal and transverse components of the wave vector are $\sqrt{K^2 - k_j^2}$ and k_j, respectively.

As in [24], interpreting the straight channel to consist also of periodicity cells of unit length with flat bottom, we may write the problem, using the Gelfand transform, in the bounded periodicity cell with $\varepsilon = 0$

$$\omega_0 = \{(x, y, z) : |x| < l/2,\ |y| < 1/2,\ z \in (-d, 0)\}$$

as a family of spectral problems. Namely, for each $\eta \in [0, 2\pi)$ we obtain a spectral problem

$$-\Delta U^0 = 0 \quad \text{in } \omega_0, \tag{2.11}$$

$$\partial_z U^0 = \Lambda^0(\eta) U^0 \quad \text{on } \sigma_0, \tag{2.12}$$

$$\partial_x U^0 = 0 \quad \text{for } x = \pm\frac{l}{2}, \tag{2.13}$$

$$\partial_z U^0 = 0 \quad \text{for } z = -d, \tag{2.14}$$

$$U^0\Big(x, -\frac{1}{2}, z; \eta\Big) = e^{-i\eta} U^0\Big(x, \frac{1}{2}, z; \eta\Big), \tag{2.15}$$

$$\partial_y U^0\Big(x, -\frac{1}{2}, z; \eta\Big) = e^{-i\eta} \partial_y U^0\Big(x, \frac{1}{2}, z; \eta\Big), \tag{2.16}$$

The parameter K is represented in the form $K = \pm\sqrt{\zeta^2 + k_j^2}$, where ζ can be decomposed uniquely as $\zeta = 2\pi q + \eta$ with $q \in \mathbb{Z}$ and $\eta \in [0, 2\pi)$. Hence we may rewrite the above solution pair as

$$\Lambda^0_{q,j}(\eta) = D(\sqrt{(2\pi q + \eta)^2 + k_j^2}), \tag{2.17}$$

$$U^0_{q,j}(x, y, z; \eta) = \cos\Big(k_j\Big(x + \frac{l}{2}\Big)\Big) e^{+iy(2\pi q + \eta)} g_{q,j}(z; \eta), \tag{2.18}$$

$$g_{q,j}(z; \eta) = (e^{z\sqrt{(2\pi q+\eta)^2 + k_j^2}} + e^{-(z+2d)\sqrt{(2\pi q+\eta)^2 + k_j^2}}).$$

The range of the dispersion curves $\eta \mapsto \Lambda^0_{q,j}(\eta)$ gives us the closed segments $\Upsilon^0_k(\eta)$ which then will constitute the spectrum in the unperturbed case, which is known to be the closed positive real axis $\overline{\mathbb{R}}_+$. This can be seen from the graphs of the dispersion curves, which form the truss-structure as in Fig. 1.

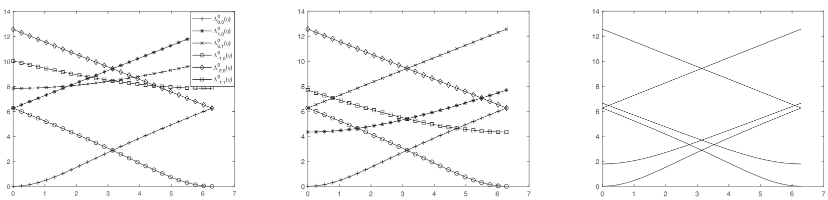

Figure 1. The dispersion curves for the straight channel: a) Channel width $l = 0.4$, b) Channel width $l = \frac{\sqrt{3}}{2}$, c) Channel width $l = \sqrt{2}$

We note here that our reduced representation of the dispersion relation differs from the conventional one, where the first Brillouin zone is the interval $[-\pi, \pi]$. But for our purposes it is more convenient to choose as the first Brillouin zone the interval $[0, 2\pi]$, so that the Bragg point is in the middle of the interval.

2.4. Statement of the main results

In this section we formulate the sufficient conditions which ensure the existence of the band gaps. For that we introduce the points $\eta_{\pm 1} = \pi \pm \frac{\pi}{4l^2}$, which are intersection points of the dispersion curves: $\Lambda^0_{-1,0}(\eta_{-1}) = \Lambda^0_{0,1}(\eta_{-1})$ and $\Lambda^0_{0,0}(\eta_1) = \Lambda^0_{-1,1}(\eta_1)$. The Fourier-coefficients of the profile function h are

$$H^y(l) = \int_{\sigma_0} e^{i2\pi y} h(x, y)\, dx\, dy$$

$$H^{xy}(l) = \int_{\sigma_0} h(x, y) \cos\Big(\frac{\pi}{l}\Big(x + \frac{l}{2}\Big)\Big) e^{i2\pi y}\, dx\, dy,$$

$$H^x(l) = \int_{\sigma_0} h(x, y) \cos\Big(\frac{2\pi}{l}\Big(x + \frac{l}{2}\Big)\Big)\, dx\, dy,$$

THEOREM 2.1. 1. *Let* $0 < l \le \frac{1}{2}$ *and assume that* $H^y(l) \neq 0$. *Then there exists* $\alpha > 0$ *and* $\varepsilon_0 > 0$ *such that for all* $\varepsilon < \varepsilon_0$

(a) $\max_{\eta \in [0,2\pi)} \Lambda^\varepsilon_1(\eta) < \min_{\eta \in [0,2\pi)} \Lambda^\varepsilon_2(\eta)$.

(b) *For all* $\lambda^\varepsilon \in\,]D(\pi) - \varepsilon^\alpha, D(\pi) + \varepsilon^\alpha[$ *the problem* (2.2)–(2.4) *has only the trivial solution* $u^\varepsilon = 0$.

2. *Let* $\frac{1}{2} < l < 1$ *and assume that* $H^y(l) \neq 0$, $H^{xy}(l) \neq 0$. *Then there exists* $\alpha > 0$ *and* $\varepsilon_0 > 0$ *such that for all* $\varepsilon < \varepsilon_0$

(c) *the following inequalities*

$$\max_{\eta \in [0,2\pi)} \Lambda^\varepsilon_1(\eta) < \min_{\eta \in [0,2\pi)} \Lambda^\varepsilon_2(\eta)$$

and

$$\max_{\eta \in [0,2\pi)} \Lambda^\varepsilon_2(\eta) < \min_{\eta \in [0,2\pi)} \Lambda^\varepsilon_3(\eta)$$

hold.

(d) *For every*

$$\lambda^\varepsilon \in\,]D(\eta_1) - \varepsilon^\alpha, D(\eta_1) + \varepsilon^\alpha[\, \cup\,]D(\pi) - \varepsilon^\alpha, D(\pi) + \varepsilon^\alpha[$$

the problem (2.2)–(2.4) *has only the trivial solution.*

3. *Let* $l = 1$. *Assume that* $H^y(l) \neq 0$, $H^{xy}(l) \neq 0$ *and* $H^x(l) \ge 0$. *Then the statements* (c)–(d) *occur.*

3. Splitting of dispersion curves

3.1. *Asymptotic analysis of eigenvalues*

In the following sections we will describe the splitting phenomenon of dispersion curves for small $\varepsilon > 0$ leading to the band gap structure in the dispersion relations. The main tool is an asymptotic analysis of the eigenvalues $\Lambda^0_{q,j}(\eta)$ under perturbations of the bottom. Especially, we are interested in the eigenvalues $\Lambda^0_{q,j}(\eta)$ which have algebraic multiplicity higher than one. Those eigenvalues are the intersection points of two or more dispersion curves. We will show that, when the bottom of the channel is perturbed, exactly at those points the dispersion curves differ from each other and a small gap opens between them, which in some cases gives raise to a spectral gap for our spectral problem.

As in our previous paper [5], we follow the approach adopted by Nazarov [24]. In order to see whether a gap is opened near the intersection point η_0, we introduce the deviation parameter δ, replacing η by $\eta_0 + \varepsilon\delta$. The deviation parameter will be used to describe the behaviour of eigenvalues $\Lambda^\varepsilon_k(\eta)$ in a small neighbourhood of the intersection point: a suitable choice of $\delta = \delta(\varepsilon)$ will be done in the proof of Theorem 2.1 (see (4.56)). Outside this small neighbourhood, where the eigenvalues $\Lambda^0_k(\eta)$ are simple, the classical perturbation theory is then used to show that the perturbed eigenvalues $\Lambda^\varepsilon_k(\eta)$ satisfy the condition

$$|\Lambda^\varepsilon_k(\eta) - \Lambda^0_k(\eta)| < c\varepsilon$$

for some constant $c > 0$ independent on ε.

For the eigenvalues and functions $\Lambda^\varepsilon_k(\eta_0 + \varepsilon\delta)$ and $U^\varepsilon(\cdot;\eta)$, we use the asymptotic expansion around an intersection point $\eta_0 \in [0, 2\pi)$ as follows:

$$\Lambda^\varepsilon_k(\eta_0 + \varepsilon\delta) = \Lambda^0_{q,j}(\eta_0) + \varepsilon\Lambda'_{q,j}(\delta) + \widetilde{\Lambda_{q,j}}, \tag{3.19}$$

$$U^\varepsilon_k = U^0 + \varepsilon U'_{q,j} + \widetilde{U_{q,j}}, \tag{3.20}$$

where $\Lambda^0_k(\eta_0) = \Lambda^0_{q,j}(\eta_0)$ is the double eigenvalue of the problem (2.11)–(2.16) and it is given by (2.17) for suitable choices of q and j. The function

$$U^0 = a_+ U^0_+ + a_- U^0_-$$

belongs to the two-dimensional eigenspace spanned by the corresponding eigenfunctions $U^0_\pm$. We choose them to be as in (2.18). For U^0_+ the integers q and j differ from those of U^0_-, obviously. In order to simplify the notation we fix q, j and omit them in the rest of the present section. The coefficients $a_\pm$ are to be determined alongside the first order correction terms $\Lambda'(\delta)$ and U'.

In order to find the correct problem for U', we insert the right-hand side of (3.19) and (3.20) into (2.5), (2.6) and, setting the terms corresponding to identical powers of ε equal, we obtain

$$\Delta U' = 0 \quad \text{in } \omega_0,$$
$$\partial_z U' = \Lambda^0 U' + \Lambda' U^0, \quad \text{on } \sigma_0$$

In order to deal with the boundary condition (2.7), we have to expand $\partial_n U_\varepsilon$ at $\partial\omega_\varepsilon \backslash \sigma_0$. Since the bottom is represented by the equation $-d + \varepsilon h(x, y) - z = 0$, then, for any smooth function $F(x, y, z)$ the normal derivative at the bottom has the following expansion

$$\begin{aligned}\partial_n F &= (1 + \varepsilon^2 |\nabla h|^2)^{-1/2} (\varepsilon \partial_x h \partial_x F + \varepsilon \partial_y h \partial_y F - \partial_z F)_{|z=-d+\varepsilon h(x,y)} \\ &= -\partial_z F + \varepsilon(\nabla_{xy} h \cdot \nabla_{xy} F - \partial_z^2 F \cdot h(x))_{|z=-d} + O(\varepsilon^2).\end{aligned}$$

Using the above formula for $F = U^0 + \varepsilon U' + \tilde{U}$ in equation (2.7), equating, again, the terms corresponding to identical powers of ε, and using (2.11), we get the boundary condition

$$\partial_z U' = \nabla_{xy} h \cdot \nabla_{xy} U^0 + (\Delta_{xy} U^0) h, \quad \text{if } z = -d.$$

At the lateral walls of the periodicity cell the normal derivative $\partial_n = \pm\partial_x$ and thus we obtain a boundary condition

$$\partial_x U'\Big(\pm\frac{l}{2}, y, z\Big) = 0, \quad |y| < \frac{1}{2}, \quad -d < z < 0.$$

Finally, since $e^{-i(\eta_0+\varepsilon\delta)} = e^{-i\eta_0}(1 - i\varepsilon\delta + O(\varepsilon^2))$, inserting (3.20) into (2.8), (2.9), we get

$$U'\Big(x, -\frac{1}{2}, z; \eta_0\Big) = e^{-i\eta_0}(-i\delta U^0 + U')\Big(x, \frac{1}{2}, z; \eta_0\Big), \tag{3.21}$$

$$\partial_y U'\Big(x, -\frac{1}{2}, z; \eta_0\Big) = e^{-i\eta_0}(-i\delta \partial_y U^0 + \partial_y U')\Big(x, \frac{1}{2}, z; \eta_0\Big), \tag{3.22}$$

Thus, finally, the problem for the first correction term U' can be written as a mixed boundary value problem

$$\Delta U' = 0 \quad \text{in } \omega_0, \tag{3.23}$$

$$\partial_z U' = \Lambda^0 U' + \Lambda' U^0 \quad \text{on } \sigma_0 \tag{3.24}$$

$$\partial_z U' = \nabla_{xy} h \cdot \nabla_{xy} U^0 + (\Delta_{xy} U^0) h, \quad z = -d, \tag{3.25}$$

$$\partial_x U' = 0, \quad x = \pm\frac{l}{2} \tag{3.26}$$

$$U'\Big(x, -\frac{1}{2}, z; \eta_0\Big) = e^{-i\eta_0}(-i\delta U^0 + U')\Big(x, \frac{1}{2}, z; \eta_0\Big), \tag{3.27}$$

$$\partial_y U'\Big(x, -\frac{1}{2}, z; \eta_0\Big) = e^{-i\eta_0}(-i\delta \partial_y U^0 + \partial_y U')\Big(x, \frac{1}{2}, z; \eta_0\Big). \tag{3.28}$$

Note that the third equation above can be equivalently written as

$$\partial_z U' = \operatorname{div}_{xy}(h\nabla_{xy} U^0), \quad \text{if } z = -d. \tag{3.29}$$

Since Λ^0 is a double eigenvalue of problem (2.11)–(2.16), according to the Fredholm alternative, the formally self-adjoint elliptic boundary value problem (3.23)–(3.28) has a solution U' if and only if two compatibility conditions are satisfied. To derive these conditions, one may directly insert the eigenfunctions $U^0_\pm$ and the solution U' into the Green formula on ω_0, to obtain

$$\int_{\partial\omega_0} \partial_n \overline{U}^0_\pm U' - \int_{\partial\omega_0} \partial_n U' \overline{U}^0_\pm = 0. \tag{3.30}$$

In the following we split the boundary of the periodicity cell ω_0 into the top surface σ_0, the bottom σ_d, and the lateral surfaces $\sigma^x_\pm$, $\sigma^y_\pm$.

Choosing $U^0_\pm$ to be any of the functions in (2.18) (with $\pm$ related to the sign of propagation of the wave $e^{+iy(2\pi q+\eta)}$), and using equations (2.11)–(2.16) for $U^0_\pm$ and (3.23)–(3.26) for U', we get the following integrals on each face of the periodicity cell ω_0.

$$\begin{aligned}\int_{\sigma_0} (\partial_n \overline{U^0_\pm} U' - \partial_n U' \overline{U^0_\pm}) &= \int_{\sigma_0} (\partial_z \overline{U}^0_\pm U' - \partial_z U' \overline{U^0_\pm}) \\ &= -\Lambda' \int_{\sigma_0} U^0 \overline{U^0_\pm}.\end{aligned}$$

$$\begin{aligned}\int_{\sigma_d} (\partial_n \overline{U^0_\pm} U' - \partial_n U' \overline{U}^0_\pm) &= -\int_{\sigma_d} (\partial_z \overline{U^0_\pm} U' - \partial_z U' \overline{U^0_\pm}) \\ &= \int_{\sigma_d} \operatorname{div}_{xy}(h\nabla_{xy} U^0) \overline{U^0_\pm} \\ &= -\int_{\sigma_d} (h\nabla_{xy} U^0 \cdot \nabla_{xy} \overline{U^0_\pm})\end{aligned}$$

where the boundary terms along the boundary of σ_d are zero under the assumption that $h(x, y)$ has compact support, i.e.,

$$\operatorname{supp} h \subset\subset (-l/2, l/2) \times (-1/2, 1/2). \tag{3.31}$$

This is a technical rectriction that simplifies the calculations. Moreover, we have

$$\int_{\sigma^x_\pm} (\partial_n \overline{U^0_\pm} U' - \partial_n U' \overline{U^0_\pm}) = 0$$

$$\int_{\sigma^y_+} (\partial_n \overline{U^0_\pm} U' - \partial_n U' \overline{U^0_\pm}) + \int_{\sigma^y_-} (\partial_n \overline{U^0_\pm} U' - \partial_n U' \overline{U^0_\pm})$$

$$= -i\delta \int_{\sigma^y_-} \left(\overline{U^0_\pm}\left(\frac{1}{2}\right) \partial_y U^0\left(\frac{1}{2}\right) - U^0\left(\frac{1}{2}\right) \partial_y \overline{U^0_\pm}\left(\frac{1}{2}\right) \right)$$

where we have used the shorthand notation $U\left(\frac{1}{2}\right) = U\left(x, \frac{1}{2}, z\right)$. Summing up all terms we get the following system of equations $(\pm)$

$$-\Lambda' \int_{\sigma_0} U^0 \overline{U^0_\pm} - \int_{\sigma_d} (h\nabla_{xy} U^0 \cdot \nabla_{xy} \overline{U^0_\pm}) - i\delta \int_{\sigma^y_\pm} \left(\overline{U^0_\pm}\left(\frac{1}{2}\right) \partial_y U^0\left(\frac{1}{2}\right) - U^0\left(\frac{1}{2}\right) \partial_y \overline{U^0_\pm}\left(\frac{1}{2}\right) \right) = 0. \tag{3.32}$$

In the following we will replace

$$U^0 = a_+ U^0_+ + a_- U^0_-$$

for suitable choices of the pair $U^0_\pm$.

Estimates of the remainder terms $\widetilde{\Lambda_{q,j}}$, $\widetilde{U_{q,j}}$ in (3.19), (3.20), as ε, δ are small enough, may be proved in the similar manner as in [5]. The proof will be presented in Section 4.

3.2. *A case of Bragg resonance*

To start our analysis by the simplest case, we focus on the lowest dispersion curves $\Lambda^0_{0,0}(\eta) = D(\eta)$ and $\Lambda^0_{-1,0}(\eta) = D(\eta - 2\pi)$, when the width of the channel l satisfies $0 < l \le \frac{1}{2}$ (see Fig. 1(a)), i.e., $\Lambda^0_{1,0}(0) \le \Lambda^0_{0,1}(0)$. The above curves intersect only at $\eta_0 = \pi$, e.g, at the Bragg point.

Now by choosing

$$U^0_+ = U^0_{0,0}, \quad U^0_- = U^0_{-1,0}$$

and inserting them and $U^0 = a_+ U^0_+ + a_- U^0_-$ into (3.32), we get the eigenvalue problem

$$M(a) = \Lambda'(\delta)\mathbf{a}, \quad \mathbf{a} = (a_+, a_-)^\top,$$

for the matrix

$$M = \begin{bmatrix} A\delta & BH^y(l) \\ \overline{BH^y(l)} & -A\delta \end{bmatrix},$$

where

$$A = \frac{-i \int_{\sigma^y_-} (\overline{U^0_+} \partial_y U^0_+ - U^0_+ \overline{\partial_y U^0_+})}{\int_{\sigma_0} |U^0_+|^2} = 2\pi \frac{\int_{-d}^0 g^2_{0,0}(z;\pi)\,dz}{g^2_{0,0}(0;\pi)},$$

$$B = \frac{\int_{\sigma_d} h(x,y) \nabla_{x,y} U^0_+ \cdot \overline{\nabla U^0_-}}{\int_{\sigma_0} |U^0_+|^2} = -\frac{4\pi^2 e^{-2\pi d}}{l g^2_{0,0}(0;\pi)} \tag{3.33}$$

$$H^y(l) = \int_{-\frac{l}{2}}^{\frac{l}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{i2\pi y} h(x,y)\,dx\,dy$$

and $g_{0,0}$ is as in (2.18). In the computations of the matrix elements, we have explicitly used the fact that the mean value of the profile function is zero. In other words, the first correction term $\Lambda'(\delta)$ in the neighbourhood of $\eta_0 = \pi$ is the eigenvalue of the above eigenvalue problem.

In this case, the asymptotic expansion (3.19) has the correction terms

$$\Lambda'_{\pm} = \pm\sqrt{B^2|H^y(l)|^2 + A^2\delta^2}, \tag{3.34}$$

which are non-zero provided

$$H^y(l) = \int_{-\frac{l}{2}}^{\frac{l}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}} e^{i2\pi y} h(x,y)\,dx\,dy \neq 0. \tag{3.35}$$

Under the above condition, we can prove the existence of a gap in the spectrum of our original problem (2.2)–(2.4), namely

$$\max_{\eta} \Lambda_1^{\varepsilon}(\eta) < \min_{\eta} \Lambda_2^{\varepsilon}(\eta) \tag{3.36}$$

for small enough ε.

The lowest perturbed dispersion curve $\Lambda_1^{\varepsilon}(\eta)$ takes the shape as in Fig. 2 (a). In particular, we note that, if we take $h(x,y)$ is independent of x, then the problem reduces to two-dimensional surface wave propagation, and we recover the results obtained in [5].

3.3. *Band gaps at non-Bragg points*

In this section, we consider cases where the dispersion curves intersect each other also at the non-Bragg points ($\eta \neq \pi$), i.e., far away from the edges of the first Brillouin zone. In particular, we focus on the following dispersion curves (see Fig. 1(b) and (c))

$$\begin{aligned}
\Lambda^0_{0,0}(\eta) &= D(\eta)\\
\Lambda^0_{-1,0}(\eta) &= D(\eta - 2\pi)\\
\Lambda^0_{0,1}(\eta) &= D(\sqrt{\eta^2 + \pi^2 l^{-2}})\\
\Lambda^0_{-1,1}(\eta) &= D(\sqrt{(\eta - 2\pi)^2 + \pi^2 l^{-2}}).
\end{aligned}$$

Here we assume that $\frac{1}{2} < l < 1$. Then,

$$\Lambda^0_{0,0}(\pi) < \Lambda^0_{0,1}(0) < \Lambda^0_{-1,0}(0),$$

and the curves $\Lambda^0_{0,1}$ and $\Lambda^0_{-1,0}$ intersect at the point $\eta_{-1} = \pi - \frac{\pi}{4l^2}$; $\Lambda^0_{0,0}$ and $\Lambda^0_{-1,1}$ intersect at $\eta_1 = \pi + \frac{\pi}{4l^2}$; $\Lambda^0_{0,0}$ and $\Lambda^0_{-1,0}$ intersect at $\eta_0 = \pi$ (see Fig. 1(b)).

We expand $\Lambda_1^{\varepsilon}(\eta_{-1} + \varepsilon\delta)$ and U^{ε} as in (3.19), (3.20) with $\eta = \eta_{-1}$. The eigenfunctions

$$U^0_+ = U^0_{0,1} \quad \text{and} \quad U^0_- = U^0_{-1,0}, \tag{3.37}$$

correspond to the eigenvalues $\Lambda^0_{0,1}(\eta_{-1}) = \Lambda^0_{-1,0}(\eta_{-1})$, where $U^{0,\pm}_{q,j}$ are given by (2.18). Inserting $U^0_\pm$ and $U^0 = a_+ U^0_+ + a_- U^0_-$ into the compatibility condition (3.32) we obtain an eigenvalue problem for the correction term Λ':

$$\begin{bmatrix} C + D\delta & B \\ \bar{B} & A\delta \end{bmatrix} \begin{bmatrix} a_+ \\ a_- \end{bmatrix} = \Lambda' 2l \mathrm{e}^{-2\eta_1 d} \cosh^2(\eta_1 d) \begin{bmatrix} a_+ \\ a_- \end{bmatrix}. \tag{3.38}$$

Introducing the shorthand notations

$$H^{xy}(l) = \int_\omega h(x,y) \cos\Big(\frac{\pi}{l}\Big(x + \frac{l}{2}\Big)\Big) e^{i2\pi y}\, dx\, dy,$$

$$H^{x}(l) = \int_\omega h(x,y) \cos\Big(\frac{2\pi}{l}\Big(x + \frac{l}{2}\Big)\Big)\, dx\, dy,$$

$$G = \|g_{0,1}\|^2_{L^2(-d,0)},$$

the elements of the matrices in (3.38) are

$$A = 2(\eta_{-1} - 2\pi) l G, \quad B = 4(\eta_{-1} - 2\pi)\eta_{-1} \mathrm{e}^{-2\eta_1 d} H^{xy}(l),$$

$$C = 2\Big(\frac{\pi^2}{l^2} - \eta_1^2\Big) e^{-2\eta_1 d} H^x(l), \quad D = \eta_{-1} l G.$$

Since the matrix on the left in (3.38) is Hermitian symmetric, the eigenvalues $\mu_\pm = 2l \mathrm{e}^{-2\eta_1 d} \cosh^2(\eta_1 d) \Lambda'_\pm$ are real, where

$$\mu_\pm = C + (A + D)\delta \pm \sqrt{(C + (A + D)\delta)^2 + 4|B|^2}. \tag{3.39}$$

For sufficiently small δ, the eigenvalue problem (3.38) has two non-zero eigenvalues $\mu_+ > 0$ and $\mu_- < 0$, provided the condition

$$H^{xy}(l) \neq 0 \tag{3.40}$$

is satisfied. In this case, the perturbation splits the intersection of the graphs of $\Lambda^0_{0,1}$ and $\Lambda^0_{-1,0}$ at η_{-1} into two non-intersecting curves. This gives the possibility for a spectral gap. Same conditions arise at the point $\eta_1 = \pi + \frac{\pi}{4l^2}$, due to symmetry of the dispersion curves with respect to $\eta = \pi$.

At the point $\eta = \pi$, corresponding to the intersection $\Lambda^0_{0,0}(\pi) = \Lambda^0_{-1,0}(\pi)$, the expansion of $\Lambda^\varepsilon_1(\pi + \varepsilon\delta)$ performed in Section 3.2 remains valid also here, when $1/2 < l < 1$. In particular, the correction terms $\Lambda'_\pm$ are given by (3.34), and they are non-zero, with opposite signs, under the condition (3.35).

As a conclusion, if both conditions (3.35) and (3.40) take place, then the lowest dispersion curves of the problem (2.2)–(2.4) separate as follows

$$\Lambda^\varepsilon_1(\eta) < \Lambda^\varepsilon_2(\eta) \quad \text{for all } \eta \in [0, 2\pi[, \tag{3.41}$$

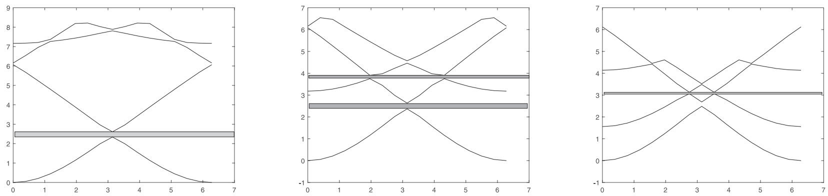

Figure 2. The perturbed dispersion curves: a) $l = \frac{\sqrt{2}}{4}$, b) $l = \frac{\sqrt{3}}{2}$, c) $l = \sqrt{2}$

and a gap occurs at the higher level, namely

$$\max_{\eta} \Lambda_2^{\varepsilon}(\eta) < \min_{\eta} \Lambda_3^{\varepsilon}(\eta). \tag{3.42}$$

The lowest perturbed dispersion curves $\Lambda_1^{\varepsilon}(\eta)$, $\Lambda_2^{\varepsilon}(\eta)$ and $\Lambda_3^{\varepsilon}(\eta)$ are shown in Fig. 2(b). Rigorous proofs of the above inequalities (3.41), (3.42), can be obtained, following the approach presented in [5] and [4]. Our analysis shows for the first time, by choosing the periodic bottom profile appropriately, that in addition to the band gap created by the Bragg resonances at the ends of the first Brillouin zone also non-Bragg gaps appear far away from the edges of the Brillouin zones. Previously this phenomenon has been detected experimentally for surface gravity waves in a channel by periodic walls [32].

3.4. The combined case

Let us assume for the moment that $l = 1$ and investigate the perturbation of the lowest dispersion curves $M_p(\eta)$, $p = 0, 1, 2$, which are defined as follows

$$M_0(\eta) = \begin{cases} \Lambda_{0,0}^0(\eta), & 0 \le \eta < \pi, \\ \Lambda_{-1,0}^0(\eta), & \pi \le \eta < 2\pi, \end{cases}$$

$$M_1(\eta) = \begin{cases} \Lambda_{0,1}^0(\eta), & 0 \le \eta < \eta_{-1}, \\ \Lambda_{-1,0}^0(\eta), & \eta_{-1} \le \eta < \pi, \\ M_1(2\pi - \eta), & \pi \le \eta < 2\pi, \end{cases}$$

$$M_2(\eta) = \begin{cases} \Lambda_{-1,0}^0(\eta), & 0 \le \eta < \eta_{-1}, \\ \Lambda_{1,0}^0(\eta), & \eta_{-1} \le \eta \le \pi, \\ M_2(2\pi - n), & \pi \le \eta < 2\pi \end{cases}$$

(see also Fig. 3, for the corresponding perturbed curves). In this case we have $\Lambda_{0,1}^0(0) = \Lambda_{0,0}^0(\pi)$. As it was shown in Section 3.3, at the points $\eta_0 = \pi$, $\eta_{\pm 1}$ the graphs of the dispersion curves split into two parts, under the conditions (3.35) and (3.40), forming two non-intersecting dispersion curves $\Lambda_1^{\varepsilon}(\eta)$ and $\Lambda_2^{\varepsilon}(\eta)$ such that

$$\Lambda_1^{\varepsilon}(\eta) < \Lambda_2^{\varepsilon}(\eta) \quad \forall \eta \in [0, 2\pi).$$

The spectral gap appears, if the following stronger inequality takes place:

$$\Lambda_1^\varepsilon(\pi) = \max_{\eta\in[0,2\pi)} \Lambda_1^\varepsilon(\eta) < \min_{\eta\in[0,2\pi)} \Lambda_2^\varepsilon(\eta) = \Lambda_2^\varepsilon(0). \tag{3.43}$$

Since, in this case, $\Lambda_{0,1}^0(0) = \Lambda_{0,0}^0(\pi)$, in order to understand the situation, we have to take into account also the perturbation of the simple eigenvalue $\Lambda_{0,1}^0(0)$. This is performed as in (3.19) for the double eigenvalues, i.e., by setting

$$\Lambda_2^\varepsilon(0+\varepsilon\delta) = \Lambda_{0,1}^0(0) + \varepsilon\Lambda_{0,1}'(\delta) + \widetilde{\Lambda_{0,1}}(\varepsilon\delta). \tag{3.44}$$

The formula for the correction term is now

$$\Lambda_{0,1}'(\delta) = -\|U_+^0\|_{L^2(\sigma_0)}^{-2} \int_{\sigma_d} h|\nabla_{xy} U_+^0|^2.$$

Inserting $U_+^0 = U_{0,1}^0$ in this equation we get

$$\Lambda_{0,1}' = \pi^2 \int_{\sigma_d} \cos\Big(2\pi\Big(x+\frac{1}{2}\Big)\Big) h(x,y)\,dx\,dy.$$

Assuming that the Fourier coefficient

$$H^x(1) = \int_{\sigma_d} \cos\Big(2\pi\Big(x+\frac{1}{2}\Big)\Big) h(x,y)\,dx\,dy \ge 0, \tag{3.45}$$

the correction term $\Lambda_{0,1}'(\delta)$ is non-negative. If we take $\Lambda_{0,0}'(\delta) = \Lambda_-'$ given in (3.34) and insert into the expansion

$$\Lambda_1^\varepsilon(\pi+\varepsilon\delta) = \Lambda_{0,0}^0(\pi) + \varepsilon\Lambda_{0,0}'(\delta) + \widetilde{\Lambda_{0,0}}(\pi+\varepsilon\delta) \tag{3.46}$$

and assume that also condition (3.35) is satisfied, we can prove that (3.43) takes place. In other words, the spectral gap opens between the dispersion curves $\Lambda_1^\varepsilon(\eta)$ and $\Lambda_2^\varepsilon(\eta)$. The two lowest dispersion curves are shown in Fig. 3(b).

For example, assuming that the profile function is odd in x-variable for every $y \in \left(-\frac{1}{2}, \frac{1}{2}\right)$ the condition (3.45) is valid with $H^x = 0$.

However, even if the condition (3.45) is violated, then the spectral gap may still appear as in Fig. 3. In that example $H^x < 0$.

Note that, increasing the width l of the channel, the first eigenvalue of the problem (2.5)–(2.9) with $\eta = 0$ (or $\eta = 2\pi$) decreases faster than the first eigenvalue for the same problem with $\eta = \pi$. Then the second dispersion curve could shadow the lowest dispersion curve and there would not be a band gap as in Fig. 3. However, as in this example, the band gap still exists between the second and third dispersion curve.

3.5. Example

Here we present a simple example in the case when the channel width $l = 1$, depth $d = 0.5$ and a periodic arrangement of boxes is mounted at the bottom of the straight channel. The height of the box is $\varepsilon = 0.2$ and the bottom is a square $S = \{(y,z) : |y| \le \frac{1}{4}, |x| \le \frac{1}{4}\}$. Hence the bottom of the periodicity cell is given by

$$z = -d + \varepsilon\chi_S(x,y),$$

where χ_S is the characteristic function of S:

$$\chi_S(x,y) = \begin{cases} 1, & |x| < \frac{l}{4},\ |y| < \frac{1}{4} \\ 0, & \text{otherwise} \end{cases} \tag{3.47}$$

In this case, the Fourier coefficients of the profile function are

$$H^y(l) = \frac{\sin(\frac{\pi}{2})}{\pi} = \frac{1}{\pi} > 0,$$

$$H^x(l) = \frac{1}{5\pi}\cos\left(\frac{\pi}{4}\right) \neq 0,$$

$$H^{xy}(l) = \frac{8}{15\pi^2}\cos\left(\frac{\pi}{4}\right)\sin\left(\frac{3\pi}{8}\right) \neq 0.$$

Hence the assumptions of Theorem 2.1 is satisfied and the gaps are opened both at the Bragg point $\eta_0 = \pi$ and at the non-Bragg point $\eta_{-1} = \frac{3\pi}{4}$, as seen at Fig. 3. The estimated width W of the band gap at non-Bragg point is given by the formula

$$W = 2\sqrt{C^2 + 4B^2} \approx 0.1,$$

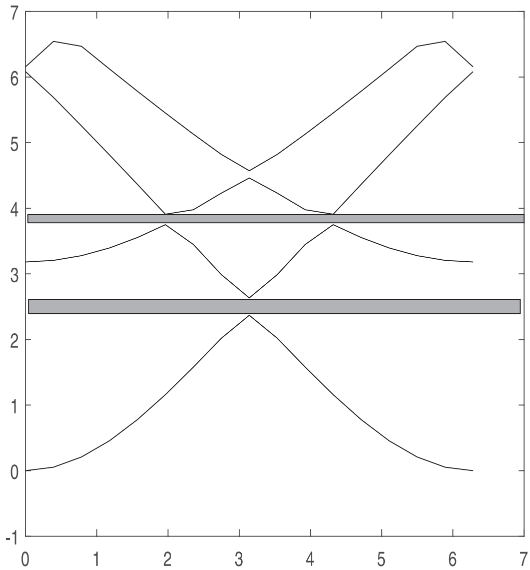

Figure 3. The perturbed dispersion curves

where C and B are the constants in (3.39). The computed results are in good agreement with the results of the asymptotic analysis.

The dispersion curves shown in figures 2 and 3 for the perturbed channels are computed with the open source software Freefem++.

Note that h in (3.47) is not smooth, actually discontinuous. However, profile functions of this type have the same asymptotic formulae for the existence of gaps (see [5], Fig. 5(a) and related remarks).

4. Proof of the main theorem

To prove the appearance of the band gaps we have to investigate the behaviour of the perturbed eigenvalues $\Lambda^{\varepsilon}_m(\eta)$, $\eta \in [0, 2\pi)$, $m = 1, 2, 3$, in the periodicity cell ω_ε. This will be divided in two steps. First, we will show that outside a neighbourhood of the intersection points η_p, $p = -1, 0, 1$ the eigenvalues $\Lambda^{\varepsilon}_m(\eta)$ do not deviate too much from the eigenvalues $\Lambda^0_m(\eta)$ of the unperturbed problem. In the next step, we estimate the remainder terms $\widetilde{\Lambda^{\varepsilon}_{q,j}}(\eta)$ in the vicinity of the intersection points η_p, $p = -1, 0, 1$. Essentially the proof is given already in our previous paper [5, Section 4], but we provide a condensed presentation of it for readers convenience. Since the case $0 < l \le \frac{1}{2}$ is the same as in our previous paper [5], we concentrate on the case $\frac{1}{2} < l < 1$.

For the proper functional analytic setting we introduce the space $H^1_\eta(\omega^\varepsilon)$ which is the closed subspace of the Sobolev space $H^1(\omega^\varepsilon)$ satisfying the quasi-periodicity conditions (2.8) and (2.9). Furthermore, we define in $H^1_\eta(\omega^\varepsilon)$ the scalar product

$$\langle U^\varepsilon, V^\varepsilon \rangle_\eta = (\nabla U^\varepsilon, \nabla V^\varepsilon)_{\omega^\varepsilon} + (U^\varepsilon, V^\varepsilon)_{\sigma_0},$$

and the operator $T^\varepsilon(\eta)$

$$\langle T^\varepsilon(\eta) U^\varepsilon, V^\varepsilon \rangle_\eta = (U^\varepsilon, V^\varepsilon)_{\sigma_0} \; \forall U^\varepsilon, \quad V^\varepsilon \in H^1_\eta(\omega^\varepsilon). \tag{4.48}$$

Now the spectral problem (2.5)–(2.9) becomes equivalent with the eigenvalue problem

$$T^\varepsilon(\eta) U^\varepsilon = \tau^\varepsilon(\eta) U^\varepsilon \quad \text{in } H^1_\eta(\omega^\varepsilon)$$

with the spectral parameter

$$\tau^\varepsilon(\eta) = (1 + \Lambda^\varepsilon(\eta))^{-1}. \tag{4.49}$$

Obviously, the operator $T^\varepsilon(\eta)$ is positive, self-adjoint and compact due to the compact embedding of $L^2(\sigma_0)$ into $H^1(\omega^\varepsilon)^*$ (the dual space of $H^1(\omega^\varepsilon)$).

In comparing the eigenvalues outside a neighbourhood of the intersection point, we rely on the analytic perturbation theory of self-adjoint operators [10, Ch.VII.6.2]. Since the perturbation is compact and small, we conclude that the eigenvalues of the model problems in ω^ε and ω^0 have the relationship

$$|\Lambda_m^\varepsilon(\eta) - \Lambda_m^0(\eta)| \le c_m\varepsilon \quad \text{for } \varepsilon \in (0, \varepsilon_m), \tag{4.50}$$

where the positive numbers c_m and ε_m depend on the eigenvalue number m but are independent of $\varepsilon \in (0, \varepsilon_m)$ and $\eta \in [0, 2\pi)$.

On the other hand, take for a while $q = 0$, $j = 0$ and $0 \le \eta < \pi$, the function $\eta \mapsto D(\eta) = \Lambda_{0,0}^0(\eta)$ is convex and increasing. Then we observe that

$$D(\pi) - \Lambda_{0,0}^0(\eta) \ge C_0(\pi - \eta)$$

for some positive constant $C_0 > 0$. Combining this with (4.50), the eigenvalues $\Lambda_{0,0}^\varepsilon(\eta) = \Lambda_1^\varepsilon(\eta)$ satisfy the estimate

$$\Lambda_1^\varepsilon(\eta) \le D(\pi) - C_0\varepsilon^{\frac{3}{4}} < D(\eta_1), \quad \text{when } \eta < \pi - \varepsilon^{\frac{3}{4}}.$$

By the same reasoning we then conclude that the eigenvalues $\Lambda_m^\varepsilon(\eta)$, $m = 1, 2$ fulfil the following inequalities: if $|\eta - \eta_{\pm 1}| \ge \varepsilon^{\frac{3}{4}}$, $|\eta - \pi| \ge \varepsilon^{\frac{3}{4}}$, then

$$\begin{aligned} &\Lambda_1^\varepsilon(\eta) < D(\pi) - C_0\varepsilon^{\frac{3}{4}}, \\ &D(\pi) + C_0\varepsilon^{\frac{3}{4}} < \Lambda_2^\varepsilon(\eta) < D(\eta_1) - C_0\varepsilon^{\frac{3}{4}}, \\ &\Lambda_3^\varepsilon(\eta) > D(\eta_1) + C_0\varepsilon^{\frac{3}{4}}. \end{aligned} \tag{4.51}$$

To prove that the dispersion curves will split at the points η_p, we will need the following lemma on "almost eigenvalues" (see, e.g., [3, Ch. 6]).

LEMMA 4.1. *Let $u^\varepsilon \in H_\eta^1(\omega^\varepsilon)$ and $t^\varepsilon \in \mathbb{R}_+$ be such that*

$$\|u^\varepsilon; H_\eta^1(\omega^\varepsilon)\| = 1 \quad \text{and} \quad \|T^\varepsilon(\eta)u^\varepsilon - t^\varepsilon u^\varepsilon; H_\eta^1(\omega^\varepsilon)\|^{1/2} = \kappa^\varepsilon \in (0, t^\varepsilon), \tag{4.52}$$

Then there exists an eigenvalue $\tau_m^\varepsilon(\eta)$ of the operator $T^\varepsilon(\eta)$ subject to the inequality

$$|\tau_m^\varepsilon(\eta) - t^\varepsilon| \le \kappa^\varepsilon.$$

In what follows, we replace the subscript q, j with $p+$ or $p-$, meaning that $\Lambda_{p+}^0(\eta)$, $\Lambda_{p-}^0(\eta)$ are the curves intersecting at η_p $(p = -1, 0, 1)$, with $\Lambda_{p+}^0(\eta)$ increasing and $\Lambda_{p-}^0(\eta)$ decreasing. This notation is adopted for all related quantities in the asymptotic expansions.

To apply the above Lemma 4.1 we choose the approximating eigenpair $(t^\varepsilon, u^\varepsilon)$ as follows:

$$t_{p\pm}^\varepsilon = (1 + \Lambda_{p\pm}^0(\eta_p) + \varepsilon\Lambda'_{p\pm}(\delta))^{-1}, \quad u_{p\pm}^\varepsilon = \langle \mathscr{U}_{p\pm}^\varepsilon, \mathscr{U}_{p\pm}^\varepsilon \rangle_{\eta_p+\varepsilon\delta}^{-1/2} \mathscr{U}_{p\pm}^\varepsilon, \tag{4.53}$$

where

$$\mathscr{U}_{p\pm}^\varepsilon(y, z) = \mathscr{U}_{p\pm}^0(y, z; \eta_p) + \varepsilon\mathscr{U}'_{p\pm}(x, y, z; \delta) + \varepsilon^2\tilde{\mathscr{U}}_{p\pm}^\varepsilon(x, y, z). \tag{4.54}$$

In (4.54) the function $\mathscr{U}^0_{p\pm}$ is the linear combination of the eigenfunctions $U^0_\pm$:

$$\mathscr{U}^0_{p\pm}(x,y,z) = a^\pm_+(\delta)\,U^0_+(x,y,z;\eta_p) + a^\pm_-(\delta)\,U^0_-(x,y,z;\eta_p),$$

where the vector $\mathbf{a}^\pm(\delta) = (a^\pm_+(\delta), a^\pm_-(\delta))$ is the normalized eigenvector of the problem (3.38), i.e., $\|\mathbf{a}^\pm(\delta)\| = 1$. The second term $\mathscr{U}'_{p\pm}(x,y,z;\delta)$ in (4.54) is the smooth extension of the solution of the problem (3.23)–(3.28) satisfying the estimate

$$\|\mathscr{U}'_{p\pm}; H^3(\omega^e)\| \le c_p(1+|\delta|).$$

The last term $\tilde{\mathscr{U}}^\varepsilon_{p\pm}$ in (4.54) we fix to compensate the discrepancies of the sum $\mathscr{U}^0_{p\pm} + \varepsilon\mathscr{U}'_{p\pm}$ in the quasi-periodicity conditions (2.8) and (2.9) for $\eta = \eta_p + \varepsilon\delta$. As in [5, Sect. 4(c)], we can find a function $\tilde{\mathscr{U}}^\varepsilon_{p\pm} \in H^3(\omega^\varepsilon)$ which compensates the discrepancies and satisfies the estimate

$$\|\tilde{\mathscr{U}}^\varepsilon_{p\pm}; H^3(\omega^\varepsilon)\| \le c_p\delta(1+\delta).$$

Furthermore, since $\mathscr{U}'_{p\pm}$ is the solution of (3.23)–(3.28), in the Steklov boundary condition (2.6) we have

$$\begin{aligned} g^\varepsilon_0(x,y) &:= \partial_z\mathscr{U}^\varepsilon_{p\pm}(x,y,0) - (\Lambda^0_{p\pm}(\eta_p) + \varepsilon\Lambda'_{p\pm}(\delta))\mathscr{U}^\varepsilon_{p\pm}(x,y,0) \\ &= \varepsilon^2(\partial_z\tilde{\mathscr{U}}^\varepsilon_{p\pm}(x,y,0) - (\Lambda^0_{p\pm}(\eta_p) + \varepsilon\Lambda'_{p\pm}(\delta)))\tilde{\mathscr{U}}^\varepsilon_{p\pm}(x,y,0) \\ &\quad - \varepsilon^2\Lambda'_{p\pm}(\delta)\mathscr{U}'_{p\pm}(x,y,0;\delta) \end{aligned}$$

and in the Neumann condition (2.7) at the bottom

$$\begin{aligned} g^\varepsilon_d(x,y) &:= \partial_n\mathscr{U}^\varepsilon_{p\pm}(x,y,-d+\varepsilon h(x,y)) \\ &= ((1+\varepsilon^2|\nabla_{x,y}h(x,y)|^2)^{-1/2} - 1)\partial_n\mathscr{U}^\varepsilon_{p\pm}(x,y,-d+\varepsilon h(x,y)) \\ &\quad + \varepsilon^2(-\partial_z\tilde{\mathscr{U}}^\varepsilon_{p\pm}(x,y,-d+\varepsilon h(y)) \\ &\quad + \varepsilon\nabla_{x,y}h(x,y)\cdot\nabla_{x,y}\tilde{\mathscr{U}}^\varepsilon_{p\pm}(x,y,-d+\varepsilon h(x,y))) \\ &\quad - (\partial_z\mathscr{U}^0_{p\pm}(x,y,-d+\varepsilon h(y);\eta_p) - \partial_z\mathscr{U}^0_{p\pm}(x,y,-d;\eta_p) \\ &\quad + \varepsilon h(x,y)\partial_z^2\mathscr{U}^0_{p\pm}(x,y,-d;\pi)) \\ &\quad + \varepsilon\nabla_{x,y}h(x,y)\cdot(\nabla_{x,y}\mathscr{U}^0_{p\pm}(x,y,-d+\varepsilon h(y);\eta_p) - \nabla_{x,y}\mathscr{U}^0_{p\pm}(y,-d;\eta_p)) \\ &\quad - \varepsilon(\partial_z\mathscr{U}'_{p\pm}(x,y,-d+\varepsilon h(x,y);\delta) - \partial_z\mathscr{U}'_{p\pm}(x,y,-d;\delta)) \\ &\quad + \varepsilon^2\nabla_{x,y}h(x,y)\cdot\nabla_{x,y}\mathscr{U}'_{p\pm}(x,y,-d+\varepsilon h(x,y);\delta) \end{aligned}$$

These formulae imply the estimate

$$\|g^\varepsilon_0; L^2(\gamma)\| + \|g^\varepsilon_d; L^2(\gamma^\varepsilon_d)\| \le c_p\varepsilon^2(1+\delta^2)(1+\varepsilon|\delta|).$$

We finally mention that $\mathscr{U}^0_{p\pm}$ satisfies the equation (2.5) in ω^ε but $\mathscr{U}'_{p\pm}$ does it only in ω^0. Therefore, recalling the smooth extension of $\mathscr{U}'_{p\pm}$, we obtain

$$\begin{aligned}\varepsilon\|\Delta\mathscr{U}'_{p\pm};L^2(\omega^\varepsilon)\| &= \varepsilon\|\Delta\mathscr{U}'_{p\pm};L^2(\omega^\varepsilon\backslash\omega^0)\|\\ &\le c\varepsilon^{3/2}\|\mathscr{U}'_{p\pm};H^3(\omega^\varepsilon)\|\\ &\le c_p\varepsilon^{3/2}(1+|\delta|).\end{aligned}$$

Here we have taken into account that $\omega^\varepsilon\backslash\omega^0$ is a thin set of width $O(\varepsilon)$.

For the computation of $\kappa^\varepsilon=\kappa^\varepsilon_{p\pm}$ in (4.52) we use the definitions of $t^\varepsilon_{p\pm}$ and $u^\varepsilon_{p\pm}$ in (4.53) to obtain

$$\begin{aligned}\kappa^\varepsilon_{p\pm} &= \langle\mathscr{U}^\varepsilon_{p\pm},\mathscr{U}^\varepsilon_{p\pm}\rangle^{-1/2}_{\pi+\varepsilon\delta}t^\varepsilon_{p\pm}\sup|(1+\Lambda^0_{p\pm}(\pi)+\varepsilon\Lambda'_{p\pm}(\delta)(\mathscr{U}^\varepsilon_{p\pm},v^\varepsilon)_\gamma\\ &\quad-(\nabla\mathscr{U}^\varepsilon_{p\pm},\nabla v^\varepsilon)_{\omega^\varepsilon}-(\mathscr{U}^\varepsilon_{p\pm},v^\varepsilon)_{\sigma_0}|\\ &= \langle\mathscr{U}^\varepsilon_{p\pm},\mathscr{U}^\varepsilon_{p\pm}\rangle^{-1/2}_{\pi+\varepsilon\delta}t^\varepsilon_{p\pm}\sup|(-\Delta\mathscr{U}^\varepsilon_{p\pm},v^\varepsilon)_{\omega^\varepsilon}+(g^\varepsilon_0,v^\varepsilon)_{\sigma_0}+(g^\varepsilon_d,v^\varepsilon)_{\sigma^\varepsilon_d}|.\end{aligned}\tag{4.55}$$

Here the supremum is calculated over all functions $v^\varepsilon\in H^1_{\pi+\varepsilon\delta}(\omega^\varepsilon)$ such that $\langle v^\varepsilon,v^\varepsilon\rangle_{\pi+\varepsilon\delta}=1$. Clearly,

$$\|v^\varepsilon;L^2(\omega^\varepsilon)\|+\|v^\varepsilon;L^2(\sigma_0)\|+\|v^\varepsilon;L^2(\sigma^\varepsilon_d)\|\le c.$$

In the sequel, we assume that

$$|\delta|\le c_p\varepsilon^{5/4}.\tag{4.56}$$

We then observe that

$$\begin{aligned}&|t^\varepsilon_{p\pm}|\le c_p(1+\varepsilon|\delta|)\le C_p,\\ &\langle\mathscr{U}^\varepsilon_{p\pm},\mathscr{U}^\varepsilon_{p\pm}\rangle\ge c_p(1-\varepsilon(1+|\delta|)-\varepsilon^2\delta(1+\delta))\ge\frac{1}{2}c_p>0.\end{aligned}$$

where c_p, C_p stand for different positive constants which may depend on p but are independent of $\varepsilon\in(0,\varepsilon_p)$. Collecting the above estimates we convert the relation (4.55) into

$$\kappa^\varepsilon_{p\pm}\le(\varepsilon^{3/2}(1+|\delta|)+\varepsilon^2(1+\delta^2)(1+\varepsilon|\delta|))\le c_p\varepsilon^{5/4}.$$

Hence by Lemma 4.1 there exist eigenvalues $\tau^\varepsilon_{p\pm}(\eta_p+\varepsilon\delta)$ of the operator $T^\varepsilon(\eta_p+\varepsilon\delta)$ such that

$$|\tau^\varepsilon_{p\pm}(\eta_p+\varepsilon\delta)-(1+\Lambda^0_{p\pm}(\eta_p)+\varepsilon\Lambda'_{p\pm}(\delta))^{-1}|\le c_p\varepsilon^{5/4},$$

or, in view of (4.49),

$$|\Lambda^\varepsilon_{q\pm}(\eta_p+\varepsilon\delta)-\Lambda^0_{p\pm}(\eta_p)-\varepsilon\Lambda'_{p\pm}(\delta)|\le c_p\varepsilon^{5/4}.\tag{4.57}$$

Due to the formula (3.39) and the assumption (3.40) the eigenvalues $\Lambda^{\varepsilon}_{p+}(\eta_p + \varepsilon\delta)$ and $\Lambda^{\varepsilon}_{p-}(\eta_p + \varepsilon\delta)$ are different from each other. Moreover they stay in a $c\varepsilon$-neighbourhood of the point $\Lambda^{0}_{p\pm}(\eta_p)$ which, according to (4.50), contains only the eigenvalues $\Lambda^{\varepsilon}_{2}(\eta_p + \varepsilon\delta)$ and $\Lambda^{\varepsilon}_{3}(\eta_p + \varepsilon\delta)$ if $p = \pm 1$, $\Lambda^{\varepsilon}_{1}(\eta_p + \varepsilon\delta)$ and $\Lambda^{\varepsilon}_{2}(\eta_p + \varepsilon\delta)$ if $p = 0$. Thus, these eigenvalues are distinct and satisfy the relation (4.57). In view of (2.17), (3.39) and (3.40) this observation together with inequalities (4.51) proves Theorem 2.1.

Acknowledgments. This research is supported by INDAM-GNAMPA. The second author was supported by the Russian Foundation of Basic Research grant 18-01-00325. The third author was supported by the Jenny and Antti Wihuri foundation. The authors gratefully acknowledge the hospitality of University of Oulu and of Politecnico di Torino.

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Received 30 July 2016,
and in revised form 21 December 2017.

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