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Partial Differential Equations — *Small perturbations of nonlocal biharmonic problems with variable exponent and competing nonlinearities*, by Khaled Kefi and Vicenţiu D. Rădulescu, communicated on February 9, 2018.

Abstract. — The present paper deals with the analysis of combined effects of an absorption term and a small perturbation of the reaction term in a $p(x)$-biharmonic Kirchhoff problem with Navier boundary condition. The main result in this work establishes the existence of a continuous spectrum consisting in an interval. The proofs combine variational methods with energy estimates.

Key words: $p(x)$-biharmonic Kirchhoff problem, Navier boundary condition, Ekeland variational principle, generalized Sobolev spaces, continuous spectrum

Mathematics Subject Classification: 35D05, 35D30, 35J58, 35J60, 58E05

1. Introduction

The qualitative analysis of nonlinear problems with one or several variable exponents started with the pioneering papers of Halsey [19] and Zhikov [38], in strong relationship with the behavior of strongly anisotropic materials. Their work is an important contribution to the refined mathematical analysis of nonlinear problems with one or more variable exponents, mainly because it allows the understanding of some classes of nonlinear problems with possible lack of uniform convexity. Nonlinear problems with this structure are motivated by numerous models in the applied sciences that are driven by partial differential equations with one or more variable exponents. In some circumstances, the standard analysis based on the theory of usual Lebesgue and Sobolev function spaces, L^p and $W^{1,p}$, is not appropriate in the framework of material that involve nonhomogeneities. For instance, both electro-rheological "smart" fluids and phenomena arising in image processing are described in a correct way by nonlinear models in which the exponent p is not necessarily constant. The variable exponent describe the geometry of a material which is allowed to change its hardening exponent according to the point. This leads to the analysis of variable exponents Lebesgue and Sobolev function spaces (denoted by $L^{p(x)}$ and $W^{1,p(x)}$), where p is a real-valued (non-constant) function. This is a common abstract framework in homogenization and nonlinear elasticity. We refer here to the monograph by Rădulescu and Repovš [36], which includes a thorough variational and topological analysis of several classes of problems with variable exponent (see also the survey paper Rădulescu [35] and the important contributions by Pucci et al. [8, 32, 34]).

In this paper, we first discuss the existence of a continuous spectrum consisting in an interval for the following nonlocal biharmonic problem with variable exponent:

$$(1)\qquad \begin{cases} M(t)(\Delta^2_{p(x)}u + a(x)|u|^{p(x)-2}u) = \lambda V_1(x)|u|^{q(x)-2}u, & \text{in } \Omega \\ u = \Delta u = 0, & \text{on } \partial\Omega, \end{cases}$$

where $\Delta^2_{p(x)}u = \Delta(|\Delta u|^{p(x)-2}\Delta u)$ is the $p(x)$-biharmonic operator and Ω is a smooth bounded domain in $\mathbb{R}^N$. We assume that $M(t)$ is a continuous function and

$$t := \int_\Omega \frac{1}{p(x)}(|\Delta u|^{p(x)}\,dx + a(x)|u|^{p(x)})\,dx,$$

λ is a positive parameter, p, q are continuous functions on $\overline{\Omega}$ and $a \in L^\infty(\Omega)$ such that $\operatorname{essinf}_{x\in\Omega} a(x) > 0$. We assume that V_1 is a weight function in a generalized Lebesgue space such that $V_1 > 0$ in an open set $\Omega_0 \subset\subset \Omega$, where $|\Omega_0| > 0$.

Next, we focus on the following perturbed problem

$$(2)\qquad \begin{cases} M(t)(\Delta^2_{p(x)}u + a(x)|u|^{p(x)-2}u) \\ \quad = \lambda(V_1(x)|u|^{q(x)-2}u - V_2(x)|u|^{\alpha(x)-2}u) & \text{in } \Omega \\ u = \Delta u = 0, & \text{on } \partial\Omega, \end{cases}$$

where α is a continuous function on $\overline{\Omega}$ and V_2 is a nonnegative one in a generalized Lebesgue spaces.

Problems (1) and (2) are *nonlocal problems* because of the presence of the term M, which implies that the equations in (1) and (2) are no longer pointwise. This provokes some mathematical difficulties which make the study of such a problem particulary interesting.

In 1883, Kirchhoff [24] introduced a model given by the equation

$$(3)\qquad \rho\frac{\partial^2 u}{\partial t^2} - \left(\frac{\rho_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2} = 0,$$

which extends the classical d'Alembert's wave equation by considering the effects of the changes in the length of the strings during the vibrations. The parameters in the above equation have the following meanings: L is the length of the string, h is the area of the cross-section, E is the young modulus of the material, ρ is the mass density and ρ_0 is the initial tension. A feature of problem (3) is that the equation contains a nonlocal coefficient $\frac{\rho_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx$, which depends on the average $\frac{1}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx$. Nonlocal effects also find various applications in biological systems.

After the work of Lions [25], various equations of Kirchhoff type have been investigated, see [2, 12]. Moreover, Kirchhoff-type equations involving p-Laplacian and $p(x)$-Laplacian have been studied in many papers; see, [11, 15, 18, 21, 30]. A parabolic version of problem (3) can be used to describe the growth and movement of a particular species. The movement, modeled by the integral term, is assumed to be dependent on the energy of the entire system with u being its population density. Alternatively, the movement of particular species may be subject to the total population density within the domain (for instance, the spreading of bacteria), which gives rise to nonlocal parabolic equations. We refer to [33] for details.

We also mention that fourth-order elliptic equations arise in many domains like micro-electro-mechanical systems, surface diffusion on solids, thin film theory, flow in Hele–Shaw cells and phase field models of multiphasic systems, see [3, 17, 29]. Recent contributions concerning a fourth order elliptic problems with $p(x)$ biharmonic operators can be found in [23].

In the present paper, we study problem (1) under the following assumptions:

(H_0) $M : \mathbb{R} \to [m_0, +\infty)$ is a continuous function, with $m_0 > 0$;

(H_1) there exists $0 < \theta < 1$ such that

$$\hat{M}(t) \geq (1-\theta)M(t)t \quad \text{for all } t \geq 0, \quad \text{where } \hat{M}(t) = \int_0^t M(s)\,ds;$$

(H_2) $1 < q(x) < p(x) < \frac{N}{2} < s_1(x)$, for all $x \in \overline{\Omega}$, $V_1 \in L^{s_1(x)}(\Omega)$ and $V_1 > 0$ in $\Omega_0 \subset\subset \Omega$, with $|\Omega_0| > 0$.

We point out that Kefi [22] was the first to introduce assumption like (H_2) to study problems involving Lebesgue and Sobolev spaces with variable exponents.

2. Terminology and abstract setting

To study $p(x)$-biharmonic problems, we need some results on the spaces $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$ and $W^{k,p(x)}(\Omega)$; see [20, 35, 36] for details, complements and proofs.

Let

$$C_+(\overline{\Omega}) := \{h : h \in C(\overline{\Omega}),\, h(x) > 1, \text{ for all } x \in \overline{\Omega}\}.$$

For any $p \in C_+(\overline{\Omega})$, we denote $1 < p^- := \min_{x\in\overline{\Omega}} p(x) \leq p^+ = \max_{x\in\overline{\Omega}} p(x) < \infty$ and

$$L^{p(x)}(\Omega) = \left\{u : \Omega \to \mathbb{R} \text{ measurable and } \int_\Omega |u(x)|^{p(x)}\,dx < \infty\right\}.$$

The spaces $L^{p(x)}(\Omega)$ have been introduced by Orlicz [31].

The space $L^{p(x)}(\Omega)$ is endowed with the Luxemburg norm, which is defined by

$$|u|_{p(x)} = \inf\left\{\mu > 0 : \int_\Omega \left|\frac{u(x)}{\mu}\right|^{p(x)} dx \leq 1\right\}.$$

Clearly, when $p(x) \equiv p$, the space $L^{p(x)}(\Omega)$ reduces to the classical Lebesgue space $L^p(\Omega)$ and the norm $|u|_{p(x)}$ reduces to the standard norm $\|u\|_{L^p} = (\int_\Omega |u|^p\,dx)^{\frac{1}{p}}$ in $L^p(\Omega)$.

For any positive integer k, let

$$W^{k,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : D^\alpha u \in L^{p(x)}(\Omega), |\alpha| \le k\},$$

where $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_N)$ is a multi-index, $|\alpha| = \sum_{i=1}^N \alpha_i$ and $D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial^{\alpha_1}_{x_1} \cdots \partial^{\alpha_N}_{x_N}}$. Then $W^{k,p(x)}(\Omega)$ is a separable and reflexive Banach space equipped with the norm

$$\|u\|_{k,p(x)} = \sum_{|\alpha| \le k} |D^\alpha u|_{p(x)}.$$

The space $W_0^{k,p(x)}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{k,p(x)}(\Omega)$.

Let $L^{p'(x)}(\Omega)$ be the conjugate space of $L^{p(x)}(\Omega)$ with $\frac{1}{p} + \frac{1}{p'} = 1$. Then the following Hölder-type inequality

$$\left|\int_\Omega uv\,dx\right| \le \Big(\frac{1}{p^-} + \frac{1}{(p')^-}\Big)|u|_{p(x)}|v|_{p'(x)}, \quad \text{for all } u \in L^{p(x)}(\Omega),\ v \in L^{p'(x)}(\Omega) \tag{4}$$

holds. Moreover, if h_1, h_2 and $h_3 : \overline{\Omega} \to (1, \infty)$ are Lipschitz continuous functions such that $1/h_1(x) + 1/h_2(x) + 1/h_3(x) = 1$, then for any $u \in L^{h_1(x)}(\Omega)$, $v \in L^{h_2(x)}(\Omega)$ and $w \in L^{h_3(x)}(\Omega)$ the following inequality holds (see [16, Proposition 2.5]):

$$\left|\int_\Omega uvw\,dx\right| \le \Big(\frac{1}{h_1^-} + \frac{1}{h_2^-} + \frac{1}{h_3^-}\Big)|u|_{h_1(x)}|v|_{h_2(x)}|w|_{h_3(x)}. \tag{5}$$

Inequality (4) and its generalized version (5) are due to Orlicz [31].

The modular on the space $L^{p(x)}(\Omega)$ is the map $\rho_{p(x)} : L^{p(x)}(\Omega) \to \mathbb{R}$ defined by

$$\rho_{p(x)}(u) := \int_\Omega |u|^{p(x)}\,dx.$$

Proposition 1 (See [28]). *For all $u, v \in L^{p(x)}(\Omega)$, we have*

1. $|u|_{p(x)} < 1$ (*resp.* $= 1, > 1$) $\Leftrightarrow \rho_{p(x)}(u) < 1$ (*resp.* $= 1, > 1$).
2. $\min(|u|^{p^-}_{p(x)}, |u|^{p^+}_{p(x)}) \le \rho_{p(x)}(u) \le \max(|u|^{p^-}_{p(x)}, |u|^{p^+}_{p(x)})$.
3. $\rho_{p(x)}(u - v) \to 0 \Leftrightarrow |u - v|_{p(x)} \to 0$.

Proposition 2 (See [13]). *Let p and q be measurable functions such that $p \in L^\infty(\Omega)$, and $1 \le p(x)q(x) \le \infty$, for a.e. $x \in \Omega$. Let $u \in L^{q(x)}(\Omega)$, $u \neq 0$. Thens*

$$\min(|u|^{p^+}_{p(x)q(x)}, |u|^{p^-}_{p(x)q(x)}) \le |\,|u|^{p(x)}|_{q(x)} \le \max(|u|^{p^-}_{p(x)q(x)}, |u|^{p^+}_{p(x)q(x)}).$$

DEFINITION 1. Assume that spaces E, F are Banach spaces, we define the norm on the space $X := E \cap F$ as $\|u\|_X = \|u\|_E + \|u\|_F$.

In order to discuss problems (1) and (2), we need some properties of the space $X := W_0^{1,p(x)}(\Omega) \cap W^{2,p(x)}(\Omega)$. From Definition 1, we know that for any $u \in X$ we have $\|u\| = \|u\|_{1,p(x)} + \|u\|_{2,p(x)}$, thus $\|u\| = |u|_{p(x)} + |\nabla u|_{p(x)} + \sum_{|\alpha|=2} |D^\alpha u|_{p(x)}$. In Zang and Fu [37], the equivalence of the norms was proved, and it was even proved that the norm $|\Delta u|_{p(x)}$ is equivalent to the norm $\|u\|$ (see [37, Theorem 4.4]). Note that $(X, \|.\|)$ is a separable and reflexive Banach space.

Let

$$\|u\|_a := \inf\left\{ \mu > 0 : \int_\Omega \left(\left|\frac{\Delta u}{\mu}\right|^{p(x)} + a(x)\left|\frac{u}{\mu}\right|^{p(x)} \right) dx \le 1 \right\} \quad \text{for } u \in X.$$

Since $a \in L^\infty(\Omega)$ and $\operatorname{essinf}_{x\in\Omega} a > 0$, we deduce that $\|u\|_a$ is equivalent to the norms $\|u\|$ and $|\Delta u|_{p(x)}$ in X. In our paper, we will use the norm $\|u\|_a$ and the modular is defined as $\rho_{p(x)} : X \to \mathbb{R}$ by

$$\rho_{p(x)}(u) = \int_\Omega |\Delta u|^{p(x)} + a(x)|u|^{p(x)}\, dx,$$

which satisfies the same properties as Proposition 2. Accordingly, we have the following property.

PROPOSITION 3. *For all $u \in L^{p(x)}(\Omega)$, we have*

1. $\|u\|_a < 1$ *(resp.* $= 1, > 1) \Leftrightarrow \rho_{p(x)}(u) < 1$ *(resp.* $= 1, > 1)$.
2. $\min(\|u\|_a^{p^-}, \|u\|_a^{p^+}) \le \rho_{p(x)}(u) \le \max(\|u\|_a^{p^-}, \|u\|_a^{p^+})$.
3. $\|u_n\|_a \to 0$ *(respectively,* $\to \infty) \Leftrightarrow \rho_{p(x)}(u_n) \to 0$ *(respectively,* $\to \infty)$.

PROPOSITION 4. *Let* $L(u) = \int_\Omega \frac{1}{p(x)} (|\Delta u|^{p(x)}\, dx + a(x)|u|^{p(x)})\, dx$, *then*

1. $L : X \to \mathbb{R}$ *is sequentially weakly lower semi continuous,* $L \in C^1(X, \mathbb{R})$.
2. *The mapping* $L' : X \to X^*$ *is a strictly monotone, bounded homeomorphism and is of type* (S_+), *that is, if* $u_n \rightharpoonup u$ *and* $\limsup_{n\to+\infty} L'(u_n)(u_n - u) \le 0$, *then* $u_n \to u$.

We recall that the critical Sobolev exponent is defined as follows:

$$\begin{cases} p^*(x) = \dfrac{Np(x)}{N - 2p(x)}, & p(x) < \dfrac{N}{2}, \\ p^*(x) = +\infty, & p(x) \ge \dfrac{N}{2}. \end{cases}$$

We point out that if $q \in C^+(\overline{\Omega})$ and $q(x) < p^*(x)$ for all $x \in \overline{\Omega}$, then X is continuously and compactly embedded in $L^{q(x)}(\Omega)$.

The Lebesgue and Sobolev spaces with variable exponents coincide with the usual Lebesgue and Sobolev spaces provided that p is constant. According to [36, pp. 8–9], the function spaces $L^{p(x)}$ and $W^{1,p(x)}$ have some non-usual properties, such as:

(i) If $p \geq 1$ is a real number, then the following co-area formula

$$\int_\Omega |u(x)|^p\, dx = p \int_0^\infty t^{p-1} |\{x \in \Omega; |u(x)| > t\}|\, dt$$

has no analogue in the framework of variable exponents (namely, if $p : \overline{\Omega} \to [1, \infty)$ is a nonconstant smooth function).

(ii) Spaces $L^{p(x)}$ do *not* satisfy the *mean continuity property*. More exactly, if p is nonconstant and continuous in an open ball B, then there is some $u \in L^{p(x)}(B)$ such that $u(x+h) \notin L^{p(x)}(B)$ for every $h \in \mathbb{R}^N$ with arbitrary small norm.

(iii) Function spaces with variable exponent are *never* invariant with respect to translations. The convolution is also limited. For instance, the classical Young inequality

$$|f * g|_{p(x)} \leq C |f|_{p(x)} \|g\|_{L^1}$$

remains true if and only if p is constant.

3. Auxiliary properties and main result for problem (1)

Throughout this section, the letters $c, c_i, i = 1, 2, \ldots.$ denote positive constants which may change from line to line. Let $s_1'(x)$ denote the conjugate exponent of the function $s_1(x)$ and set $r_1(x) := \frac{s_1(x)q(x)}{s_1(x)-q(x)}$. Then we have the following embedding property.

Remark 1. Assume that assumption (H_2) is fulfilled, then $\max(r_1(x), s_1'(x)q(x)) < p^*(x)$, for all $x \in \overline{\Omega}$, consequently the embeddings $X \hookrightarrow L^{s_1'(x)q(x)}(\Omega)$ and $X \hookrightarrow L^{r_1(x)}(\Omega)$ are compact and continuous.

Definition 2. We say that $\lambda \in \mathbb{R}$ is an eigenvalue of problem (1), if there exists $u \in X \backslash \{0\}$ such that $\Delta u = 0$ on $\partial\Omega$ and

$$M\Big(\int_\Omega \frac{1}{p(x)} (|\Delta u|^{p(x)}\, dx + a|u|^{p(x)}\, dx)\Big) \int_\Omega (|\Delta u|^{p(x)-2} \Delta u \Delta v + a|u|^{p(x)-2} uv)\, dx$$
$$= \lambda \int_\Omega V_1 |u|^{q(x)-2} uv\, dx,$$

for any $v \in X$. If λ is an eigenvalue of problem (1), then the corresponding $u \in X \backslash \{0\}$ is a weak solution of problem (1).

The first main result of this part is the following.

THEOREM 1. *Assume that hypotheses* (H_0), (H_1), (H_2) *are fulfilled. Then there exists* $\lambda^* > 0$, *such that any* $\lambda \in (0, \lambda^*)$ *is an eigenvalue of problem* (1).

In order to describe the variational framework associated to (1), we define the functionals $\Phi, J : X \to \mathbb{R}$ defined as follows:

$$\Phi(u) = \hat{M}\Big(\int_\Omega \frac{1}{p(x)}(|\Delta u|^{p(x)} + a(x)|u|^{p(x)})\,dx\Big) \quad \text{and} \quad J(u) = \int_\Omega \frac{V_1(x)}{q(x)}|u|^{q(x)}\,dx.$$

By Proposition 2 and Remark 1, J is well defined and for all $u \in X$

$$|J(u)| \le \frac{1}{q^-}|V_1|_{s_1(x)}\,\big||u|^{q(x)}\big|_{s_1'(x)} \le \begin{cases} \frac{1}{q^-}|V_1|_{s_1(x)}|u|^{q^-}_{s_1'(x)q(x)}, & \text{if } |u|_{s_1'(x)q(x)} \le 1, \\ \frac{1}{q^-}|V_1|_{s_1(x)}|u|^{q^+}_{s_1'(x)q(x)}, & \text{if } |u|_{s_1'(x)q(x)} > 1. \end{cases}$$

The Euler–Lagrange functional corresponding to problem (1) is $\Psi_\lambda : X \to \mathbb{R}$ and is defined by

$$\Psi_\lambda(u) := \Phi(u) - \lambda J(u).$$

3.1. *Proof of Theorem 1*

We start with the following auxiliary property.

PROPOSITION 5. *Assume that hypotheses* (H_0), (H_1) *and* (H_2) *are fulfilled. Then* $\Psi_\lambda \in C^1(X, \mathbb{R})$ *is weakly lower semi-continuous and* $u \in X$ *is a critical point of* Ψ_λ *if and only if* u *is a weak solution of problem* (1).

PROOF. To show that $\Psi_\lambda \in C^1(X, \mathbb{R})$, we establish that for all $\varphi \in X$,

$$\lim_{t\to 0^+} \frac{\Psi_\lambda(u + t\varphi) - \Psi_\lambda(u)}{t} = \langle d\Psi_\lambda(u), \varphi\rangle,$$

and $d\Psi_\lambda : X \to X^*$ continuous, where we denote by X^* the dual space of X.

For all $\varphi \in X$ we have

$$\begin{aligned} \lim_{t\to 0^+} \frac{J(u + t\varphi) - J(u)}{t} &= \frac{d}{dt}J(u + t\varphi)|_{t=0} = \frac{d}{dt}\int_\Omega \frac{V_1(x)}{q(x)}|u + t\varphi|^{q(x)}\,dx|_{t=0} \\ &= \int_\Omega \frac{\partial}{\partial t}\Big(\frac{V_1(x)}{q(x)}|u + t\varphi|^{q(x)}\Big)\Big|_{t=0}\,dx \\ &= \int_\Omega V_1(x)|u + t\varphi|^{q(x)-1}\,sgn(u + t\varphi)\varphi|_{t=0}\,dx \\ &= \int_\Omega V_1(x)|u + t\varphi|^{q(x)-2}(u + t\varphi)\varphi|_{t=0}\,dx \end{aligned}$$

$$= \int_\Omega V_1(x)|u|^{q(x)-2}u\varphi\,dx$$
$$= \langle dJ(u), \varphi\rangle.$$

The differentiation under the integral is allowed, since for all $|t| < 1$ we have:

$$|V_1(x)|u + t\varphi|^{q(x)-2}(u + t\varphi)\varphi| \le |V_1(x)|(|u| + |\varphi|)^{q(x)-1}|\varphi| \in L^1(\Omega).$$

Since $u, \varphi \in X$ we have

$$|u|, |\varphi| \in X \hookrightarrow L^{q(x)}(\Omega) \quad \text{and} \quad |\varphi| \in X \hookrightarrow L^{r_1(x)}(\Omega).$$

Due to the fact that $V_1 \in L^{s_1(x)}(\Omega)$, the conclusion is an immediate consequence of the generalized Hölder inequality (5).

Next, we show that for all $u \in X$, $dJ(u)$ is in X^*. We first observe that $dJ(u)$ is linear. Since there is a continuous embedding $X \hookrightarrow L^{r_1(x)}(\Omega)$, we have

$$|v|_{r_1(x)} \le c\|v\|_a, \quad \text{for all } v \in X. \tag{6}$$

Using inequalities (5) and (6) we obtain

$$\begin{aligned} |\langle dJ(u), \varphi\rangle| &= \left|\int_\Omega V_1(x)|u|^{q(x)-2}u\varphi\,dx\right| \\ &\le \int_\Omega |V_1(x)|\,|u|^{q(x)-1}|\varphi|\,dx \\ &\le |V_1|_{s_1(x)}|\,|u|^{q(x)-1}|_{\frac{q(x)}{q(x)-1}}|\varphi|_{r_1(x)} \\ &\le c|V_1|_{s_1(x)}|\,|u|^{q(x)-1}|_{\frac{q(x)}{q(x)-1}}\|\varphi\|_a. \end{aligned}$$

Thus, there exists $c_1 := c|V_1(x)|\,|\,|u|^{q(x)-1}|_{\frac{q(x)}{q(x)-1}} > 0$ such that

$$|\langle dJ(u), \varphi\rangle| \le c_1\|\varphi\|_a.$$

Using the linearity of $dJ(u)$ and the above inequality we deduce that $dJ(u) \in X^*$. For the Fréchet differentiability we need the following auxiliary property.

LEMMA 1 (See [5]). *The map* $u \in L^{q(x)}(\Omega) \mapsto |u|^{q(x)-2}u \in L^{\frac{q(x)}{q(x)-1}}(\Omega)$ *is continuous.*

We conclude that J is Fréchet differentiable.

The functional Φ is well defined, is continuously Gâteaux differentiable and its Gâteaux derivative is given by

$$\langle d\Phi(u), v\rangle = M\Big(\int_\Omega \frac{1}{p(x)}(|\Delta u|^{p(x)}\,dx + a(x)|u|^{p(x)}\,dx)\Big) \times \int_\Omega (|\Delta u|^{p(x)-2}\Delta u\Delta v + a(x)|u|^{p(x)-2}uv)\,dx,$$

for all $u, v \in X$.

We deduce that $\Psi_\lambda \in C^1(X, \mathbb{R})$ because $\Phi, J \in C^1(X, \mathbb{R})$. Moreover

$$\langle d\Psi_\lambda(u), v\rangle = M\Big(\int_\Omega \frac{1}{p(x)}(|\Delta u|^{p(x)}\,dx + a(x)|u|^{p(x)}\,dx)\Big) \times \int_\Omega (|\Delta u|^{p(x)-2}\Delta u\Delta v + a(x)|u|^{p(x)-2}uv)\,dx - \lambda\int_\Omega V_1(x)|u|^{q(x)-2}uv\,dx$$

for all $u, v \in X$.

Let u be a critical point of Ψ_λ. Then we have $d\Psi_\lambda(u) = 0_{X^*}$, that is,

$$\langle d\Psi_\lambda(u), v\rangle = 0, \quad \text{for all } v \in X.$$

Therefore

$$M\Big(\int_\Omega \frac{1}{p(x)}(|\Delta u|^{p(x)}\,dx + a(x)|u|^{p(x)}\,dx)\Big)\int_\Omega (|\Delta u|^{p(x)-2}\Delta u\Delta v + a(x)|u|^{p(x)-2}uv)\,dx = \lambda\int_\Omega V_1(x)|u|^{q(x)-2}uv\,dx,$$

for all $v \in X$. It follows that u is a weak solution of problem (1).

Now we assume that u is a weak solution of (1). By Definition 2 we deduce that $\langle d\Psi_\lambda(u), v\rangle = 0$, for all $v \in X$. We obtain $d\Psi_\lambda(u) = 0_{X^*}$, hence u is a critical point of Ψ_λ. This completes the proof of Proposition 5. □

The following property shows the existence of a *mountain* for Ψ_λ near the origin.

LEMMA 2. *Suppose that the hypotheses of Theorem* 1 *are fulfilled. Then for all* $\rho \in (0,1)$, *there exist* $\lambda^* > 0$ *and* $b > 0$ *such that for all* $u \in X$ *with* $\|u\|_a = \rho$

$$\Psi_\lambda(u) \ge b > 0 \quad \textit{for all } \lambda \in (0, \lambda^*).$$

PROOF. Since the embedding $X \hookrightarrow L^{s_1'(x)q(x)}(\Omega)$ is continuous, we have

$$|u|_{s_1'(x)q(x)} \le c_2\|u\|_a, \quad \text{for all } u \in X. \tag{7}$$

Let us assume that $\|u\|_a < \min(1, 1/c_2)$, where c_2 is the positive constant of inequality (7). It follows that $|u|_{s_1'(x)q(x)} < 1$. Moreover, by hypothesis (H_0), we have $\hat{M}(t) \geq m_0 t$. Consequently, by combining Hölder's inequality (4), Proposition 3 and inequality (7), we deduce that for all $u \in X$ with $\|u\|_a = \rho$,

$$\begin{aligned}
\Psi_\lambda(u) &= \hat{M}\Big(\int_\Omega \frac{1}{p(x)}(|\Delta u|^{p(x)} + a(x)|u|^{p(x)})\,dx\Big) - \frac{\lambda}{q^-}\int_\Omega V_1(x)|u|^{q(x)}\,dx \\
&\geq \frac{m_0}{p^+}\|u\|_a^{p^+} - \frac{\lambda}{q^-}|V_1|_{s_1(x)}\big|\,|u|^{q(x)}\big|_{s'(x)} \\
&\geq \frac{m_0}{p^+}\|u\|_a^{p^+} - \frac{\lambda}{q^-}|V_1|_{s_1(x)}|u|^{q^-}_{s'(x)q(x)} \\
&\geq \frac{m_0}{p^+}\|u\|_a^{p^+} - \frac{\lambda}{q^-}|V_1|_{s_1(x)}c_2^{q^-}\|u\|_a^{q^-} \\
&= \frac{m_0}{p^+}\rho^{p^+} - \frac{\lambda}{q^-}c_2^{q^-}|V_1|_{s_1(x)}\rho^{q^-} = \rho^{q^-}\Big(\frac{m_0}{p^+}\rho^{p^+-q^-} - \frac{\lambda}{q^-}c_2^{q^-}|V_1|_{s_1(x)}\Big).
\end{aligned}$$

By the above inequality, we remark that if we define

$$\lambda^* = \frac{m_0\rho^{p^+-q^-}}{2p^+}\,\frac{q^-}{c_2^{q^-}|V_1|_{s_1(x)}}, \tag{8}$$

then for any $\lambda \in (0, \lambda^*)$ and $u \in X$ with $\|u\|_a = \rho$ there exists $b > 0$ such that

$$\Psi_\lambda(u) \geq b > 0.$$

The proof of Lemma 2 is complete. □

The following result asserts the existence of a *valley* for Ψ_λ near the origin.

LEMMA 3. *There exists $\varphi \in X \backslash \{0\}$ such that $\varphi \geq 0$ and $\Psi_\lambda(t\varphi) < 0$, for $t > 0$ small enough.*

PROOF. By hypothesis (H_1), there exists $t_0 > 0$ such that for all $t > t_0$, we have

$$\hat{M}(t) \leq \frac{\hat{M}(t_0)}{t_0^{\frac{1}{1-\theta}}}\,t^{\frac{1}{1-\theta}} = c_3 t^{\frac{1}{1-\theta}}.$$

Moreover, by (H_2), we have $q(x) < p(x)$, for all $x \in \overline{\Omega}_0$.

In the sequel, we denote

$$q_0^- := \inf_{\Omega_0} q(x) \quad \text{and} \quad p_0^- := \inf_{\Omega_0} p(x).$$

Let ε_0 be such that $q_0^- + \varepsilon_0 < p_0^-$. Since $q \in C(\overline{\Omega}_0)$, there exists an open set $\Omega_1 \subset \Omega_0$ such that $|q(x) - q_0^-| < \varepsilon_0$, for all $x \in \Omega_1$. It follows that $q(x) \leq q_0^- + \varepsilon_0 < p_0^-$, for all $x \in \Omega_1$.

Let $\varphi \in C_0^\infty(\Omega)$ be such that $\operatorname{supp}(\varphi) \subset \Omega_1 \subset \Omega_0$, $\varphi = 1$ in a subset $\Omega_1' \subset \operatorname{supp}(\varphi)$, and $0 \le \varphi \le 1$ in Ω_1. It follows that

$$\begin{aligned}
\Psi_\lambda(t\varphi) &= \hat{M}\Big(\int_\Omega \frac{1}{p(x)}(|\Delta(t\varphi)|^{p(x)} + a(x)|t\varphi|^{p(x)})\,dx\Big) - \lambda\int_\Omega \frac{V_1(x)}{q(x)}|t\varphi|^{q(x)}\,dx \\
&\le c_3\Big(\int_{\Omega_0} \frac{t^{p(x)}}{p(x)}(|\Delta\varphi|^{p(x)} + a(x)|\varphi|^{p(x)})\,dx\Big)^{\frac{1}{1-\theta}} - \lambda\int_{\Omega_1} \frac{V_1(x)}{q(x)}t^{q(x)}V(x)|\varphi|^{q(x)}\,dx \\
&\le \frac{t^{\frac{p_0^-}{1-\theta}}}{(p_0^-)^{\frac{1}{1-\theta}}}\Big(\int_{\Omega_0}(|\Delta\varphi|^{p(x)} + a(x)|\varphi|^{p(x)})\,dx\Big)^{\frac{1}{1-\theta}} - \frac{\lambda t^{q_0^-+\varepsilon_0}}{q_0^+}\int_{\Omega_1} V_1(x)|\varphi|^{q(x)}\,dx \\
&\le c_4 t^{p_0^-}\Big(\int_{\Omega_0}(|\Delta\varphi|^{p(x)} + a(x)|\varphi|^{p(x)})\,dx\Big)^{\frac{1}{1-\theta}} - \frac{\lambda t^{q_0^-+\varepsilon_0}}{q_0^+}\int_{\Omega_1} V_1(x)|\varphi|^{q(x)}\,dx.
\end{aligned}$$

Therefore

$$\Psi_\lambda(t\varphi) < 0$$

for $t < \delta^{1/(p_0^- - q_0^- - \varepsilon_0)}$ with

$$0 < \delta < \min\left\{1, \frac{\frac{\lambda}{c_4 q_0^+}\int_{\Omega_1} V_1(x)|\varphi|^{q(x)}\,dx}{(\int_{\Omega_0}(|\Delta\varphi|^{p(x)} + a(x)|\varphi|^{p(x)})\,dx)^{\frac{1}{1-\theta}}}\right\}.$$

Finally, we point out that

$$\int_{\Omega_0}(|\Delta\varphi|^{p(x)} + a(x)|\varphi|^{p(x)})\,dx > 0.$$

Indeed, supposing the contrary we have $\int_\Omega(|\Delta\varphi|^{p(x)} + a(x)|u|^{p(x)})\,dx = 0$. By Proposition 3, we deduce that $\|\varphi\|_a = 0$ and consequently $\varphi = 0$ in Ω, a contradiction. The proof of Lemma 3 is complete. □

Proof of Theorem 1 completed. Let $\lambda^* > 0$ be defined as in (8) and $\lambda \in (0, \lambda^*)$. By Lemma 2 it follows that on the boundary of the ball centered at the origin and of radius ρ in X, denoted by $B_\rho(0)$, we have

$$\inf_{\partial B_\rho(0)} \Psi_\lambda > 0. \tag{9}$$

On the other hand, by Lemma 3, there exists $\varphi \in X$ such that $\Psi_\lambda(t\varphi) < 0$ for all $t > 0$ small enough. Moreover, by hypothesis (H_0), Hölder's inequality (4), Proposition 3 and inequality (7), we deduce that for any $u \in B_\rho(0)$ we have

$$\Psi_\lambda(u) \ge \frac{m_0}{p^+}\|u\|_a^{p^+} - \frac{\lambda}{q^-}c_2^{q^-}|V_1|_{s_1(x)}\|u\|_a^{q^-}.$$

It follows that

$$-\infty < \underline{c} := \inf_{\overline{B_\rho(0)}} \Psi_\lambda < 0.$$

Let $0 < \varepsilon < \inf_{\partial B_\rho(0)} \Psi_\lambda - \inf_{B_\rho(0)} \Psi_\lambda$. Due to the above information, the functional $\Psi_\lambda : \overline{B_\rho(0)} \to \mathbb{R}$ is lower bounded on $\overline{B_\rho(0)}$ and $\Psi_\lambda \in C^1(\overline{B_\rho(0)}, \mathbb{R})$. Thus, by Ekeland's variational principle [14], there exists $u_\varepsilon \in \overline{B_\rho(0)}$ such that

$$\begin{cases} \underline{c} \le \Psi_\lambda(u_\varepsilon) \le \underline{c} + \varepsilon \\ 0 < \Psi_\lambda(u) - \Psi_\lambda(u_\varepsilon) + \varepsilon \cdot \|u - u_\varepsilon\|_a, \quad u \ne u_\varepsilon. \end{cases}$$

Since

$$\Psi_\lambda(u_\varepsilon) \le \inf_{\overline{B_\rho(0)}} \Psi_\lambda + \varepsilon \le \inf_{B_\rho(0)} \Psi_\lambda + \varepsilon < \inf_{\partial B_\rho(0)} \Psi_\lambda,$$

we deduce that $u_\varepsilon \in B_\rho(0)$. Now, we define $I_\lambda : \overline{B_\rho(0)} \to \mathbb{R}$ by $I_\lambda(u) = \Psi_\lambda(u) + \varepsilon \cdot \|u - u_\varepsilon\|_a$. It is clear that u_ε is a minimum point of I_λ and thus

$$\frac{I_\lambda(u_\varepsilon + t \cdot v) - I_\lambda(u_\varepsilon)}{t} \ge 0$$

for small $t > 0$ and any $v \in B_1(0)$. The above relation yields

$$\frac{\Psi_\lambda(u_\varepsilon + t \cdot v) - \Psi_\lambda(u_\varepsilon)}{t} + \varepsilon \cdot \|v\|_a \ge 0$$

Letting $t \to 0$ it follows that $\langle d\Psi_\lambda(u_\varepsilon), v\rangle + \varepsilon \cdot \|v\|_a \ge 0$ and we infer that $\|d\Psi_\lambda(u_\varepsilon)\|_a \le \varepsilon$.

We deduce that there exists a sequence $\{w_n\} \subset B_\rho(0)$ such that

$$\Psi_\lambda(w_n) \to \underline{c} \quad \text{and} \quad d\Psi_\lambda(w_n) \to 0_{X^*}. \tag{10}$$

The sequence $\{w_n\}$ is bounded in X. Thus, there exists w in X such that, up to a subsequence, $\{w_n\}$ converges weakly to w in X. Since $r_1(x) < p^*(x)$ for all $x \in \overline{\Omega}$ we deduce that there exists a compact embedding $E \hookrightarrow L^{r_1(x)}(\Omega)$ and consequently $\{w_n\}$ converges strongly in $L^{r_1(x)}(\Omega)$. In order to have strong convergence, we need the following auxiliary result.

Proposition 6. *We have*

$$\lim_{n\to\infty} \int_\Omega V_1(x)|w_n|^{q(x)-2} w_n(w_n - w)\, dx = 0.$$

PROOF. Using Hölder's inequality (4) we have

$$\begin{aligned}\int_\Omega V_1(x)|w_n|^{q(x)-2}w_n(w_n - w)\,dx &\le |V_1|_{s_1(x)}\,\big||w_n|^{q(x)-2}w_n(w_n - w)\big|_{s_1'(x)}\\ &\le |V_1|_{s_1(x)}\,\big||w_n|^{q(x)-2}w_n\big|_{\frac{q(x)}{q(x)-1}}|w_n - w|_{r_1(x)}.\end{aligned}$$

If $\big||w_n|^{q(x)-2}w_n\big|_{\frac{q(x)}{q(x)-1}} > 1$, by Proposition 2, we have

$$\big||w_n|^{q(x)-2}w_n\big|_{\frac{q(x)}{q(x)-1}} \le |w_n|_{q(x)}^{q^+}.$$

Using now the compact embedding $X \hookrightarrow L^{q(x)}(\Omega)$, we conclude the proof. □

Since $d\Psi_\lambda(w_n) \to 0$ and $\{w_n\}$ is bounded in X we have

$$\begin{aligned}|\langle d\Psi_\lambda(w_n), w_n - w\rangle| &\le |\langle d\Psi_\lambda(w_n), w_n\rangle| + |\langle d\Psi_\lambda(w_n), w\rangle|\\ &\le \|d\Psi_\lambda(w_n)\|_a\|w_n\|_a + \|d\Psi_\lambda(w_n)\|_a\|w\|_a.\end{aligned}$$

Moreover, by Proposition 6, we have

$$\lim_{n\to\infty}\langle d\Psi_\lambda(w_n), w_n - w\rangle = 0.$$

Therefore

$$\begin{aligned}&\lim_{n\to\infty} M\Big(\int_\Omega \frac{1}{p(x)}(|\Delta w_n|^{p(x)} + a(x)|w_n|^{p(x)})\,dx\Big)\\ &\quad\times \int_\Omega (|\Delta w_n|^{p(x)-2}\Delta w_n(\Delta w_n - \Delta w) + a(x)|w_n|^{p(x)-2}w_n(w_n - w))\,dx = 0.\end{aligned}$$

Combining hypothesis (H_0) and Proposition 4, we deduce that $\{w_n\}$ converges strongly to w in X. Since $\Psi_\lambda \in C^1(X, \mathbb{R})$, we conclude that

$$d\Psi_\lambda(w_n) \to d\Psi_\lambda(w), \quad \text{as } n \to \infty. \tag{11}$$

Now, relations (10) and (11) yield

$$\Psi_\lambda(w) = \underline{c} < 0 \quad \text{and} \quad d\Psi_\lambda(w) = 0. \tag{12}$$

In order to show that w is a solution of problem (1), it remains to show that $\Delta w = 0$ on $\partial\Omega$. Due to relation (12), $w \in X\backslash\{0\}$ is a critical point of Ψ_λ, so

$$M(t)\int_\Omega |\Delta w|^{p(x)-2}\Delta w\Delta v\,dx = \int_\Omega m(x)v\,dx \quad \text{for all } v \in X, \tag{13}$$

where

$$m(x) = \lambda V_1(x)|w|^{q(x)-2}w - M(t)a(x)|w|^{p(x)-2}w.$$

Relation (13) implies that

$$(14)\qquad M(t)\int_\Omega |\Delta w|^{p(x)-2}\Delta w \Delta v\,dx = \int_\Omega m(x)v\,dx \quad \text{for all } v \in C_0^\infty(\Omega).$$

Let ζ be the unique solution of the problem

$$\begin{cases} \Delta\zeta = m(x) & \text{in } \Omega \\ \zeta = 0 & \text{on } \partial\Omega. \end{cases}$$

Relation (14) yields

$$M(t)\int_\Omega |\Delta w|^{p(x)-2}\Delta w \Delta v\,dx = \int_\Omega (\Delta\zeta)v\,dx \quad \text{for all } v \in C_0^\infty(\Omega).$$

Using the Green formula we have

$$\int_\Omega (\Delta\zeta)v\,dx = \int_\Omega \zeta\Delta v\,dx.$$

Therefore

$$(15)\qquad M(t)\int_\Omega |\Delta w|^{p(x)-2}\Delta w \Delta v\,dx = \int_\Omega \zeta\Delta v\,dx \quad \text{for all } v \in C_0^\infty(\Omega).$$

On the other hand, for all $\tilde{w} \in C_0^\infty(\Omega)$ there exists a unique $v \in C_0^\infty(\Omega)$ such that $\Delta v = \tilde{w}$ in Ω. Thus, relation (15) can be rewritten as

$$\int_\Omega (M(t)|\Delta w|^{p(x)-2}\Delta w - \zeta)\tilde{w}\,dx = 0 \quad \text{for all } \tilde{w} \in C_0^\infty(\Omega).$$

Applying the fundamental lemma of the calculus of variations, we deduce that

$$M(t)|\Delta w|^{p(x)-2}\Delta w - \zeta = 0 \quad \text{in } \Omega.$$

According to assertion (H_0) and since $\zeta = 0$ on $\partial\Omega$, we conclude that $\Delta w = 0$ on $\partial\Omega$. Thus, w is a nontrivial weak solution of problem (1) such that $\Delta w = 0$. Since $\Psi_\lambda(|w|) = \Psi_\lambda(w)$ then problem (1) has a non-negative solution. The proof is now complete. □

4. Main result for problem (2)

In what follows assume we the following hypothesis:

(H_3) $1 < q(x) < \alpha(x) < p(x) < \frac{N}{2} < \min(s_1(x), s_2(x))$, for all $x \in \overline{\Omega}$, where $s_2 \in C(\overline{\Omega})$ and $V_2 \in L^{s_2(x)}(\Omega)$ such that $V_2 \geq 0$ in Ω.

Let $s_2'(x)$ denote the conjugate exponent of the function $s_2(x)$ and $r_2(x) := \frac{s_2(x)\alpha(x)}{s_2(x)-\alpha(x)}$. Then the following embedding properties hold.

REMARK 2. Under assumption (H_3), we have $\max(r_2(x)s_2'(x)\alpha(x)) < p^*(x)$, for all $x \in \overline{\Omega}$. Consequently, the embeddings $X \hookrightarrow L^{s_2'(x)\alpha(x)}(\Omega)$ and $X \hookrightarrow L^{r_2(x)}(\Omega)$ are compact and continuous.

DEFINITION 3. We say that $\lambda \in \mathbb{R}$ is an eigenvalue of problem (2), if there exists $u \in X\backslash\{0\}$ such that $\Delta u = 0$ on $\partial\Omega$ and

$$M(t)\int_\Omega (|\Delta u|^{p(x)-2}\Delta u\Delta v + a(x)|u|^{p(x)-2}uv)\,dx$$
$$= \lambda\int_\Omega (V_1(x)|u|^{q(x)-2} - V_2(x)|u|^{\alpha(x)-2})uv\,dx,$$

for any $v \in X$. If λ is an eigenvalue of problem (2), then the corresponding $u \in X\backslash\{0\}$ is a weak solution of problem (2).

The main result on this part of the paper is the following.

THEOREM 2. *Assume that hypotheses* (H_0), (H_1) *and* (H_3) *are fulfilled. Then there exists* $\lambda^* > 0$, *such that any* $\lambda \in (0, \lambda^*)$ *is an eigenvalue of problem* (2).

In order to describe the variational framework associated to (2), we define the functional $\gamma : X \to \mathbb{R}$ as follows:

$$\gamma(u) = \int_\Omega \frac{V_2(x)}{\alpha(x)}|u|^{\alpha(x)}\,dx.$$

By Proposition 2 and Remark 2, γ is well defined and for all $u \in X$

$$|\gamma(u)| \le \frac{1}{\alpha^-}|V_2|_{s_2(x)}|\,|u|^{\alpha(x)}|_{s_2'(x)} \le \begin{cases} \frac{1}{\alpha^-}|V_2|_{s(x)}|u|^{\alpha^-}_{s_2'(x)\alpha(x)}, & \text{if } |u|_{s_2'(x)\alpha(x)} \le 1, \\ \frac{1}{\alpha^-}|V_2|_{s(x)}|u|^{\alpha^+}_{s_2'(x)\alpha(x)}, & \text{if } |u|_{s_2'(x)\alpha(x)} > 1. \end{cases}$$

The Euler–Lagrange functional corresponding to problem (2) is $\Psi_\lambda : X \to \mathbb{R}$ and is defined by

$$\Psi_\lambda(u) := \Phi(u) - \lambda J(u) + \lambda\gamma(u).$$

According to Proposition 4, one has $\Phi \in C^1(X, \mathbb{R})$. Moreover, under assumption (H_3) and Proposition 2 in [6] one has $J, \gamma \in C^1(X, \mathbb{R})$, so $\Psi_\lambda \in C^1(X, \mathbb{R})$ and

$$\langle d\Psi_\lambda(u), v\rangle = M\Big(\int_\Omega \frac{1}{p(x)}(|\Delta u|^{p(x)}\,dx + a(x)|u|^{p(x)}\,dx)\Big)$$
$$\times \int_\Omega (|\Delta u|^{p(x)-2}\Delta u\Delta v + a(x)|u|^{p(x)-2}uv)\,dx$$
$$- \lambda\int_\Omega V_1(x)|u|^{q(x)-2}uv\,dx + \lambda\int_\Omega V_2(x)|u|^{\alpha(x)-2}uv\,dx,$$

for all $u, v \in X$.

4.1. *Proof of Theorem 2*

The following property shows the existence of a *mountain* for Ψ_λ near the origin.

LEMMA 4. *Suppose that the hypotheses of Theorem 2 are fulfilled. Then for all* $\rho \in (0,1)$, *there exists* $\lambda^* > 0$ *and* $b > 0$ *such that for all* $u \in X$ *with* $\|u\|_a = \rho$

$$\Psi_\lambda(u) \geq b > 0 \quad \textit{for all } \lambda \in (0, \lambda^*).$$

PROOF. Let us assume that $\|u\|_a < \min(1, 1/c_2)$, where c_2 is the positive constant of inequality (7). It follows that $|u|_{s_1'(x)q(x)} < 1$. Since $V_2 \geq 0$ on Ω, we use the same steps as the proof of Lemma 4 to deduce that for all $u \in X$ with $\|u\|_a = \rho$,

$$\begin{aligned}
\Psi_\lambda(u) &\geq \hat{M}\Big(\int_\Omega \frac{1}{p(x)}(|\Delta u|^{p(x)} + a(x)|u|^{p(x)})\,dx\Big) \\
&\quad - \frac{\lambda}{q^-}\int_\Omega V_1(x)|u|^{q(x)}\,dx + \frac{\lambda}{\alpha^+}\int_\Omega V_2(x)|u|^{\alpha(x)}\,dx \\
&\geq \frac{m_0}{p^+}\|u\|_a^{p^+} - \frac{\lambda}{q^-}|V_1|_{s_1(x)}\big|\,|u|^{q(x)}\big|_{s_1'(x)} \\
&= \frac{m_0}{p^+}\rho^{p^+} - \frac{\lambda}{q^-}c_2^{q^-}|V_1|_{s_1(x)}\rho^{q^-} = \rho^{q^-}\Big(\frac{m_0}{p^+}\rho^{p^+-q^-} - \frac{\lambda}{q^-}c_2^{q^-}|V_1|_{s_1(x)}\Big).
\end{aligned}$$

By the above inequality, we remark that if we define

$$\lambda^* = \frac{m_0\rho^{p^+-q^-}}{2p^+}\frac{q^-}{c_2^{q^-}|V_1|_{s_1(x)}}, \tag{16}$$

then for any $\lambda \in (0, \lambda^*)$ and $u \in X$ with $\|u\|_a = \rho$ there exists $b > 0$ such that

$$\Psi_\lambda(u) \geq b > 0.$$

The proof of Lemma 4 is complete. □

The following result asserts the existence of a *valley* for Ψ_λ near the origin.

LEMMA 5. *There exists* $\varphi \in X\backslash\{0\}$ *such that* $\varphi \geq 0$ *and* $\Psi_\lambda(t\varphi) < 0$, *for* $t > 0$ *small enough.*

PROOF. By assumption (H_3), one has $q(x) < \alpha(x)$, for all $x \in \overline{\Omega}_0$. In the sequel, we denote

$$q_0^- := \inf_{\Omega_0} q(x), \quad p_0^- := \inf_{\Omega_0} p(x) \quad \text{and} \quad \alpha_0^- := \inf_{\Omega_0} \alpha(x).$$

Let ε_0 be such that $q_0^- + \varepsilon_0 < \alpha_0^-$. Since $q \in C(\overline{\Omega}_0)$, there exists an open set $\Omega_1 \subset \Omega_0$ such that $|q(x) - q_0^-| < \varepsilon_0$, for all $x \in \Omega_1$. It follows that $q(x) \leq q_0^- + \varepsilon_0 < \alpha_0^-$, for all $x \in \Omega_1$.

Let $\varphi \in C_0^\infty(\Omega)$ be such that $\mathrm{supp}(\varphi) \subset \Omega_1 \subset \Omega_0$, $\varphi = 1$ in a subset $\Omega_1' \subset \mathrm{supp}(\varphi)$, and $0 \leq \varphi \leq 1$ in Ω_1. It follows that

$$
\begin{aligned}
\Psi_\lambda(t\varphi) &= \hat{M}\Big(\int_\Omega \frac{1}{p(x)}(|\Delta(t\varphi)|^{p(x)} + a(x)|t\varphi|^{p(x)})\,dx\Big) \\
&\quad - \lambda \int_\Omega \frac{V_1(x)}{q(x)}|t\varphi|^{q(x)}\,dx + \lambda \int_\Omega \frac{t^{\alpha(x)}}{\alpha(x)} V_2(x)|\varphi|^{\alpha(x)}\,dx \\
&\leq c_3\Big(\int_{\Omega_0} \frac{t^{p(x)}}{p(x)}(|\Delta\varphi|^{p(x)} + a(x)|\varphi|^{p(x)})\,dx\Big)^{\frac{1}{1-\theta}} \\
&\quad - \lambda \int_{\Omega_1} \frac{V_1(x)}{q(x)} t^{q(x)}|\varphi|^{q(x)}\,dx + \lambda \int_{\Omega_0} \frac{t^{\alpha(x)}}{\alpha(x)} V_2(x)|\varphi|^{\alpha(x)}\,dx \\
&\leq c_3 \frac{t^{\frac{p_0^-}{1-\theta}}}{(p_0^-)^{\frac{1}{1-\theta}}}\Big(\int_{\Omega_0}(|\Delta\varphi|^{p(x)} + a(x)|\varphi|^{p(x)})\,dx\Big)^{\frac{1}{1-\theta}} \\
&\quad - \frac{\lambda t^{q_0^- + \varepsilon_0}}{q_0^+}\int_{\Omega_1} V_1(x)|\varphi|^{q(x)}\,dx + \lambda \frac{t^{\alpha_0^-}}{\alpha_0^-}\int_{\Omega_0} V_2(x)|\varphi|^{\alpha(x)}\,dx \\
&\leq c_3 \frac{t^{p_0^-}}{(p_0^-)^{\frac{1}{1-\theta}}}\Big(\int_{\Omega_0}(|\Delta\varphi|^{p(x)} + a(x)|\varphi|^{p(x)})\,dx\Big)^{\frac{1}{1-\theta}} \\
&\quad - \frac{\lambda t^{q_0^- + \varepsilon_0}}{q_0^+}\int_{\Omega_1} V_1(x)|\varphi|^{q(x)}\,dx + \lambda \frac{t^{\alpha_0^-}}{\alpha_0^-}\int_{\Omega_0} V_2(x)|\varphi|^{\alpha(x)}\,dx \\
&\leq \frac{\max(c_3, 1)t^{\alpha_0^-}}{\min(\alpha_0^-, (p_0^-)^{\frac{1}{1-\theta}})}\Big[\int_{\Omega_0}(|\Delta\varphi|^{p(x)} + a(x)|\varphi|^{p(x)})\,dx)^{\frac{1}{1-\theta}} \\
&\quad + \lambda \int_{\Omega_0} V_2(x)|\varphi|^{\alpha(x)}\,dx\Big] - \frac{\lambda t^{q_0^- + \varepsilon_0}}{q_0^+}\int_{\Omega_1} V_1(x)|\varphi|^{q(x)}\,dx
\end{aligned}
$$

Therefore

$$\Psi_\lambda(t\varphi) < 0$$

for $t < \delta^{1/(\alpha_0^- - q_0^- - \varepsilon_0)}$ with

$$0 < \delta < \min\left\{1, \frac{\frac{\lambda \min(\alpha_0^-, (p_0^-)^{\frac{1}{1-\theta}})}{\max(c_3, 1)q_0^+}\int_{\Omega_1} V_1(x)|\varphi|^{q(x)}\,dx}{(\int_{\Omega_0}(|\Delta\varphi|^{p(x)} + a(x)|\varphi|^{p(x)})\,dx)^{\frac{1}{1-\theta}} + \int_{\Omega_0} V_2(x)|\varphi|^{\alpha(x)}\,dx}\right\}.$$

Finally, we point out that

$$\left(\int_{\Omega_0}(|\Delta\varphi|^{p(x)}+a(x)|\varphi|^{p(x)})\,dx\right)^{\frac{1}{1-\theta}}+\int_{\Omega_0}V_2(x)|\varphi|^{\alpha(x)}\,dx>0.$$

Indeed, supposing the contrary we have $(\int_\Omega(|\Delta\varphi|^{p(x)}+a(x)|u|^{p(x)}\,dx))^{\frac{1}{1-\theta}}+\int_{\Omega_0}V_2(x)|\varphi|^{\alpha(x)}\,dx=0$. By Proposition 3, we deduce that $\|\varphi\|_a=0$ and consequently $\varphi=0$ in Ω, a contradiction. The proof of Lemma 5 is complete. □

Proof of Theorem 2 completed. Let $\lambda^*>0$ be defined as in (16) and $\lambda\in(0,\lambda^*)$. By Lemma 4 it follows that on the boundary of the ball centered at the origin and of radius ρ in X, denoted by $B_\rho(0)$, we have

$$\inf_{\partial B_\rho(0)}\Psi_\lambda>0. \tag{17}$$

On the other hand, by Lemma 5, there exists $\varphi\in X$ such that $\Psi_\lambda(t\varphi)<0$ for all $t>0$ small enough. Moreover, by hypothesis (H_0), Hölder's inequality (4), Proposition 3 and inequality (7), we deduce that for any $u\in B_\rho(0)$ we have

$$\Psi_\lambda(u)\geq\frac{m_0}{p^+}\|u\|_a^{p^+}-\frac{\lambda}{q^-}c_2^{q-}|V_1|_{s_1(x)}\|u\|_a^{q^-}.$$

It follows that

$$-\infty<\underline{c}:=\inf_{\overline{B_\rho(0)}}\Psi_\lambda<0.$$

Let $0<\varepsilon<\inf_{\partial B_\rho(0)}\Psi_\lambda-\inf_{B_\rho(0)}\Psi_\lambda$. Due to the above information, the functional $\Psi_\lambda:\overline{B_\rho(0)}\to\mathbb{R}$ is lower bounded on $\overline{B_\rho(0)}$ and $\Psi_\lambda\in C^1(\overline{B_\rho(0)},\mathbb{R})$. Thus, by Ekeland's variational principle, there exists $u_\varepsilon\in\overline{B_\rho(0)}$ such that

$$\begin{cases}\underline{c}\leq\Psi_\lambda(u_\varepsilon)\leq\underline{c}+\varepsilon\\ 0<\Psi_\lambda(u)-\Psi_\lambda(u_\varepsilon)+\varepsilon\cdot\|u-u_\varepsilon\|_a,\quad u\neq u_\varepsilon.\end{cases}$$

Since

$$\Psi_\lambda(u_\varepsilon)\leq\inf_{\overline{B_\rho(0)}}\Psi_\lambda+\varepsilon\leq\inf_{B_\rho(0)}\Psi_\lambda+\varepsilon<\inf_{\partial B_\rho(0)}\Psi_\lambda,$$

we deduce that $u_\varepsilon\in B_\rho(0)$. Now, we define $I_\lambda:\overline{B_\rho(0)}\to\mathbb{R}$ by $I_\lambda(u)=\Psi_\lambda(u)+\varepsilon\cdot\|u-u_\varepsilon\|_a$. It is clear that u_ε is a minimum point of I_λ and thus

$$\frac{I_\lambda(u_\varepsilon+t\cdot v)-I_\lambda(u_\varepsilon)}{t}\geq 0$$

for small $t>0$ and any $v\in B_1(0)$. The above relation yields

$$\frac{\Psi_\lambda(u_\varepsilon+t\cdot v)-\Psi_\lambda(u_\varepsilon)}{t}+\varepsilon\cdot\|v\|_a\geq 0$$

Letting $t \to 0$ it follows that $\langle d\Psi_\lambda(u_\varepsilon), v\rangle + \varepsilon \cdot \|v\|_a \geq 0$ and we infer that $\|d\Psi_\lambda(u_\varepsilon)\|_a \leq \varepsilon$.

We deduce that there exists a sequence $\{w_n\} \subset B_\rho(0)$ such that

$$\Psi_\lambda(w_n) \to \underline{c} \quad \text{and} \quad d\Psi_\lambda(w_n) \to 0_{X^*}. \tag{18}$$

The sequence $\{w_n\}$ is bounded in X. Thus, there exists w in X such that, up to a subsequence, $\{w_n\}$ converges weakly to w in X. Since $\max(r_1(x), r_2(x)) < p^*(x)$, for all $x \in \overline{\Omega}$, then X is compactly embedded in $L^{r_1(x)}(\Omega)$ and $L^{r_2(x)}(\Omega)$. In order to establish the strong convergence of $\{u_n\}$ on X, we use proposition 6 and the following auxiliary property.

PROPOSITION 7. *We have*

$$\lim_{n\to\infty} \int_\Omega V_2(x)|u_n|^{\alpha(x)-2}u_n(u_n - u)\,dx = 0.$$

PROOF. Using Hölder's inequality (4) we have

$$\int_\Omega V_2(x)|u_n|^{\alpha(x)-2}u_n(u_n - u)\,dx \leq |V_2|_{s_2(x)}\,\big|\,|u_n|^{\alpha(x)-2}u_n\big|_{\frac{\alpha(x)}{\alpha(x)-1}}|u_n - u|_{r_2(x)},$$

Next, by Proposition 2, if

$$\big|\,|u_n|^{\alpha(x)-2}u_n\big|_{\frac{\alpha(x)}{\alpha(x)-1}} > 1$$

then

$$\big|\,|u_n|^{\alpha(x)-2}u_n\big|_{\frac{\alpha(x)}{\alpha(x)-1}} \leq |u_n|_{\alpha(x)}^{\alpha^+}.$$

Using the compact embedding $X \hookrightarrow L^{\alpha(x)}(\Omega)$, we conclude the proof of the proposition. □

Since $d\Psi_\lambda(w_n) \to 0$ and $\{w_n\}$ is bounded in X we have

$$\begin{aligned}|\langle d\Psi_\lambda(w_n), w_n - w\rangle| &\leq |\langle d\Psi_\lambda(w_n), w_n\rangle| + |\langle d\Psi_\lambda(w_n), w\rangle| \\ &\leq \|d\Psi_\lambda(w_n)\|_a\|w_n\|_a + \|d\Psi_\lambda(w_n)\|_a\|w\|_a.\end{aligned}$$

Moreover, by Propositions 6 and 7, we have

$$\lim_{n\to\infty} \langle d\Psi_\lambda(w_n), w_n - w\rangle = 0.$$

Therefore

$$\begin{aligned}&\lim_{n\to\infty} M\Big(\int_\Omega \frac{1}{p(x)}(|\Delta w_n|^{p(x)} + a(x)|w_n|^{p(x)})\,dx\Big) \\ &\quad\times \int_\Omega (|\Delta w_n|^{p(x)-2}\Delta w_n(\Delta w_n - \Delta w) + a(x)|w_n|^{p(x)-2}w_n(w_n - w))\,dx = 0.\end{aligned}$$

Combining hypothesis (H_0) and Proposition 4, we deduce that $\{w_n\}$ converges strongly to w in X. Since $\Psi_\lambda \in C^1(X, \mathbb{R})$, we conclude that

$$d\Psi_\lambda(w_n) \to d\Psi_\lambda(w), \quad \text{as } n \to \infty. \tag{19}$$

Now, relations (18) and (19) yield

$$\Psi_\lambda(w) = \underline{c} < 0 \quad \text{and} \quad d\Psi_\lambda(w) = 0. \tag{20}$$

In order to show that w is a solution of problem (2), it remains to show that $\Delta w = 0$ on $\partial\Omega$. Due to relation (20), $w \in X \setminus \{0\}$ is a critical point of Ψ_λ, so

$$M(t) \int_\Omega |\Delta w|^{p(x)-2} \Delta w \Delta v \, dx = \int_\Omega m(x) v \, dx \quad \text{for all } v \in X, \tag{21}$$

where

$$m(x) = \lambda(V_1(x)|w|^{q(x)-2} w - V_2(x)|w|^{\alpha(x)-2} w) - M(t) a(x) |w|^{p(x)-2} w.$$

Relation (21) implies that

$$M(t) \int_\Omega |\Delta w|^{p(x)-2} \Delta w \Delta v \, dx = \int_\Omega m(x) v \, dx \quad \text{for all } v \in C_0^\infty(\Omega). \tag{22}$$

Let ζ be the unique solution of the problem

$$\begin{cases} \Delta\zeta = m(x) & \text{in } \Omega \\ \zeta = 0 & \text{on } \partial\Omega. \end{cases}$$

Relation (22) yields

$$M(t) \int_\Omega |\Delta w|^{p(x)-2} \Delta w \Delta v \, dx = \int_\Omega (\Delta\zeta) v \, dx \quad \text{for all } v \in C_0^\infty(\Omega).$$

Using the Green formula we have

$$\int_\Omega (\Delta\zeta) v \, dx = \int_\Omega \zeta \Delta v \, dx.$$

Therefore

$$M(t) \int_\Omega |\Delta w|^{p(x)-2} \Delta w \Delta v \, dx = \int_\Omega \zeta \Delta v \, dx \quad \text{for all } v \in C_0^\infty(\Omega). \tag{23}$$

On the other hand, for all $\tilde{w} \in C_0^\infty(\Omega)$ there exists a unique $v \in C_0^\infty(\Omega)$ such that $\Delta v = \tilde{w}$ in Ω. Thus, relation (23) can be rewritten as

$$\int_\Omega (M(t)|\Delta w|^{p(x)-2} \Delta w - \zeta) \tilde{w} \, dx = 0 \quad \text{for all } \tilde{w} \in C_0^\infty(\Omega).$$

Applying the fundamental lemma of the calculus of variations, we deduce that

$$M(t)|\Delta w|^{p(x)-2}\Delta w - \zeta = 0 \quad \text{in } \Omega.$$

According to assertion (H_0) and since $\zeta = 0$ on $\partial\Omega$, we conclude that $\Delta w = 0$ on $\partial\Omega$. Thus, w is a nontrivial weak solution of problem (2) such that $\Delta w = 0$. Since $\Psi_\lambda(|w|) = \Psi_\lambda(w)$ then problem (2) has a non-negative solution. The proof is now complete. □

4.2. *Example*

Let $M(t) = \beta + \gamma t$, where β, γ are positive constants and $t := \int_\Omega \frac{1}{p(x)}(|\Delta u|^{p(x)}\,dx + a(x)|u|^{p(x)})\,dx$. We first observe that

$$M(t) \geq \beta > 0.$$

Taking $\theta = \frac{1}{2}$, we have

$$\hat{M}(t) = \int_0^t M(s)\,ds = \beta t + \frac{1}{2}\gamma t^2 \geq \frac{1}{2}(\beta + \gamma t)t = (1-\theta)M(t)t.$$

Consider the nonlocal problem

$$\begin{cases} M(t)(\Delta^2_{p(x)}u + a(x)|u|^{p(x)-2}u) = \lambda V_1(x)|u|^{q(x)-2}u, & x \in \Omega \\ u = \Delta u = 0, & x \in \partial\Omega, \end{cases} \tag{24}$$

where $a \in L^\infty(\Omega)$ such that $\operatorname{essinf}_{x\in\Omega} a(x) > 0$, the functions p, q and V_1 satisfy hypothesis assertion (H_2). Then, by Ekeland's variational principle, there exists λ^* such that any $\lambda \in (0, \lambda^*)$ is eigenvalue for problem (24). Moreover if we consider the problem

$$\begin{cases} M(t)(\Delta^2_{p(x)}u + a(x)|u|^{p(x)-2}u) \\ \quad = \lambda(V_1(x)|u|^{q(x)-2}u - V_2(x)|u|^{\alpha(x)-2}u), & x \in \Omega \\ u = \Delta u = 0, & x \in \partial\Omega, \end{cases} \tag{25}$$

where the functions p, q, α and V_2 satisfy hypothesis assertion (H_3), then there exists λ_1^* such that any $\lambda \in (0, \lambda_1^*)$ is eigenvalue for problem (25).

4.3. *Final comments*

(i) Problems (1) and (2) correspond to a *subcritical* setting, as described by Remark 1 and Remark 2. We consider that valuable research directions correspond either to the *critical* or to the *supercritical* framework (in the sense of Sobolev variable exponents). No results are known even for the *almost critical* case with *lack of compactness*. More precisely, with the same notations as in Remark 1 and 2, a very interesting open problem is to study the qualitative analysis of solutions of problem (1), provided that there exists $z_1 \in \Omega$ such that

$$\max(r_1(z_1), s_1'(z_1)q(z_1)) = p^*(z_1),$$

but

$$\max(r_1(x), s_1'(x)q(x)) < p^*(x) \quad \text{for all } z \in \Omega\backslash\{z_1\},$$

and those of problem (2) provided that there exists $z_1, z_2 \in \Omega$ such that

$$\max(r_1(z_1), s_1'(z_1)q(z_1)) = p^*(z_1) \quad \text{and} \quad \max(r_2(z_2)s_2'(z_2)\alpha(z_2)) = p^*(z_2)$$

but

$$\max(r_1(x), s_1'(x)q(x)) < p^*(x) \quad \text{for all } z \in \Omega\backslash\{z_1\}$$

and

$$\max(r_2(x)s_2'(x)\alpha(x)) < p^*(x) \quad \text{for all } z \in \Omega\backslash\{z_2\}.$$

(ii) Another very interesting research direction is to extend the approach developed in this paper to the abstract setting recently studied by Mingione et al. [4, 9, 10], namely *double phase* problems with associated energies of the type

$$u \mapsto \int_\Omega [|\Delta u|^{p_1(x)} + V(x)|\Delta u|^{p_2(x)}]\, dx$$

and

$$u \mapsto \int_\Omega [|\Delta u|^{p_1(x)} + V(x)|\Delta u|^{p_2(x)} \log(e + |x|)]\, dx,$$

where $p_1(x) \leq p_2(x)$, $p_1 \neq p_2$, and $V(x) \geq 0$. Considering two different materials with power hardening exponents $p_1(x)$ and $p_2(x)$ respectively, the coefficient $V(x)$ dictates the geometry of a composite of the two materials. When $V(x) > 0$ then $p_2(x)$-material is present, otherwise the $p_1(x)$-material is the only one making the composite.

These problems extend to a bi-harmonic setting with variable exponents the pioneering papers by Marcellini [26, 27] on (p, q)-problems, which involve integral functionals of the type

$$u \mapsto \int_\Omega F(x, \nabla u)\, dx.$$

The integrand $F : \Omega \times \mathbb{R}^N \to \mathbb{R}$ satisfied unbalanced polynomial growth conditions of the type

$$|\xi|^p \lesssim F(x, \xi) \lesssim |\xi|^q \quad \text{with } 1 < p < q,$$

for every $x \in \Omega$ and $\xi \in \mathbb{R}^N$.

(iii) We suggest to extend the methods developed in this paper to the more general framework of Musielak–Orlicz–Sobolev spaces (see [36, Chaper 4] for a collection of stationary problems studied in these function spaces).

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