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Partial Differential Equations — *$W^{2,2}$-solvability of the Dirichlet problem for a class of elliptic equations with discontinuous coefficients,* by Flavia Giannetti and Gioconda Moscariello, communicated on April 20, 2018.

Abstract. — We study the Dirichlet problem for the second order elliptic equation

$$-\sum_{i,j=1}^{N} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} = f(x)$$

in a bounded regular domain $\Omega \subset \mathbb{R}^N$, $N > 2$. We assume that $f \in L^2$ and that the coefficients a_{ij} are measurable and bounded functions with the first derivatives in the Marcinkiewicz class weak-L^N and having a sufficiently small distance to L^∞. Under these assumptions we prove the solvability of the problem in $W^{2,2} \cap W_0^{1,2^*}$, where $2^* = \frac{2N}{N-2}$. An higher integrability result for the gradient of the solution is achieved when $f \in L^p$, $p > 2$.

Key words: Nondivergence elliptic equations, Dirichlet problem

Mathematics Subject Classification: 35J25, 35B45

1. Introduction

We consider the Dirichlet problem

$$(1.1) \qquad \begin{cases} Lu(x) = -\displaystyle\sum_{i,j=1}^{N} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} = f(x) & \text{in } \Omega \\ u(x) = 0 & \text{on } \partial\Omega \end{cases}$$

where Ω is a bounded domain sufficiently regular in $\mathbb{R}^N$, $N > 2$ (see Section 2), $f \in L^2(\Omega)$ and a_{ij} are measurable functions satisfying the following conditions

$$(1.2) \qquad \begin{cases} a_{ij}(x) = a_{ji}(x) \\ |\xi|^2 \le \displaystyle\sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \le M|\xi|^2 \end{cases}$$

for every $\xi \in \mathbb{R}^N$ and for $x \in \Omega$ a.e.

The study of the existence and uniqueness in $W^{2,2} \cap W_0^{1,2}$ of the solution for problem (1.1) requires a particular attention if $N > 2$ since, in this case, some additional regularity of the coefficients are needed.

In [14] and [15] the problem has been solved by C. Miranda under the hypothesis $a_{ij} \in W^{1,N}$ and this assumption has been replaced in [7] with the condition $\dfrac{\partial a_{ij}}{\partial x_s} \in L^N$, for $1 \le s \le N-1$.

Later, a significant improvement has been given by A. Alvino and G. Trombetti (see [2]) assuming that the first derivatives of the coefficients belong to the Marcinkiewicz class weak-L^N with a suitable controll on the norms.

It is worth pointing out that, in [6], the authors proved a well-posedness result in the class $W^{2,p} \cap W_0^{1,p}$, $1 < p < +\infty$, for the Dirichlet problem (1.1) with $f \in L^p$, assuming a condition on the coefficients themselves, i.e. $a_{ij} \in VMO \cap L^\infty$. Finally, in [10], the problem has been faced under a smallness condition on the BMO-norm of a_{ij}. Observe that in [10] the coefficients could be unbounded.

On the other hand more recently, in [8] and in [11], some second order elliptic equations with lower order terms in divergence form, respectively linear and nonlinear, have been considered and the existence, the uniqueness and the regularity of the solutions have been studied, assuming that the coefficients of the lower order terms, lying in $L^{N,\infty}$, have a distance to L^∞ sufficiently small. Note that this condition doesn't imply the smallness of the $L^{N,\infty}$-norm of the coefficients themselves (see Section 2).

In the present paper, motivated by these last results and by the fact that the equation in (1.1) written in the variational form presents lower order terms (see Section 2), we study the solvability in $W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ of the problem (1.1), assuming that the first derivatives of the functions a_{ij} belong to the class $L^{N,\infty}(\Omega)$, $N > 2$, and satisfy some smallness conditions on their distances to L^∞.

Actually, we solve the problem in $W^{2,2}(\Omega) \cap W_0^{1,2^*}(\Omega)$ and our main result can be stated as follows

Theorem 1.1. *Let us assume that the functions a_{ij} satisfy* (1.2) *and that their first derivatives $\dfrac{\partial a_{ij}}{\partial x_s}$, $s = 1, \dots, N$, belong to the class $L^{N,\infty}(\Omega)$. Set $\mathbf{b} = \left(\sum_j \frac{\partial a_{ij}}{\partial x_j}\right)_i$ and $E = \sum_{irs}\left[\sum_k \frac{\partial(a_{ir}a_{ks} - a_{ik}a_{rs})}{\partial x_k}\right]^2$. If the following conditions hold*

$$\text{(1.3)} \qquad \operatorname{dist}_{L^{N,\infty}}(\mathbf{b}, L^\infty) < \frac{N-2}{4}, \quad \operatorname{dist}_{L^{\frac{N}{2},\infty}}(E, L^\infty) < \left(\frac{N-2}{4} \cdot 2^{-\frac{1}{N}}\right)^2$$

and $f \in L^2(\Omega)$, then the Dirichlet problem (1.1) *admits a unique solution $u \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$. Moreover $\nabla u \in L^{2^*}(\Omega)$ and there exists a positive constant C, depending on N and the distances in* (1.3), *such that*

$$\text{(1.4)} \qquad \|\nabla u\|_{L^{2^*}(\Omega)} + \|D^2 u\|_{L^2(\Omega)} \le C(\|f\|_{L^2(\Omega)} + \|\mathbf{b}\|_{L^2(\Omega)}).$$

Example 1.2. Consider the following Dirichlet problem in the cube $Q = (0,1]^N$

$$(1.5)\qquad \begin{cases} -\displaystyle\sum_{i,j=1}^{N}\left(\delta_{i,j}+\frac{Ax_ix_j}{|x|^2}+\varphi(x)\right)u_{x_ix_j}=f(x) & \text{in } Q\\ u(x)=0 & \text{on } \partial Q. \end{cases}$$

with $\varphi \in C^1(\bar{Q})$, $\varphi \geq 0$, and $f \in L^2(Q)$. Observe that the coefficients of the equation in (1.5) verify all the assumptions of Theorem 1.1. In particular, their derivatives are in $L^{N,\infty}(Q)$, but they do not belong to the Lebesgue space $L^N(Q)$ (for more details we refer to [17]). It follows that our result cannot be deduced by the one obtained by C. Miranda in [14] and [15]. Remark also that conditions in (1.3) are verified provided the constant A is not too large.

The basic idea in the proof of Theorem 1.1 is to combine the result of the existence and uniqueness in $W_0^{1,2}(\Omega)$ proved in [8] with Miranda's tools. This combination will be possible thanks to the fact that, as observed above, the equation in (1.1) can be opportunely written in divergence form. First of all, our strategy will consist in establishing the estimate in (1.4) for regular solutions. Once such a priori estimate will be proved, we shall consider regularized problems, whose solvability is known, and show that the limit of the regularized solutions solves problem (1.1).

In Section 4, assuming conditions on the coefficients a_{ij} similar to that of Theorem 1.1, we also obtain higher integrability of the gradient of the solution. More precisely, under the hypothesis $f \in L^p(\Omega)$, $p > 2$, we prove that $\nabla u \in L^{p^*}(\Omega)$.

We conclude underlining that our conditions on the distances are clearly satisfied if the derivatives $\dfrac{\partial a_{ij}}{\partial x_s}$ belong to any space in which L^∞ is dense, and then in particular if they belong to $L^{N,q}$ with $1 < q < \infty$, since such distances are null.

2. Notation and preliminary results

Our assumption on the domain Ω will be expressed in terms of validity of C. Miranda's tools. More precisely, we shall consider domains of class C^3 which are defined as follows

Definition 2.1. A bounded domain $\Omega \subset \mathbb{R}^N$ is of class C^3 if at each point $x_0 \in \partial\Omega$ there is a ball $B = B(x_0)$ and a one-to-one mapping ψ of B onto $D = \psi(B) \subset \mathbb{R}^N$ such that

$$\psi(B \cap \Omega) \subset \mathbb{R}^N_+;\quad \psi(B \cap \partial\Omega) \subset \partial\mathbb{R}^N_+;\quad \psi \in C^3(B);\quad \psi^{-1} \in C^3(D).$$

2.1. *Lorentz spaces*

In order to frame our problem, we recall some definitions and results useful in the sequel.

For $1 < p, q < +\infty$, the *Lorentz space* $L^{p,q}(\Omega)$ consists of all measurable functions f defined on Ω such that

$$\|f\|_{p,q}^q = p \int_0^{+\infty} |\Omega_t|^{\frac{q}{p}} t^{q-1}\, dt < +\infty$$

where we have used the notation $\Omega_t = \{x \in \Omega : |f(x)| > t\}$, for $t \geq 0$, and $|\Omega_t|$ for the Lebesgue measure of the set Ω_t. For $p = q$, the space $L^{p,q}(\Omega)$ coincides with the Lebesgue space $L^p(\Omega)$.

Finally, the class $L^{p,\infty}(\Omega)$, also known as the Marcinkiewicz class weak-$L^p(\Omega)$, consists of all functions f such that

$$|f|_{p,\infty}^p = \sup_{t>0} t^p |\Omega_t| < +\infty$$

and it is a Banach space equipped with the norm

$$\|f\|_{p,\infty} = \sup_{E \subset \Omega} |E|^{\frac{1}{p}-1} \int_E |f|\, dx. \tag{2.1}$$

Since it holds that

$$\frac{(p-1)^p}{p^{p+1}} \|f\|_{p,\infty}^p \leq |f|_{p,\infty}^p \leq \|f\|_{p,\infty}^p, \tag{2.2}$$

(see [3], Lemma A.2) we shall use the notation $L^{p,\infty}$ or weak-L^p, with the norm (2.1), indifferently.

It is useful for our aims to observe that for f belonging to weak-$L^p(\mathbb{R}^N)$ and $g \in L^1(\mathbb{R}^N)$, the convolution $f * g$ belongs to weak-$L^p(\mathbb{R}^N)$ and

$$\|f * g\|_{L^{p,\infty}} \leq \|f\|_{L^{p,\infty}} \|g\|_{L^1} \tag{2.3}$$

(see [18], Theorem 8, p. 119 and [3], Lemma A.4).

Note that the following inclusions hold

$$L^{p,1} \subset L^{p,q} \subset L^{p,r} \quad \text{for } 1 < q < r < +\infty$$
$$L^{p,q} \subset L^{p,\infty} \subset L^r \quad \text{for } 1 < r < p,\ 1 < q < +\infty$$

and that the distance of a given $f \in$ weak-L^p to L^∞ is defined as

$$\mathrm{dist}_{L^{p,\infty}}(f, L^\infty) = \inf_{g \in L^\infty} \|f - g\|_{p,\infty}.$$

For an exaustive discussion on the distance to L^∞ we refer to [5]. Here, we only stress that if we consider the truncation operator, defined for $h > 0$ as

$$T_h f = \frac{f}{|f|} \min\{|f|, h\},$$

we have that

$$\operatorname{dist}_{L^{p,\infty}}(f, L^{\infty}) = \lim_{h\to\infty} \|f - T_h f\|_{p,\infty}. \tag{2.4}$$

We remark that for any $p \in (1, \infty)$, L^{∞} is not dense in $L^{p,\infty}$. Moreover, assuming that $\operatorname{dist}_{L^{p,\infty}}(f, L^{\infty})$ is small does not give any smallness control on the norm in $L^{p,\infty}$ (see [8]).

The Sobolev Embedding theorem in Lorentz spaces will be useful for us (see [1], [9]).

THEOREM 2.2. *Let us assume* $1 < p < N$, $q \geq 1$. *Then every function* $u \in W_0^{1,1}(\Omega)$ *verifying* $|\nabla u| \in L^{p,q}$ *actually belongs to* $L^{p^*,q}$, *where* $p^* = \frac{Np}{N-p}$ *and*

$$\|u\|_{p^*,q} \leq C\|\nabla u\|_{p,q}$$

where $C = \omega_N^{-1/N} \frac{p}{N-p}$.

We shall need also the following *local estimate* near curved boundaries proved in [2].

THEOREM 2.3. *For any* $\tau > 0$ *there is some* $R_\tau > 0$ *such that, if* $u \in W^{1,2}(B_{R_\tau} \cap \Omega)$ *and* $\operatorname{supp} u \subset \overline{\Omega} \cap B_{R_\tau}$ *then*

$$\|u\|_{L^{2^*,2}(\Omega\cap B_{R_\tau})} \leq C(1+\tau)\|\nabla u\|_{L^2(\Omega\cap B_{R_\tau})} \tag{2.5}$$

where $C = \frac{2^{\frac{N+1}{N}}}{N-2}$.

Note that we used the notation B_{R_τ} to indicate a ball centered in the origin with radius R_τ.

2.2. *A useful estimate*

As observed in the Introduction, our approach is based on the fact that the equation in (1.1) can be written in the variational form

$$-\sum_j \frac{\partial}{\partial x_j}\Big(\sum_i a_{ij}\frac{\partial u}{\partial x_i}\Big) + \sum_i b_i \frac{\partial u}{\partial x_i} = f \tag{2.6}$$

provided

$$b_i(x) = \sum_j \frac{\partial a_{ij}}{\partial x_j}. \tag{2.7}$$

For this reason, we start with the following result in the variational context.

Theorem 2.4. *Let $\mathscr{A}(x)$ a matrix-valued, bounded function on Ω satisfying the ellipticity condition*

$$\mathscr{A}(x)\xi \cdot \xi \geq |\xi|^2$$

for all $(x, \xi) \in \Omega \times \mathbb{R}^N$, and let $\mathbf{b} \in L^{N,\infty}(\Omega, \mathbb{R}^N)$ such that

$$\mathrm{dist}_{L^{N,\infty}}(\mathbf{b}, L^\infty) < \frac{N-2}{4}.$$

Then

(1) *if $f \in L^{\frac{2N}{N+2}}$, there exists a unique solution $u \in W_0^{1,2}$ of the equation*

$$-\mathrm{div}(\mathscr{A}(x)\nabla u) + \mathbf{b}(x) \cdot \nabla u = f(x). \tag{2.8}$$

Moreover there exists a positive constant c depending on N and $\mathrm{dist}_{L^{N,\infty}}(\mathbf{b}, L^\infty)$ *such that*

$$\|\nabla u\|_2 \leq c(\|f\|_{\frac{2N}{N+2}} + \|\mathbf{b}\|_2); \tag{2.9}$$

(2) *if $f \in L^p$, $\frac{2N}{N+2} < p < \frac{N}{2}$, the solution $u \in L^{(p^*)^*}$ and the following estimate holds*

$$\|u\|_{(p^*)^*} \leq c(\|f\|_p + \|u\|_2); \tag{2.10}$$

for some constant $c = c(p, N)$;

(3) *if $f \in L^p$, $p > \frac{N}{2}$, the solution $u \in L^\infty$.*

Actually the real novelty in the previous result is the estimate (2.9), since the second and the third assertion can be easily deduced arguing respectively as in Theorem 4.1 and Theorem 4.9 of [8].

The estimate (2.9), which will be proved below, reveals to be a key tool in the proof of the a priori bound for the solution $u \in W^{2,2} \cap W_0^{1,2}$ of the problem (1.1).

Indeed, by virtue of the observation at the beginning of this section, we can easily deduce by Theorem 2.4 the following

Theorem 2.5. *Let $u \in W_0^{1,2}$ be the solution of the Dirichlet problem associated to* (2.6), *where the functions a_{ij} satisfy* (1.2), *$f \in L^2(\Omega)$ and $\mathbf{b} = \left(\sum_j \frac{\partial a_{ij}}{\partial x_j}\right)_i \in L^{N,\infty}$ is such that*

$$\mathrm{dist}_{L^{N,\infty}}(\mathbf{b}, L^\infty) < \frac{N-2}{4}.$$

Then there exist a positive constant c depending on N and $\mathrm{dist}_{L^{N,\infty}}(\mathbf{b}, L^\infty)$ *such that*

$$\|\nabla u\|_2 \leq c(\|f\|_2 + \|\mathbf{b}\|_2).$$

For the proof of Theorem 2.4, we shall need the following version of the Sobolev–Poincaré inequality contained in [12].

LEMMA 2.6. *For each matrix field* $H \in L^1_{loc}(\Omega, \mathbb{R}^N)$ *with* $\operatorname{div} H \in L^r(\Omega)$, $1 < r < N$, *there exists a divergence free matrix field* $H_0 \in L^1_{loc}(\Omega)$ *such that*

$$\left(\int_B |H - H_0|^{\frac{Nr}{N-r}} dx\right)^{\frac{N-r}{Nr}} \leq c(N,r)\left(\int_B |\operatorname{div} H|^r \, dx\right)^{\frac{1}{r}} \tag{2.11}$$

for every ball B strictly contained in Ω.

PROOF OF THEOREM 2.4. Fix $F \in L^2(\Omega)$ such that $\operatorname{div} F(x) = f(x)$ and observe that obviously we can write the variational equation in (2.8) as

$$-\operatorname{div}(\mathscr{A}(x)\nabla u) + \mathbf{b}(x) \cdot \nabla u = \operatorname{div} F(x). \tag{2.12}$$

Note now that the associated Dirichlet problem is dual to the following

$$\begin{cases} \operatorname{div}(\mathscr{A}(x)\nabla w + w\mathbf{b}(x)) = \operatorname{div} G(x) & \text{in } \Omega \\ w(x) = 0 & \text{on } \partial\Omega \end{cases} \tag{2.13}$$

with $G \in L^2(\Omega, \mathbb{R}^N)$. If we suppose that $\mathbf{b}_0 \in L^\infty$ is such that $\|\mathbf{b} - \mathbf{b}_0\|_{N,\infty} < \frac{N-2}{4}$, arguing as in [8] and [11], we rewrite the equation in (2.13) as

$$\operatorname{div}(\mathscr{A}\nabla w + w(\mathbf{b} - \mathbf{b}_0)) = \operatorname{div}(G - w\mathbf{b}_0) \tag{2.14}$$

and obtain the existence of a unique solution $w \in W^{1,2}_0$ of the problem (2.13).

Since, in particular, w solves equation (2.14), we obviously have

$$\int_\Omega \mathscr{A}\nabla w \cdot \nabla w + w(\mathbf{b} - \mathbf{b}_0) \cdot \nabla w \, dx = \int_\Omega (G - w\mathbf{b}_0)\nabla w \, dx.$$

By using Hölder's inequality and Sobolev's embedding, we get

$$\int_\Omega |w(\mathbf{b} - \mathbf{b}_0) \cdot \nabla w| \, dx \leq \frac{2}{N-2} \|\mathbf{b} - \mathbf{b}_0\|_{N,\infty} \|\nabla w\|_2^2,$$

and therefore, recalling the ellipticity assumption and that $\|\mathbf{b} - \mathbf{b}_0\|_{N,\infty} < \frac{N-2}{4}$, we get

$$\int_\Omega \mathscr{A}\nabla w \cdot \nabla w + w(\mathbf{b} - \mathbf{b}_0) \cdot \nabla w \, dx \geq \frac{1}{2} \|\nabla w\|_2^2. \tag{2.15}$$

On the other hand, setting for any constant $k > 0$

$$\Omega_k = \{x \in \Omega : |w(x)| > k\},$$

it holds

$$\begin{aligned}\int_\Omega (G - w\mathbf{b}_0)\nabla w\,dx &\le \|G\|_2\|\nabla w\|_2 + \int_\Omega |w|\,|\mathbf{b}_0|\,|\nabla w|\,dx\\ &= \|G\|_2\|\nabla w\|_2 + \int_{\Omega\setminus\Omega_k} |w|\,|\mathbf{b}_0|\,|\nabla w|\,dx + \int_{\Omega_k} |w|\,|\mathbf{b}_0|\,|\nabla w|\,dx\\ &\le \|G\|_2\|\nabla w\|_2 + k\|\mathbf{b}_0\|_2\|\nabla w\|_2 + \|\mathbf{b}_0\|_{L^{N,\infty}(\Omega_k)}\|w\|_{2^*,2}\|\nabla w\|_2\\ &\le \|G\|_2\|\nabla w\|_2 + k\|\mathbf{b}_0\|_2\|\nabla w\|_2 + \frac{2}{N-2}\|\mathbf{b}_0\|_{L^{N,\infty}(\Omega_k)}\|\nabla w\|_2^2\\ &\le \|G\|_2\|\nabla w\|_2 + k\|\mathbf{b}_0\|_2\|\nabla w\|_2 + c(N)\|\mathbf{b}_0\|_\infty|\Omega_k|\,\|\nabla w\|_2^2\end{aligned}$$

where we used once again Hölder's inequality and Sobolev's embedding.

By last inequality and (2.15), it follows that

$$\frac{1}{2}\|\nabla w\|_2 \le \|G\|_2 + k\|\mathbf{b}_0\|_2 + c(N)\|\mathbf{b}_0\|_\infty|\Omega_k|\,\|\nabla w\|_2. \tag{2.16}$$

Since by Lemma 3.2 in [16], we have that

$$\left[\int_\Omega |\log(1+|w|)|^{2^*}\right]^{\frac{2}{2^*}} \le 2\left(\frac{2}{N-2}\right)^2(\|\mathbf{b}\|_2^2 + \|G\|_2^2),$$

we can deduce that

$$|\Omega_k|^{\frac{2^*}{2}} < \frac{2}{\log^2(1+k)}\left(\frac{2}{N-2}\right)^2(\|\mathbf{b}\|_2^2 + \|G\|_2^2).$$

It follows that there exists a constant $\bar{k}$ such that $\forall k > \bar{k}$ it is

$$|\Omega_k| < \frac{1}{4(c(N)\|\mathbf{b}_0\|_\infty)}$$

so that, by (2.16), we have $\forall k > \bar{k}$

$$\frac{1}{2}\|\nabla w\|_2 \le \|G\|_2 + k\|\mathbf{b}_0\|_2 + \frac{1}{4}\|\nabla w\|_2 \tag{2.17}$$

and therefore

$$\frac{1}{4}\|\nabla w\|_2 \le \|G\|_2 + k\|\mathbf{b}_0\|_2.$$

Now observe that the assumption $\mathrm{dist}_{L^{N,\infty}}(\mathbf{b}, L^\infty) < \frac{N-2}{4}$, thanks to the equality (2.4), gives the existence of a constant $h = h(\mathbf{b}, N)$ such that $\|\mathbf{b} - T_h\mathbf{b}\|_{N,\infty} < \frac{N-2}{4}$, so it is legitimate to choose $\mathbf{b}_0 = T_h\mathbf{b}$ in the previous arguments getting

$$\|\nabla w\|_2 \le c(\|G\|_2 + k\|\mathbf{b}\|_2) \tag{2.18}$$

where the constant c obviously depends only on N. Clearly the constant k depends on the $\mathrm{dist}_{L^{N,\infty}}(\mathbf{b}, L^\infty)$.

Assume now that u is a solution of the Dirichlet problem associated to the equation (2.12) and that $w \in W_0^{1,2}$ is a solution of (2.13) with $G = \nabla u$. It follows that

$$\int_\Omega \langle \mathscr{A}(x)\nabla u, \nabla\varphi\rangle + (\mathbf{b}(x)\cdot\nabla u)\varphi = -\int_\Omega F\nabla\varphi \quad \forall\varphi \in W_0^{1,2} \tag{2.19}$$

and

$$\int_\Omega \langle \mathscr{A}(x)\nabla w, \nabla\varphi\rangle + (w\mathbf{b}(x))\nabla\varphi = \int_\Omega \nabla u\nabla\varphi \quad \forall\varphi \in W_0^{1,2}. \tag{2.20}$$

Therefore, by taking $\varphi = w$ in the first equation and $\varphi = u$ in the second equation, we obtain that

$$\int_\Omega |\nabla u|^2\,dx = -\int_\Omega F\nabla w\,dx. \tag{2.21}$$

Hence, by Young's inequality

$$\|\nabla u\|_2^2 \le \varepsilon\|\nabla w\|_2^2 + c(\varepsilon)\|F\|_2^2$$

and by using the estimate in (2.18), we have for a constant $c = c(N)$ that

$$\|\nabla u\|_2^2 \le \varepsilon c(\|\nabla u\|_2^2 + k\|\mathbf{b}\|_2^2) + c(\varepsilon)\|F\|_2^2.$$

By a suitable choice of ε, it follows that

$$\|\nabla u\|_2 \le c'(\|F\|_2 + k\|\mathbf{b}\|_2). \tag{2.22}$$

Replacing F with $F - F_0$, where F_0 is chosen with divergence free and such that Lemma 2.6 holds, we can easily deduce from (2.22) and inequality (2.11) the following estimate

$$\|\nabla u\|_2 \le c(\|\mathrm{div}\,F\|_{\frac{2N}{N+2}} + \|\mathbf{b}\|_2),$$

that is

$$\|\nabla u\|_2 \le c(\|f\|_{\frac{2N}{N+2}} + \|\mathbf{b}\|_2), \tag{2.23}$$

with $c = c(N, \mathrm{dist}_{L^{N,\infty}}(\mathbf{b}, L^\infty))$. □

3. The main result

This section is devoted to the proof of Theorem 1.1. As already observed in the Introduction, the existence and the uniqueness will follow by approximation arguments once obtained an a priori estimate for regular solutions of the problem (1.1). To this aim, we shall prove first an interior estimate (see Theorem 3.1) and then we shall extend it to the entire domain. In order to obtain this extension, we shall need to assume that the boundary is sufficiently smooth. Following Miranda's arguments (see [15]) we shall require Ω of class C^3 (see Definition 2.1 above).

Let us start proving the following interior estimate

THEOREM 3.1. *Under the assumptions of Theorem* 1.1*, if* $u \in C^\infty(\Omega)$ *is a solution of the equation* $Lu = f$*, then for every subdomain* $\Omega' \subset\subset \Omega$*, there exists a positive constant* C *depending on* N *and the distances in* (1.3) *such that*

$$\|\nabla u\|_{L^{2^*}(\Omega')} + \|D^2 u\|_{L^2(\Omega')} \le C(\|f\|_{L^2(\Omega)} + \|\mathbf{b}\|_{L^2(\Omega)}). \tag{3.1}$$

PROOF. For a fixed $\varphi \in C_0^\infty(\Omega)$ such that $\varphi \equiv 1$ on $\Omega' \subset\subset \Omega$, consider the function $v = \varphi u$. We get

$$Lv = \varphi f - uL\varphi - \sum_{i,j} a_{ij}\Big(\frac{\partial\varphi}{\partial x_i}\frac{\partial u}{\partial x_j} + \frac{\partial u}{\partial x_i}\frac{\partial\varphi}{\partial x_j}\Big) := F \tag{3.2}$$

and therefore the following inequality proved in [14]

$$\sum_{i,k}\Big(\frac{\partial^2 v}{\partial x_i \partial x_k}\Big)^2 \le F^2 + \sum_{irks} A_{irks}\frac{\partial}{\partial x_k}\Big(\frac{\partial v}{\partial x_i}\frac{\partial^2 v}{\partial x_r \partial x_s}\Big), \tag{3.3}$$

where we have used the notation $A_{irks} = a_{ir}a_{ks} - a_{ik}a_{rs}$.

It follows that

$$\sum_{i,k}\int_\Omega \Big(\frac{\partial^2 v}{\partial x_i \partial x_k}\Big)^2 dx \le \int_\Omega F^2\,dx - \sum_{irks}\int_\Omega \Big(\frac{\partial v}{\partial x_i}\frac{\partial^2 v}{\partial x_r \partial x_s}\Big)\frac{\partial}{\partial x_k}A_{irks}\,dx \tag{3.4}$$
$$:= I_1 + I_2$$

We immediately deduce from (3.2) that, in order to estimate $|I_1|$, we can use the bound of $\|u\|_{W_0^{1,2}}$. Therefore, by mean of Theorem 2.5 we get

$$|I_1| \le c(\|f\|_2 + \|\mathbf{b}\|_2)^2$$

with $c = c(N, \mathrm{dist}_{L^{N,\infty}}(\mathbf{b}, L^\infty))$.

Let us carry on with an estimate for $|I_2|$.

By using Schwartz inequality, we immediately obtain that

$$|I_2| \le \Big(\sum_{r,s}\int_\Omega \Big(\frac{\partial^2 v}{\partial x_r \partial x_s}\Big)^2 dx\Big)^{\frac{1}{2}}\Big(\int_\Omega \Big(\sum_i \Big(\frac{\partial v}{\partial x_i}\Big)^2\Big)\Big(\sum_{irs}\Big(\sum_k \frac{\partial A_{irks}}{\partial x_k}\Big)^2\Big)\,dx\Big)^{\frac{1}{2}}$$

or equivalently that

$$|I_2| \le \|D^2 v\|_{L^2(\Omega)} \Big(\int_\Omega E(x) |\nabla v|^2 \, dx \Big)^{\frac{1}{2}}, \tag{3.5}$$

where we have used the notation

$$E = \sum_{irs} \Big[\sum_k \frac{\partial A_{irks}}{\partial x_k} \Big]^2 .$$

Now, by virtue of the second assumption in (1.3), it is legitimate to choose $E_0 \in L^\infty(\Omega)$ such that $\|E - E_0\|_{L^{N/2,\infty}(\Omega)} < \big(\frac{N-2}{4}\big)^2$. We have that

$$\begin{aligned}
|I_2| &\le \|D^2 v\|_{L^2(\Omega)} \Big(\int_\Omega |E - E_0| \, |\nabla v|^2 \, dx + \int_\Omega |E_0| \, |\nabla v|^2 \, dx \Big)^{\frac{1}{2}} \\
&\le \|D^2 v\|_{L^2(\Omega)} \Big[\Big(\int_\Omega |E - E_0| \, |\nabla v|^2 \, dx \Big)^{\frac{1}{2}} + \Big(\int_\Omega |E_0| \, |\nabla v|^2 \, dx \Big)^{\frac{1}{2}} \Big] \\
&\le \|D^2 v\|_{L^2(\Omega)} \Big[\Big(\int_\Omega |E - E_0| \, |\nabla v|^2 \, dx \Big)^{\frac{1}{2}} + \|E_0\|^{\frac{1}{2}}_{L^\infty(\Omega)} \|\nabla v\|_{L^2(\Omega)} \Big] \\
&\le \|D^2 v\|_{L^2(\Omega)} [(\|E - E_0\|_{L^{N/2,\infty}(\Omega)} \|\nabla v\|^2_{L^{2^*,2}})^{\frac{1}{2}} + \|E_0\|^{\frac{1}{2}}_{L^\infty(\Omega)} \|\nabla v\|_{L^2(\Omega)}] \\
&\le \|D^2 v\|_{L^2(\Omega)} \Big[\Big(\Big(\frac{2}{N-2} \Big)^2 \|E - E_0\|_{L^{N/2,\infty}(\Omega)} \|D^2 v\|^2_{L^2(\Omega)} \Big)^{\frac{1}{2}} \\
&\qquad + \|E_0\|^{\frac{1}{2}}_{L^\infty(\Omega)} \|\nabla v\|_{L^2(\Omega)} \Big]
\end{aligned}$$

where, in the last two lines we used Hölder's inequality and the Embedding Theorem 2.2.

Combining the estimate of $|I_1|$ and $|I_2|$ with (3.4), we have

$$\begin{aligned}
\|D^2 v\|^2_{L^2(\Omega)} \le \|D^2 v\|_{L^2(\Omega)} \Big[\Big(\Big(\frac{2}{N-2} \Big)^2 \|E - E_0\|_{L^{N/2,\infty}(\Omega)} \|D^2 v\|^2_{L^2(\Omega)} \Big)^{\frac{1}{2}} \\
+ \|E_0\|^{\frac{1}{2}}_{L^\infty(\Omega)} \|\nabla v\|_{L^2(\Omega)} \Big] + c(\|f\|_{L^2(\Omega)} + \|\mathbf{b}\|_{L^2(\Omega)})^2
\end{aligned}$$

and hence, using that $\|E - E_0\|_{L^{N/2,\infty}(\Omega)} \le \big(\frac{N-2}{4}\big)^2$,

$$\frac{1}{2} \|D^2 v\|^2_{L^2(\Omega)} \le \|D^2 v\|_2 \|E_0\|^{\frac{1}{2}}_\infty \|\nabla v\|_2 + c(\|f\|_2 + \|\mathbf{b}\|_2)^2.$$

At this point, the use of Young's inequality gives

$$\frac{1}{2} \|D^2 v\|^2_{L^2(\Omega)} \le \varepsilon \|D^2 v\|^2_2 + c(\varepsilon) \|E_0\|_\infty \|\nabla v\|^2_2 + c(\|f\|_2 + \|\mathbf{b}\|_2)^2.$$

A suitable choice of ε yields to reabsorb the first term in the right hand side of the previous estimate to the left hand side and recalling Theorem 2.5, we get

$$\|D^2 v\|^2_{L^2(\Omega)} \le C(\|f\|_{L^2(\Omega)} + \|\mathbf{b}\|_{L^2(\Omega)})^2$$

with a constant $C = C(N, \mathrm{dist}_{L^{N,\infty}}(\mathbf{b}, L^\infty), \mathrm{dist}_{L^{\frac{N}{2},\infty}}(E, L^\infty))$.

Since $\varphi \equiv 1$ on Ω', we conclude that

$$\|D^2 u\|_{L^2(\Omega')} \le C(\|f\|_{L^2(\Omega)} + \|\mathbf{b}\|_{L^2(\Omega)}) \tag{3.6}$$

where $C = C(N, \mathrm{dist}_{L^{N,\infty}}(\mathbf{b}, L^\infty), \mathrm{dist}_{L^{\frac{N}{2},\infty}}(E, L^\infty))$.

Observe now that, combining the second assertion of Theorem 2.4 with the well known interpolation inequality by Gagliardo–Nirenberg, we deduce that $\nabla u \in L^{2^*}$ and that the following inequality holds

$$\|\nabla u\|_{L^{2^*}(\Omega')} \le c\|D^2 u\|^{1/2}_{L^2(\Omega')}\|u\|^{1/2}_{L^{(2^*)^*}(\Omega')}.$$

The use of Young's inequality yields

$$\|\nabla u\|_{L^{2^*}(\Omega')} \le \varepsilon\|u\|_{L^{(2^*)^*}(\Omega')} + c(\varepsilon)\|D^2 u\|_{L^2(\Omega')}$$

and therefore the estimate (3.1) can be easily derived by a suitable choice of ε and by inequality (3.6). □

The following theorem concerns the regularity up to the boundary of the domain Ω.

THEOREM 3.2. *Under the same assumptions of Theorem* 1.1, *assume that* $u \in C^\infty(\Omega) \cap C^0(\overline{\Omega})$ *is a solution of the Dirichlet problem* (1.1). *Then there exists a positive constant* C, *depending on* N, M *and the distances in* (1.3), *such that*

$$\|\nabla u\|_{L^{2^*}(\Omega)} + \|D^2 u\|_{L^2(\Omega)} \le C(\|f\|_{L^2(\Omega)} + \|\mathbf{b}\|_{L^2(\Omega)})$$

PROOF. Let us fix $0 < \tau < \frac{1}{2}$. According to Theorem 2.3, let us cover $\partial\Omega$ with m balls B_j centered in $x_j \in \partial\Omega$ and having radius R_τ such that the estimate (2.5) holds.

Moreover, let us suppose that B_0 is an open subset such that $B_0 \subset\subset \Omega$ and $\Omega \subset \bigcup_{j=0}^m B_j$.

Let $\{\xi_j\}_{0\le j\le m}$ be a partition of unity subordinate to the covering $\{B_j\}$ of Ω and set $v_j = \xi_j u$.

Arguing as in Theorem 3.1, we have in $\Omega_j = B_j \cap \Omega$

$$Lv_j = \xi_j f - uL\xi_j - \sum_{rs} a_{rs}\Big(\frac{\partial \xi_j}{\partial x_r}\frac{\partial u}{\partial x_s} + \frac{\partial \xi_j}{\partial x_s}\frac{\partial u}{\partial x_r}\Big) := F_j$$

and, by Miranda's inequality, we get

$$\sum_{i,k}\int_{\Omega_j}\Big(\frac{\partial^2 v_j}{\partial x_i\partial x_k}\Big)^2 dx \le \int_{\Omega_j} F_j^2\,dx - \sum_{irks}\int_{\Omega_j}\Big(\frac{\partial v_j}{\partial x_i}\,\frac{\partial^2 v_j}{\partial x_r\partial x_s}\Big)\frac{\partial}{\partial x_k}A_{irks}\,dx + \sum_{irks}\int_{\partial\Omega\cap B_j}\Big(\frac{\partial v_j}{\partial x_i}\,\frac{\partial^2 v_j}{\partial x_r\partial x_s}\Big)A_{irks}\nu_k\,d\sigma = J_1+J_2+J_3 \tag{3.7}$$

where $(\nu_1,\dots\nu_n)$ denotes the unit outward normal to $\partial\Omega$.

Proceeding exactly as in Theorem 3.1 for $|I_1|$, we obtain for $|J_1|$ the following estimate

$$|J_1| \le c(\|f\|_{L^2(\Omega_j)} + \|\mathbf{b}\|_{L^2(\Omega_j)})^2$$

with $c = c(N, \mathrm{dist}_{L^{N,\infty}}(\mathbf{b}, L^\infty))$ while, replacing Theorem 2.2 with Theorem 2.3 in the arguments used to estimate $|I_2|$, for $E_0 \in L^\infty(\Omega)$ such that $\|E - E_0\|_{L^{N/2,\infty}(\Omega)} < \Big(\frac{N-2}{2^{2+\frac{1}{N}}}\Big)^2$, we get

$$|J_2| \le \|D^2 v_j\|_{L^2(\Omega_j)}\Bigg[\bigg(\Big(\frac{2^{\frac{N+1}{N}}}{N-2}(1+\tau)\Big)^2\|E-E_0\|_{L^{N/2,\infty}(\Omega_j)}\|D^2 v_j\|^2_{L^2(\Omega_j)}\bigg)^{\frac{1}{2}} + \|E_0\|_\infty^{\frac{1}{2}}\|\nabla v_j\|_{L^2(\Omega_j)}\Bigg] \tag{3.8}$$

Finally, it remains to estimate $|J_3|$. It can be checked that for $C_0 = C_0(M)$

$$|J_3| \le C_0\int_{\partial\Omega\cap B_j}\sum_i\Big(\frac{\partial v_j}{\partial x_i}\Big)^2 d\sigma$$

(see (2.12) in [15]) and therefore that

$$|J_3| \le c(\|\nabla v_j\|^2_{L^2(\Omega_j)} + \|D^2 v_j\|_{L^2(\Omega_j)}\|\nabla v_j\|_{L^2(\Omega_j)}) \tag{3.9}$$

with $c = c(N, M)$.

Hence, for any $\eta > 0$

$$|J_3| \le c'\|\nabla v_j\|^2_{L^2(\Omega_j)} + \eta\|D^2 v_j\|^2_{L^2(\Omega_j)},$$

that is

$$|J_3| \le c(\|f\|_{L^2(\Omega_j)} + \|\mathbf{b}\|_{L^2(\Omega_j)})^2 + \eta\|D^2 v_j\|^2_{L^2(\Omega_j)}$$

where $c = c(N, M, \eta, \mathrm{dist}_{L^{N,\infty}}(\mathbf{b}, L^\infty))$.

Combining the estimates obtained above for $|J_1|$, $|J_2|$, $|J_3|$ with (3.7), we get

$$\|D^2 v_j\|^2_{L^2(\Omega_j)} \le \frac{2^{\frac{N+1}{N}}}{N-2}(1+\tau)\|E-E_0\|^{\frac{1}{2}}_{L^{\frac{N}{2},\infty}(\Omega_j)}\|D^2 v_j\|^2_{L^2(\Omega_j)} + \|E_0\|_\infty^{\frac{1}{2}}\|\nabla v_j\|_{L^2(\Omega_j)}\|D^2 v_j\|_{L^2(\Omega_j)} + c(\|f\|_{L^2(\Omega_j)} + \|\mathbf{b}\|_{L^2(\Omega_j)})^2 + \eta\|D^2 v_j\|^2_{L^2(\Omega_j)}.$$

Reasoning analogously we have done in Theorem 3.1 and choosing η opportunely, we first obtain

$$\|D^2 v_j\|_{L^2(\Omega_j)} \le c(\|f\|_{L^2(\Omega_j)} + \|\mathbf{b}\|_{L^2(\Omega_j)})$$

and then

$$\|\nabla u\|_{L^{2^*}(\Omega_j)} + \|D^2 u\|_{L^2(\Omega_j)} \le C(\|f\|_{L^2(\Omega_j)} + \|\mathbf{b}\|_{L^2(\Omega_j)})$$

with $C = C(N, M, \mathrm{dist}_{L^{N,\infty}}(\mathbf{b}, L^\infty), \mathrm{dist}_{L^{\frac{N}{2},\infty}}(E, L^\infty))$.

The conclusion follows observing that the inequality obtained also holds for B_0 by Theorem 3.1 and by observing that $\Omega \subset \bigcup_{j=0}^m B_j$. □

Combining Theorem 3.1 and Theorem 3.2 and using density arguments, we deduce the desired global estimate for the solution $u \in W^{2,2}(\Omega) \cap W_0^{1,2^*}(\Omega)$ of (1.1).

The result of existence and uniqueness can be now easily obtained.

Proof of Theorem 1.1. Let us extend the matrix of the coefficients a_{ij} to $\mathbb{R}^N$ putting zero outside Ω and consider a sequence of mollifiers ρ_ε.

Denoting by $a_{ij}^\varepsilon = \rho_\varepsilon * a_{ij}$, $f^\varepsilon = \rho_\varepsilon * f$, we have that $a_{ij}^\varepsilon \in C^\infty(\overline{\Omega}) \cap L^\infty(\Omega)$, $a_{ij}^\varepsilon \to a_{ij}$ in L^2 and $f^\varepsilon \to f$ in L^2. Moreover one can easily verify that

$$|\xi|^2 \le \sum_{i,j=1}^N a_{ij}^\varepsilon(x)\xi_i\xi_j \le M|\xi|^2. \tag{3.10}$$

and that the derivatives Da_{ij}^ε satisfy the assumptions of Theorem 1.1.

Indeed, thank to (2.3), one can observe that for $\mathbf{b}_0^\varepsilon$ and $(\mathbf{b} - \mathbf{b}_0)^\varepsilon$, it holds that

$$\|(\mathbf{b} - \mathbf{b}_0)^\varepsilon\|_{N,\infty} = \|\mathbf{b}^\varepsilon - \mathbf{b}_0^\varepsilon\|_{N,\infty} \le \|\mathbf{b} - \mathbf{b}_0\|_{N,\infty} \quad \text{and} \quad \|\mathbf{b}_0^\varepsilon\|_\infty \le \|\mathbf{b}_0\|_\infty.$$

Obviously, analogous inequalities hold for $\|E_0^\varepsilon\|_\infty$ and $\|(E - E_0)^\varepsilon\|_{N/2,\infty}$.

Hence the unique solution u^ε of the Dirichlet problem associated to the equation

$$-\sum_{i,j=1}^N a_{ij}^\varepsilon(x)\frac{\partial^2 u^\varepsilon}{\partial x_i \partial x_j} = f^\varepsilon(x)$$

verifies the estimate of Theorem 1.1 with a constant C independent of ε.

It follows that, up to a subsequence, u^ε converges strongly in $W^{1,2}$ and weakly in $W^{2,2}$ to a function u. It remains to show that u is a solution of the problem (1.1). To this aim, let us observe that u^ε solves the variational equation

$$-\sum_j \frac{\partial}{\partial x_j}\Big(\sum_i a_{ij}^\varepsilon \frac{\partial u^\varepsilon}{\partial x_i}\Big) + \sum_i b_i^\varepsilon \frac{\partial u^\varepsilon}{\partial x_i} = f^\varepsilon \tag{3.11}$$

with

$$b_i^\varepsilon(x) = \sum_j \frac{\partial a_{ij}^\varepsilon}{\partial x_j}, \tag{3.12}$$

that is

$$\int_\Omega \langle a_{ij}^\varepsilon(x)\nabla u^\varepsilon, \nabla\varphi\rangle + (b_i^\varepsilon(x)\nabla u^\varepsilon)\varphi = \int_\Omega f^\varepsilon\varphi \quad \forall\varphi \in C_0^\infty(\Omega). \tag{3.13}$$

Since $a_{ij}^\varepsilon \to a_{ij}$ in $L^2(\Omega)$, $b_i^\varepsilon \to b_i$ in $L^2(\Omega)$ and $f^\varepsilon \to f$ in $L^2(\Omega)$, the function u is the unique solution of (1.1). □

4. The higher integrability

The aim of this section is to prove that it is possible to get a bound for the solution in $W^{2,2}(\Omega)$ of the problem (1.1) more significant than the one obtained in Theorem 1.1, as well as we assume an higher integrability of the right hand side. More precisely we shall prove the following

Theorem 4.1. *Let us assume that the functions a_{ij} satisfy* (1.2) *and that their first derivatives $\frac{\partial a_{ij}}{\partial x_s}$, $s = 1, \dots, N$, belong to the class $L^{N,\infty}$. Set*

$$\tilde{E} = \sum_{j,h}\Big(\sum_k \frac{\partial a_{jh}}{\partial x_k}\Big)^2, \quad E = \sum_{irs}\left[\sum_k \frac{\partial(a_{ir}a_{ks} - a_{ik}a_{rs})}{\partial x_k}\right]^2.$$

and assume $f \in L^p$, $2 < p < N$. Then there exists $\sigma_0 > 0$ such that, if

$$\operatorname{dist}_{L^{\frac{N}{2},\infty}}(\tilde{E}, L^\infty) < \sigma_0, \quad \operatorname{dist}_{L^{\frac{N}{2},\infty}}(E, L^\infty) < \sigma_0, \tag{4.1}$$

the unique solution $u \in W^{2,2}(\Omega)$ of the problem (1.1) *satisfies the following inequality*

$$\|\nabla u\|_{L^{\frac{Np}{N-p}}(\Omega)} + \|D^2u\|_{L^2(\Omega)} \le C(\|f\|_{L^p(\Omega)} + \|\mathbf{b}\|_{L^2(\Omega)}) \tag{4.2}$$

for a positive constant C depending on N, p, M and the distances in (4.1).

Remark 4.2. Observe that the smallness of the $\operatorname{dist}_{L^{\frac{N}{2},\infty}}(\tilde{E}, L^\infty)$ implies the smallness of the $\operatorname{dist}_{L^{N,\infty}}(\mathbf{b}, L^\infty)$ and hence the assumptions of Theorem 1.1 are satisfied for a suitable choice of σ_0.

Proof. We confine ourselves to the case $u \in C^\infty(\Omega) \cap C^0(\overline{\Omega})$. Once we will show that (4.2) holds, it will be necessary to use density arguments.

As we have done in Theorem 3.1, we start from the inequality (3.3) applied to the function $v = \varphi u$ for a fixed $\varphi \in C_0^\infty(\Omega)$ such that $\varphi \equiv 1$ on $\Omega' \subset\subset \Omega$. For a positive function $\vartheta \in C^1(\overline{\Omega})$ we get

$$
(4.3)\qquad \int_\Omega \vartheta \sum_{i,k}\Big(\frac{\partial^2 v}{\partial x_i \partial x_k}\Big)^2 dx \le \int_\Omega \vartheta F^2\,dx - \int_\Omega \sum_{irks}\Big(\frac{\partial v}{\partial x_i}\frac{\partial^2 v}{\partial x_r \partial x_s}\Big)\frac{\partial \vartheta}{\partial x_k} A_{irks}\,dx
$$

$$
- \int_\Omega \vartheta \sum_{irks}\Big(\frac{\partial v}{\partial x_i}\frac{\partial^2 v}{\partial x_r \partial x_s}\Big)\frac{\partial}{\partial x_k} A_{irks}\,dx
$$

$$
= I_1' + I_2' + I_3'.
$$

Let $\mu > 0$ a real number such that

$$
(4.4)\qquad \frac{2N(\mu+1)}{N-2} = \frac{2\mu p}{p-2} = \frac{Np}{N-p},
$$

set

$$
\vartheta = \vartheta_\mu = \Big(\sum_{i,k=1}^N a_{ik}\frac{\partial v}{\partial x_i}\frac{\partial v}{\partial x_k}\Big)^\mu, \quad P_\mu = \int_\Omega \Big(\sum_i \Big(\frac{\partial v}{\partial x_i}\Big)^2\Big)^\mu \sum_{i,k}\Big(\frac{\partial^2 v}{\partial x_i \partial x_k}\Big)^2 dx
$$

and let's give an estimate for each of the integrals above.

By a simple use of Hölder's inequality, we have for the first integral

$$
|I_1'| \le \|F\|_p^2 \|\vartheta\|_{\frac{p}{p-2}} \le c_1 \|F\|_p^2 \|\nabla v\|_{\frac{2\mu p}{p-2}}^{2\mu} = c_1 \|F\|_p^2 \|\nabla v\|_{\frac{Np}{N-p}}^{2\mu},
$$

where the constant $c_1 = c_1(N, M, p)$.

For I_2' we first observe that

$$
(4.5)\qquad I_2' = \mu \int_\Omega \sum_{ikrs}\Big(\frac{\partial v}{\partial x_i}\frac{\partial^2 v}{\partial x_r \partial x_s}\Big) A_{irks}\vartheta_{\mu-1}\frac{\partial}{\partial x_k}\Big(\sum_{jh} a_{jh}\frac{\partial v}{\partial x_j}\frac{\partial v}{\partial x_h}\Big)
$$

and then, by using the second assumption in (1.2) and the Cauchy–Schwartz inequality, we obtain

$$
|I_2'| \le \mu C(M)\Bigg[\int_\Omega \Big(\sum_i \Big(\frac{\partial v}{\partial x_i}\Big)^2\Big)^\mu \sum_{rs}\Big(\frac{\partial^2 v}{\partial x_r \partial x_s}\Big)^2\Bigg]^{\frac12}
$$

$$
\times \Bigg[\int_\Omega \Big(\sum_i \Big(\frac{\partial v}{\partial x_i}\Big)^2\Big)^{\mu+1} \sum_{jh}\Big(\sum_k \frac{\partial a_{jh}}{\partial x_k}\Big)^2\Bigg]^{\frac12} + \mu C(M)\Bigg[\int_\Omega F^2 \Big(\sum_i \Big(\frac{\partial v}{\partial x_i}\Big)^2\Big)^\mu\Bigg]^{\frac12}
$$

$$
\times \Bigg[\int_\Omega \Big(\sum_i \Big(\frac{\partial v}{\partial x_i}\Big)^2\Big)^\mu \sum_{rs}\Big(\frac{\partial^2 v}{\partial x_r \partial x_s}\Big)^2\Bigg]^{\frac12}
$$

$$
= \mu C(M) P_\mu^{\frac12}\Bigg[\Big(\int_\Omega \tilde{E}|\nabla v|^{2(\mu+1)}\Big)^{\frac12} + \Big(\int_\Omega F^2 \Big(\sum_i \Big(\frac{\partial v}{\partial x_i}\Big)^2\Big)^\mu\Big)^{\frac12}\Bigg].
$$

Let $\tilde{E}_0 \in L^\infty(\Omega)$ be such that $\|\tilde{E} - \tilde{E}_0\|_{L^{N/2,\infty}(\Omega)} < \sigma_0$, where σ_0 will be chosen later. By using Hölder's inequality and recalling (4.4), we get

$$|I_2'| \leq CP_\mu^{\frac{1}{2}}\Big[\Big(\int_\Omega |\tilde{E} - \tilde{E}_0\|\nabla v|^{2(\mu+1)}\,dx + \int_\Omega |\tilde{E}_0|\,|\nabla v|^{2(\mu+1)}\,dx\Big)^{\frac{1}{2}} + \|F\|_p\|\nabla v\|^{\mu}_{\frac{Np}{N-p}}\Big]$$
$$\leq CP_\mu^{\frac{1}{2}}[(\|\tilde{E} - \tilde{E}_0\|_{\frac{N}{2},\infty}\|\nabla v\|^{2(\mu+1)}_{\frac{Np}{N-p},2(\mu+1)} + \|\tilde{E}_0\|_\infty\|\nabla v\|^{2(\mu+1)}_{2(\mu+1)})^{\frac{1}{2}} + \|F\|_p\|\nabla v\|^{\mu}_{\frac{Np}{N-p}}]$$

where $C = C(N, M, p)$.

Using now the shortened notation $E = \sum_{irs}\Big(\sum_k \dfrac{\partial A_{irks}}{\partial x_k}\Big)^2$ and arguing similarly we have done for I_2', we have for I_3'

$$|I_3'| \leq P_\mu^{\frac{1}{2}}(\|E - E_0\|_{\frac{N}{2},\infty}\|\nabla v\|^{2(\mu+1)}_{\frac{Np}{N-p},2(\mu+1)} + \|E_0\|_\infty\|\nabla v\|^{2(\mu+1)}_{2(\mu+1)})^{\frac{1}{2}}.$$

where $E_0 \in L^\infty(\Omega)$ is such that $\|E - E_0\|_{L^{N/2,\infty}(\Omega)} < \sigma_0$.

Recalling the estimates of $|I_i'|$, $i = 1, \dots, 3$, we have that

$$P_\mu \leq c_1\|F\|_p^2\|\nabla v\|^{2\mu}_{\frac{Np}{N-p}} + CP_\mu^{\frac{1}{2}}[(\|\tilde{E} - \tilde{E}_0\|_{\frac{N}{2},\infty}\|\nabla v\|^{2(\mu+1)}_{\frac{Np}{N-p},2(\mu+1)} + \|\tilde{E}_0\|_\infty\|\nabla v\|^{2(\mu+1)}_{2(\mu+1)})^{\frac{1}{2}}$$
$$+ \|F\|_p\|\nabla v\|^{\mu}_{\frac{Np}{N-p}}] + P_\mu^{\frac{1}{2}}(\|E - E_0\|_{\frac{N}{2},\infty}\|\nabla v\|^{2(\mu+1)}_{\frac{Np}{N-p},2(\mu+1)} + \|E_0\|_\infty\|\nabla v\|^{2(\mu+1)}_{2(\mu+1)})^{\frac{1}{2}}$$

where c_1 and C depend on N, M and p.

Hence we deduce with the aid of Young's inequality that

$$P_\mu \leq c_1\|F\|_p^2\|\nabla v\|^{2\mu}_{\frac{Np}{N-p}} + \varepsilon P_\mu + c(\varepsilon)C(\|\tilde{E} - \tilde{E}_0\|_{\frac{N}{2},\infty}\|\nabla v\|^{2(\mu+1)}_{\frac{Np}{N-p},2(\mu+1)}$$
$$+ \|\tilde{E}_0\|_\infty\|\nabla v\|^{2(\mu+1)}_{2(\mu+1)} + \|F\|_p^2\|\nabla v\|^{2\mu}_{\frac{Np}{N-p}}) + \varepsilon P_\mu$$
$$+ c(\varepsilon)(\|E - E_0\|_{\frac{N}{2},\infty}\|\nabla v\|^{2(\mu+1)}_{\frac{Np}{N-p},2(\mu+1)} + \|E_0\|_\infty\|\nabla v\|^{2(\mu+1)}_{2(\mu+1)}). \tag{4.6}$$

Choosing $\varepsilon = \dfrac{1}{4}$, we obtain

$$\frac{1}{2}P_\mu \leq c_1\|F\|_p^2\|\nabla v\|^{2\mu}_{\frac{Np}{N-p}} + C(\|\tilde{E} - \tilde{E}_0\|_{\frac{N}{2},\infty}\|\nabla v\|^{2(\mu+1)}_{\frac{Np}{N-p},2(\mu+1)}$$
$$+ \|\tilde{E}_0\|_\infty\|\nabla v\|^{2(\mu+1)}_{2(\mu+1)} + \|F\|_p^2\|\nabla v\|^{2\mu}_{\frac{Np}{N-p}})$$
$$+ c(\|E - E_0\|_{\frac{N}{2},\infty}\|\nabla v\|^{2(\mu+1)}_{\frac{Np}{N-p},2(\mu+1)} + \|E_0\|_\infty\|\nabla v\|^{2(\mu+1)}_{2(\mu+1)}) \tag{4.7}$$

where $C = C(M, N, p)$. Now observing that

$$|\nabla(\nabla v|\nabla v|^\mu)| \leq (\mu + 1)|D^2 v| \cdot |\nabla v|^\mu,$$

the use of Sobolev's inequality yields

$$\|\nabla v|\nabla v|^{\mu}\|_{2^*,2} \le (\mu+1)\frac{2}{N-2}\|\,|D^2v|\,|\nabla v|^{\mu}\|_2$$

that is

$$\|\nabla v\|^{2(\mu+1)}_{2^*(\mu+1),2(\mu+1)} \le (\mu+1)^2\Big(\frac{2}{N-2}\Big)^2\|\,|D^2v|\,|\nabla v|^{\mu}\|_2^2$$

i.e.

$$\|\nabla v\|^{2(\mu+1)}_{\frac{Np}{N-p},2(\mu+1)} \le (\mu+1)^2\Big(\frac{2}{N-2}\Big)^2 P_{\mu}. \tag{4.8}$$

Combining (4.7) and (4.8), it follows that

$$\frac{1}{8}\Big(\frac{N-2}{\mu+1}\Big)^2\|\nabla v\|^{2(\mu+1)}_{\frac{Np}{N-p},2(\mu+1)} \le C[\|F\|_p^2\|\nabla v\|^{2\mu}_{\frac{Np}{N-p}} + (\|\tilde{E}_0\|_\infty + \|E_0\|_\infty)\|\nabla v\|^{2(\mu+1)}_{2(\mu+1)}]$$
$$+ C(\|\tilde{E}-\tilde{E}_0\|_{\frac{N}{2},\infty} + \|E-E_0\|_{\frac{N}{2},\infty})\|\nabla v\|^{2(\mu+1)}_{\frac{Np}{N-p},2(\mu+1)}.$$

If we choose

$$\sigma_0 < \min\Big\{\frac{1}{16C}\Big(\frac{N-2}{\mu+1}\Big)^2, \Big(\frac{N-2}{2^{2+\frac{1}{N}}}\Big)^2\Big\}$$

and recall that $\|\tilde{E}-\tilde{E}_0\|_{\frac{N}{2},\infty} < \sigma_0$ and that $\|E-E_0\|_{\frac{N}{2},\infty} < \sigma_0$ we get

$$\|\nabla v\|^{2(\mu+1)}_{\frac{Np}{N-p},2(\mu+1)} \le C[\|F\|_p^2\|\nabla v\|^{2\mu}_{\frac{Np}{N-p}} + (\|\tilde{E}_0\|_\infty + \|E_0\|_\infty)\|\nabla v\|^{2(\mu+1)}_{2(\mu+1)}] \tag{4.9}$$

where $C = C(M,N,p)$.

On the other hand, since (4.4) holds, we easily obtain that $2(\mu+1) < p^*$ and therefore that

$$\|\nabla v\|^{2(\mu+1)}_{\frac{Np}{N-p}} \le c\|\nabla v\|^{2(\mu+1)}_{\frac{Np}{N-p},2(\mu+1)}.$$

Hence, by estimate (4.9), we get

$$\|\nabla v\|^{2(\mu+1)}_{\frac{Np}{N-p}} \le C[\|F\|_p^2\|\nabla v\|^{2\mu}_{\frac{Np}{N-p}} + (\|\tilde{E}_0\|_\infty + \|E_0\|_\infty)\|\nabla v\|^{2(\mu+1)}_{2(\mu+1)}] \tag{4.10}$$

where $C = C(M,N,p)$.

At this point, the use of Young's inequality with exponents $\mu+1$ and $\frac{\mu+1}{\mu}$ and the fact that, for a convenient choice of $0 < a < 1$, the following inequality holds

$$\|\nabla v\|_{2(\mu+1)} \le \|\nabla v\|^{a}_{\frac{Np}{N-p}}\|\nabla v\|_2^{1-a},$$

we obtain

$$\|\nabla v\|_{\frac{Np}{N-p}}^{2(\mu+1)} \le \varepsilon \|\nabla v\|_{\frac{Np}{N-p}}^{2(\mu+1)} + c(\varepsilon)\|F\|_p^{2(\mu+1)} + C(\|\tilde{E}_0\|_\infty + \|E_0\|_\infty)(\|\nabla v\|_{\frac{Np}{N-p}}^{a}\|\nabla v\|_2^{1-a})^{2(\mu+1)}. \tag{4.11}$$

Using again Young's inequality with exponents $\frac{1}{a}$ and $\frac{1}{1-a}$, we get

$$\|\nabla v\|_{\frac{Np}{N-p}}^{2(\mu+1)} \le \varepsilon \|\nabla v\|_{\frac{Np}{N-p}}^{2(\mu+1)} + c(\varepsilon)\|F\|_p^{2(\mu+1)} + c(\varepsilon)[(\|\tilde{E}_0\|_\infty + \|E_0\|_\infty)^{\frac{1}{1-a}}\|\nabla v\|_2]^{2(\mu+1)} + \varepsilon \|\nabla v\|_{\frac{Np}{N-p}}^{2(\mu+1)}.$$

Choosing $\varepsilon < \dfrac{1}{2}$ and taking into account that Theorem 2.5 holds, we conclude that

$$\|\nabla u\|_{L^{\frac{Np}{N-p}}(\Omega')} \le C(\|f\|_{L^p(\Omega)} + \|\mathbf{b}\|_{L^2(\Omega)}) \tag{4.12}$$

with $C = C(N, p, M, \mathrm{dist}_{L^{\frac{N}{2},\infty}}(E, L^\infty), \mathrm{dist}_{L^{\frac{N}{2},\infty}}(\tilde{E}, L^\infty))$.

For an analogous estimate up to the boundary, we obviously have to argue similarly as we have done in Theorem 3.2 and start from the estimate

$$\begin{aligned} \sum_{i,k}\int_{\Omega_j} \vartheta\Big(\frac{\partial^2 v_j}{\partial x_i \partial x_k}\Big)^2 dx \le{}& \int_{\Omega_j} \vartheta F_j^2\, dx - \sum_{irks}\int_{\Omega_j} \vartheta\Big(\frac{\partial v_j}{\partial x_i}\frac{\partial^2 v_j}{\partial x_r \partial x_s}\Big)\frac{\partial}{\partial x_k}A_{irks}\, dx \\ &+ \sum_{irks}\int_{\Omega_j}\Big(\frac{\partial v_j}{\partial x_i}\frac{\partial^2 v_j}{\partial x_r \partial x_s}\Big)\frac{\partial \vartheta}{\partial x_k}A_{irks}\, dx \\ &+ \sum_{irks}\int_{\partial\Omega\cap B_j}\vartheta\Big(\frac{\partial v_j}{\partial x_i}\frac{\partial^2 v_j}{\partial x_r \partial x_s}\Big)A_{irks}\nu_k\, d\sigma \\ ={}& J_1' + J_2' + J_3' + J_4' \end{aligned} \tag{4.13}$$

Since the estimates of $|J_i'|$, $i = 1,2,3$ are similar to the estimates of $|I_i'|$, $i = 1,2,3$ obtained above, we shall give only the estimate of $|J_4'|$.

To this aim, we recall estimate (2.11) in [15] (see also [13]) and have

$$|J_4'| \le c(\|\nabla v_j\|_{L^{2(\mu+1)}(\Omega_j)}^{2(\mu+1)} + P_\mu^{\frac{1}{2}}\|\nabla v_j\|_{L^{2(\mu+1)}(\Omega_j)}^{\mu+1})$$

with $c = c(N, M)$. Therefore similar calculations we have done for (4.6) give

$$\begin{aligned} P_\mu \le{}& c_1\|F_j\|_p^2\|\nabla v_j\|_{\frac{Np}{N-p}}^{2\mu} + \varepsilon P_\mu + c(\varepsilon)C(\|\tilde{E} - \tilde{E}_0\|_{\frac{N}{2},\infty}\|\nabla v_j\|_{\frac{Np}{N-p},2(\mu+1)}^{2(\mu+1)} \\ &+ \|\tilde{E}_0\|_\infty\|\nabla v_j\|_{2(\mu+1)}^{2(\mu+1)} + \|F_j\|_p^2\|\nabla v_j\|_{\frac{Np}{N-p}}^{2\mu}) \\ &+ \varepsilon P_\mu + c(\varepsilon)(\|E - E_0\|_{\frac{N}{2},\infty}\|\nabla v_j\|_{\frac{Np}{N-p},2(\mu+1)}^{2(\mu+1)} \\ &+ \|E_0\|_\infty\|\nabla v_j\|_{2(\mu+1)}^{2(\mu+1)}) + c(\|\nabla v_j\|_{2(\mu+1)}^{2(\mu+1)} + P_\mu^{\frac{1}{2}}\|\nabla v_j\|_{2(\mu+1)}^{\mu+1}) \end{aligned}$$

and applying Young's inequality to the last term

$$
\begin{aligned}
P_\mu \le{}& c_1\|F_j\|_p^2\|\nabla v_j\|_{\frac{Np}{N-p}}^{2\mu} + \varepsilon P_\mu + c(\varepsilon)C(\|\tilde{E}-\tilde{E}_0\|_{\frac{N}{2},\infty}\|\nabla v_j\|_{\frac{Np}{N-p},2(\mu+1)}^{2(\mu+1)}\\
&+\|\tilde{E}_0\|_\infty\|\nabla v_j\|_{2(\mu+1)}^{2(\mu+1)} + \|F_j\|_p^2\|\nabla v_j\|_{\frac{Np}{N-p}}^{2\mu})\\
&+\varepsilon P_\mu + c(\varepsilon)(\|E-E_0\|_{\frac{N}{2},\infty}\|\nabla v_j\|_{\frac{Np}{N-p},2(\mu+1)}^{2(\mu+1)}\\
&+\|E_0\|_\infty\|\nabla v_j\|_{2(\mu+1)}^{2(\mu+1)}) + \varepsilon P_\mu + c(\varepsilon)\|\nabla v_j\|_{2(\mu+1)}^{2(\mu+1)}
\end{aligned}
$$

Choosing $\varepsilon = \frac{1}{6}$ we get

$$
\begin{aligned}
\frac{1}{2}P_\mu \le{}& c_1\|F_j\|_p^2\|\nabla v_j\|_{\frac{Np}{N-p}}^{2\mu} + C(\|\tilde{E}-\tilde{E}_0\|_{\frac{N}{2},\infty}\|\nabla v_j\|_{\frac{Np}{N-p},2(\mu+1)}^{2(\mu+1)}\\
&+\|\tilde{E}_0\|_\infty\|\nabla v_j\|_{2(\mu+1)}^{2(\mu+1)} + \|F_j\|_p^2\|\nabla v_j\|_{\frac{Np}{N-p}}^{2\mu})\\
&+c(\|E-E_0\|_{\frac{N}{2},\infty}\|\nabla v_j\|_{\frac{Np}{N-p},2(\mu+1)}^{2(\mu+1)}\\
&+\|E_0\|_\infty\|\nabla v_j\|_{2(\mu+1)}^{2(\mu+1)}) + c\|\nabla v_j\|_{2(\mu+1)}^{2(\mu+1)}
\end{aligned}
$$

that is similar to (4.7). Hence, if we carry on arguing as for (4.12), we obtain

$$
\|\nabla v_j\|_{L^{\frac{Np}{N-p}}(\Omega_j)} \le C(\|f\|_{L^p(\Omega_j)} + \|\mathbf{b}\|_{L^2(\Omega_j)}) \tag{4.14}
$$

and therefore, taking into account that $\Omega \subset \bigcup_{j=0}^m B_j$, the estimate up to the boundary follows. The global estimate (4.2) can be therefore obtained recalling also the inequality (1.4). □

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Flavia Giannetti
Dipartimento di Matematica e Applicazioni "R. Caccioppoli"
Università di Napoli "Federico II"
Via Cintia, Monte S. Angelo
80126 Napoli, Italy
giannett@unina.it

Gioconda Moscariello
Dipartimento di Matematica e Applicazioni "R. Caccioppoli"
Università di Napoli "Federico II"
Via Cintia, Monte S. Angelo
80126 Napoli, Italy
gmoscari@unina.it