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Partial Differential Equations — *Poisson equation beyond its natural domain of definition*, by Luigi D'Onofrio, Luigi Greco, Tadeusz Iwaniec and Roberta Schiattarella, communicated on December 15, 2017.

Abstract. — This note is concerned with the Poisson equation in the unit disk of the complex plane. Our setting is in the Sobolev space $\mathscr{W}^{1,p}(\mathbb{D})$ with exponent $1 < p < \infty$. Such a setting with $p \neq 2$ is referred to as beyond the natural domain of definition. The novelty lies in the use of a singular integral called Beurling Transform.

Key words: Laplace and Poisson equation, PDEs in the complex plane, Beurling transform, Riemann–Hilbert problem

Mathematics Subject Classification (primary; secondary): 35J05, 35J25; 35J46, 35J56

1. Introduction

Throughout this note $\mathbb{D}$ is the unit disk in the complex plane $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, and $\partial\mathbb{D} = \mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$. To every exponent $1 < p < \infty$ there corresponds its Hölder conjugate exponent defined by the rule $pq = p + q$.

We will make use of the Cauchy-Riemann derivatives:

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right), \quad \text{where } z = x + iy.$$

In general, to every PDE (linear or nonlinear) there corresponds the so-called natural domain of its definition. For the Poisson equation

$$\Delta u = \operatorname{div} f \quad \text{in } \mathbb{D},$$

the natural domain consists of a given vector field $f \in \mathscr{L}^2(\mathbb{D}, \mathbb{R}^2)$ and unknown function u in the Sobolev space $\mathscr{W}_0^{1,2}(\mathbb{D})$. In such setting the unique solution is found by minimizing the energy functional

$$\mathcal{E}[u] = \int_{\mathbb{D}} (|\nabla u|^2 - 2\langle \nabla u \,|\, f \rangle), \quad \langle \cdot \,|\, \cdot \rangle - \text{the usual scalar product in } \mathbb{R}^2.$$

However, this variational approach is unavailable when $f \in \mathscr{L}^p(\mathbb{D}, \mathbb{R}^2)$, $p \neq 2$. The singular integrals come into play.

An appropriate formulation of the Poisson equation takes the form:

$$\Delta u = \varphi \in \mathscr{W}^{-1,p}(\mathbb{D}) \quad \text{for } u \in \mathscr{W}^{1,p}(\mathbb{D}),\ 1 < p < \infty. \tag{1.1}$$

Here $\mathscr{W}^{-1,p}(\mathbb{D}) = [\mathscr{W}_0^{1,q}(\mathbb{D})]^*$ stands for the dual space of $\mathscr{W}_0^{1,q}(\mathbb{D})$, $pq = p + q$. The existence, uniqueness and estimates are well known in even more general context. We refer the interested reader to [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [15], [16], [17], [18], [19], [20], for fuller discussion.

We shall make use of the Riemann–Hilbert first order system of PDE's in the complex plane. The $\mathscr{L}^p$-estimates are reduced to a singular integral operator widely known as Beurling Transform. This method gains additional interest because the Beurling transform connects two homotopy classes of the first order elliptic PDEs; precisely, it takes $\frac{\partial h}{\partial \bar{z}}$ into $\frac{\partial h}{\partial z}$ for all $h \in \mathscr{W}_0^{1,p}(\mathbb{D})$, $1 < p < \infty$.

2. Setting Poisson equation for $f \in \mathscr{W}_0^{-1,p}(\mathbb{D})$

The Laplace operator takes the Sobolev space $\mathscr{W}_0^{1,p}(\mathbb{D})$ into $\mathscr{W}^{-1,p}(\mathbb{D})$, $1 < p < \infty$ in symbols;

$$\Delta u = \phi \in \mathscr{W}^{-1,p}(\mathbb{D}), \quad \text{for } u \in \mathscr{W}_0^{1,p}(\mathbb{D}). \tag{2.1}$$

In fact the space $\mathscr{W}^{-1,p}(\mathbb{D})$ consists of Schwartz distributions of the form

$$\phi = \operatorname{div} f, \quad \text{where } f \in \mathscr{L}^p(\mathbb{D}, \mathbb{R}^2).$$

The action of a linear functional

$$\phi : \mathscr{W}_0^{1,q}(\mathbb{D}) \to \mathbb{R} \quad \text{on the test function } v \in \mathscr{W}_0^{1,q}(\mathbb{D})$$

is given by the rule

$$\phi[v] = -\int_{\mathbb{D}} \langle f \mid \nabla v \rangle.$$

This integral does not depend on the choice of the vector field f once it represents $\phi = \operatorname{div} f$. The norm of such ϕ is defined as follows:

$$\|\phi\|_{\mathscr{W}^{-1,p}(\mathbb{D})} = \inf\{\|f\|_{\mathscr{L}^p(\mathbb{D})};\ \operatorname{div} f = \phi\}$$

It is clear that

$$\Delta : \mathscr{W}_0^{1,p}(\mathbb{D}) \to \mathscr{W}^{-1,p}(\mathbb{D})$$

THEOREM 2.1. *For every* $\phi \in \mathscr{W}^{-1,p}(\mathbb{D})$, $1 < p < \infty$, *there exists a unique* $u \in \mathscr{W}_0^{1,p}(\mathbb{D})$ *such that*

$$\Delta u = \phi \quad \textit{in } \mathbb{D}. \tag{2.2}$$

Moreover, we have a uniform bound

$$\|\nabla u\|_{\mathscr{L}^p(\mathbb{D})} \leq C_p \|\phi\|_{\mathscr{W}^{-1,p}(\mathbb{D})}. \tag{2.3}$$

This gives rise to the Poisson operator

$$\mathcal{P} : \mathscr{W}^{-1,p}(\mathbb{D}) \to \mathscr{L}^p(\mathbb{D}, \mathbb{R}^2),$$

which assigns to every $\phi \in \mathscr{W}^{-1,p}(\mathbb{D})$ the gradient field $\nabla u \in \mathscr{L}^p(\mathbb{D}, \mathbb{R}^2)$. The Poisson operator is continuous.

The proof goes in three steps:

- *Step 1* Uniqueness of the solution to (2.2)
- *Step 2* Existence of the solution to (2.2)
- *Step 3* Estimate (2.3)

3. UNIQUENESS

Surprisingly, *step 1* is not obvious when $1 < p < 2$. The problem reduces to showing that the Laplace equation

$$\Delta u = 0, \quad \text{for } u \in \mathscr{W}_0^{1,p}(\mathbb{D}), \tag{3.1}$$

admits only trivial solution $u \equiv 0$. The weak form of the Laplace equation reads as follows

$$\int_{\mathbb{D}} \langle \nabla u \,|\, \nabla v \rangle = 0 \quad \text{for all } v \in \mathscr{C}_0^\infty(\mathbb{D}).$$

Clearly, this equation also holds for all $v \in \mathscr{W}_0^{1,q}(\mathbb{D})$, by an approximation argument. To prove the uniqueness we need two Lemmas; the second one concludes the proof of the uniqueness.

LEMMA 3.1. *For every* $v \in \mathscr{W}_0^{1,p}(\mathbb{D})$ *and* $\psi \in \mathscr{C}^\infty(\bar{\mathbb{D}})$ *it holds:*

$$\int_{\mathbb{D}} (v_z \psi_{\bar{z}} - v_{\bar{z}} \psi_z)\, d\sigma(z) = 0, \quad d\sigma(z) = dx\, dy \tag{3.2}$$

PROOF. In view of density of $\mathscr{C}_0^\infty(\mathbb{D})$ in $\mathscr{W}_0^{1,p}(\mathbb{D})$, it suffices to prove (3.2) for $v \in \mathscr{C}_0^\infty(\mathbb{D})$. In this case the equation is immediate from Stokes' Theorem:

$$\int_{\mathbb{D}} (v_z \psi_{\bar z} - v_{\bar z} \psi_z)\, d\sigma(z) = \frac{i}{2} \int_{\mathbb{D}} d[v\psi_{\bar z}\, d\bar z + v\psi_z\, dz]$$
$$= \frac{i}{2} \int_{\partial\mathbb{D}} [v\psi_{\bar z}\, d\bar z + v\psi_z\, dz] = 0. \qquad \square$$

LEMMA 3.2. *The Laplace Equation*

$$\Delta u = 0$$

for $u \in \mathscr{W}_0^{1,p}(\mathbb{D})$, $1 \le p < \infty$, *has only trivial solution,* $u \equiv 0$.

PROOF. Let us begin with the case $2 \le p < \infty$ in which $\mathscr{W}_0^{1,p}(\mathbb{D}) \subset \mathscr{W}_0^{1,2}(\mathbb{D})$. It is a simple matter of integration by parts to see that the Dirichlet energy of u vanishes. Indeed, the weak form of the Laplace equation reads as,

$$\int_{\mathbb{D}} \langle \nabla u \,|\, \nabla \chi \rangle \, \mathrm{d}x\, \mathrm{d}y = 0 \quad \text{for all } \chi \in \mathscr{W}_0^{1,2}(\mathbb{D}).$$

Letting $\chi = u$ we conclude with $\nabla u \equiv 0$. Thus u is constant and, in view of the zero boundary condition, $u \equiv 0$.

The case $1 \le p < 2$ requires more work. The idea is to show that u actually belongs to $\mathscr{W}_0^{1,2}(\mathbb{D})$. In fact we are going to show that u extends as $\mathscr{C}^\infty$-smooth function on the entire complex plane. For this purpose, consider a measurable function defined in $\mathbb{C}$ by the rule:

$$g(z) = \begin{cases} u_z(z) & \textit{for } |z| < 1 \\ z^{-2} u_{\bar z}(1/\bar z) & \textit{for } |z| > 1. \end{cases} \tag{3.3}$$

It is immaterial what values of $g(z)$ are prescribed on the circle $\mathbb{S}^1$. Notice that g is analytic in $\mathbb{D}$, because

$$\frac{\partial}{\partial \bar z} g(z) = u_{z\bar z} = \frac{1}{4} \Delta u = 0. \tag{3.4}$$

For all $z \in \mathbb{C} \backslash \mathbb{S}^1$ we have the following relation

$$g(z) = z^{-2} \overline{g(1/\bar z)}, \quad \text{equivalently } g(1/\bar z) = \bar z^2 \overline{g(z)}. \tag{3.5}$$

This shows that g is also analytic in $\mathbb{C} \backslash \overline{\mathbb{D}}$. With the aid of Weyl's Lemma we shall show that g is analytic in $\mathbb{C}$. Weyl's Lemma asserts that a function $g \in \mathscr{L}^1_{\mathrm{loc}}(\mathbb{C})$ is analytic if and only if

$$\int_{\mathbb{C}} g(z) \eta_{\bar z}(z)\, \mathrm{d}\sigma(z) = 0, \quad \text{for every test function } \eta \in \mathscr{C}_0^\infty(\mathbb{C}). \tag{3.6}$$

Since g is analytic in both $\mathbb{D}$ and $\mathbb{C} \backslash \overline{\mathbb{D}}$ we may, and do, assume that η is supported in a neighborhood of $\mathbb{S}^1 = \partial \mathbb{D}$. Precisely, $\eta \in \mathscr{C}_0^\infty(\triangle)$ where $\triangle = \{z \in \mathbb{C} :$

$0 < r < |z| < \frac{1}{r} < \infty\}$. We decompose η as $\eta(z) = \varphi(z) + \psi(z)$, where

$$\varphi(z) = \frac{1}{2}[\eta(z) - \eta(1/\bar{z})] \quad \text{and} \quad \psi(z) = \frac{1}{2}[\eta(z) + \eta(1/\bar{z})]$$

Here both φ and ψ belong to $\mathscr{C}_0^\infty(\triangle)$. In addition, we note that $\varphi = 0$ on $\mathbb{S}^1$. In particular, $\varphi \in \mathscr{W}_0^{1,q}(\mathbb{D}) \cap \mathscr{W}_0^{1,q}(\mathbb{C}\backslash\overline{\mathbb{D}})$, with $2 < q \le \infty$. This yields

$$\int_{\mathbb{C}} g\varphi_{\bar{z}} = \int_{\mathbb{D}} g\varphi_{\bar{z}} + \int_{\mathbb{C}\backslash\bar{\mathbb{D}}} g\varphi_{\bar{z}} = 0 + 0 = 0.$$

The computation of the term $\int_{\mathbb{C}} g\psi_{\bar{z}}$ is more involved. As before, we split this integral into two integrals; one over $\mathbb{D}$ and the other over $\mathbb{C}\backslash\overline{\mathbb{D}}$. We transform the integral over $\mathbb{C}\backslash\overline{\mathbb{D}}$ into an integral over $\mathbb{D}$, simply by changing the variable z into $1/\bar{z}$. The Jacobian of this transformation equals $|z|^{-4}$, so we have

$$\begin{aligned} \int_{\mathbb{C}} g(z)\psi_{\bar{z}}(z)\,d\sigma(z) &= \int_{\mathbb{D}} g(z)\psi_{\bar{z}}(z)\,d\sigma(z) + \int_{\mathbb{C}\backslash\bar{\mathbb{D}}} g(z)\psi_{\bar{z}}(z)\,d\sigma(z) \\ &= \int_{\mathbb{D}} g(z)\psi_{\bar{z}}(z)\,d\sigma(z) + \int_{\mathbb{D}} g(1/\bar{z})\psi_{\bar{z}}(1/\bar{z})\frac{d\sigma(z)}{|z|^4}. \end{aligned} \tag{3.7}$$

Analogously to (3.5), we have the following identity

$$\psi_{\bar{z}}(1/\bar{z}) = -z^2\psi_z(z), \quad \text{for every } z \in \mathbb{C}$$

Hence the latter integral takes the form

$$-\int_{\mathbb{D}} \bar{z}^2\overline{g(z)}z^2\psi_z(z)\frac{d\sigma(z)}{|z|^4}.$$

Therefore, by the definition of g,

$$\begin{aligned} \int_{\mathbb{C}} g(z)\psi_{\bar{z}}(z)\,d\sigma(z) &= \int_{\mathbb{D}} [g(z)\psi_{\bar{z}}(z) - \bar{g}(z)\psi_z(z)]\,d\sigma(z) \\ &= \int_{\mathbb{D}} [u_z(z)\psi_{\bar{z}}(z) - \overline{u_z}(z)\psi_z(z)]\,d\sigma(z) \\ &= \int_{\mathbb{D}} [u_z(z)\psi_{\bar{z}}(z) - u_{\bar{z}}(z)\psi_z(z)]\,d\sigma(z) \end{aligned} \tag{3.8}$$

because $\overline{u_z} = u_{\bar{z}}$ for real functions. Applying Lemma 3.1 we get

$$\int_{\mathbb{C}} g\eta_{\bar{z}} = 0 \quad \text{for every } \eta \in \mathscr{C}_0^\infty(\mathbb{C}).$$

In conclusion, by Weyl's Lemma, g is an entire function. It follows from the definition of g that $\lim_{z\to\infty} g(z) = 0$. Liouville's Theorem tells us that $g \equiv 0$, implying that $u_z \equiv 0$ in $\mathbb{D}$. Since $u \in \mathscr{W}_0^{1,p}(\mathbb{D})$, it follows that $u \equiv 0$. □

4. Existence of the solution

Given $\phi \in \mathscr{W}^{-1,p}(\mathbb{D})$ we shall construct $u \in \mathscr{W}_0^{1,p}(\mathbb{D})$ which satisfies the equation

$$\begin{cases} \Delta u = \phi \\ u \in \mathscr{W}_0^{1,p}(\mathbb{D}). \end{cases}$$

Choose and fix any representation of ϕ in the form

$$\phi = \operatorname{div} f, \quad \text{where } f = (A, B) \in \mathscr{L}^p(\mathbb{D}, \mathbb{R}^2).$$

It is convenient to look at f as complex valued function $f = A + iB \in \mathscr{L}^p(\mathbb{D}, \mathbb{C})$, note that

$$f_z = \frac{1}{2}(f_x - if_y) = \frac{1}{2}(A_x + iB_x - iA_y + B_y).$$

Thus

$$\operatorname{div} f = 2 \operatorname{Re} f_z.$$

Next we solve the Riemann–Hilbert boundary value problem:

$$\begin{cases} h_{\bar z} = f & \text{for } h \in \mathscr{W}^{1,p}(\mathbb{D}, \mathbb{C}) \\ \operatorname{Re} h \in \mathscr{W}_0^{1,p}(\mathbb{D}) & 1 < p < \infty. \end{cases}$$

This problem admits a solution of the form

$$h(z) = \frac{1}{\pi} \int_{\mathbb{D}} \left(\frac{f(\tau)}{z - \tau} - \frac{z\overline{f(\tau)}}{1 - z\bar\tau} \right) d\sigma(\tau)$$

see Section 4.8.1 in [3], formula (4.134). We have

$$h_{\bar z} = f \quad \text{and} \quad h_z = S_{\mathbb{D}} f,$$

where the singular integral operator $S_{\mathbb{D}} : \mathscr{L}^p(\mathbb{D}) \to \mathscr{L}^p(\mathbb{D})$, $1 < p < \infty$, is known as Beurling transform in $\mathbb{D}$. The explicit formula reads as

$$S_{\mathbb{D}} f(z) = -\frac{1}{\pi} \int_{\mathbb{D}} \left(\frac{f(\tau)}{(z - \tau)^2} - \frac{\overline{f(\tau)}}{(1 - z\bar\tau)^2} \right) d\sigma(\tau).$$

We have

$$\|S_{\mathbb{D}} f\|_{\mathscr{L}^p} \le C_p \|f\|_{\mathscr{L}^p}$$

for some constant $C_p \geq 1$. This constant can be computed in terms of the norm of Berling transform, see [3] Section 4.8.2.

Now we define

$$u = \frac{1}{2}\operatorname{Re} h \in \mathscr{W}_0^{1,p}(\mathbb{D}).$$

The following complex derivatives are understood in the sense of Schwartz distributions:

$$\begin{aligned}\Delta u = 4u_{z\bar{z}} &= 2(\operatorname{Re} h)_{z\bar{z}}\\ &= (h+\bar{h})_{z\bar{z}} = (h_{z\bar{z}} + \overline{h_{z\bar{z}}})\\ &= (f_z + \overline{f_z}) = 2\operatorname{Re} f_z = \operatorname{div} f\end{aligned}$$

as desired.

Remark 4.1. Uniqueness may also be deduced from the existence result, by so-called duality argument. Indeed, we want to prove that if v solves the homogeneous boundary value problem

$$\begin{cases}\Delta v = 0\\ v \in \mathscr{W}_0^{1,q}(\mathbb{D})\end{cases} \tag{4.1}$$

then $v \equiv 0$. To this end, we solve the problem

$$\begin{cases}\Delta u = \operatorname{div}(|\nabla v|^{q-2}\nabla v)\\ u \in \mathscr{W}_0^{1,p}(\mathbb{D})\end{cases} \tag{4.2}$$

We can use u as a test function for (4.1) and v as a test function for (4.2). This yields

$$0 = \int_{\mathbb{D}} \langle \nabla v \,|\, \nabla u\rangle = \int_{\mathbb{D}} |\nabla v|^q.$$

5. The $\mathscr{L}^p$-estimate

Inequality (2.3) is immediate

$$\begin{aligned}\|\nabla u\|_{\mathscr{L}^p(\mathbb{D})} &= 2\|u_z\|_{\mathscr{L}^p(\mathbb{D})}\\ &= \frac{1}{2}\|(h+\bar{h})_z\|_{\mathscr{L}^p(\mathbb{D})}\\ &= \frac{1}{2}\|h_z + \overline{h_{\bar{z}}}\|_{\mathscr{L}^p(\mathbb{D})} = \frac{1}{2}\|S_{\mathbb{D}}f + \bar{f}\|_{\mathscr{L}^p(\mathbb{D})}\\ &\leq \frac{1}{2}\|S_{\mathbb{D}}f\|_{\mathscr{L}^p(\mathbb{D})} + \frac{1}{2}\|f\|_{\mathscr{L}^p(\mathbb{D})} \leq C_p\|f\|_{\mathscr{L}^p(\mathbb{D})}.\end{aligned}$$

In conclusion, the Poisson operator $\mathcal{P} : \mathscr{W}^{-1,p}(\mathbb{D}) \to \mathscr{L}^p(\mathbb{D})$ is continuous.

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