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Partial Differential Equations — *Stability of evolution equations with small commutators in a Banach space*, by Michael Gil', communicated on April 20, 2018.

Abstract. — Let A be a generator of a C_0-semigroup in a Banach space $\mathscr{X}$, and $B(t)$ $(t \geq 0)$ be a variable bounded piece-wise strongly continuous operator in $\mathscr{X}$. We consider the equation $dy(t)/dt = (A + B(t))y(t)$ $(t \geq 0)$. It is assumed that the commutator $K(t) = AB(t) - B(t)A$ is a bounded operator. Under that condition, exponential stability conditions are derived in terms of the commutator.

Key words: Banach space, linear differential equation, stability

Mathematics Subject Classification: 47D06, 35B35, 34G10

1. Introduction and statement of the main result

Let $\mathscr{X}$ be a Banach space with a norm $\|.\|$ and the identity operator I. By $\mathscr{B}(\mathscr{X})$ the set of all bounded operators in $\mathscr{X}$ is denoted. For a linear operator C, $\mathrm{Dom}(C)$ is the domain. If $C \in \mathscr{B}(\mathscr{X})$, then $\|C\|$ is its operator norm.

Throughout this paper A is a generator of a C_0-semigroup e^{At} on $\mathscr{X}$, and $B(t)$ $(t \geq 0)$ is a variable bounded piece-wise strongly continuous operator mapping $\mathrm{Dom}(A)$ *into itself* for each $t \geq 0$. We will investigate exponential stability conditions for the equation

$$\frac{dy(t)}{dt} = (A + B(t))y(t) \quad (t \geq 0). \tag{1.1}$$

It is assumed that the commutator $K(t) = AB(t) - B(t)A$ is uniformly bounded on $[0, \infty)$, i.e. the norms of $(AB(t) - BA(t))x$ $(x \in \mathrm{Dom}(A), \|x\| = 1)$ are uniformly bounded. So $AB(t) - B(t)A$ can be extended to $\mathscr{X}$ as a bounded operator. That extension is denoted by $K(t)$. Moreover,

$$\kappa := \sup_{t \geq 0} \|K(t)\| < \infty. \tag{1.2}$$

A solution to (1.1) for given $y_0 \in \mathrm{Dom}(A)$ is a function $y : [0, \infty) \to \mathrm{Dom}(A)$ having at each point $t > 0$ a strong derivative, at zero the right strong derivative, and satisfying (1.1) for all $t > 0$ and $y(0) = y_0$. Since $B(t)$ is bounded, maps the domain of A into itself, and A is a generator of a C_0-semigroup, the existence, uniqueness and continuous dependence on initial vectors of solutions is due to the well-known Theorem II.3.4 [10].

Equation (1.1) is said to be exponentially stable, if there are positive constants M and ε, such that $\|y(t)\| \le M \exp[-\varepsilon t]\|y(0)\|$ $(t \ge 0)$ for any solution $y(t)$ of (1.1).

Equation (1.1) can be considered as the equation

$$\frac{dy(t)}{dt} = C(t)y(t) \tag{1.3}$$

with a variable linear operator $C(t)$. Observe that $C(t)$ in the considered case has a special form: it is the sum of operators A and $B(t)$. This fact allows us to use the information about the coefficients more completely than the theory of differential equations (1.3) containing an arbitrary operator $C(t)$.

The literature on stability theory of differentievolution equations in a Banach space is very rich, cf. [1]–[6], [11]–[14] and the references given therein, but to the best of our knowledge, stability conditions for equation (1.1) in terms of the commutator were not investigated. Note that equation (1.1) is usually considered as a perturbation of a stable semigroup generated by A (see e.g. [7, 8, 9], and references given in these papers). At the same time as we will see, stability conditions in terms of the commutator enable us to investigate equations with an unstable semigroup e^{At}. This fact, in particular, is important for stabilization of systems with distributed parameters.

Let $U_B(t,s)$ $(t \ge s \ge 0)$ be the evolution operator of the equation

$$\dot{u}(t) = B(t)u(t) \quad (t \ge 0). \tag{1.4}$$

Suppose that there are constants $c_0 \ge 1$ and $b_0 \in \mathbb{R}$, such that

$$\|U_B(t,s)\| \le c_0 \exp[b_0(t-s)] \quad (t \ge s \ge 0). \tag{1.5}$$

Now we are in a position to formulate the main result of the paper.

Theorem 1.1. *Let conditions* (1.2) *and* (1.5) *hold, and the operator* $A + b_0 I$ *generate an exponentially stable semigroup* $e^{(A+b_0 I)t}$. *In addition, let*

$$\zeta := \kappa c_0 \left(\int_0^\infty \|e^{(A+b_0 I)t}\|\, dt \right)^2 < 1. \tag{1.6}$$

Then equation (1.1) *is exponentially stable.*

This theorem is proved in the next section. It is sharp: if $B(t)f = b_0 f$ $(f \in \mathcal{X})$, then $\zeta = 0$ and Theorem 1.1 gives us necessary and sufficient exponential stability conditions for (1.1): $e^{(A+b_0 I)t}$ should be exponentially stable.

2. Proof of Theorem 1.1

Put $[e^{At}, B(r)] := e^{tA}B(r) - B(r)e^{At}$.

LEMMA 2.1. *Let A generate a C_0-semigroup e^{At}, $B(r) \in \mathscr{B}(\mathscr{X})$ $(r \geq 0)$ map* Dom(A) *into itself and condition* (1.2) *hold. Then for all $r, t \geq 0$ the operator* $[e^{At}, B(r)] := e^{tA}B(r) - B(r)e^{At}$ *is bounded and*

$$[e^{At}, B(r)] = \int_0^t e^{sA}K(r)e^{(t-s)A}\,ds.$$

Moreover, $[e^{At}, B(r)]$ *maps* Dom(A) *into itself.*

PROOF. On Dom(A) we have

$$\begin{aligned}
\int_0^t e^{sA}K(r)e^{(t-s)A}\,ds &= \int_0^t e^{sA}(AB(r) - B(r)A)e^{(t-s)A}\,ds \\
&= \int_0^t (e^{sA}AB(r)e^{(t-s)A} - e^{sA}B(r)Ae^{(t-s)A})\,ds \\
&= \int_0^t \Big(\frac{\partial}{ds}e^{sA}B(r)e^{(t-s)A} + e^{sA}B(r)\frac{\partial}{ds}e^{(t-s)A}\Big)\,ds \\
&= \int_0^t \frac{\partial}{ds}(e^{sA}B(r)e^{(t-s)A})\,ds = e^{sA}B(r)e^{(t-s)A}\big|_{s=0}^t \\
&= e^{At}B(r) - B(r)e^{At}.
\end{aligned}$$

Since the operator $\int_0^t e^{sA}K(r)e^{(t-s)A}\,ds$ is bounded, and $B(r)$ maps Dom(A) into itself, we get the required result. □

Denote by $X(t,s)$ the evolution operator of (1.1) and put $Y(t,s) = e^{A(t-s)}U_B(t,s)$, and

$$\|Z\|_C := \sup_{t \geq s \geq 0} \|Z(t,s)\|$$

for an operator function $Z(t,s)$ uniformly bounded on $0 \leq s \leq t < \infty$.

LEMMA 2.2. *With the notation* $F(t,s) := [e^{A(t-s)}, B(t)]U_B(t,s)$ $(t \geq s \geq 0)$, *let* $\|Y\|_C < \infty$ *and*

$$\gamma(F) := \sup_s \int_s^\infty \|F(t,s)\|\,dt < 1. \tag{2.1}$$

Then

$$\|X\|_C \leq \frac{\|Y\|_C}{1 - \gamma(F)} \tag{2.2}$$

and

$$\|X - Y\|_C \leq \frac{\gamma(F)\|Y\|_C}{1 - \gamma(F)}. \tag{2.3}$$

PROOF. Note that

$$\frac{dX(t,s)h}{dt} = (A + B(t))X(t,s)h \quad (h \in \mathrm{Dom}(A)) \tag{2.4}$$

and

$$\begin{aligned} \frac{dY(t,s)h}{dt} &= (Ae^{A(t-s)}U_B(t,s) + e^{A(t-s)}B(t)U_B(t,s))h \\ &= ((A + B(t))e^{A(t-s)}U_B(t,s) + e^{A(t-s)}B(t)U_B(t,s) \\ &\quad - B(t)e^{A(t-s)}U_B(t,s))h \\ &= (A + B(t))Y(t,s)h + F(t,s)h \quad (h \in \mathrm{Dom}(A)). \end{aligned} \tag{2.5}$$

Due to Lemma 2.1, operator $F(t,s)$ is bounded for all finite t, s and maps $\mathrm{Dom}(A)$ into itself. Subtracting (2.4) from (2.5), on $\mathrm{Dom}(A)$ we get

$$\frac{d(Y(t) - X(t))}{dt} = (A + B(t))(Y(t,s) - X(t,s)) + F(t,s). \tag{2.6}$$

By the differentiation we obtain

$$(Y(t,s) - X(t,s))h = \int_s^t X(t,s_1)F(s_1,s)h\,ds_1 \quad (h \in \mathrm{Dom}(A)).$$

Since $\mathrm{Dom}(A)$ is dense, and $Y(t,s)$, $X(t,s)$ and $F(t,s)$ are bounded, we can write

$$Y(t,s) - X(t,s) = \int_s^t X(t,s_1)F(s_1,s)\,ds_1.$$

Consequently,

$$\|Y(t,s) - X(t,s)\| \le \int_s^t \|X(t,s_1)\|\,\|F(s_1,s)\|\,ds_1, \tag{2.7}$$

and therefore,

$$\|X(t,s)\| \le \|Y(t,s)\| + \int_s^t \|X(t,s_1)\|\,\|F(s_1,s)\|\,ds_1. \tag{2.8}$$

Hence, for any finite $t > s$ we obtain

$$\sup_{0 \le s \le v \le t} \|X(v,s)\| \le \|Y\|_C + \sup_{0 \le s \le v \le t} \|X(v,s)\|\gamma(F).$$

Now (2.1) implies

$$\sup_{0\le s\le v\le t} \|X(v,s)\| \le \|Y\|_C/(1-\gamma(F)). \tag{2.9}$$

This proves (2.2). From (2.7) and (2.2), inequality (2.3) follows. This proves the lemma. □

PROOF OF THEOREM 1.1. By Lemma 2.1,

$$\begin{aligned}\|F(t,s)\| &\le \|[e^{A(t-s)}, B(t)]\|\,\|U_B(t,s)\| \\ &\le \kappa\|U_B(t,s)\| \int_s^t \|e^{A(v-s)}\|\,\|e^{A(t-v)}\|\,dv.\end{aligned}$$

Hence,

$$\int_s^\infty \|F(t,s)\|\,dt \le \hat{\gamma}(s),$$

where

$$\hat{\gamma}(s) := \kappa \int_s^\infty \|U_B(t,s)\| \int_s^t \|e^{A(v-s)}\|\,\|e^{A(t-v)}\|\,dv\,dt.$$

So $\gamma(F) \le \sup_s \hat{\gamma}(s)$. From (1.5) and stability of $e^{(A+b_0I)t}$ it follows $\|Y\|_C < \infty$ and

$$\begin{aligned}\hat{\gamma}(s) &\le \kappa c_0 \int_s^\infty e^{b_0(t-s)} \int_s^t \|e^{A(t-v)}\|\,\|e^{A(v-s)}\|\,dv\,dt \\ &= \kappa c_0 \int_s^\infty \|e^{A(v-s)}\| \int_v^\infty \|e^{A(t-v)}\| e^{b_0(t-s)}\,dt\,dv \\ &= \kappa c_0 \int_s^\infty \|e^{A(v-s)}\| \int_0^\infty \|e^{At_1}\| e^{b_0(t_1+v-s)}\,dt_1\,dv \\ &= \int_s^\infty \|e^{(A+b_0)(v-s)}\|\,dv \int_0^\infty \|e^{(A+b_0)t_1}\|\,dt_1.\end{aligned}$$

Thus

$$\gamma(F) \le \kappa c_0 \|e^{(A+b_0I)t}\|^2_{L^1(0,\infty)}.$$

Here

$$\|e^{(A+b_0I)t}\|_{L^1(0,\infty)} = \int_0^\infty \|e^{(A+b_0I)t}\|\,dt.$$

Now (1.6) and Lemma 2.2 prove relations (2.2) and (2.3). Inequality (2.2) means that (1.1) is Lyapunov stable. Furthermore, substitute

$$y(t) = u_\varepsilon(t)e^{-\varepsilon t} \quad (\varepsilon > 0) \tag{2.10}$$

into (1.1). Then

$$du_\varepsilon(t)/dt = (A + B(t) + \varepsilon I)u_\varepsilon(t). \tag{2.11}$$

If ε is small enough, then conditions (1.2), (1.5) and (1.6) hold with $B(t) + \varepsilon I$ instead of $B(t)$.

Applying our above arguments to equation (2.11) we can assert that (2.11) is Lyapunov stable. So due to (2.10) equation (1.1) is exponentially stable. This proves the theorem. □

3. Example

Consider the problem

$$\frac{\partial}{\partial t}u = \frac{\partial^2 u}{\partial x^2} + T(x)u + M(t)u \quad (u = u(t,x),\ 0 < x < 1), \tag{3.1}$$

$$u(t,0) = u(t,1) = 0 \quad (t > 0), \tag{3.2}$$

where is $T(x)$ is a twice continuously differentiable in x $n \times n$-matrix function defined on $[0,1]$; $M(t)$ is a piece-wise continuous $n \times n$-matrix, independent of x and uniformly bounded on $[0,\infty)$.

Take $\mathscr{X} = L^2([0,1];\mathbb{C}^n)$ – the Hilbert space of n-vector valued functions defined on $[0,1]$ with the scalar product

$$(v,w) = \int_0^1 (v(x),w(x))_n\,dx \quad (v,w \in L^2([0,1];\mathbb{C}^n)),$$

where $(\cdot\,,\cdot)_n$ is the scalar product in $\mathbb{C}^n$. For brevity put $L^2([0,1];\mathbb{C}^n) = L^2_n$. Take

$$(Af)(x) = f''(x) + T(x)f(x) \quad \text{and} \quad (B(t)f)(x) = M(t)f(x)$$
$$(f \in \mathrm{Dom}(A);\ 0 \le x \le 1,\ t \ge 0)$$

with

$$\mathrm{Dom}(A) = \{h \in L^2_n : h'' \in L^2_n, h(0) = h(1) = 0\}.$$

Then $K(t) = T(x)M(t) - M(t)T(x)$ and

$$\kappa = \sup_t \|K(t)\| = \sup_{x,t} \|T(x)M(t) - M(t)T(x)\|_n.$$

Here $\|\cdot\|_{=}\sqrt{(\cdot,\cdot)}$ is the norm in L^2_n and $\|\cdot\|_n$ is the norm in $\mathbb{C}^n$. Simple calculations show that $\|e^{At}\| \le \exp[-(\pi - \Lambda(T))t]$ $(t \ge 0)$, where

$$\Lambda(T) := \sup_x \sup_{h \in \mathbb{C}^n} \Re(T(x)h, h)_n/(h, h)_n.$$

Assume that the evolution operator $U_M(t, s)$ of the matrix differential equation

$$\dot{v}(t) = M(t)v(t) \quad (t \ge 0)$$

satisfies the inequality

$$\|U_M(t, s)\|_n \le c_0 \exp[b_0(t - s)] \quad (t \ge s \ge 0). \tag{3.3}$$

Then condition (1.5) holds. If, in addition,

$$\pi - \Lambda(T) - b_0 > 0, \tag{3.4}$$

then

$$\int_0^\infty e^{-(\pi-\Lambda(T)-b_0)t}\, dt = \frac{1}{\pi - \Lambda(T) - b_0}.$$

Now Theorem 1.1 implies the following result.

Let the conditions (3.3), (3.4) *and*

$$\frac{\kappa c_0}{(\pi - \Lambda(T) - b_0)^2} < 1$$

hold. Then equation (3.1) *is exponentially stable.*

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