Rend. Lincei Mat. Appl. 29 (2018), 431–437
DOI 10.4171/RLM/815

Calculus of Variations — *BMO-type norms and anisotropic surface measures*, by Giovanni E. Comi, communicated on February 9, 2018.[1]

Abstract. — The purpose of this note is to present an anisotropic variant of the *BMO*-type norm introduced in [4] and to show its relation with a surface measure, which is indeed a multiple of the perimeter in the isotropic case. This is done in the spirit of the new characterization of the perimeter of a measurable set in $\mathbb{R}^n$ recently studied by Ambrosio, Bourgain, Brezis and Figalli in [2].

Key words: Sets of finite perimeter, *BMO*-type norms, anisotropic perimeter, sphere packing problem

Mathematics Subject Classification: 42B35

1. Introduction

Ambrosio, Bourgain, Brezis and Figalli recently studied in [1] and [2] a new characterization of the perimeter of a set in $\mathbb{R}^n$ by considering the following functionals originating from a *BMO*-type seminorm (defined at first in [4]):

$$\mathsf{I}_\varepsilon(f) = \varepsilon^{n-1} \sup_{\mathcal{G}_\varepsilon} \sum_{Q' \in \mathcal{G}_\varepsilon} \fint_{Q'} \left| f(x) - \fint_{Q'} f \right| dx, \tag{1.1}$$

where $\mathcal{G}_\varepsilon$ is any disjoint collection of ε-cubes Q' with arbitrary orientation and cardinality not exceeding ε^{1-n}.

In particular, they focused on the case $f = \mathbf{1}_A$; that is, the characteristic function of a measurable set A, and proved that

$$\lim_{\varepsilon \to 0} \mathsf{I}_\varepsilon(\mathbf{1}_A) = \frac{1}{2} \min\{1, \mathsf{P}(A)\}. \tag{1.2}$$

Moreover, if we remove the cardinality bound on $\mathcal{G}_\varepsilon$ from the definition of I_ε, by scaling we obtain

$$\lim_{\varepsilon \to 0} \mathsf{I}_\varepsilon(\mathbf{1}_A) = \frac{1}{2} \mathsf{P}(A). \tag{1.3}$$

[1] This paper is related to a talk given at "XXVII Convegno Nazionale di Calcolo delle Variazioni" – Levico Terme (Trento) 6–10 February, 2017.

This theme has been further investigated in [6], where the authors considered the general case of a BV function f, with a particular attention to the SBV space. We also refer to [7] for a variant of this construction that led to Sobolev and fractional Sobolev norms and spaces.

In [3], Ambrosio and the author consider the more general case of anisotropic coverings formed by copies of the ε-dilation of a bounded open set with Lipschitz boundary C; that is, not allowing for arbitrary orientations. In doing so, we also remove the upper bound on cardinality that seems to be very specific of the case of cubes. Thus, we define

$$H_\varepsilon^C(A) := \varepsilon^{n-1} \sup_{\mathcal{H}_\varepsilon} \sum_{C' \in \mathcal{H}_\varepsilon} \fint_{C'} \left| \mathbf{1}_A(x) - \fint_{C'} \mathbf{1}_A \right| dx, \tag{1.4}$$

where $\mathcal{H}_\varepsilon$ is any disjoint family of translations C' of the set εC with no bounds on cardinality.

2. The main results

The main result of [3] is the following:

Theorem 2.1. *There exists $\varphi^C : \mathbb{S}^{n-1} \to (0, +\infty)$, bounded and lower semicontinuous, such that, for any set of finite perimeter A, one has*

$$\lim_{\varepsilon \to 0} H_\varepsilon^C(A) = \int_{\mathscr{F}A} \varphi^C(\nu_A(x))\, d\mathscr{H}^{n-1}(x), \tag{2.1}$$

where $\mathscr{F}A$ and ν_A are respectively the reduced boundary of A and the approximate unit normal to $\mathscr{F}A$. Moreover, if A is measurable and $\mathsf{P}(A) = \infty$, one has

$$\lim_{\varepsilon \to 0} H_\varepsilon^C(A) = +\infty. \tag{2.2}$$

We give here just a short sketch of the proof, with a highlight on the key steps.

At first, we define suitable localized versions $H_\varepsilon(A, \Omega)$ of the functionals, by taking a covering inside Ω; and we set $H_\pm(A, \Omega)$ to be the lim sup and the lim inf as $\varepsilon \to 0$. We notice that we have the scaling property $H_\varepsilon^{\lambda C}(A, \Omega) = \lambda^{1-n} H_{\varepsilon\lambda}^C(A, \Omega)$ and $H_\pm^{\lambda C}(A, \Omega) = \lambda^{1-n} H_\pm^C(A, \Omega)$, for any $\lambda > 0$.

A simple comparison argument based on the results of [2] leads to the proof of (2.2). Indeed, one can show that, if $D \subset C$, there exists a constant $\theta(C, D) > 0$ such that

$$H_\varepsilon^D(A, \Omega) \le \frac{|C|^2}{|D|^2} \theta H_\varepsilon^C(A, \Omega). \tag{2.3}$$

We notice now that, without loss of generality, we can assume $B(0, r) \subset C$ for some $r > 0$, and that we can pack a cube of side length $2/\sqrt{n}$ in a unit ball.

Hence, it is enough to compare the functional H_ε^B, defined using covering with ε-balls, and the functional I_ε without the cardinality bound on the covering families, thus obtaining

$$\liminf_{\varepsilon\to 0} H_\varepsilon^C(A) \geq c_{n,r} \liminf_{\varepsilon\to 0} H_\varepsilon^B(A) \geq \tilde{c}_{n,r} \liminf_{\varepsilon\to 0} \mathsf{I}_{\frac{2}{\sqrt{n}}\varepsilon}(\mathbf{1}_A) = +\infty,$$

for any measurable set A of infinite perimeter, by (1.3).

As for the rectifiable case, we fix C, dropping the superscript, and we assume that $\mathrm{diam}(C) = 1$, without loss of generality by the scaling property. Then, we observe that, for any set E of finite perimeter, $H_\varepsilon(E, \cdot)$ and $H_\pm(E, \cdot)$ are increasing set functionals defined on the family of open sets, which are traslation invariant and $(n-1)$-homogeneous; that is, for any $x \in \mathbb{R}^n$,

$$H_\varepsilon(x+E, x+\Omega) = H_\varepsilon(E, \Omega) \quad \text{and} \quad H_\pm(x+E, x+\Omega),$$

and, for any $t > 0$,

$$H_{t\varepsilon}(tE, t\Omega) = t^{n-1} H_\varepsilon(E, \Omega) \quad \text{and} \quad H_\pm(tE, t\Omega) = t^{n-1} H_\pm(E, \Omega).$$

In addition, $H_-(E, \cdot)$ is superadditive and $H_+(E, \cdot)$ is almost subadditive, in the sense that

$$H_+(E, \Omega_1 \cup \Omega_2) \leq H_+(E, W_1) + H_+(E, W_2),$$

for any open sets $W_i \supset \{x \in \mathbb{R}^n : \mathrm{dist}(x, \Omega_i) < \delta\}$, $i = 1, 2$, for some $\delta > 0$. Moreover, by the relative isoperimetric inequality which holds in the open bounded set C with Lipschitz boundary, we obtain an upper bound for H_+:

$$H_+(E, \Omega) \leq 2\gamma \mathsf{P}(E, \Omega), \tag{2.4}$$

where γ is the relative isoperimetric constant of C.

We then define the upper and lower density of $H_\pm$ by setting

$$\varphi_\pm(v) := H_\pm(S_v, Q_v),$$

where $v \in \mathbb{S}^{n-1}$, $S_v := \{x \in \mathbb{R}^n : x \cdot v \geq 0\}$ and Q_v is a unit cube centered in the origin having one face orthogonal to v and bisected by the hyperplane ∂S_v. It is possible to show that φ is bounded from above and from below, by (2.4) and by a comparison argument employing (2.3) and (1.3), respectively. In addition, the superadditivity, homogeneity and translation invariance of $H_-(S_v, \cdot)$ imply that

$$\varphi_-(v) \geq \sup_{t>0} H_t(S_v, Q_v),$$

which then shows that φ_- is lower semicontinuous and $\varphi_- = \varphi_+$.

Therefore, we can define the density

$$\varphi(v) := \lim_{\varepsilon\to 0} H_\varepsilon(S_v, Q_v),$$

and it is clear that, if $x \in \partial S_\nu$ and $Q_\nu(x,r)$ is a cube of side length r centered in x and with one face orthogonal to ν, by translation invariance and homogeneity we have

$$\lim_{r\to 0} \frac{H_\pm(S_\nu, Q_\nu(x,r))}{r^{n-1}} = H_\pm(S_\nu, Q_\nu(x,1)) = \varphi(\nu).$$

By an argument employing a modulus of continuity of the map $E \to H_\varepsilon(E,\Omega)$ and the fine properties of the sets of finite perimeter, we can prove a result on the lower and upper density of $H_\pm(E,\cdot)$ with respect to the measure $\mathsf{P}(E,\cdot)$.

Theorem 2.2. *Let E be a set of finite perimeter and ν_E be its measure theoretic interior normal. Then, for $\mathscr{H}^{n-1}$-a.e. $x \in \mathscr{F}E$, we have*

$$D^-_{\mathsf{P}} H_-(x) := \liminf_{r\to 0} \frac{H_-(E, Q_{\nu_E(x)}(x,r))}{\mathsf{P}(E, Q_{\nu_E(x)}(x,r))} \geq \varphi(\nu_E(x)), \tag{2.5}$$

$$D^+_{\mathsf{P}} H_+(x) := \limsup_{r\to 0} \frac{H_+(E, Q_{\nu_E(x)}(x,r))}{\mathsf{P}(E, Q_{\nu_E(x)}(x,r))} \leq \varphi(\nu_E(x)). \tag{2.6}$$

In particular, it follows that

$$D^-_{\mathsf{P}} H_-(x) = D^+_{\mathsf{P}} H_+(x) = \varphi(\nu_E(x)) \quad \text{for } \mathscr{H}^{n-1}\text{-a.e. } x \in \mathscr{F}E.$$

To proceed, we apply to the nondecreasing set functions $H_\pm(E,\cdot)$ an argument similar to the classical density theorems for measures, for which we need the Vitali covering theorem for cubes and properties which replace the additivity. Indeed, H_- is superadditive, and this is sufficient to achieve a lower bound; however, $H_+(E,\cdot)$ is not a subadditive set function on the family of open sets, hence we consider its inner regular envelope

$$H^*_+(E,\Omega) := \sup\{H_+(E,\Omega') : \Omega' \Subset \Omega\},$$

which is actually σ-subadditive.

Theorem 2.3. *For any Borel set $B \subset \mathscr{F}E$ and $t > 0$, we have that*

$$\liminf_{r\to 0} \frac{H_-(E, Q_{\nu_E(x)}(x,r))}{\mathsf{P}(E, Q_{\nu_E(x)}(x,r))} \geq t \tag{2.7}$$

for all $x \in B$ implies $H_-(E,U) \geq t\mathscr{H}^{n-1}(B)$ for any open set $U \supset B$. On the other hand, we have that

$$\limsup_{r\to 0} \frac{H_+(E, Q_{\nu_E(x)}(x,r))}{\mathsf{P}(E, Q_{\nu_E(x)}(x,r))} \leq t \tag{2.8}$$

for all $x \in B$ *implies* $H_+^*(E, U) \le t\mathsf{P}(E, U) + 2\gamma\mathsf{P}(E, U \setminus B)$ *for any open set* $U \supset B$.

We now use the previous results to adapt the classical proofs of the differentiation theorem for Radon measures to $H_\pm(E, \cdot)$.

The key idea is to partition $\mathscr{F}E$ in the family of sets $\{x \in \mathscr{F}E : \varphi(\nu_E(x)) \in (t^k, t^{k+1}]\}$ for some $t > 1$ fixed and $k \in \mathbb{Z}$, and then use the density theorems. Letting $t \downarrow 1$, we obtain

$$\int_{\mathscr{F}E} \varphi(\nu_E)\, d\mathscr{H}^{n-1} \le H_-(E, \mathbb{R}^n) \le H_+^*(E, \mathbb{R}^n) \le \int_{\mathscr{F}E} \varphi(\nu_E)\, d\mathscr{H}^{n-1}.$$

Indeed, the superadditivity of $H_-(E, \cdot)$ ensures the lower estimate and the σ-subadditivity of $H_+^*(E, \cdot)$, together with (2.4), provides the upper estimate.

Then, it is easy to show that $H_+^*(E, \mathbb{R}^n) = H_+(E, \mathbb{R}^n)$, and so we get (2.1).

In addition, it is possible to achieve a localized version of the main results for $H_\varepsilon^C(E, A)$ on a rich family of open sets A.

We notice that the right hand side of (2.1) can be seen as an anisotropic version of the perimeter, $\mathsf{P}_\varphi(A)$. However, the anisotropic perimeter is lower semicontinuous w.r.t. the convergence in measure if and only if the density φ is the restriction to the unit sphere of a positively 1-homogeneous and convex function. Hence, even though the particular geometry of the covering sets is not essential to prove the existence of the limit, one might ask if there are conditions under which φ has indeed that property. The problem is nontrivial since we can show that, if C is the unit square $(0,1)^2$ in $\mathbb{R}^2$, then the positively 1-homogeneous extension of φ^C is not convex. In particular, the convexity of C is not a sufficient condition, and actually no sufficient condition is presently known.

It is however not difficult to see that φ is a constant if we allow for arbitrary rotations of the covering sets, or if we choose as C a set invariant under rotations. In particular, if C is the unit open cube in this isotropic setting, we recover $\varphi \equiv 1/2$, as in [2].

2.1. Covering with balls

If $B = B(0,1)$ is the unit ball, for any set E of finite perimeter one has

$$\lim_{\varepsilon \to 0} H_\varepsilon^B(E) = \xi \mathsf{P}(E),$$

for some dimensional constant $\xi = \xi(n)$. It would be of interest to estimate the value of such constant, which can be also seen as $\xi_n = H_\pm^B(S_\nu, Q_\nu)$, for any $\nu \in \mathbb{S}^{n-1}$.

By a result due to Cianchi ([5]), we know that the sharp isoperimetric constant in the unit ball is $1/(4\omega_{n-1})$, where ω_n is the volume of the unit ball in $\mathbb{R}^n$. This helps us finding an upper bound for ξ_n, by (2.4).

On the other hand, the derivation of a lower bound is related to the well-known Kepler's problem (see for instance [8], [10]). This problem, also called "packing problem", consists in looking for the best way to place finite unions of disjoint open balls with the same (small) radius inside a unit cube in $\mathbb{R}^n$ in order to cover as much volume as possible. As the radius tends to 0, it is possible to show that the ratio of volume covered converges to the best volume fraction $\rho_n \in (0, 1]$. Kepler's problem is highly non trivial, and the value of the constant ρ_n is presently known only in dimensions 2 and 3 ([11], [9]).

Since we can choose a covering family of ε-balls which are inside Q_ν and are bisected by ∂S_ν, our aim is to give a lower estimate of the cardinality of such covering. In this way, it is clear that we are looking for the optimal fraction of the volume of the $(n-1)$ unit cube $Q_\nu \cap \partial S_\nu$ which can be covered by a finite union of disjoint ε-balls as $\varepsilon \to 0$. Then, the number N_ε of $(n-1)$-dimensional ε-balls of such an optimal covering will satisfy

$$N_\varepsilon \omega_{n-1} \varepsilon^{n-1} \sim \rho_{n-1}.$$

These $(n-1)$-dimensional ε-balls can be seen as the sections $\partial S_\nu \cap B'$ for some disjoint n-dimensional ε-balls B' which are bisected by the hyperplane ∂S_ν and lie inside the cube Q_ν. Therefore, we get

$$\xi_n \geq \lim_{\varepsilon \to 0} \varepsilon^{n-1} \frac{1}{2} N_\varepsilon = \frac{\rho_{n-1}}{2\omega_{n-1}}.$$

Using these observations, we can find the following lower and upper bounds for the constants ξ_n:

$$\frac{\rho_{n-1}}{2\omega_{n-1}} \leq \xi_n \leq \frac{1}{2\omega_{n-1}}.$$

Detailed proofs and other examples can be found in [3].

References

[1] L. Ambrosio - H. Brezis - J. Bourgain - A. Figalli, *Perimeter of sets and BMO-type norms*, CRAS, 352 (2014), 697–698.

[2] L. Ambrosio - H. Brezis - J. Bourgain - A. Figalli, *BMO-type norms related to the perimeter of sets*, Comm. Pure Appl. Math., 69 (2016), 1062–1086.

[3] L. Ambrosio - G. E. Comi, *Anisotropic surface measures as limits of volume fractions*, cvmgt preprint, to appear in the volume "Nonlinear Analysis", dedicated to H. Brezis e L. Nirenberg, 2016.

[4] J. Bourgain - H. Brezis - P. Mironescu, *A new function space and applications*, Journal of the EMS, 17 (2015), 2083–2101.

[5] A. Cianchi, *A sharp form of Poincaré type inequalities on balls and spheres*, Z. Angew. Math. Phys. 40 (1989), no. 4, 558–569.

[6] N. Fusco - C. Sbordone - G. Moscariello, *A formula for the total variation of SBV functions*, J. Funct. Anal., 270 (2016), 419–446.

[7] N. Fusco - C. Sbordone - G. Moscariello, *BMO-type seminorms and Sobolev functions*, ESAIM: Control, Optimisation and Calculus of Variations (2016).

[8] T. C. Hales, *Cannonballs and honeycombs*, Notices of the American Mathematical Society, 47 (2000), 440–449.

[9] T. C. Hales, *A proof of the Kepler conjecture*, Annals of Mathematics, 162 (2005), 1065–1185.

[10] G. C. Szpiro, *Kepler's conjecture*, John Wiley and Sons, Inc., 2003.

[11] L. F. Toth, *Über die dichteste Kugellagerung*, Math. Z. 48 (1943), 676–684.

Received 24 May 2017,
and in revised form 16 January 2018.

Giovanni E. Comi
Scuola Normale Superiore
Piazza Cavalieri 7
56126 Pisa, Italy
giovanni.comi@sns.it