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Partial Differential Equations — *New techniques for solving some class of singular elliptic equations*, by Michel Chipot and Linda Maria De Cave, communicated on March 9, 2018.

Abstract. — We consider existence and uniqueness of solutions to elliptic problems set in open subsets of $\mathbb{R}^N$, bounded and unbounded. These problems are characterised by the presence of a linear higher order term and a nonlinear lower order term which may blow up where the solution is zero and which involves a distribution.

Key words: Singular elliptic equations, asymptotic analysis, cylinders

Mathematics Subject Classification: 35J25, 35J75, 35B40

1. Introduction

The aim of this paper is to find u solution to

$$(1.1)\qquad \begin{cases} u \in W_0^{1,2}(\Omega), \\ \int_\Omega A(x)\nabla u \cdot \nabla v \, dx = \langle \mu, H(u)v \rangle \quad \forall v \in V, \end{cases}$$

where V is a space of functions containing the Schwartz-space $\mathscr{D}(\Omega)$ of C^∞-functions with compact support in Ω (see [2], [7], [11] for the references on the spaces used here). The open set Ω can be bounded or unbounded in $\mathbb{R}^N$, μ is a suitable distribution defined on Ω, $A(x)$ is a bounded and elliptic matrix, namely such that there exist two positive constants λ and Λ such that

$$(1.2)\qquad \lambda|\xi|^2 \le A(x)\xi \cdot \xi, \quad |A(x)\xi| \le \Lambda|\xi|$$

for all $\xi \in R^N$ and for almost every $x \in \Omega$. Finally, H is a continuous function outside the origin that may blow up at zero. Note that the symbol $\langle \cdot, \cdot \rangle$ stands for the duality product between an element of some Sobolev space and an element of its dual, a dot denotes the scalar product in $\mathbb{R}^N$, $|\ |$ the Euclidean norm.

Regarding μ, we will always assume that $\mu \in W^{-1,2}(\Omega)$, the dual of $W_0^{1,2}(\Omega)$, or that $\mu \in W_{\ell oc}^{-1,2}(\Omega)$. At some point we will suppose in addition that μ is a nonnegative measure. Recall that, if $\mu \in W^{-1,2}(\Omega)$, by the Lax–Milgram Theorem there exists one and only one element $\hat{\mu} \in W_0^{1,2}(\Omega)$ such that

$$(1.3)\qquad \langle \mu, v\rangle = \int_\Omega A(x)\nabla\hat{\mu}\cdot\nabla v\,dx \quad \forall v \in W_0^{1,2}(\Omega) \quad \text{and}$$

$$\|\hat{\mu}\|_{W_0^{1,2}(\Omega)} \leq \frac{1}{\lambda}\|\mu\|_{W^{-1,2}(\Omega)}$$

and, since

$$\langle \mu, v\rangle \leq \Big(\int_\Omega |A(x)\nabla\hat{\mu}|^2\,dx\Big)^{\frac{1}{2}}\Big(\int_\Omega |\nabla v|^2\,dx\Big)^{\frac{1}{2}},$$

we have also

$$\|\mu\|_{W^{-1,2}(\Omega)} \leq \Lambda\|\hat{\mu}\|_{W_0^{1,2}(\Omega)}.$$

Here the symbol $\| \, \|_X$ denotes the norm in a Banach space X.

Regarding the function H, we will consider the case where H is continuous and bounded on the whole $\mathbb{R}$ and the case where H admits a singularity at the origin. In the first case, we will require that H is Lipschitz continuous on $\mathbb{R}$ and such that

$$(1.4)\qquad (sH(s))' \in L^\infty(\mathbb{R}).$$

In the case where $\lim_{s\to 0^+} H(s) = +\infty$, we will instead require that H is a positive function defined on $\mathbb{R}^+$ (one can imagine that H is defined also on $\mathbb{R}^-$ and that it is identically zero on this set), nonincreasing, Lipschitz continuous on $(\varepsilon, +\infty)$ for all $\varepsilon > 0$ and such that

$$(1.5)\qquad \exists \mathscr{K} : \mathbb{R}^+ \to \mathbb{R}^+ \quad \text{s.t. } H \leq \mathscr{K} \quad \text{and} \quad \text{s.t. } (s\mathscr{K}(s)) \in W^{1,\infty}(\mathbb{R}^+).$$

Hence in this case the problem (1.1) may be singular since, by assumptions, the right hand side may blow up at zero.

In literature we can find many papers dealing with possibly singular elliptic problems on bounded domains Ω_b (the subscript b is used to underline the boundedness of the domain) whose model is

$$(1.6)\qquad \begin{cases} -\mathrm{div}(\mathscr{A}(x,\nabla u)) = H(u)\mu & \text{in } \Omega_b, \\ u > 0 & \text{in } \Omega_b, \\ u = 0 & \text{on } \partial\Omega_b, \end{cases}$$

where $\lim_{s\to 0} H(s) <$ or $= +\infty$, $\mathscr{A} : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ satisfies Leray–Lions conditions of p-Laplace type with $1 < p < N$ and μ is a suitable nonnegative datum.

In the singular case with regular positive data μ, at least Hölder continuous, and with a linear and uniformly elliptic higher order term, we recall the pioneering papers [5, 12], where the existence and uniqueness of a classical solution to (1.6) is proven under suitable assumptions on the singularity H.

In the singular case with data μ less regular, namely under the assumption that μ is a nonnegative bounded Radon measure or a nonnegative $L^1(\Omega_b)$ function, the main references are [1, 8, 10, 14, 15]. In this more general setting the strategy to solve this kind of problems is to approximate them with nonsingular ones, "truncating" in some sense the singular right hand side, and to prove a priori estimates and compactness results on the sequence of approximate solutions in order to give at least a distributional formulation to the singular problem, that appears as the limit of the approximations.

Finally recall [13] for the nonsingular case with quadratic coerciveness of the higher order term and [9] for the (possibly) singular case with generic coerciveness $p \in (1, N)$ of the higher order term. In both papers the data are nonnegative Radon measures on Ω_b.

The main idea underling this note is to solve the (possibly singular) problem (1.1) using the representation (1.3) given by the Lax–Milgram Theorem and the properties (1.4) and (1.5) to deal with the possibly singular right hand side.

We would like to point out here that we deal also with the case of Ω unbounded. We also notice that, to deal with (1.1), we avoid to use the notion of renormalized solution introduced in [6] as done, for example, in [10, 9, 13].

The plan of the paper is as follows. In Section 2 we will solve problem (1.1) in the case in which Ω is a bounded subset of $\mathbb{R}^N$ and H is a continuous and bounded function on the whole $\mathbb{R}$. In this section we will solve (1.1) with test functions in $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$. Section 3 is devoted to find a solution to (1.1) in the case in which Ω is bounded and H blows up at the origin. In Sections 4 and 5 we will analyze problem (1.1) in the case of an unbounded domain Ω and of a nonlinearity H that can be both bounded on the whole $\mathbb{R}$ and singular at the origin. In Section 5 we relax our requests on μ assuming $\mu \in W_{\ell oc}^{-1,2}(\Omega)$ at the expense of considering cylindrical unbounded open subsets Ω.

2. Ω and H bounded

We will start by the case in which Ω is a bounded subset of $\mathbb{R}^{\mathrm{N}}$ and $H : \mathbb{R} \to \mathbb{R}$ is a continuous and bounded function on the whole real line.

2.1. *Approximation of* μ

We have the following results.

Lemma 2.1. *Let* $\mu \in W^{-1,2}(\Omega)$, $A(x)$ *be a matrix such that* (1.2) *holds and* $\hat{\mu}$ *be defined by* (1.3). *Let* $\varepsilon > 0$ *and* v_ε *be the unique weak solution to the following singular perturbation problem*

$$\begin{cases} v_\varepsilon \in W_0^{1,2}(\Omega), \\ -\varepsilon \operatorname{div}(A(x)\nabla v_\varepsilon) + v_\varepsilon = \hat{\mu} \quad \text{in } \Omega. \end{cases} \tag{2.1}$$

Then, as $\varepsilon \to 0$, *one has*

$$v_\varepsilon \to \hat{\mu} \quad \text{in } W_0^{1,2}(\Omega) \tag{2.2}$$

and if μ is a nonnegative measure, it holds

$$-\mathrm{div}(A(x)\nabla v_\varepsilon) \geq 0 \quad \textit{and} \quad v_\varepsilon \leq \hat{\mu}. \tag{2.3}$$

Proof. In order to lighten the notation we will denote with $\mathscr{A}(\cdot)$ the operator $-\mathrm{div}(A(x)\nabla\cdot)$. From (2.1) one deduces that

$$\mathscr{A}(v_\varepsilon) = \frac{(\hat{\mu} - v_\varepsilon)}{\varepsilon} \in W_0^{1,2}(\Omega).$$

Applying $\mathscr{A}$ to the equation of (2.1), by (1.3) one gets.

$$\varepsilon\mathscr{A}(\mathscr{A}(v_\varepsilon)) + \mathscr{A}(v_\varepsilon) = \mu \tag{2.4}$$

This implies

$$\langle \mu - \mathscr{A}(v_\varepsilon), \varphi \rangle = \varepsilon \int_\Omega A(x)\nabla(\mathscr{A}(v_\varepsilon)) \cdot \nabla\varphi \, dx \quad \forall\varphi \in W_0^{1,2}(\Omega).$$

Thus it results

$$\|\mu - \mathscr{A}(v_\varepsilon)\|_{W^{-1,2}(\Omega)} \leq \left(\int_\Omega |A(x)\nabla(\varepsilon\mathscr{A}(v_\varepsilon))|^2 \, dx \right)^{\frac{1}{2}}. \tag{2.5}$$

If we take $\varepsilon\mathscr{A}(v_\varepsilon)$ as test function in (2.4), we arrive to

$$\int_\Omega A(x)\nabla(\varepsilon\mathscr{A}(v_\varepsilon)) \cdot \nabla(\varepsilon\mathscr{A}(v_\varepsilon)) \, dx + \varepsilon\|\mathscr{A}(v_\varepsilon)\|_{L^2(\Omega)}^2 = \langle \mu, \varepsilon\mathscr{A}(v_\varepsilon) \rangle \tag{2.6}$$

and so we deduce

$$\lambda \int_\Omega |\nabla(\varepsilon\mathscr{A}(v_\varepsilon))|^2 \, dx \leq \|\mu\|_{W^{-1,2}(\Omega)} \left(\int_\Omega |\nabla(\varepsilon\mathscr{A}(v_\varepsilon))|^2 \, dx \right)^{\frac{1}{2}}.$$

This implies that $\varepsilon\mathscr{A}(v_\varepsilon)$ is bounded in $W_0^{1,2}(\Omega)$ and thus there exists $v_0 \in W_0^{1,2}(\Omega)$ such that, up to a subsequence

$$\varepsilon\mathscr{A}(v_\varepsilon) \rightharpoonup v_0 \quad \text{in } W_0^{1,2}(\Omega), \quad \varepsilon\mathscr{A}(v_\varepsilon) \to v_0 \quad \text{in } L^2(\Omega).$$

So, going back to (2.6), one deduces also that

$$\lim_{\varepsilon\to 0} \|\varepsilon\mathscr{A}(v_\varepsilon)\|_{L^2(\Omega)}^2 \leq \lim_{\varepsilon\to 0} \varepsilon\langle \mu, \varepsilon\mathscr{A}(v_\varepsilon) \rangle = 0$$

Then $v_0 = 0$ and, by uniqueness of the limit, the convergence above holds true for the whole sequence. Back to (2.6), it results also that

$$\lim_{\varepsilon\to 0} \|\varepsilon\mathscr{A}(v_\varepsilon)\|_{W_0^{1,2}(\Omega)}^2 = 0.$$

Hence, by (2.5), we derive that

$$\mathscr{A}(v_\varepsilon) \to \mu \quad \text{in } W^{-1,2}(\Omega)$$

and, by (2.1), we derive that

$$v_\varepsilon \to \hat{\mu} \quad \text{in } W_0^{1,2}(\Omega),$$

namely (2.2).

Now we assume that μ is nonnegative. Then, taking $(\mathscr{A}(v_\varepsilon))^- \in W_0^{1,2}(\Omega)$ as test function in (2.4), we deduce

$$\int_\Omega \varepsilon A(x)\nabla(\mathscr{A}(v_\varepsilon)) \cdot \nabla(\mathscr{A}(v_\varepsilon))^- \, dx + \int_\Omega (\mathscr{A}(v_\varepsilon))(\mathscr{A}(v_\varepsilon))^- \, dx \geq 0$$

and thus we obtain

$$\int_\Omega \varepsilon A(x)|\nabla(\mathscr{A}(v_\varepsilon))^-|^2 \, dx + \int_\Omega ((\mathscr{A}(v_\varepsilon))^-)^2 \, dx \leq 0,$$

which implies that $\mathscr{A}(v_\varepsilon) \geq 0$ almost everywhere in Ω. This completes the proof of (2.3), taking into account the equation solved by v_ε. □

Lemma 2.2. *Let $\mu \in W^{-1,2}(\Omega)$ be nonnegative, $A(x)$ be a matrix such that* (1.2) *holds and $\hat{\mu}$ be defined by* (1.3)*. If $\mu \not\equiv 0$, for every domain $\omega \subset\subset \Omega$ there exists a positive constant c_ω such that*

$$\hat{\mu} \geq c_\omega \quad \text{on } \omega.$$

Proof. By Lemma 2.1 one has

$$\hat{\mu} \geq v_1$$

and v_1 satisfies

$$\begin{cases} -\operatorname{div}(A(x)\nabla v_1) = \hat{\mu} - v_1 & \text{in } \Omega \\ v_1, -\operatorname{div}(A(x)\nabla v_1) \in W_0^{1,2}(\Omega), & -\operatorname{div}(A(x)\nabla v_1) \geq 0. \end{cases}$$

Since $\hat{\mu} - v_1 \not\equiv 0$ on Ω (otherwise $\mu \equiv 0$), by the strong maximum principle, one deduces that

$$\forall \omega \subset\subset \Omega \quad \text{exists} \quad c_\omega > 0 \quad \text{s.t.} \quad v_1 \geq c_\omega.$$

This completes the proof of the Lemma. □

We denote by $\hat{\mu}_n$ the solution to (2.1) for $\varepsilon = \frac{1}{n}$, namely $\hat{\mu}_n = v_{\frac{1}{n}}$. Recall that

$$-\operatorname{div}(A(x)\nabla\hat{\mu}_n) \in W_0^{1,2}(\Omega), \quad \hat{\mu}_n \to \hat{\mu} \quad \text{in } W_0^{1,2}(\Omega).$$

In what follows we will consider $\hat{\mu}_n$ as an approximation of $\hat{\mu}$ and we will pass to the limit in $n \in \mathbb{N}$ to solve (1.1) in a suitable weak sense, assuming that Ω is a bounded subset of $\mathbb{R}^N$ and that $H : \mathbb{R} \to \mathbb{R}$ is a continuous and bounded function on the whole real line.

2.2. *Passage to the limit*

LEMMA 2.3. *Let $H : \mathbb{R} \to \mathbb{R}$ be bounded and Lipschitz continuous on $\mathbb{R}$, $\mu \in W^{-1,2}(\Omega)$, $A(x)$ be a matrix such that* (1.2) *holds, $\hat{\mu}$ be defined by* (1.3) *and $\hat{\mu}_n$ the solution to* (2.1) *for $\varepsilon = \frac{1}{n}$. Then there exists a solution u_n to the problem*

$$\begin{cases} u_n \in W_0^{1,2}(\Omega), \\ \displaystyle\int_\Omega A(x)\nabla u_n \cdot \nabla v \, dx = \int_\Omega A(x)\nabla \hat{\mu}_n \cdot \nabla(H(u_n)v) \, dx \quad \forall v \in W_0^{1,2}(\Omega). \end{cases} \tag{2.7}$$

PROOF. The equation in (2.7) can be written as follows

$$\int_\Omega A(x)\nabla u_n \cdot \nabla v \, dx = \int_\Omega -\mathrm{div}(A(x)\nabla \hat{\mu}_n)H(u_n)v \, dx \quad \forall v \in W_0^{1,2}(\Omega). \tag{2.8}$$

Note that, thanks to our assumptions on H, $H(u)v \in L^2(\Omega)$ for all $u, v \in L^2(\Omega)$. For $w \in L^2(\Omega)$ we define the following map

$$S_n : L^2(\Omega) \to L^2(\Omega), \quad S_n(w) = w_n$$

where w_n is the unique solution to the problem

$$\begin{cases} w_n \in W_0^{1,2}(\Omega), \\ \displaystyle\int_\Omega A(x)\nabla w_n \cdot \nabla v \, dx = \int_\Omega -\mathrm{div}(A(x)\nabla \hat{\mu}_n)H(w)v \, dx \quad \forall v \in W_0^{1,2}(\Omega). \end{cases} \tag{2.9}$$

To show that (2.8) has a solution, it is enough to prove that S_n has a fixed point. Taking w_n as test function in (2.9), one finds

$$\begin{aligned} \lambda \int_\Omega |\nabla w_n|^2 \, dx &\le \int_\Omega -\mathrm{div}(A(x)\nabla \hat{\mu}_n)H(w)w_n \, dx \\ &\le \|\mathrm{div}(A(x)\nabla \hat{\mu}_n)\|_{L^2(\Omega)} \|w_n\|_{L^2(\Omega)} \|H\|_{L^\infty(\mathbb{R})}. \end{aligned}$$

Using the Poincaré inequality we deduce

$$\lambda \|w_n\|^2_{L^2(\Omega)} \le \lambda C_p^2 \|\nabla w_n\|^2_{L^2(\Omega)} \le C_p^2 \|\mathrm{div}(A(x)\nabla \hat{\mu}_n)\|_{L^2(\Omega)} \|H\|_{L^\infty(\mathbb{R})} \|w_n\|_{L^2(\Omega)}$$

where C_p is the Poincaré constant. Thus

$$\|w_n\|_{L^2(\Omega)} \le \frac{C_p^2}{\lambda} \|\mathrm{div}(A(x)\nabla\hat{\mu}_n)\|_{L^2(\Omega)} \|H\|_{L^\infty(\mathbb{R})}, \tag{2.10}$$
$$\|\nabla w_n\|_{L^2(\Omega)} \le \frac{C_p}{\lambda} \|\mathrm{div}(A(x)\nabla\hat{\mu}_n)\|_{L^2(\Omega)} \|H\|_{L^\infty(\mathbb{R})}.$$

If we introduce the convex set K_n

$$K_n = \left\{ v \in L^2(\Omega) \text{ s.t. } \|v\|_{L^2(\Omega)} \le \frac{C_p^2}{\lambda} \|\mathrm{div}(A(x)\nabla\hat{\mu}_n)\|_{L^2(\Omega)} \|H\|_{L^\infty(\mathbb{R})} \right\},$$

it is clear that S_n maps the convex set K_n into itself. Moreover S_n is continuous and $S_n(K_n)$ is relatively compact in K_n, as it results from (2.10). Hence, by the Schauder fixed point Theorem, it follows that S_n admits at least a fixed point. This fixed point solves (2.8). □

THEOREM 2.1. *Let $H : \mathbb{R} \to \mathbb{R}$ be bounded, Lipschitz continuous and such that*

$$(sH(s))' \in L^\infty(\mathbb{R}).$$

If $\mu \in W^{-1,2}(\Omega)$, $A(x)$ is a matrix such that (1.2) *holds and $\hat{\mu}$ is defined by* (1.3), *then there exists u solution to*

$$\begin{cases} u \in W_0^{1,2}(\Omega), \\ \displaystyle\int_\Omega A(x)\nabla u \cdot \nabla v \, dx = \int_\Omega A(x)\nabla\hat{\mu} \cdot \nabla(H(u)v) \, dx \\ \qquad = \langle \mu, H(u)v \rangle \quad \forall v \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega). \end{cases} \tag{2.11}$$

EXAMPLE 2.1. One can easily verify that, for instance, the function $H : \mathbb{R} \to \mathbb{R}$ given by

$$H(s) := \frac{\ln\sqrt{s^2+1}}{s}$$

satisfies the assumptions above.

PROOF. Let $\hat{\mu}_n$ be the solution to (2.1) for $\varepsilon = \frac{1}{n}$ and consider u_n solution to (2.7). Taking $v = u_n$ as a test function in (2.7) we get

$$\lambda \int_\Omega |\nabla u_n|^2 \, dx \le \int_\Omega A(x)\nabla\hat{\mu}_n \cdot \nabla(H(u_n)u_n) \, dx \le \Lambda c_\infty \int_\Omega |\nabla\hat{\mu}_n| \, |\nabla u_n| \, dx,$$

where c_∞ is the L^∞ norm of the function $(H(s)s)'$. Using Cauchy–Schwarz inequality and recalling that the $W_0^{1,2}(\Omega)$-norm of $\hat{\mu}_n$ is bounded since $\hat{\mu}_n$ converges to $\hat{\mu}$ in $W_0^{1,2}(\Omega)$, we find that the $W_0^{1,2}(\Omega)$-norm of u_n is bounded and so that there exists some $u \in W_0^{1,2}(\Omega)$ such that, up to a subsequence

$$u_n \rightharpoonup u \quad \text{in } W_0^{1,2}(\Omega), \quad u_n \to u \quad \text{in } L^2(\Omega).$$

Then, if $v \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$, one has

$$H(u_n)v \rightharpoonup H(u)v \quad \text{in } W_0^{1,2}(\Omega), \quad H(u_n)v \to H(u)v \quad \text{in } L^2(\Omega)$$

and passing to the limit in (2.7) one gets (2.11). □

2.3. *The case of a nonnegative measure μ*

Let us first prove the following.

THEOREM 2.2. *Suppose that $\mu \in W^{-1,2}(\Omega)$ is nonnegative. Then, under the assumptions of Theorem* 2.1 *and if*

$$H(s) \geq 0 \quad \forall s \leq 0, \tag{2.12}$$

the solution to (2.11) *satisfies*

$$u \geq 0 \quad \textit{a.e. in } \Omega.$$

PROOF. We fix $k > 0$ and we take $T_k(u^-)$ as test function in (2.11), where the truncation function $T_k : \mathbb{R} \to \mathbb{R}$ is defined as follows

$$T_k(s) := \max(-k, \min(s, k)), \tag{2.13}$$

(note that if $s \in \mathbb{R}^+$, $T_k(s) = \min(s, k)$). We get

$$\int_\Omega A(x)\nabla u \cdot \nabla T_k(u^-)\, dx = \int_\Omega A(x)\nabla \hat{\mu} \cdot \nabla(H(u)T_k(u^-))\, dx = \langle \mu, H(u)T_k(u^-)\rangle.$$

Since by (2.12)

$$H(u)T_k(u^-) = H(-u^-)T_k(u^-) \geq 0 \quad \text{a.e. in } \Omega,$$

we derive that

$$\int_\Omega |\nabla T_k(u^-)|^2 \leq 0 \quad \Rightarrow \quad u^- = 0 \quad \text{a.e. in } \Omega,$$

as desired. □

One has also the following comparison result.

THEOREM 2.3. *Suppose that $\mu \in W^{-1,2}(\Omega)$ is nonnegative and that H_1 and H_2 are two functions which satisfy the assumptions of Theorem* 2.1 *and such that*

$$H_1 \geq H_2, \quad H_2 \textit{ is nonincreasing.}$$

Then, if u_i *is a solution to* (2.11) *corresponding to* H_i, $i = 1, 2$, *one has*

$$u_1 \geq u_2.$$

PROOF. If $v \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$, by subtraction of the equations satisfied by u_1 and u_2 one has

$$\int_\Omega A(x)\nabla(u_1 - u_2) \cdot \nabla v \, dx = \int_\Omega A(x)\nabla\hat{\mu} \cdot \nabla((H_1(u_1) - H_2(u_2))v) \, dx,$$

so that, if $v = T_k((u_1 - u_2)^-)$ for $k > 0$,

$$\int_\Omega A(x)\nabla(u_1 - u_2) \cdot \nabla T_k((u_1 - u_2)^-) \, dx$$
$$= \int_\Omega A(x)\nabla\hat{\mu} \cdot \nabla((H_1(u_1) - H_2(u_2))T_k((u_1 - u_2)^-)) \, dx.$$

Since

$$H_1(u_1) - H_2(u_2) = H_1(u_1) - H_2(u_1) + H_2(u_1) - H_2(u_2) \geq 0$$

where $u_1 \leq u_2$, from above one obtains

$$\int_\Omega |\nabla T_k((u_1 - u_2)^-)|^2 \, dx \leq 0$$

and the result follows. □

REMARK 2.1. The same result holds true if instead of assuming H_2 nonincreasing one assumes H_1 nonincreasing. Indeed it is sufficient to write

$$H_1(u_1) - H_2(u_2) = H_1(u_1) - H_1(u_2) + H_1(u_2) - H_2(u_2)$$

and to use the same argument.

As an obvious Corollary we have the following.

COROLLARY 2.1. *Let* $\mu \in W^{-1,2}(\Omega)$ *be a nonnegative measure. If, besides the assumptions of Theorem* 2.1, H *is also nonincreasing, then the solution to* (2.11) *is unique.*

3. Ω BOUNDED AND H SINGULAR AT THE ORIGIN

Now we would like to allow H to have a singularity at zero.

THEOREM 3.1. *Let* $\mu \in W^{-1,2}(\Omega)$ *be a nonnegative bounded measure,* $A(x)$ *be a matrix such that* (1.2) *holds,* $\hat{\mu}$ *be defined by* (1.3) *and let* $H : \mathbb{R}^+ \to \mathbb{R}^+$ *be a non-*

negative, nonincreasing function such that

$$\begin{cases} \lim_{s\to 0^+} H(s) = +\infty, \\ \forall \varepsilon > 0 \ H \text{ is Lipschitz continuous on } (\varepsilon, +\infty), \end{cases} \tag{3.1}$$
$$\exists \mathscr{K} : \mathbb{R}^+ \to \mathbb{R}^+ \quad s.t.\ H \le \mathscr{K} \quad and \quad s.t.\ (s\mathscr{K}(s)) \in W^{1,\infty}(\mathbb{R}^+).$$

Then there exists u solution to

$$\begin{cases} u \in W_0^{1,2}(\Omega), \quad u \ge 0, \\ \int_\Omega A(x)\nabla u \cdot \nabla v \, dx = \int_\Omega A(x)\nabla\hat{\mu} \cdot \nabla(H(u)v)\, dx \quad \forall v \in C_c^1(\Omega). \end{cases} \tag{3.2}$$

Proof. There exists a first value $n_0 \in \mathbb{N}\backslash\{0\}$ such that $H(n_0) \le n_0$. In what follows we will always assume $n \ge n_0$. Let us consider the following approximation H_n of H

$$H_n(s) = \begin{cases} n & \text{if } s < 0, \\ T_n(H(s)) & \text{if } 0 \le s \le n, \\ \min\{H(n)(n+1-s), T_n(H(s))\} & \text{if } n \le s \le n+1, \\ 0 & \text{if } s \ge n+1, \end{cases} \tag{3.3}$$

where T_n is the truncation function at level $n \in \mathbb{N}$ (see (2.13)). Then it is easy to verify that $H_n(s) \le H(s)$ for $s \ge 0$ and that the sequence H_n is nondecreasing in $n \in \mathbb{N}$. Moreover, thanks to (3.1), H_n satisfies the assumptions of Theorem 2.1 and of Theorem 2.2 and

$$\lim_{n\to\infty} H_n(s) = H(s) \quad \text{if } s > 0.$$

Thus, for each $n \in \mathbb{N}$, $n \ge n_0$ fixed, there exists u_n solution to

$$\begin{cases} u_n \in W_0^{1,2}(\Omega), \quad u_n \ge 0 \quad \text{in } \Omega, \\ \int_\Omega A(x)\nabla u_n \cdot \nabla v \, dx \\ \quad = \int_\Omega A(x)\nabla\hat{\mu} \cdot \nabla(H_n(u_n)v)\, dx \quad \forall v \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega). \end{cases} \tag{3.4}$$

If $k > 0$ is fixed, we can take $T_k(u_n)$ as test function in (3.4), obtaining

$$\begin{aligned} &\int_\Omega A(x)\nabla u_n \cdot \nabla(T_k(u_n))\, dx \\ &\quad = \int_\Omega A(x)\nabla\hat{\mu} \cdot \nabla(H_n(u_n)T_k(u_n))\, dx \\ &\quad = \int_\Omega A(x)\nabla\hat{\mu} \cdot \nabla\{H_n(u_n) - H_n(T_k(u_n)) + H_n(T_k(u_n))\}T_k(u_n)\, dx \end{aligned}$$

$$
\begin{aligned}
&= \langle \mu, \{H_n(u_n) - H_n(T_k(u_n)) + H_n(T_k(u_n))\} T_k(u_n) \rangle \\
&\leq \langle \mu, H_n(T_k(u_n)) T_k(u_n) \rangle,
\end{aligned}
$$

where, since $H_n(u_n)T_k(u_n) \in W_0^{1,2}(\Omega)$, we have been allowed to use formula (1.3) and hence the following inequality

$$
H_n(u_n) - H_n(T_k(u_n))T_k(u_n) \leq 0 \quad \text{on } \Omega.
$$

Now, since one has

$$
(s\mathscr{K}(s))' = f(s) \in L^\infty(\mathbb{R}^+) \quad \Leftrightarrow \quad s\mathscr{K}(s) = \int_0^s f(t)\, dt + c_{\mathscr{K}},
$$

we have

$$
T_k(u_n)\mathscr{K}(T_k(u_n)) - c_{\mathscr{K}} \in W_0^{1,2}(\Omega),
$$

where $c_{\mathscr{K}}$ is a constant depending on $\mathscr{K}$. From the computations above and since

$$
H_n \leq H \leq \mathscr{K},
$$

it follows that

$$
\begin{aligned}
\int_\Omega A(x)\nabla u_n \cdot \nabla T_k(u_n)\, dx &\leq \langle \mu, H_n(T_k(u_n))T_k(u_n) \rangle \\
&\leq \langle \mu, T_k(u_n)\mathscr{K}(T_k(u_n)) - c_{\mathscr{K}} + c_{\mathscr{K}} \rangle \\
&= \langle \mu, T_k(u_n)\mathscr{K}(T_k(u_n)) - c_{\mathscr{K}} \rangle + |c_{\mathscr{K}}|\mu(\Omega) \\
&= \int_\Omega A(x)\nabla\hat{\mu} \cdot \nabla\{T_k(u_n)\mathscr{K}(T_k(u_n)) - c_{\mathscr{K}}\}\, dx + |c_{\mathscr{K}}|\mu(\Omega) \\
&\leq \Lambda \|f\|_{L^\infty(\mathbb{R}^+)} \int_\Omega |\nabla\hat{\mu}|\, |\nabla T_k(u_n)|\, dx + |c_{\mathscr{K}}|\mu(\Omega).
\end{aligned}
$$

Hence

$$
\lambda \int_\Omega |\nabla T_k(u_n)|^2\, dx \leq \frac{\Lambda^2 \|f\|^2_{L^\infty(\mathbb{R}^+)}}{2\lambda} \int_\Omega |\nabla\hat{\mu}|^2\, dx + \frac{\lambda}{2} \int_\Omega |\nabla T_k(u_n)|^2\, dx + |c_{\mathscr{K}}|\mu(\Omega)
$$

so that

$$
\int_\Omega |\nabla T_k(u_n)|^2\, dx \leq \frac{\Lambda^2 \|f\|^2_{L^\infty(\mathbb{R}^+)}}{\lambda^2} \int_\Omega |\nabla\hat{\mu}|^2\, dx + \frac{2}{\lambda}|c_{\mathscr{K}}|\mu(\Omega).
$$

Letting $k \to \infty$ we get

$$
\int_\Omega |\nabla u_n|^2\, dx \leq \frac{\Lambda^2 \|f\|^2_{L^\infty(\mathbb{R}^+)}}{\lambda^2} \int_\Omega |\nabla\hat{\mu}|^2\, dx + \frac{2}{\lambda}|c_{\mathscr{K}}|\mu(\Omega),
$$

and so we deduce that the sequence u_n is bounded in $W_0^{1,2}(\Omega)$. In particular there exists a function $u \in W_0^{1,2}(\Omega)$ such that

$$u_n \rightharpoonup u \quad \text{in } W_0^{1,2}(\Omega), \quad u_n \to u \quad \text{in } L^2(\Omega)$$

up to a subsequence. Since one has $H_{n+1} \geq H_n$, it follows from Theorem 2.3 that

$$u_{n+1} \geq u_n \geq u_{n_0} \quad \text{on } \Omega \text{ for all } n \geq n_0, \tag{3.5}$$

so that for $n \geq n_0$

$$u \geq u_{n_0} \quad \text{on } \Omega.$$

Now, since $H_{n_0}(u_{n_0})v \in W_0^{1,2}(\Omega)$ for all $v \in C_c^1(\Omega)$, applying (1.3) we deduce that $u_{n_0} \in W_0^{1,2}(\Omega)$ is such that

$$-\mathrm{div}(A(x)\nabla u_{n_0}) = H_{n_0}(u_{n_0})\mu \geq 0 \quad \text{in the distributional sense in } \Omega.$$

Then, applying Lemma 2.2 with $\mu = -\mathrm{div}(A(x)u_{n_0})$ and using (3.5), we deduce that

$$\forall \omega \subset\subset \Omega \exists c_\omega > 0 \quad \text{s.t.} \quad u \geq u_n \geq u_{n_0} \geq c_\omega \quad \text{on } \omega \text{ for all } n \geq n_0.$$

Let now $v \in C_c^1(\Omega)$ be such that $\mathrm{supp}(v) \subset \omega \subset\subset \Omega$. On $\mathrm{supp}(v)$ one has

$$\begin{aligned} |\nabla(H_n(u_n)v)| &= |H_n'(u_n)v\nabla u_n + H_n(u_n)\nabla v| \\ &\leq |H_n'(u_n)v\nabla u_n| + \|H\|_{L^\infty([c_\omega,+\infty))}|\nabla v| \leq c_1|\nabla u_n| + c_2. \end{aligned}$$

The latter formula holds true since $|H_n'(u_n)v|$ is uniformly bounded with respect to $n \in \mathbb{N}$ if n is large enough and $v \in C_c^1(\Omega)$. Indeed we have that

$$|H_n'(u_n)v| \leq \|H_n'\|_{L^\infty([c_\omega,+\infty))}\|v\|_{L^\infty(\Omega)}$$

and, if $n \geq \max\{n_0, c_\omega, H(c_\omega)\}$, we have

$$H_n'(s)(s) = \begin{cases} H'(s) & \text{if } c_\omega \leq s \leq n, \\ H'(s) \text{ or } -H(n) & \text{if } n \leq s \leq n+1, \\ 0 & \text{if } s \geq n+1. \end{cases}$$

Since $|-H(n)| = H(n) \leq H(c_\omega)$, H being nonincreasing and $n \geq c_\omega$, and since $|H'(s)|$ is bounded in $[c_\omega, +\infty)$, H being Lipschitz continuous on $[c_\omega, +\infty)$, we have that $|H_n'(u_n)v|$ is uniformly bounded with respect to $n \in \mathbb{N}$ if $n \geq \max\{n_0, c_\omega, H(c_\omega)\}$ and $v \in C_c^1(\Omega)$.

Thus $H_n(u_n)v$ is bounded in $W_0^{1,2}(\Omega)$ with respect to $n \in \mathbb{N}$ and, since u_n converges almost everywhere to u, up to a subsequence one gets

$$H_n(u_n)v \rightharpoonup H(u)v \quad \text{in } W_0^{1,2}(\Omega), \quad H_n(u_n)v \to H(u)v \quad \text{in } L^2(\Omega) \quad \forall v \in C_c^1(\Omega).$$

Then, passing to the limit in (3.4), leads to (3.2) and this completes the proof of the theorem. □

4. Ω UNBOUNDED

In this section we would like to extend the existence and uniqueness results of Theorem 2.1, Theorem 2.2, Corollary 2.1, Theorem 3.1 to the case when Ω is unbounded. Our first result is the following.

THEOREM 4.1. *Let $\Omega \subset \mathbb{R}^N$ be an unbounded open set, $\mu \in W^{-1,2}(\Omega)$ be a non-negative measure, $A(x)$ be a matrix such that* (1.2) *holds and $H : \mathbb{R} \to \mathbb{R}$ be such that*

$$\begin{cases} H \text{ is nonincreasing, bounded and Lipschitz continuous on } \mathbb{R} \\ (sH(s))' \in L^\infty(\mathbb{R}), \\ H(s) \geq 0 \quad \forall s \leq 0. \end{cases}$$

Then there exists a unique u such that

$$\begin{cases} u \in W_0^{1,2}(\Omega), \quad u \geq 0 \quad a.e. \text{ in } \Omega, \\ \displaystyle\int_\Omega A(x)\nabla u \cdot \nabla v \, dx = \int_\Omega A(x)\nabla\hat{\mu} \cdot \nabla(H(u)v) \, dx \quad \forall v \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega). \end{cases}$$

PROOF. For $\ell > 0$ set $\Omega_\ell = B_\ell \cap \Omega$, where B_ℓ denotes the open ball of radius ℓ centered at 0, and $V_\ell = W_0^{1,2}(\Omega_\ell)$. Assuming that the functions of $W_0^{1,2}(\Omega_\ell)$ are extended by 0 outside Ω_ℓ, it is clear that $W_0^{1,2}(\Omega_\ell) \subset W_0^{1,2}(\Omega)$ thus $\mu \in V_\ell'$, the dual of V_ℓ, for all $\ell > 0$ and from Theorem 2.1, Theorem 2.2, Corollary 2.1 there exists a unique u_ℓ solution to

$$\text{(4.1)} \quad \begin{cases} u_\ell \in V_\ell, \quad u_\ell \geq 0 \quad \text{a.e. in } \Omega, \\ \displaystyle\int_{\Omega_\ell} A(x)\nabla u_\ell \cdot \nabla v \, dx = \langle \mu, H(u_\ell)v \rangle \quad \forall v \in W_0^{1,2}(\Omega_\ell) \cap L^\infty(\Omega_\ell). \end{cases}$$

Taking $v = T_k(u_\ell)$, $k > 0$, one derives easily

$$\begin{aligned} \lambda \int_{\Omega_\ell} |\nabla T_k(u_\ell)|^2 \, dx &\leq \int_{\Omega_\ell} A(x)\nabla u_\ell \cdot \nabla T_k(u_\ell) \, dx = \langle \mu, H(u_\ell)T_k(u_\ell) \rangle \\ &= \langle \mu, \{H(u_\ell) - H(T_k(u_\ell)) + H(T_k(u_\ell))\} T_k(u_\ell) \rangle \\ &\leq \langle \mu, H(T_k(u_\ell))T_k(u_\ell) \rangle \\ &\leq c_\infty \|\mu\|_{W^{-1,2}(\Omega)} \Big(\int_{\Omega_\ell} |\nabla T_k(u_\ell)|^2 \, dx \Big)^{\frac{1}{2}} \end{aligned}$$

where c_∞ is the L^∞-bound of $(sH(s))'$. It follows, letting $k \to \infty$, that

$$\left(\int_\Omega |\nabla u_\ell|^2\,dx\right)^{\frac{1}{2}} \le \frac{\|\mu\|_{W^{-1,2}(\Omega)} c_\infty}{\lambda}.$$

Thus, up to a subsequence, one has for some $u \in W_0^{1,2}(\Omega)$ when $\ell \to \infty$

$$u_\ell \rightharpoonup u \quad \text{in } W_0^{1,2}(\Omega), \qquad u_\ell \to u \quad \text{in } L^2_{\ell oc}(\Omega).$$

Let $v \in W_0^{1,2}(\Omega_{\ell^*}) \cap L^\infty(\Omega_{\ell^*})$, where $0 < \ell^* < \ell$, be extended by zero outside Ω_{ℓ^*}. Since H, H' are uniformly bounded one has

$$\int_\Omega |\nabla(H(u_\ell)v)|^2\,dx \le 2\int_\Omega |H'(u_\ell)v\nabla u_\ell|^2 + |H(u_\ell)\nabla v|^2\,dx \le c$$

where c is independent of ℓ. Thus, up to a subsequence

$$\nabla(H(u_\ell)v) \rightharpoonup G \quad \text{in } L^2(\Omega) \text{ and } \mathscr{D}'(\Omega).$$

Since H is Lipschitz continuous one has clearly

$$H(u_\ell)v \to H(u)v \quad \text{in } L^2_{\ell oc}(\Omega).$$

Thus by the uniqueness of the limit in $\mathscr{D}'(\Omega)$ one has

$$G = \nabla(H(u)v).$$

Passing to the limit in (4.1) written as

$$\int_{\Omega_{\ell^*}} A(x)\nabla u_\ell \cdot \nabla v\,dx = \int_{\Omega_{\ell^*}} A(x)\nabla\hat{\mu} \cdot \nabla(H(u_\ell)v)\,dx$$

one gets for every $\ell^* > 0$

$$\text{(4.2)} \qquad \int_\Omega A(x)\nabla u \cdot \nabla w\,dx = \int_\Omega A(x)\nabla\hat{\mu} \cdot \nabla(H(u)w)\,dx \qquad \forall w \in W_0^{1,2}(\Omega_{\ell^*}) \cap L^\infty(\Omega_{\ell^*}).$$

Let now $v \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ and $\theta_{\ell^*} : \mathbb{R}^{\mathrm{N}} \to \mathbb{R}$ be the function defined as

$$\theta_{\ell^*}(x) = \min(1, \operatorname{dist}(x, \mathbb{R}^N \backslash B_{\ell^*})).$$

Clearly, when $\ell^* \to \infty$

$$\theta_{\ell^*}v \to v, \quad \theta_{\ell^*}H(u)v \to H(u)v \quad \text{in } W_0^{1,2}(\Omega).$$

Using $w = \theta_{\ell^*}v$ in (4.2) and passing to the limit in ℓ^* the existence of u follows. The nonnegativity of u follows from the one of u_ℓ and the uniqueness can be proven as in Corollary 2.1. This completes the proof of the theorem. □

We consider now the case where H is singular at the origin.

THEOREM 4.2. *Let $\Omega \subset \mathbb{R}^N$ be an unbounded open set and H and $A(x)$, μ satisfying the assumptions of Theorem* 3.1. *Then there exists u solution to the following singular problem*

$$\begin{cases} u \in W^{1,2}_{\ell oc}(\overline{\Omega}), \quad u > 0 \quad in\ \Omega, \quad u = 0 \quad on\ \partial\Omega, \\ -\mathrm{div}(A(x)\nabla u) = H(u)\mu \quad in\ \Omega, \end{cases} \tag{4.3}$$

the last equation of (4.3) *being understood as*

$$\int_\Omega A(x)\nabla u \cdot \nabla v\,dx = \int_\Omega A(x)\nabla\hat{\mu} \cdot \nabla(H(u)v)\,dx \tag{4.4}$$
$$= \langle \mu, H(u)v\rangle \quad \forall v \in C^1_c(\Omega),$$

and $W^{1,2}_{\ell oc}(\overline{\Omega}) = \{v \in W^{1,2}(K)\ \forall K \subset \Omega,\ K\ bounded\}$.

PROOF. Set as in the previous proof $\Omega_\ell = B_\ell \cap \Omega$, $V_\ell = W^{1,2}_0(\Omega_\ell)$. Since $\mu \in W^{-1,2}(\Omega)$, we have that $\mu \in V'_\ell$ for all $\ell \in \mathbb{R}^+$. We consider the approximation (3.3) of the singular function H and the following problems, of the kind (3.4), as approximations of (4.3)

$$\begin{cases} u_{n,\ell} \in V_\ell, \quad u_{n,\ell} \geq 0 \quad \text{a.e. in } \Omega_\ell, \\ \displaystyle\int_{\Omega_\ell} A(x)\nabla u_{n,\ell} \cdot \nabla v = \int_{\Omega_\ell} A(x)\nabla\hat{\mu} \cdot \nabla(H_n(u_{n,\ell})v)\,dx \quad \forall v \in V_\ell \cap L^\infty(\Omega_\ell). \end{cases} \tag{4.5}$$

We proceed then as in the proof of Theorem 3.1 to arrive to

$$\int_{\Omega_\ell} |\nabla u_{n,\ell}|^2\,dx \leq \frac{\Lambda^2\|f\|^2_{L^\infty(\mathbb{R}^+)}}{\lambda^2}\int_{\Omega_\ell}|\nabla\hat{\mu}|^2\,dx + \frac{2}{\lambda}|c_{\mathscr{K}}|\mu(\Omega_\ell)$$
$$\leq \frac{\Lambda^2\|f\|^2_{L^\infty(\mathbb{R}^+)}}{\lambda^2}\int_{\Omega}|\nabla\hat{\mu}|^2\,dx + \frac{2}{\lambda}|c_{\mathscr{K}}|\mu(\Omega)$$

namely to

$$\|u_{n,\ell}\|_{V_\ell} \leq c, \tag{4.6}$$

where c is a positive constant independent of n and ℓ.

If $\ell^* \leq \ell$ we have that $V_\ell \hookrightarrow W^{1,2}(\Omega_{\ell^*})$ and so, from (4.6), we deduce that $\|u_{n,\ell}\|_{W^{1,2}(\Omega_{\ell^*})} \leq c$. In particular there exists a weak limit in $W^{1,2}(\Omega_{\ell^*})$ of $u_{n,\ell}$ with respect to ℓ, that we will denote with $u_{n,\infty}$. Since $u_{n,\ell} \in V_\ell \subset \{v \in W^{1,2}(\Omega_{\ell^*}) : v = 0 \text{ on } \partial\Omega_{\ell^*} \cap \partial\Omega\}$, space which is weakly closed, we conclude easily that

$$u_{n,\infty} \in W^{1,2}_{\ell oc}(\overline{\Omega}) \quad \text{with } u_{n,\infty} = 0 \text{ on } \partial\Omega.$$

From (4.6), by the weak lower semicontinuity of the norm, we deduce that

$$\|u_{n,\infty}\|_{V_{\ell^*}} \leq c. \tag{4.7}$$

We consider then $v \in V_{\ell^*} \cap L^\infty(\Omega_{\ell^*})$, $\ell \geq \ell^*$, and we extend v by zero in $\Omega_\ell \backslash \Omega_{\ell^*}$. Hence $v \in V_\ell \cap L^\infty(\Omega_\ell)$ and we can take it as test function in (4.5), finding

$$\int_{\Omega_{\ell^*}} A(x)\nabla u_{n,\ell} \cdot \nabla v \, dx = \int_{\Omega_{\ell^*}} A(x)\nabla\hat{\mu} \cdot \nabla(H_n(u_{n,\ell})v) \, dx$$
$$\forall v \in V_{\ell^*} \cap L^\infty(\Omega_{\ell^*}),\ \forall \ell \geq \ell^*.$$

Using the boundedness of $u_{n,\ell}$ in $W^{1,2}(\Omega_{\ell^*})$, we can pass to the limit in ℓ obtaining

$$\int_{\Omega_{\ell^*}} A(x)\nabla u_{n,\infty} \cdot \nabla v \, dx = \int_{\Omega_{\ell^*}} A(x)\nabla\hat{\mu} \cdot \nabla(H_n(u_{n,\infty})v) \, dx \tag{4.8}$$
$$\forall v \in V_{\ell^*} \cap L^\infty(\Omega_{\ell^*}),\ \forall \ell^* \in \mathbb{R}^+.$$

Taking into account (4.7) and proceeding like before for $u_{n,\ell}$ (recall (4.6)), we deduce that there exists a weak limit in $W^{1,2}(\Omega_{\ell^*})$ of $u_{n,\infty}$ with respect to n, for all $\ell^* \in \mathbb{R}^+$. We denote this weak limit with u and we have that

$$u \in W^{1,2}_{\ell oc}(\overline{\Omega}) \quad \text{with } u = 0 \text{ on } \partial\Omega.$$

Now we can conclude, as at the end of the proof of Theorem 3.1, that u is bounded from below by a positive constant on each $\omega \subset\subset \Omega$ and that $H_n(u_{n,\infty})v$ is bounded in V_{ℓ^*} for all $\ell^* \in \mathbb{R}^+$ and for all $v \in C^1_c(\Omega)$. So, using the weak convergence in V_{ℓ^*} of this sequence to its almost everywhere limit $H(u)v$, we can pass to the limit in n in (4.8) with test functions $v \in C^1_c(\Omega)$, obtaining (4.4). □

5. $\mu \in W^{-1,2}_{\ell oc}(\Omega)$, Ω CYLINDRICAL

When Ω is unbounded the condition $\mu \in W^{-1,2}(\Omega)$ is somehow restrictive since some very simple distributions – like for instance a constant function in an infinite strip – do not enjoy this property (see [4]). The goal of this section is to relax this constraint in the case of unbounded open set of cylindrical type. Let us precise our notation.

Suppose that Ω is an unbounded open set such that

$$\Omega \subseteq \mathbb{R}^q \times \omega^{N-q}$$

where $1 \leq q < N$ and ω^{N-q} is a bounded open set in $\mathbb{R}^{N-q}$.

We will split a point $x \in \mathbb{R}^N$ in two components X_1, X_2, where

$$X_1 := x_1, \ldots, x_q \in \mathbb{R}^q, \quad X_2 := x_{q+1}, \ldots, x_N \in \mathbb{R}^{N-q}.$$

Analogously, the gradient of a function u defined on $\mathbb{R}^N$ will be split in two components as follows

$$\nabla u = (\nabla_{X_1} u, \nabla_{X_2} u) \quad \text{where} \begin{cases} \nabla_{X_1} := (\partial_{x_1}, \dots, \partial_{x_q}), \\ \nabla_{X_2} := (\partial_{x_{q+1}}, \dots, \partial_{x_N}). \end{cases}$$

Moreover, for $\ell \in \mathbb{R}^+$ fixed, we will consider the following bounded subset of $\mathbb{R}^N$

$$\Omega_\ell := \Omega \cap (\ell\omega^q \times \omega^{N-q}) \tag{5.1}$$

where ω^q is an open, bounded and convex subset of $\mathbb{R}^q$ containing the origin. We will set

$$V_\ell := W_0^{1,2}(\Omega_\ell), \quad V'_\ell := (W_0^{1,2}(\Omega_\ell))', \tag{5.2}$$

where V_ℓ is equipped with the gradient norm in $L^2(\Omega_\ell)$.

Let us assume that $\mu \in (\bigcap_{\ell \in \mathbb{R}^+} V'_\ell) \backslash W^{-1,2}(\Omega)$ and that its norm in V'_ℓ blows up as a positive power of ℓ when $\ell \to \infty$. Then, inspired by Theorem 4.1 of [3] (see also Theorem 2.1 of [4]), we are able to prove the following convergence result.

THEOREM 5.1. *Let Ω be as above, $A(x)$ and $H : \mathbb{R} \to \mathbb{R}$ satisfy the assumptions of Theorem* 4.1 *and let μ be such that*

$$\begin{cases} \mu \in V'_\ell, \\ \mu \geq 0 \text{ and } \mu \not\equiv 0 \quad \text{on } \Omega_\ell, \\ \|\mu\|_{V'_\ell} = \mathcal{O}(\ell^\gamma) \quad \text{for some } \gamma > 0, \end{cases} \qquad \forall \ell \in \mathbb{R}^+,$$

where Ω_ℓ and V'_ℓ are defined in (5.1), (5.2). *Then, if $\ell \in \mathbb{R}^+$ is arbitrarily fixed and $\hat{\mu}_\ell$ is the unique element in V_ℓ such that*

$$\langle \mu, v \rangle = \int_{\Omega_\ell} A(x)\nabla\hat{\mu}_\ell \cdot \nabla v \, dx \quad \forall v \in V_\ell, \quad \|\hat{\mu}_\ell\|_{V_\ell} \leq \frac{1}{\lambda}\|\mu\|_{V'_\ell} \leq \frac{\Lambda}{\lambda}\|\hat{\mu}_\ell\|_{V_\ell}, \tag{5.3}$$

there exists a unique u_ℓ such that

$$\begin{cases} u_\ell \in V_\ell, \quad u_\ell \geq 0 \quad \text{a.e. in } \Omega_\ell, \\ \displaystyle\int_{\Omega_\ell} A(x)\nabla u_\ell \cdot \nabla v \, dx = \int_{\Omega_\ell} A(x)\nabla\hat{\mu}_\ell \cdot \nabla(H(u_\ell)v) \, dx \quad \forall v \in V_\ell \cap L^\infty(\Omega_\ell). \end{cases} \tag{5.4}$$

Moreover there exist two positive constants c, C such that

$$\|\nabla(u_\ell - u_\infty)\|_{L^2(\Omega_{\frac{\ell}{2}})} = ce^{-C\ell} \quad \forall \ell \in \mathbb{R}^+, \tag{5.5}$$

where u_∞ is the unique solution to the following problem

$$(5.6)\qquad \begin{cases} u_\infty \in W^{1,2}_{\ell oc}(\overline{\Omega}), \quad u_\infty = 0 \quad \text{on } \partial\Omega, \\ -\operatorname{div}(A(x)\nabla u_\infty) = H(u_\infty)\mu \quad \text{in } \Omega, \\ \|\nabla u_\infty\|_{L^2(\Omega_\ell)} = \mathcal{O}(\ell^\gamma), \end{cases}$$

the second equation of (5.6) *being understood as*

$$(5.7)\qquad \int_{\Omega_{\ell^*}} A(x)\nabla u_\infty \cdot \nabla v\,dx = \int_{\Omega_{\ell^*}} A(x)\nabla\hat{\mu}_{\ell^*} \cdot \nabla(H(u_\infty)v)\,dx = \langle \mu, H(u_\infty)v\rangle$$
$$\forall v \in V_{\ell^*} \cap L^\infty(\Omega_{\ell^*}),\ \forall \ell^* \in \mathbb{R}^+.$$

Remark 5.1. In the following proof, if no otherwise specified, we will denote by c_i ($i \in \mathbb{N}$) several positive constants which value may change from line to line and, sometimes, on the same line. These values will only depend on the data (for instance c_i may depend on Ω and N) but they will never depend on the indexes of the sequences we will introduce.

Proof. If $\ell \in \mathbb{R}^+$ is arbitrarily fixed, the existence of a unique u_ℓ such that (5.4) holds follows from Theorem 2.1, Theorem 2.2 and Corollary 2.1. In what follows we prove the existence of u_∞ satisfying (5.5) and solving (5.6) in the sense of (5.1). First of all, if $\ell_1 \le \ell - 1$, we denote with ρ_{ℓ_1} a function such that

$$\rho_{\ell_1}(x) = \rho_{\ell_1}(X_1), \quad 0 \le \rho_{\ell_1} \le 1, \quad \rho_{\ell_1} = \begin{cases} 1 \text{ on } \ell_1\omega^q \\ 0 \text{ outside } (\ell_1+1)\omega^q \end{cases}, \quad |\nabla_{X_1}\rho_{\ell_1}| \le c_1,$$

where c_1 is a positive constant independent of ℓ_1.

1) Estimate for $u_\ell - u_{\ell+r}$ if $0 \le r \le 1$.
Since the function $T_k((u_\ell - u_{\ell+r})^+)\rho_{\ell_1}$ belongs to $V_\ell \cap V_{\ell+r} \cap L^\infty(\Omega_{\ell+r})$, we can choose it as test function in the equations satisfied by u_ℓ and $u_{\ell+r}$ (see (5.4)). Subtracting them we get

$$\int_{\Omega_{\ell_1+1}} A(x)\nabla(u_\ell - u_{\ell+r}) \cdot \nabla[T_k((u_\ell - u_{\ell+r})^+)\rho_{\ell_1}]\,dx$$
$$= \langle \mu, (H(u_\ell) - H(u_{\ell+r}))T_k((u_\ell - u_{\ell+r})^+)\rho_{\ell_1}\rangle$$

and, since H is non-increasing, we obtain

$$\int_{\Omega_{\ell_1+1}} A(x)\nabla(u_\ell - u_{\ell+r}) \cdot \nabla[T_k((u_\ell - u_{\ell+r})^+)\rho_{\ell_1}]\,dx \le 0.$$

This implies easily (see the properties of A)

$$\begin{aligned}
&\int_{\Omega_{\ell_1+1}} |\nabla T_k((u_\ell - u_{\ell+r})^+)|^2 \rho_{\ell_1}\, dx \\
&\quad \leq \frac{\Lambda}{\lambda} \int_{D_{\ell_1}} |\nabla_{X_1} \rho_{\ell_1}| T_k((u_\ell - u_{\ell+r})^+) |\nabla (u_\ell - u_{\ell+r})^+|\, dx \\
&\quad \leq c_1 \Big(\int_{D_{\ell_1}} |\nabla (u_\ell - u_{\ell+r})^+|^2\, dx \Big)^{\frac{1}{2}} \Big(\int_{D_{\ell_1}} (T_k((u_\ell - u_{\ell+r})^+)\, dx)^2 \Big)^{\frac{1}{2}},
\end{aligned}$$

where $D_{\ell_1} := \Omega_{\ell_1+1} \backslash \Omega_{\ell_1}$. Using the Poncaré inequality on the section one gets

$$\begin{aligned}
&\int_{\Omega_{\ell_1}} |\nabla T_k((u_\ell - u_{\ell+r})^+)|^2\, dx \\
&\quad \leq c_1 c_2 \Big(\int_{D_{\ell_1}} |\nabla (u_\ell - u_{\ell+r})^+|^2\, dx \Big)^{\frac{1}{2}} \Big(\int_{D_{\ell_1}} |\nabla_{X_2} T_k((u_\ell - u_{\ell+r})^+)|^2\, dx \Big)^{\frac{1}{2}},
\end{aligned}$$

where c_2 is a positive constant that depends only on ω^{N-q}. Letting $k \to \infty$ we deduce

$$\int_{\Omega_{\ell_1}} |\nabla (u_\ell - u_{\ell+r})^+|^2\, dx \leq c_3 \int_{D_{\ell_1}} |\nabla (u_\ell - u_{\ell+r})^+|^2\, dx.$$

Hence, recalling that $D_{\ell_1} := \Omega_{\ell_1+1} \backslash \Omega_{\ell_1}$, we arrive to

$$\int_{\Omega_{\ell_1}} |\nabla (u_\ell - u_{\ell+r})^+|^2\, dx \leq \frac{c_3}{c_3+1} \int_{\Omega_{\ell_1+1}} |\nabla (u_\ell - u_{\ell+r})^+|^2\, dx.$$

Starting from $\ell_1 = \dfrac{\ell}{2}$, we iterate the last inequality $\left[\dfrac{\ell}{2}\right]$ times and, recalling that $\dfrac{\ell}{2} - 1 \leq \left[\dfrac{\ell}{2}\right] \leq \dfrac{\ell}{2}$, we arrive easily to

$$\int_{\Omega_{\frac{\ell}{2}}} |\nabla (u_\ell - u_{\ell+r})^+|^2\, dx \leq c_4^{\frac{\ell}{2}-1} \int_{\Omega_\ell} |\nabla (u_\ell - u_{\ell+r})^+|^2\, dx \tag{5.8}$$

where $c_4 = \dfrac{c_3}{c_3+1} < 1$.

Analogously, if we choose

$$T_k((u_\ell - u_{\ell+r})^-) \rho_{\ell_1} \in V_\ell \cap V_{\ell+r} \cap L^\infty(\Omega_{\ell+r})$$

as test function in the problems solved by u_ℓ and $u_{\ell+r}$, we find

$$\int_{\Omega_{\frac{\ell}{2}}} |\nabla (u_\ell - u_{\ell+r})^-|^2\, dx \leq c_4^{\frac{\ell}{2}-1} \int_{\Omega_\ell} |\nabla (u_\ell - u_{\ell+r})^-|^2\, dx. \tag{5.9}$$

Summing up (5.8) and (5.9), we arrive to

$$\int_{\Omega_{\frac{\ell}{2}}} (|\nabla(u_\ell - u_{\ell+r})^+|^2 + |\nabla(u_\ell - u_{\ell+r})^-|^2)\, dx$$
$$\le c_4^{\frac{\ell}{2}-1} \int_{\Omega_\ell} (|\nabla(u_\ell - u_{\ell+r})^+|^2 + |\nabla(u_\ell - u_{\ell+r})^-|^2)\, dx$$

and since

$$|\nabla(u_\ell - u_{\ell+r})|^2 = |\nabla(u_\ell - u_{\ell+r})^+|^2 + |\nabla(u_\ell - u_{\ell+r})^-|^2$$

we conclude that

$$\int_{\Omega_{\frac{\ell}{2}}} |\nabla(u_\ell - u_{\ell+r})|^2\, dx \le c_4^{\frac{\ell}{2}-1} \int_{\Omega_\ell} |\nabla(u_\ell - u_{\ell+r})|^2\, dx. \tag{5.10}$$

2) Estimate for u_ℓ.
Taking $T_k(u_\ell)$ as test function in (5.4) we find

$$\lambda \int_{\Omega_\ell} |\nabla T_k(u_\ell)|^2\, dx$$
$$\le \int_{\Omega_\ell} A(x)\nabla\hat{\mu}_\ell \cdot \nabla\{[H(u_\ell) - H(T_k(u_\ell)) + H(T_k(u_\ell)]T_k(u_\ell)\}\, dx$$
$$= \langle \mu, \{H(u_\ell) - H(T_k(u_\ell)) + H(T_k(u_\ell))\}T_k(u_\ell)\rangle \le \langle \mu, H(T_k(u_\ell))T_k(u_\ell)\rangle$$
$$\le \int_{\Omega_\ell} A(x)\nabla\hat{\mu}_\ell \cdot \nabla[H(T_k(u_\ell))T_k(u_\ell)]\, dx$$

where we have used formula (5.3) (note that $H(u_\ell)T_k(u_\ell), H(T_k(u_\ell))T_k(u_\ell) \in V_\ell$) and that, by the monotonicity of H, it results $\{H(u_\ell) - H(T_k(u_\ell))\}T_k(u_\ell) \le 0$ on Ω_ℓ. Then we obtain

$$\lambda \int_{\Omega_\ell} |\nabla T_k(u_\ell)|^2\, dx \le \Lambda c_\infty \Big(\int_{\Omega_\ell} |\nabla\hat{\mu}_\ell|^2\, dx\Big)^{\frac{1}{2}} \Big(\int_{\Omega_\ell} |\nabla T_k(u_\ell)|^2\, dx\Big)^{\frac{1}{2}}$$

where c_∞ is the L^∞ bound of $(sH(s))'$. Thus, recalling that $\|\hat{\mu}\|_{V_\ell} \le \frac{1}{\lambda}\|\mu\|_{V'_\ell} \le \frac{1}{\lambda}\|\mu\|_{W^{-1,2}(\Omega)}$, we obtain

$$\Big(\int_{\Omega_\ell} |\nabla T_k(u_\ell)|^2\, dx\Big)^{\frac{1}{2}} \le \frac{\Lambda c_\infty}{\lambda^2}\|\mu\|_{V'_\ell} = \mathcal{O}(\ell^\gamma)$$

so that, letting $k \to \infty$, we arrive to

$$\int_{\Omega_\ell} |\nabla u_\ell|^2 \le c_5 \ell^{2\gamma}. \tag{5.11}$$

3) u_ℓ is a Cauchy sequence.
Using (5.10) and (5.11) we find

$$\int_{\Omega_{\frac{\ell}{2}}} |\nabla(u_\ell - u_{\ell+r})|^2\,dx \le c_6 e^{-\frac{\ell}{2}\ln(\frac{1}{c_4})}\Big(\int_{\Omega_\ell} |\nabla u_\ell|^2\,dx + \int_{\Omega_{\ell+r}} |\nabla u_{\ell+r}|^2\,dx\Big)$$
$$\le c_7 e^{-\frac{\ell}{2}\ln(\frac{1}{c_4})}(\ell^{2\gamma} + (\ell+r)^{2\gamma}) \le c_8^2 e^{-\alpha\ell}$$

where $0 < \alpha < \frac{1}{2}\ln\big(\frac{1}{c_4}\big)$. Hence

$$\|u_\ell - u_{\ell+r}\|_{V_{\frac{\ell}{2}}} \le c_8 e^{-\frac{\alpha}{2}\ell}.$$

If $t > 0$ is arbitrary, we deduce

$$\begin{aligned}(5.12)\qquad \|u_\ell - u_{\ell+t}\|_{V_{\frac{\ell}{2}}} &\le \|u_\ell - u_{\ell+1}\|_{V_{\frac{\ell}{2}}} + \|u_{\ell+1} - u_{\ell+2}\|_{V_{\frac{\ell+1}{2}}} \\ &\quad + \cdots + \|u_{\ell+[t]} - u_{\ell+t}\|_{V_{\frac{\ell+[t]}{2}}} \\ &\le c_8 e^{-\frac{\alpha}{2}\ell} + c_8 e^{-\frac{\alpha}{2}(\ell+1)} + \cdots + c_8 e^{-\frac{\alpha}{2}(\ell+[t])} \\ &\le c_8 e^{-\frac{\alpha}{2}\ell}(1 + e^{-\frac{\alpha}{2}} + e^{-2\frac{\alpha}{2}} + \cdots) \le c_8 e^{-\frac{\alpha}{2}\ell}\frac{1}{1-e^{-\frac{\alpha}{2}}}\end{aligned}$$

independently of $t > 0$. So, if $\ell^* \le \dfrac{\ell}{2}$, from (5.12) we deduce that u_ℓ is a Cauchy sequence in $W^{1,2}(\Omega_{\ell^*})$, and so it has a strong limit in this space, that we will denote by u_∞.

4) Limit problem.
Since $u_\ell \in V_\ell$ for all $\ell \in \mathbb{R}^+$ and $\ell^* \le \dfrac{\ell}{2}$, we have that $u_\ell = 0$ on $\partial\Omega_{\ell^*} \cap \partial\Omega$ for all $\ell \ge 2\ell^*$. Then

$$u_\ell \in \tilde{V}_{\ell^*} := \{v \in W^{1,2}(\Omega_{\ell^*}) \text{ s.t. } v = 0 \text{ on } \partial\Omega_{\ell^*} \cap \partial\Omega\} \quad \forall \ell \ge 2\ell^*.$$

Since $\tilde{V}_{\ell^*}$ is weakly closed and u_ℓ is strongly convergent to u_∞ in $W^{1,2}(\Omega_{\ell^*})$, we conclude that $u_\infty = 0$ on $\partial\Omega_{\ell^*} \cap \partial\Omega$ and this for all $\ell^* \in \mathbb{R}^+$. Hence

$$u_\infty \in W^{1,2}_{\ell oc}(\overline{\Omega}) \quad \text{with } u_\infty = 0 \text{ on } \partial\Omega.$$

Letting $t \to \infty$ in (5.12), we obtain also (5.5).

Now, we will prove that u_∞ satisfies (5.1). If $v \in V_{\ell^*} \cap L^\infty(\Omega_{\ell^*})$, $\ell \ge \ell^*$ and if we extend v by zero in $\Omega_\ell \backslash \Omega_{\ell^*}$, then $v \in V_\ell \cap L^\infty(\Omega_\ell)$. In particular we can take v as test function in the problem solved by u_ℓ, namely in (5.4). Hence we have

$$\int_{\Omega_{\ell^*}} A(x)\nabla u_\ell \cdot \nabla v\,dx = \int_{\Omega_{\ell^*}} A(x)\nabla\hat{\mu}_\ell \cdot \nabla(H(u_\ell)v)\,dx$$
$$= \int_{\Omega_{\ell^*}} A(x)\nabla\hat{\mu}_{\ell^*} \cdot \nabla(H(u_\ell)v)\,dx$$
$$\forall v \in V_{\ell^*} \cap L^\infty(\Omega_{\ell^*}),\ \forall \ell \ge \ell^*.$$

If we pass to the limit in ℓ, using the regularity of H and the fact that $u_\ell \to u_\infty$ in $W^{1,2}(\Omega_{\ell^*})$, we obtain

$$\int_{\Omega_{\ell^*}} A(x)\nabla u_\infty \cdot \nabla v = \int_{\Omega_{\ell^*}} A(x)\nabla \hat{\mu}_{\ell^*} \cdot \nabla(H(u_\infty)v)\,dx$$
$$= \langle \mu, H(u_\infty)v\rangle \quad \forall v \in V_{\ell^*} \cap L^\infty(\Omega_{\ell^*}).$$

Of course this holds for all $\ell^* \in \mathbb{R}^+$ and (5.1) is proved.

5) Estimate for $\|\nabla u_\infty\|_{L^2(\Omega_\ell)}$.
From (5.12) we deduce

$$\|u_{2\ell} - u_{2\ell+t}\|_{V_\ell} \le c_9 e^{-\alpha\ell}.$$

Since, in particular, c_9 is independent of t, letting $t \to \infty$ we find

$$\|u_{2\ell} - u_\infty\|_{V_\ell} \le c_9 e^{-\alpha\ell}.$$

Hence, using also (5.11), we deduce

$$\|u_\infty\|_{V_\ell} \le c_9 e^{-\alpha\ell} + \|u_{2\ell}\|_{V_\ell} \le c_9 e^{-\alpha\ell} + \|u_{2\ell}\|_{V_{2\ell}} \le c_9 e^{-\alpha\ell} + \mathcal{O}((2\ell)^\gamma) = \mathcal{O}(\ell^\gamma),$$

this completes the proof of (5.6).

6) Uniqueness.
Finally we want to prove that u_∞ is unique. Let us assume that u_∞ and u'_∞ are two solutions to (5.6). Then, from (5.1), we deduce

$$\int_{\Omega_{\ell^*}} A(x)\nabla(u_\infty - u'_\infty) \cdot \nabla v\,dx = \int_{\Omega_{\ell^*}} A(x)\nabla\hat{\mu}_{\ell^*} \cdot \nabla\{(H(u_\infty) - H(u'_\infty)v\}\,dx \tag{5.13}$$
$$\forall v \in V_{\ell^*} \cap L^\infty(\Omega_{\ell^*}),\ \forall \ell^* \in \mathbb{R}^+.$$

If ℓ^* is fixed and $\ell_1 \le \ell^* - 1$, we take $v^{+,-} := T_k((u_\infty - u'_\infty)^{+,-})\rho_{\ell_1} \in V_{\ell^*} \cap L^\infty(\Omega_{\ell^*})$ as test functions in (5.13) and we argue as in the first step of the proof, finding

$$\int_{\Omega_{\frac{\ell^*}{2}}} |\nabla(u_\infty - u'_\infty)|^2\,dx \le c_{10} e^{-c_{11}\ell^*} \int_{\Omega_{\ell^*}} |\nabla(u_\infty - u'_\infty)|^2\,dx.$$

Thus, recalling (5.5), we deduce

$$\int_{\Omega_{\frac{\ell^*}{2}}} |\nabla(u_\infty - u'_\infty)|^2\,dx$$
$$\le c_{12} e^{-c_{11}\ell^*}\Big(\int_{\Omega_{\ell^*}} |\nabla(u_\infty - u_{2\ell^*})|^2\,dx + \int_{\Omega_{\ell^*}} |\nabla(u_{2\ell^*} - u'_\infty)|^2\,dx\Big)$$
$$\le c_{12} e^{-c_{11}\ell^*} c^2 e^{-2C\ell^*}.$$

Finally, letting $\ell^* \to \infty$, we end up with

$$u_\infty = u'_\infty \quad \text{a.e in } \Omega,$$

as desired. □

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REFERENCES

[1] L. BOCCARDO - L. ORSINA, *Semilinear elliptic equations with singular nonlinearities*, Calc. Var. Partial Differential Equations, 37 (2009), pp. 363–380.

[2] M. CHIPOT, *Elliptic Equations: An Introductory Course*, Birkhäuser, Basel, Birkhäuser Advanced Texts, 2009.

[3] M. CHIPOT, *ℓ goes to plus infinity: an update*, J. Korean Soc. Ind. Appl. Math., 18 (2014), pp. 107–127.

[4] M. CHIPOT, *Asymptotic Issues for Some Partial Differential Equations*, Imperial College Press, 2016.

[5] M. G. CRANDALL - P. H. RABINOWITZ - L. TARTAR, *On a dirichlet problem with a singular nonlinearity*, Comm. Partial Differential Equations, 2 (1977), pp. 193–222.

[6] G. DAL MASO - F. MURAT - L. ORSINA - A. PRIGNET, *Renormalized solutions of elliptic equations with general measure data*, Ann. Scuola Norm. Sup. Pisa Cl. Sci., 28 (1999), pp. 741–808.

[7] R. DAUTRAY - J.-L. LIONS, *Analyse mathématique et calcul numérique pour les sciences et les techniques*, Tome 1, Masson, Paris, 1984.

[8] L. M. DE CAVE, *Nonlinear elliptic equations with singular nonlinearities*, Asymptot. Anal., 84 (2013), pp. 181–195.

[9] L. M. DE CAVE - R. DURASTANTI - F. OLIVA, *Existence and uniqueness results for possibly singular nonlinear elliptic equations with measure data*, Nonlinear Differ. Equ. Appl. (2018) 25: 18. https://doi.org/10.1007/s00030-018-0509-7.

[10] L. M. DE CAVE - F. OLIVA, *Elliptic equations with general singular lower order terms and measure data*, Nonlinear Anal., 128 (2015), pp. 391–411.

[11] D. GILBARG - N. S. TRUDINGER, *Elliptic Partial Differential Equations of Second Order*. Springer, Berlin, Heidelberg, New-York, Classics in Mathematics, Reprint of the 1998 edition, 2001.

[12] A. C. LAZER - P. J. MCKENNA, *On a singular nonlinear elliptic boundary-value problem*, Proc. Amer. Math. Soc., 111 (1991), pp. 721–730.

[13] F. MURAT - A. PORRETTA, *Stability properties, existence and nonexistence of renormalized solutions for elliptic equations with measure data*, Comm. Partial Differential Equations, 27 (2002), pp. 2267–2310.

[14] F. OLIVA - F. PETITTA, *Finite and Infinite energy solutions of singular elliptic problems: Existence and Uniqueness*, J. Differential Equations, 264 (2018), pp. 311–340.

[15] L. Orsina - F. Petitta, *A Lazer-Mckenna type problem with measures*, Differential Integral Equations, 29 (2016), pp. 19–36.

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