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Number Theory — *An application of the value distribution theory for semi-abelian varieties to problems of Ax–Lindemann and Manin–Mumford types*, by JUNJIRO NOGUCHI, communicated on March 9, 2018.[1]

ABSTRACT. — The aim of this paper is to prove a theorem of Ax–Lindemann type for complex semi-abelian varieties as an application of a big Picard theorem proved by the author in 1981, and then apply it to prove a theorem of classical Manin–Mumford Conjecture for semi-abelian varieties, which was proved by M. Raynaud 1983, M. Hindry 1988, ..., and Pila–Zannier 2008 by a different method from others, which is most relevant to ours. The present result might be a first instance of a *direct connection at the proof level* between the value distribution theory of holomorphic maps and the arithmetic (Diophantine) theory over algebraic number fields, while there have been many *analogies* between them.

KEY WORDS: Ax–Lindemann, Manin–Mumford, torsion points, big Picard, Nevanlinna theory, semi-abelian variety

MATHEMATICS SUBJECT CLASSIFICATION: 11J95, 32H30, 03C64

1. INTRODUCTION AND MAIN RESULTS

The purpose of this paper is to prove a theorem of Ax–Lindemann type for complex semi-abelian varieties as an application of a big Picard theorem obtained in [6] for those varieties. We then apply it to prove a theorem of Manin–Mumford type for the distribution of torsion points on a subvariety of a semi-abelian variety defined over a number field, combined with extending a part of the arguments in Pila–Zannier [11] for abelian varieties (cf. §3). The statement for abelian varieties had been called the Manin–Mumford Conjecture and proved by M. Raynaud [12], M. Hindry [3] in the generalized form for abelian algebraic groups, ..., and Pila–Zannier [11]; cf. [11], Introduction, S. Lang [4], Chap. I §6, and e.g., P. Tzermias [13] for surveys of the Manin–Mumford Conjecture.

In the course of the proof the Kawamata structure theorem for semi-abelian varieties by [6], Lemma (4.1) works quite effectively (see §3 (b)).

THEOREM 1.1 (Ax–Lindemann type). *Let* $\exp : \mathbf{C}^n \to A$ *be an exponential map of a complex semi-abelian variety* A. *Let* $V \subset \mathbf{C}^n$ *be a complex irreducible affine algebraic subvariety with the restricted map* $\exp|_V : V \to A$. *Then the Zariski closure* $X(\exp|_V)$ *of the image of* $\exp|_V$ *in* A *is a translate of a complex semi-abelian subvariety of* A.

[1] Presented by Prof. U. Zannier.

Theorem 1.2 (Manin–Mumford type). *Let $X \subset A$ be a proper algebraic reduced subvariety of a semi-abelian variety A defined over an algebraic number field. Then, the Zariski closure $\overline{X}_{\mathrm{tor}}^{\mathrm{Zar}}$ of the set X_{tor} of all torsion points on X is a finite union of translates of semi-abelian subvarieties by torsion points on X.*

In view of the above two theorems, we are naturally led to study a denseness property of the value-distribution of $\exp|_V$ in Theorem 1.1. In fact, for an algebraic divisor D on A such that its closure in a projective compactification of A is ample, we will prove that $D \cap \exp(V)$ is Zariski dense in $D \cap X(\exp|_V)$ (cf. Theorem 4.2 in §4).

In §2 we will introduce a new notion of "*strictly transcendental*" holomorphic maps into semi-abelian varieties (see Definition 2.4). By making use of a Big Picard Theorem due to [6] we prove the image structure stated in Theorem 1.1 for strictly transcendental holomorphic maps into semi-abelian varieties (see Theorem 2.5). We then prove that the map $\exp|_V$ in Theorem 1.1 is strictly transcendental (Proposition 2.6).

In §3 we will prove Theorem 1.2 by induction on $\dim X$, in which we will use [6], Lemma (4.1). The proof of Theorem 1.2 roughly consists of two parts: The first is a decomposition of the torsion points on X to an algebraic part and its complement, done by the arguments due to Pila–Zannier [11], Theorem 2.1, Step 1, being extended to the semi-abelian case. The second part is the application of Theorem 1.1 to the algebraic part of torsion points, and for its complement we use the lower and upper estimates due to Masser and Pila–Wilkie (cf. [11], §3) to deduce the statement of Theorem 1.2.

In the last §5 we give some example of a transcendental but not strictly transcendental map into an abelian variety, and study its image structure.

The present result might be a first instance of a *direct connection at the proof level* between the value distribution theory of holomorphic maps and the arithmetic (Diophantine) theory over algebraic number fields, although there have been many *analogies* between them. It is a point of interest of this paper to observe that the direct connection is provided by the theory of o-minimal structures in model theory (Pila–Wilkie [9]; this approach evolved from Bombieri–Pila [1] (cf. Zannier [14]). This aspect should be of some interest for both sides of the theories.

2. Big Picard and Ax–Lindemann

(i) *Big Picard.* Let Δ^m be the unit polydisk of $\mathbf{C}^m$ with center at 0 and let $E \subset \Delta^m$ be a complex analytic proper subset. Let Y be a complex quasi-projective algebraic variety with reduced structure, and let $\overline{Y}$ denote a projective compactification of Y. In this paper we always assume that "varieties" are irreducible with reduced structure.

Let $f : \Delta^m \backslash E \to Y$ be a holomorphic map. If there is a meromorphic map $\bar{f} : \Delta^m \to \overline{Y}$ with the restriction to $\Delta^m \backslash E$, $\bar{f}|_{(\Delta^m \backslash E)} = f$, we say that E is *removable* for f or f is (meromorphically) *extendable* over E: Otherwise, f is said to be *transcendental*.

We denote by $Z(f)(\subset \overline{Y})$ the Zariski closure of the image $f(\Delta^m \backslash E)$ in $\overline{Y}$ and set

$$X(f) = Z(f) \cap Y. \tag{2.1}$$

In case $Y = \overline{Y}$, $X(f) = Z(f)$. Note that $X(f)$ is irreducible.

REMARK 2.2. (i) Just for the definition of the transcendence of $f : \Delta^m \backslash E \to Y$, the assumption of Y or $\overline{Y}$ being algebraic is not necessary, but we prefer to assume it for simplicity.

(ii) (Hartogs extension theorem) Let Y be as above. If $\operatorname{codim} E \geq 2$, E is always removable for any holomorphic map $f : \Delta^m \backslash E \to Y$. Therefore, as the removability is concerned with quasi-projective Y, it suffices to deal with smooth E of codimension 1.

Let A be a complex semi-abelian variety with a smooth projective compactification $\bar{A}$, and let $X \subset A$ be a non-empty complex algebraic subset. We denote by $\mathrm{St}^0(X)$ the connected component of 0 of the subgroup $\{a \in A : a + X = X\}$, and simply call it the *stabilizer* of X. Then, the stabilizer $\mathrm{St}^0(X)$ is an algebraic subgroup of A and a semi-abelian subvariety by itself. Note that X is of general type if and only if $\mathrm{St}^0(X) = \{0\}$.

We recall:

THEOREM 2.3 ([6], Corollary (4.7)). *Let $f : \Delta^m \backslash E \to A$ be a holomorphic map with $X(f)$ defined by* (2.1). *If* $\mathrm{St}^0(X(f)) = \{0\}$, *then f is extendable over E; in the other words, if f is transcendental, then* $\dim \mathrm{St}^0(X(f)) > 0$.

DEFINITION 2.4 (strictly transcendental). A transcendental holomorphic map $f : \Delta^m \backslash E \to A$ is said to be *strictly transcendental* if for every semi-abelian subvariety $B \subset A$ the composite $q_B \circ f : \Delta^m \backslash E \to A/B$ with the quotient map $q_B : A \to A/B$ is either constant or transcendental.

THEOREM 2.5. *Let $f : \Delta^m \backslash E \to A$ be a strictly transcendental holomorphic map. Then, $X(f) = f(c) + \mathrm{St}^0(X(f))$ with $c \in \Delta^m \backslash E$; i.e., $X(f)$ is a translate of a semi-abelian subvariety of A.*

PROOF. Set $B = \mathrm{St}^0(X(f))$ and $g = q_B \circ f : \Delta^m \backslash E \to A/B$. It suffices to show that g is constant. Otherwise, g would be transcendental by the assumption for f. Since $\mathrm{St}^0(X(g)) = \{0\}$, it follows from Theorem 2.3 that g is extendable over E; this is a contradiction. $\square$

(ii) *Ax–Lindemann.* We keep the notation used above. Let

$$\exp : \mathbf{C}^n \to A$$

be an exponential map. A map from a complex algebraic variety into another complex algebraic variety is called a *transcendental* map if it is *not* a rational map. We then define a *strictly transcendental map* from a complex algebraic variety into A as in Definition 2.4.

PROPOSITION 2.6. *Let $V \subset \mathbf{C}^n$ be a positive dimensional irreducible complex algebraic subvariety of $\mathbf{C}^n$. Then, the restricted map $\exp|_V : V \to A$ of $\exp$ to V is strictly transcendental.*

PROOF. Let $B \subsetneq A$ be a complex semi-abelian subvariety. Suppose that

$$q_B \circ \exp|_V : V \to A_1 = A/B$$

is a non-constant rational map. We shall deduce a contradiction.

There is an algebraic curve $C \subset V$ (irreducible and $\dim C = 1$) such that the restriction

$$f := q_B \circ \exp|_C : C \to A_1$$

is non-constant rational.

(a) Case of $B = \{0\}$: There is an exact sequence

$$0 \to (\mathbf{C}^*)^t \to A \to A_0 \to 0, \tag{2.7}$$

where A_0 is an abelian variety. Set $L = \exp^{-1}(\mathbf{C}^*)^t$. Then L is a t-dimensional vector subspace of $\mathbf{C}^n$. Let $p_0 : \mathbf{C}^n \to \mathbf{C}^n/L \cong \mathbf{C}^m$ $(m = n - t)$ and $q_0 : A \to A_0$ be the quotient maps. Then exp naturally induces an exponential map

$$\exp_0 : \mathbf{C}^m \to A_0.$$

We are going to infer

CLAIM 2.8. *$q_0 \circ \exp(C)$ is a point.*

Suppose that it is not the case. Then, the Zariski closure C_0 of the image $p_0(C)$ in $\mathbf{C}^m$ is an algebraic curve in $\mathbf{C}^m$. Let ω_0 be a flat Kähler metric on A_0 such that $\exp_0^* \omega_0 = \alpha$, where $\alpha = \sum_{j=1}^m \frac{i}{2\pi} dz_j \wedge d\bar{z}_j$ with the natural coordinate system $(z_1, \dots, z_m)$ of $\mathbf{C}^m$. Since $\exp|_C$ is rational, $\exp_0|_{C_0}$ is non-constant rational. Then, the Zariski closure $W := \overline{\exp_0(C_0)}^{\mathrm{Zar}}$ in A_0 is an algebraic curve in A_0 with

$$\int_{\exp_0(C_0)} \omega_0 = \int_W \omega_0 = M < \infty. \tag{2.9}$$

If k denotes the degree of the rational map $\exp_0|_{C_0} : C_0 \to W$, we have

$$\int_{C_0} \alpha = kM. \tag{2.10}$$

Let $B(r) \subset \mathbf{C}^m$ be an open ball of radius $r(>0)$ with center at a point $a \in C_0$ and set $C_0(r) = C_0 \cap B(r)$. Wirtinger's inequality implies

$$\int_{C_0(r)} \alpha \geq v(a; C_0) r^2,$$

where $v(a; C_0)(\geq 1)$ denotes the order of C_0 at a. Letting $r \to \infty$, we have a contradiction to (2.10). Therefore, Claim 2.8 follows.

Thus, we have a non-constant rational map after a translation:

$$\exp|_C : C(\subset L \cong \mathbf{C}^t) \to (\mathbf{C}^*)^t,$$

which is a restriction of exponential map

$$(\zeta_1, \dots, \zeta_t) \in \mathbf{C}^t \to (e^{\zeta_1}, \dots, e^{\zeta_t}) \in (\mathbf{C}^*)^t.$$

Let $\bar{C}$ be the closure of C in $(\mathbf{P}^1(\mathbf{C}))^t \supset (\mathbf{C}^*)^t$. Here we write $\mathbf{P}^1(\mathbf{C}) = \mathbf{C} \cup \{\infty\}$. Then there is a point $b = (b_1, \dots, b_t) \in \bar{C} \backslash C$ with some $b_j = \infty$. Since e^{ζ_j} has an isolated *essential* singularity at $\zeta_j = \infty$, $\exp|_C$ cannot be rational.

(b) Case of general B: Set

$$\begin{aligned} &F = \exp^{-1} B, \\ &p' : \mathbf{C}^n \to \mathbf{C}^n / F \cong \mathbf{C}^{n'}, \\ &q' : A \to A/B = A'. \end{aligned}$$

Let $\exp' : \mathbf{C}^{n'} \to A'$ be the naturally induced exponential from $\exp : \mathbf{C}^n \to A$. Let C' be the Zariski closure of $p'(C)$ in $\mathbf{C}^{n'}$. Then, it follows from the assumption that $\pi'|_{C'} : C' \to A'$ would be a non-constant rational map: This contradicts what was proved in (a) above. □

Proof of Theorem 1.1. This is now immediate by Proposition 2.6 and Theorem 2.5. □

Example 2.11. Here we give a simple example of a strictly transcendental map. Set

$$\begin{aligned} &\exp : (z, w) \in \mathbf{C}^2 \to (e^z, e^w) \in (\mathbf{C}^*)^2 = A, \\ &V = \{(z, w) \in \mathbf{C}^2 : zw = 1\} = \{(z, 1/z) : z \in \mathbf{C}^*\} \cong \mathbf{C}^*. \end{aligned}$$

Then V is affine algebraic,

$$\exp_V : (z, 1/z) \in V \to (e^z, e^{1/z}) \in A$$

is strictly transcendental, and $X(\exp|_V) = A$.

3. Proof of Theorem 1.2

Let K be a number field over which A is defined. Let

$$\exp : \mathbf{C}^n \to A$$

be an exponential map. The reduction of $\exp^{-1} X_{\mathrm{tor}}$ to the algebraic part is done in parallel to the proof of Pila–Zannier [11] §3, relying on the o-minimal structure theory, and then we apply Theorem 1.1 to conclude the proof.

The proof is done by induction on $v = \dim X \geq 0$. If $v = 0$, it is trivial. Suppose that the case of $\dim X \leq v - 1$ $(v \geq 1)$ holds. Let $\dim X = v$.

We consider the stabilizer $\mathrm{St}^0(X)$.

(a) The case of $\dim \mathrm{St}^0(X) > 0$. We set the quotient map $q : A \to A_1 = A/\mathrm{St}^0(X)$ and $X_1 = q(X) = X/\mathrm{St}^0(X)$. Since $\dim X_1 < v$, the induction hypothesis implies that there are at most finitely many semi-abelian subvarieties $B_j \subset A_1$, $1 \leq j \leq l$, and torsion points $b_j \in X_{1\,\mathrm{tor}}$ such that

$$X_{1\,\mathrm{tor}} = \bigcup_{j=1}^{l} (b_j + B_{j\,\mathrm{tor}}).$$

Taking any elements $a_j \in q^{-1} b_j \cap X_{\mathrm{tor}}$, $1 \leq j \leq l$, we have

$$X_{\mathrm{tor}} \subset \bigcup_{j=1}^{l} (a_j + (q^{-1} B_j)_{\mathrm{tor}}).$$

Since $X \supset a_j + q^{-1} B_j$, the opposite inclusion above also holds; thus,

$$\overline{X}_{\mathrm{tor}}^{\mathrm{Zar}} = \bigcup_{j=1}^{l} (a_j + q^{-1} B_j).$$

(b) The case of $\mathrm{St}^0(X) = \{0\}$. Note that X is of (log) general type. We put $\Lambda = \exp^{-1}\{0\}$ $(\subset \mathbf{C}^n)$ and $Z = \exp^{-1} X$ which is a Λ-periodic analytic subset of $\mathbf{C}^n$. We consider a basis of Λ as $\mathbf{Z}$-module, which is also a basis of the real vector space $\mathbf{R}\Lambda$ of dimension $d = 2n - t$. Thus we have identifications

$$\mathbf{Z}^d \cong \mathbf{Z}\Lambda \subset \mathbf{R}\Lambda \cong \mathbf{R}^d.$$

The torsion points on X lift to points on $\mathbf{Q}\Lambda \cong \mathbf{Q}^d \subset \mathbf{R}^d$. We consider the set $\check{Z} = Z \cap \mathbf{Q}^d$ of the preimage of all torsion points on X and its restriction $\check{Z}_1 := \check{Z} \cap [0,1]^d$ of $\check{Z}$ to the closed fundamental domain $[0,1]^d$.

For a subset $W \subset \mathbf{Q}^d$ and a real number $T \geq 1$, we denote by $N_W(T)$ the number of rational points in W whose denominators are at most T.

By Pila–Wilkie [9] there is a so-called algebraic part $\check{Z}_1^{\mathrm{alg}}$ of $\check{Z}_1$, satisfying the following properties:

(i) We have

$$\check{Z}_1^{\mathrm{alg}} = \bigcup_{V \subset Z} (V \cap \{t = (t_j) \in [0,1]^d : t_j \in \mathbf{Q}\}), \tag{3.1}$$

where V runs over all positive dimensional affine algebraic subsets of $\mathbf{C}^n$, contained in Z (cf. [11], Proposition 2.1; here, it is noted that the periodicity condition of Proposition 2.1 is not used in the proof, so that it can be applied to our Z.).

(ii) For every $\varepsilon > 0$ there is a positive constant $c_1 (= c_1(\check{Z}_1, \varepsilon)$, depending on $\check{Z}_1$ and ε) such that

$$N_{\check{Z}_1'}(T) \leq c_1 T^{\varepsilon}, \quad T \geq 1, \tag{3.2}$$

where $\check{Z}_1' := \check{Z}_1 \setminus \check{Z}_1^{\mathrm{alg}}$, called the transcendental part of $\check{Z}_1$ ([9]).

We first analyze the algebraic part $\check{Z}_1^{\mathrm{alg}}$, assuming $\check{Z}_1^{\mathrm{alg}} \neq \emptyset$. Let V be as in (3.1). It follows from Theorem 1.1 that $X(\exp|_V)$ $(\subset X)$ is a translate of a positive dimensional semi-abelian subvariety of A. Let Y denote the union of all translates $a + B'$ $(a \in X)$ of positive dimensional semi-abelian subvarieties $B' \subset A$ such that $a + B' \subset X$. It follows from [6], Lemma (4.1) (cf. also [7], §5.6.4, and [11], p. 160) that Y $(\subset X)$ is an algebraic subset of dimension $< v$ and for every irreducible component Y' of Y, $\dim \mathrm{St}^0(Y') > 0$. We may assume that Y and Y' are defined over K. Applying the induction hypothesis to Y, we have finitely many positive dimensional semi-abelian subvarieties $B_j' \subset A$, $1 \leq j \leq l'$, and $P_j \in Y_{\mathrm{tor}}$ such that

$$Y_{\mathrm{tor}} = \bigcup_{j=1}^{l'} (P_j + B'_{j\,\mathrm{tor}}).$$

Therefore, we have

$$\exp(\check{Z}_1^{\mathrm{alg}}) = \bigcup_{j=1}^{l'} (P_j + B'_{j\,\mathrm{tor}}), \quad \overline{\exp(\check{Z}_1^{\mathrm{alg}})}^{\mathrm{Zar}} = \bigcup_{j=1}^{l'} (P_j + B_j').$$

We may assume that all P_j and B_j' are defined over K after a finite extension of K. Then, we have:

3.3 (Invariance). For a torsion point $P \in \exp(\check{Z}_1^{\mathrm{alg}})$ (resp. $\exp(\check{Z}_1')$), all of its conjugates over K lie in $\exp(\check{Z}_1^{\mathrm{alg}})$ (resp. $\exp(\check{Z}_1')$).

To analyze the part $\check{Z}_1'$ we prepare:

Lemma 3.4. *Let the notation be as above. Let $P \in A$ be a torsion point of exact order N. Then, there is a number $\rho > 0$ depending only on* $\dim A$ *such that*

$$[K(P) : K] \geq c_2 N^{\rho}, \quad N \geq 1, \tag{3.5}$$

where $c_2 = c_2(A, K)$ is a positive constant depending only on A and K.

Proof. For the semi-abelian variety A we have the following exact sequence

$$0 \to \mathbf{G}_m^t \to A \xrightarrow{\pi} A_0 \to 0,$$

where $\mathbf{G}_m^t$ is an algebraic torus and A_0 is an abelian variety; we may assume that all of the above morphisms and algebraic groups are defined over K. Then, $P_0 := \pi(P)$ is a torsion point of A_0, whose exact order is denoted by N_0. Note that $N_0 P \in \mathbf{G}_m^t$ and $N = N_1 N_0$, where N_1 is the order of $N_0 P$ $(\in \mathbf{G}_m^t)$. Then,

$$c_3 \varphi(N_1) \leq [K(N_0 P) : K] \leq [K(P) : K],$$

where $c_3 = c_3(t, K)$ is a positive constant and φ is the Euler function. It is known that

$$\varphi(N_1) \gg \frac{N_1}{\log \log N_1}.$$

If $N_1 \geq \sqrt{N}$, the proof is finished. Otherwise, we have $N_0 \geq \sqrt{N}$. We then apply Masser's estimate ([5]) for the following second inequality:

$$c_4 N^{\rho'/2} \leq c_4 N_0^{\rho'} \leq [K(P_0) : K] \leq [K(P) : K],$$

where $c_4 = c_4(A_0, K)$ is a positive constant. Thus, we have the lower estimate (3.5). □

Finish of the proof of Theorem 1.2. Let now $P \in \exp(\check{Z}_1')$ be a torsion point of exact order N. It follows from Invariance 3.3 that $N_{\check{Z}_1'}(N) \geq [K(P) : K]$. By Lemma 3.4 we see that

$$N_{\check{Z}_1'}(N) \geq c_2 N^{\rho}. \tag{3.6}$$

On the other hand, we have by (3.2)

$$N_{\check{Z}_1'}(N) \leq c_1 N^{\varepsilon}. \tag{3.7}$$

Taking $\varepsilon = \rho/2$ with ρ in (3.6), we conclude the boundedness of N by (3.6) and (3.7). Therefore, $\check{Z}_1'$ is a finite set. □

4. Distribution of $\exp(V)$ on divisors

Let A be a semi-abelian variety with exponential map

$$\exp : \mathbf{C}^n \to A,$$

and let $V \subset \mathbf{C}^g$ be an irreducible affine algebraic subvariety. Let $\bar{A}$ be a projective compactification of A. Because of the results of the previous sections, it might be of some interest to look at the actual value-distribution of $\exp|_V(V)$ in its Zariski closure $X(\exp|_V)$.

We first deal with a transcendental holomorphic map $f : \Delta^* \to A$ from a punctured disk Δ^* into A. In [8] we dealt with entire holomorphic maps from the whole plane $\mathbf{C}$ into A. Combining the method of [6] with the result and the arguments explored in [8] and [7] we see that the results of [8] (and [7], Chap. 6) hold for transcendental holomorphic maps from Δ^* into A. In particular, we have (cf., also [2], Theorem 5.2)

Theorem 4.1. *Let $f : \Delta^* \to A$ be a transcendental holomorphic map with $X(f)(\subset A)$ (cf. (2.1)). Let D be an effective algebraic reduced divisor on A such that the closure of D in $\bar{A}$ is ample and $D \not\supset f(\Delta^*)$ (where D stands also for the support of D). Then $f(\Delta^*) \cap D$ is Zariski dense in $X(f) \cap D$.*

By making use of this we prove:

Theorem 4.2 *Let $V \subset \mathbf{C}^n$ be an irreducible complex affine algebraic subvariety, and let D be as in Theorem* 4.1. *Then, the intersection $D \cap \exp(V)$ is Zariski dense in $D \cap X(\exp|_V)$.*

Proof. Let $\zeta_0 \in V$ be fixed. We consider a pencil of affine algebraic curves $C_\gamma \subset V$, $\gamma \in \Gamma$, passing through ζ_0, such that $\bigcup_\gamma C_\gamma$ contains a non-empty open subset of V in the sense of differential topology. By Theorem 1.1 $X(\exp|V)$ and $X(\exp|_{C_\gamma})$ are all translates of semi-abelian subvarieties of A passing through $\exp(\zeta_0)$. Since there are at most countably many such semi-abelian subvarieties, one finds a curve $C_0 = C_\gamma$ such that $X(\exp|_{C_0}) = X(\exp|_V)$. Then it suffices to show the theorem for C_0. Let $C_1 \to C_0$ be the normalization and let $\bar{C}_1$ be its smooth compactification. Then there is an analytic neighborhood U $(\subset \bar{C}_1)$ of a point Q of $\bar{C}_1 \backslash C_1$ such that $U \backslash \{Q\}$ is biholomorphic to a punctured disk Δ^*. Then our assertion is immediate by Theorem 4.1. □

5. An example of a transcendental but not strictly transcendental map

Let $\bar{C}$ be a smooth complex projective algebraic curve of genus $g \geq 1$ and let $q : \bar{C} \to J(\bar{C})$ be the Jacobian embedding; here, when $g = 1$, we simply take $q : \bar{C} \to \bar{C}(= J(\bar{C}))$ as the identity map. We set $A_1 = J(\bar{C})$, which is an abelian variety of dimension g. Let $Q \in \bar{C}$ be a point and set $C = \bar{C} \backslash \{Q\}$. Then C is affine algebraic and there is a finite map $p : C \to \mathbf{C}$.

Let $\exp : \mathbf{C}^g \to A_1$ be an exponential map. We take a linear embedding $\lambda : \mathbf{C} \to \mathbf{C}^g$ that is in sufficiently generic direction with respect to the period lattice of $\exp : \mathbf{C}^g \to A_1$. Then, $X(\exp \circ \lambda) = A_1$. We put

$$f : x \in C \to (q(x), \exp \circ \lambda \circ p(x)) \in A_1 \times A_1 =: A. \tag{5.1}$$

PROPOSITION 5.2. *Let $f : C \to A$ be as above.*

(i) *The holomorphic map f is transcendental but not strictly transcendental.*
(ii) *If $g \geq 2$, the Zariski closure $X(f)$ of the image $f(C)$ is* not *a translate of an abelian subvariety of A.*
(iii) *If $g = 1$, $X(f) = A$.*

PROOF. (i) The first half is clear. For the latter, note that $X(f) \subset \bar{C} \times A_1$. With a subgroup $\{0\} \times A_1 \subset A_1 \times A_1 = A$ we consider the quotient map $\mu : A \to A/\{0\} \times A_1 \cong A_1$. Then, $\mu \circ f = q|_C : C \to A_1$ is rational. Therefore, f is not strictly transcendental.

(ii) Since $\mu(X(f)) = q(\bar{C})$, $X(f)$ is not a translate of an abelian subvariety.

(iii) Since $\dim X(f) = 1$, or 2, it suffices to deduce a contradiction with assuming $\dim X(f) = 1$. If so, $X(f)$ is a translate of an abelian subvariety of A. We consider an effective divisor $D = \bar{C} \times \{w\}$ with $w \in \bar{C}$. We infer from the definition of f that $X(f) \cap D$ is infinite. Therefore, $X(f) = D$; this is a contradiction. □

REMARK 5.3. Păun–Sibony [10] deals with a similar application of the Bloch–Ochiai Theorem to the abelian Ax–Lindemann statement ([10], Theorem 5.2). But with regard to Proposition 5.2 above, in Theorems 5.2 of [10] one might be able to have only the non-triviality of the stabilizer of the Zariski closure of the image as in Theorem 2.3, obtained in [6] (Corollary (4.7)).

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Junjiro Noguchi
Graduate School of Mathematical Sciences
The University of Tokyo
Komaba, Meguro-ku
Tokyo 153-8914, Japan
noguchi@ms.u-tokyo.ac.jp