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Calculus of Variations — *Convex relaxation and variational approximation of functionals defined on* 1*-dimensional connected sets*, by Mauro Bonafini, Giandomenico Orlandi and Édouard Oudet, communicated on April 20, 2018.[1]

Abstract. — In this short note we announce the main results of [2] about variational problems involving 1-dimensional connected sets in the Euclidean plane, such as for example the Steiner tree problem and the irrigation (Gilbert–Steiner) problem.

Key words: Calculus of variations, geometric measure theory, gamma-convergence, convex relaxation, Steiner problem

Mathematics Subject Classification: 49J45, 49Q15, 49Q20, 49M20, 65K10

In this note we describe the main results obtained in [2] concerning problems involving 1-dimensional structures, focusing our attention on the two classical examples given by the Euclidean Steiner tree problem and Gilbert–Steiner (or irrigation type) problems. The Steiner tree problem in $\mathbb{R}^d$ can be stated as follows: given N distinct points $P_1, \dots, P_N$ in $\mathbb{R}^d$, find the shortest connected graph containing the points P_i, or equivalently

$$\text{(STP)} \qquad \inf\{\mathscr{H}^1(L),\ L \text{ connected},\ L \supset \{P_1, \dots, P_N\}\}.$$

An optimal (not necessarily unique) graph L always exists and also the structure of L is known: a union of segments connecting the endpoints, possibly meeting at 120° in at most $N-2$ further branch points, called Steiner points. On the other hand, the Gilbert–Steiner problem for $N-1$ unit sources and one sink fits within the realm of optimal transportation problems and can be formulated as follows: identifying networks L connecting the P_i as streamlines of a vector measure $\mu = \theta(x)\tau(x) \cdot \mathscr{H}^1 \llcorner L$ flowing unit masses located at P_i, $i < N$, to P_N, find an optimal network minimizing a transport cost which is a sublinear (concave) function of the mass density, to favour branching (see [11]). More precisely, for $0 < \alpha \leq 1$, we have

$$(I_\alpha) \qquad \inf\left\{\int_{\mathbb{R}^d} |\theta(x)|^\alpha \, d\mathscr{H}^1 \llcorner L,\ \operatorname{div}\mu = \sum_{i=1}^{N-1} \delta_{P_i} - (N-1)\delta_{P_N}\right\}.$$

[1] This paper is related to a talk given at "XXVII Convegno Nazionale di Calcolo delle Variazioni" – Levico Terme (Trento) 6–10 February, 2017.

In the recent years many different variational approximations have been proposed for the treatment of (STP) or (I_α), see for example [10, 7, 4, 5], but mainly for the two dimensional setting using a phase field based approach together with some coercive regularization. Here, following [8, 9], we analyse this kind of optimization problems on Euclidean graphs with fixed endpoints set A, like (STP) or irrigation-type problems, and we rephrase them as optimal partition-type problems in the planar case $\mathbb{R}^2$, obtaining a variational approximation in the sense of Γ-convergence (Theorem 3.2). Moreover we propose a convex relaxation and identify numerically the optimal networks. The corresponding analysis in $\mathbb{R}^d$, $d \geq 3$, is contained in the companion paper [3].

1. Acyclic graphs and rank one tensor valued measures

Fix a set of N distinct points $A = \{P_1, \dots, P_N\} \subset \mathbb{R}^d$, $d \geq 2$. Define $\mathscr{G}(A)$ to be the set of (connected) *acyclic graphs* L that can be described as the union $L = \bigcup_{i=1}^{N-1} \lambda_i$, where λ_i are simple rectifiable curves connecting P_i to P_N and oriented by $\mathscr{H}^1$-measurable unit vector fields τ_i satisfying $\tau_i(x) = \tau_j(x)$ for $\mathscr{H}^1$-a.e. $x \in \lambda_i \cap \lambda_j$ (i.e. the orientation of λ_i is coherent with that of λ_j on their intersection).

For each $L \in \mathscr{G}(A)$, we identify the curves λ_i with the vector measures $\Lambda_i = \tau_i \cdot \mathscr{H}^1 \llcorner \lambda_i$, so that all the information concerning this acyclic graph L is encoded in the rank one tensor valued measure $\Lambda = \tau \otimes g \cdot \mathscr{H}^1 \llcorner L$, where $\tau \in \mathbb{R}^d$ is the $\mathscr{H}^1$-measurable unit vector field carrying the orientation of the graph L, with $\operatorname{spt} \tau = L$, $\tau = \tau_i$ $\mathscr{H}^1$-a.e. on λ_i, and $g \in \mathbb{R}^{N-1}$ is an $\mathscr{H}^1$-measurable vector map over L whose components g_i are defined by $g_i \cdot \mathscr{H}^1 \llcorner L = \mathscr{H}^1 \llcorner \lambda_i = |\Lambda_i|$, with $|\Lambda_i|$ the total variation measure of Λ_i. Observe that $g_i \in \{0, 1\}$ a.e. for any i and the measures Λ_i verify the property

$$\operatorname{div} \Lambda_i = \delta_{P_i} - \delta_{P_N}. \tag{1.1}$$

Definition 1.1. Given $L \in \mathscr{G}(A)$, we call the above constructed measure $\Lambda = \tau \otimes g \cdot \mathscr{H}^1 \llcorner L$ the canonical (rank one) tensor valued measure representation of the acyclic graph L.

To any connected set $S \supset A$ with $\mathscr{H}^1(S)$ finite, i.e. to any candidate minimizer for (STP), we can associate in a canonical way (see e.g. Lemma 2.1 in [8]) an acyclic graph $L \supset A$ such that $\mathscr{H}^1(L) \leq \mathscr{H}^1(S)$ and $L \in \mathscr{G}(A)$. Given now a graph $L \in \mathscr{G}(A)$ canonically represented by the tensor valued measure Λ, we observe that $\mathscr{H}^1 \llcorner L = \sup_i (\mathscr{H}^1 \llcorner \lambda_i) = \sup_i |\Lambda_i|$, i.e. the measure $\mathscr{H}^1 \llcorner L$ corresponds to the smallest positive measure dominating the family of measures $|\Lambda_i| = \mathscr{H}^1 \llcorner \lambda_i$. We thus have

$$\mathscr{H}^1(L) = \int_{\mathbb{R}^d} d\mathscr{H}^1 \llcorner L = \int_{\mathbb{R}^d} \sup_i |\Lambda_i| =: \mathscr{F}^0(\Lambda),$$

and we recognize that minimizing $\mathscr{F}^0$ among graphs $L \in \mathscr{G}(A)$, i.e. among rank one tensor valued measures $\Lambda = \tau \otimes g \cdot \mathscr{H}^1 \llcorner L$ which are the canonical representation of acyclic graphs $L \in \mathscr{G}(A)$, solves (STP) in $\mathbb{R}^d$.

Following [9], we can address in a similar way irrigation problems from the point sources $\{P_1, \ldots, P_{N-1}\}$ to the target point P_N by using the canonical representation as rank one tensor valued measures of graphs $L \in \mathscr{G}(A)$. For doing so we need some additional notation: for a measure $\vec{\mu} = (\mu_1, \ldots, \mu_M)$, with μ_i positive measures on $\mathbb{R}^d$, let $|\vec{\mu}|_1 = \sum_i \mu_i$, so that $\vec{\mu} = g|\vec{\mu}|_1$ with $g \in \mathbb{R}^M$, $0 \le g_i \le 1$ for $1 \le i \le M$, $\sum_i g_i = 1$. Accordingly, we denote $|\vec{\mu}|_\infty$ the supremum measure $|\vec{\mu}|_\infty = \sup_i \mu_i = (\sup_i g_i)|\vec{\mu}|_1$ and for generic $p \ge 1$ we define the measure $|\vec{\mu}|_p := |g|_p |\vec{\mu}|_1$, with $|g|_p = (\sum_i g_i^p)^{1/p}$ the ℓ^p norm of g. More generally, for Ψ a norm on $\mathbb{R}^M$, we define the measure $\Psi(\vec{\mu}) := \Psi(g)|\vec{\mu}|_1$, characterized as

$$\Psi(\vec{\mu})(\mathbb{R}^d) = \sup\left\{ \sum_{i=1}^M \int_{\mathbb{R}^d} \varphi_i \, d\mu_i,\ 0 \le \varphi_i \in C_c^0(\mathbb{R}^d)\ \forall 1 \le i \le M,\ \Psi^*(\vec{\varphi}) \le 1 \right\}$$

with Ψ^* the dual norm to Ψ w.r.t. the Euclidean structure on $\mathbb{R}^M$.

Let now $\Lambda = \tau \otimes \vec{\mu} = \tau \otimes g|\vec{\mu}|_1$ be a rank one $\mathbb{R}^d \otimes \mathbb{R}^{N-1}$-valued measure with $|\tau| = 1$. For $0 < \alpha \le 1$ define

$$\mathscr{F}^\alpha(\Lambda) = \int_{\mathbb{R}^d} |g|_{1/\alpha} \, d|\vec{\mu}|_1 = |\vec{\mu}|_{1/\alpha}(\mathbb{R}^d) \tag{1.2}$$

and

$$\mathscr{F}^0(\Lambda) = \int_{\mathbb{R}^d} |g|_\infty \, d|\vec{\mu}|_1 = \int_{\mathbb{R}^d} \Big(\sup_{1 \le i \le N-1} \mu_i \Big) = |\vec{\mu}|_\infty(\mathbb{R}^d). \tag{1.3}$$

In other words, $\mathscr{F}^\alpha(\Lambda) = \|\Lambda\|_{\Psi_\alpha}$, $\mathscr{F}^0(\Lambda) = \|\Lambda\|_{\Psi_0}$ are total variation-type functionals, with respect to the norms $\Psi_\alpha = |\cdot|_{\ell^{1/\alpha}}$ and $\Psi_0 = |\cdot|_{\ell^\infty}$. When $\Lambda = \tau \otimes g \cdot \mathscr{H}^1 \llcorner L$ is the canonical representation of an acyclic graph $L \in \mathscr{G}(A)$, so that in particular we have $|\tau| = 1$ and $g_i \in \{0, 1\}$ for $1 \le i \le N-1$, we deduce

$$\mathscr{F}^0(\Lambda) = \int_{\mathbb{R}^d} |g|_\infty \, d\mathscr{H}^1 \llcorner L = \mathscr{H}^1(L),$$

$$\mathscr{F}^\alpha(\Lambda) = \int_{\mathbb{R}^d} |g|_{1/\alpha} \, d\mathscr{H}^1 \llcorner L = \int_L |\theta|^\alpha \, d\mathscr{H}^1,$$

where $\theta(x) = \sum_i g_i(x)^{1/\alpha} = \sum_i g_i(x) \in \mathbb{Z}$, and $0 \le \theta(x) \le N-1$. We thus recognize that minimizing the functional $\mathscr{F}^\alpha$ among graphs L connecting $P_1, \ldots, P_{N-1}$ to P_N solves the irrigation problem with sources $P_1, \ldots, P_{N-1}$ and target P_N (see [9]), while minimizing $\mathscr{F}^0$ among graphs L with endpoints set $\{P_1, \ldots, P_N\}$ solves (STP) in $\mathbb{R}^d$.

2. Acyclic graphs and partitions of $\mathbb{R}^2$

Our two-dimensional analysis is based on following lemma which states that two acyclic graphs having the same endpoints set give rise to a partition of $\mathbb{R}^2$, or equivalently that their oriented difference corresponds to the orthogonal distributional gradient of a piecewise integer valued function having bounded total variation.

Lemma 2.1. *Let $\{P, R\} \subset \mathbb{R}^2$ and let λ, γ be simple rectifiable curves from P to R oriented by $\mathscr{H}^1$-measurable unit vector fields τ', τ''. Define $\Lambda = \tau' \cdot \mathscr{H}^1 \llcorner \lambda$ and $\Gamma = \tau'' \cdot \mathscr{H}^1 \llcorner \gamma$. Then there exists a function $u \in SBV(\mathbb{R}^2; \mathbb{Z})$ such that, denoting Du and $Du^\perp$ respectively the measures representing the gradient and the orthogonal gradient of u, we have $Du^\perp = \Gamma - \Lambda$.*

The idea is now to take advantage of this lemma to reformulate the minimization of $\mathscr{F}^\alpha$ as an optimization problem over the class of integer valued SBV functions. For this purpose fix once and for all an acyclic graph G connecting the endpoints set A, for example

$$G = \bigcup_{i=1}^{N-1} \gamma_i, \quad \text{with } \gamma_i \text{ the segment joining } P_i \text{ to } P_N, \tag{2.1}$$

denote $\tau_i = \frac{P_N - P_i}{|P_N - P_i|}$ the orientation of each segment and identify γ_i with the vector measure $\Gamma_i = \tau_i \cdot \mathscr{H}^1 \llcorner \gamma_i$, so that $\mathscr{H}^1(G) = |\Gamma|(\mathbb{R}^2)$ and $|\Gamma| = \sup_i |\Gamma_i|$. Taking into account Lemma 2.1, and in the simplest case of $\mathscr{F}^0$, we have

Corollary 2.2. *Let $A = \{P_1, \dots, P_N\} \subset \mathbb{R}^2$ be a set of terminal points and $G \in \mathscr{G}(A)$ as above. For any acyclic graph $L \in \mathscr{G}(A)$, denoting Γ (resp. Λ) the canonical tensor valued representation of G (resp. L), we have*

$$\mathscr{H}^1(L) = \int_{\mathbb{R}^2} \sup_i |\Lambda_i| = \int_{\mathbb{R}^2} \sup_i |Du_i^\perp - \Gamma_i|$$

for suitable $u_i \in SBV(\mathbb{R}^2; \mathbb{Z})$, $1 \le i \le N - 1$.

Thus, having fixed the family of measures $\Gamma = (\Gamma_1, \dots, \Gamma_{N-1})$, we are led to consider the minimization for $U = (u_1, \dots, u_{N-1}) \in SBV(\mathbb{R}^2; \mathbb{Z}^{N-1})$ of the functional

$$\mathscr{F}^0(U) \equiv \mathscr{F}^0(DU^\perp - \Gamma) = \int_{\mathbb{R}^2} \sup_i |Du_i^\perp - \Gamma_i|.$$

We have already seen that to each acyclic graph $L \in \mathscr{G}(A)$ we can associate a $U \in SBV(\mathbb{R}^2; \mathbb{Z}^{N-1})$ such that $\mathscr{H}^1(L) = \mathscr{F}^0(U)$. Moreover we can also prove that for each minimizer $U \in SBV(\mathbb{R}^2; \mathbb{Z}^{N-1})$ of $\mathscr{F}^0$ we can identify, in the support of the family of measures $\{\Gamma_i - Du_i^\perp\}_i$, an acyclic graph L connecting the terminal points $P_1, \dots, P_N$ and such that $\mathscr{F}^0(U) = \mathscr{H}^1(L)$. This means that we have a relationship between (STP) and the minimization of $\mathscr{F}^0$ over functions in $SBV(\mathbb{R}^2; \mathbb{Z}^{N-1})$, and a similar connection can be made between the α-irrigation problem and minimization over the same space of $\mathscr{F}^\alpha(U) \equiv \mathscr{F}^\alpha(DU^\perp - \Gamma)$. To this family of functionals $\mathscr{F}^\alpha$ we now provide an approximation in the sense of Γ-convergence through Modica–Mortola type energies.

3. Γ-CONVERGENCE

We first consider Modica–Mortola functionals for functions having a prescribed jump part: for any given $i \in \{1, \dots, N-1\}$ consider the measure $\Gamma_i = \tau_i \cdot \mathscr{H}^1 \llcorner \gamma_i$ defined for G as in (2.1), and let

$$F^i_\varepsilon(u, B) = \int_B e^i_\varepsilon(u)\, dx = \int_B |Du^\perp - \Gamma_i|^2 + \frac{1}{\varepsilon^2} W(u)\, dx, \tag{3.1}$$

defined for $u \in H_i \equiv W^{1,2}_{loc}(\mathbb{R}^2 \backslash \gamma_i) \cap BV(\mathbb{R}^2)$ and $B \subset \mathbb{R}^2 \backslash \{P_1, \dots, P_N\}$ open, where $W(u) \geq 0$ is a smooth 1-periodic potential vanishing on $\mathbb{Z}$ and let $c_0 = 2\int_0^1 \sqrt{W(s)}\, ds$ (we can take for example $W(u) = \sin^2(\pi u)$, then $c_0 = 2/\pi$). Observe that due to summability issues for the absolutely continuous part of the gradient we work in local spaces.

For each fixed $i \in \{1, \dots, N-1\}$ functionals $\{\varepsilon F^i_\varepsilon\}_\varepsilon$, which are just "shifted" variants of the classical Modica–Mortola ones, Γ-converge as $\varepsilon \to 0$ to the limiting functional $F^i(u, B) = c_0 \int_B d|Du^\perp - \Gamma_i|$ (Section 3.1 in [2] or see also [1]). Based on this "component-wise" result we can now address the general problem involving total variation type functionals such as $\mathscr{F}^\alpha$.

COROLLARY 3.1 (Γ-convergence). *Let $\Psi : \mathbb{R}^{N-1} \to [0, +\infty)$ be a norm on $\mathbb{R}^{N-1}$, define $H = H_1 \times \cdots \times H_{N-1}$ and consider the functionals*

$$\mathscr{F}^\Psi_\varepsilon(U, B) = \int_B \varepsilon \Psi(\vec{e}_\varepsilon(U))\, dx, \quad \textit{for } U = (u_1, \dots, u_{N-1}) \in H, \tag{3.2}$$

$$\mathscr{F}^\Psi(U, B) = \int_B \Psi(g) d|DU^\perp - \Gamma|_1, \quad \textit{for } U \in SBV(\mathbb{R}^2; \mathbb{Z}^{N-1}), \tag{3.3}$$

for $B \subset \mathbb{R}^2 \backslash \{P_1, \dots, P_N\}$ open, where we set $\vec{e}_\varepsilon(U) = (e^1_\varepsilon(u_1), \dots, e^{N-1}_\varepsilon(u_{N-1}))$, $g = (g_1, \dots, g_{N-1})$ and, for $1 \leq i \leq N-1$, $|Du_i^\perp - \Gamma_i| = g_i |DU^\perp - \Gamma|_1$, with $|DU^\perp - \Gamma|_1 := \sum_{i=1}^{N-1} |Du_i^\perp - \Gamma_i|$. Then we have

(1) **(Compactness and lower bound inequality)** *For all $U_\varepsilon \in H$ such that $\mathscr{F}^\Psi_\varepsilon(U_\varepsilon, B) \leq C(B)$, $B \subset \mathbb{R}^2 \backslash \{P_1, \dots, P_N\}$, there exists $U \in SBV(\mathbb{R}^2; \mathbb{Z}^{N-1})$ such that (up to a subsequence) $U_\varepsilon \to U$ in $L^1(\mathbb{R}^2; \mathbb{R}^{N-1})$. Moreover,*

$$\liminf_{\varepsilon \to 0} \mathscr{F}^\Psi_\varepsilon(U_\varepsilon, B) \geq c_0 \mathscr{F}^\Psi(U, B) \tag{3.4}$$

(2) **(Upper bound (in)-equality)** *Let $\Lambda = \tau \otimes g \cdot \mathscr{H}^1 \llcorner L$ be a rank one tensor valued measure canonically representing an acyclic graph L connecting $P_1, \dots, P_N$, and let $U \in SBV(\mathbb{R}^2; \mathbb{Z}^{N-1})$ such that $Du_i^\perp = \Gamma_i - \Lambda_i$ for any*

$i = 1, \dots, N-1$. Then there exists a sequence $U_\varepsilon \in H$ such that $U_\varepsilon \to U$ in $L^1(\mathbb{R}^2; \mathbb{R}^{N-1})$ and

$$\limsup_{\varepsilon \to 0} \mathscr{F}_\varepsilon^\Psi(U_\varepsilon, B) \le c_0 \mathscr{F}^\Psi(U, B) \tag{3.5}$$

for any open subset $B \subset \mathbb{R}^2 \backslash \{P_1, \dots, P_N\}$.

Corollary 3.1, together with the convergence of minimizers (see [2]), may be summarized, in case $\mathscr{F}^\Psi$ corresponds respectively to $\mathscr{F}^0$ and $\mathscr{F}^\alpha$ for $0 < \alpha \le 1$, in the following

Theorem 3.2. *Let $A = \{P_1, \dots, P_N\} \subset \mathbb{R}^2$ and $\Gamma_i = \tau \cdot \mathscr{H}^1 \llcorner \gamma_i$, for $1 \le i \le N-1$, as in Remark 2.1. For $U = (u_1, \dots, u_{N-1})$, $u_i \in W^{1,2}_{loc}(\mathbb{R}^2 \backslash \gamma_i) \cap BV(\mathbb{R}^2)$, $0 < \alpha \le 1$, define*

$$\mathscr{F}_\varepsilon^0(U, B) = \int_B \varepsilon \sup_i e_\varepsilon^i(u_i)\, dx \quad \text{and} \quad \mathscr{F}_\varepsilon^\alpha(U, B) = \int_B \varepsilon \Big(\sum_{i=1}^{N-1} e_\varepsilon^i(u_i)^{1/\alpha} \Big)^\alpha dx,$$

where $B \subset \mathbb{R}^2 \backslash \{P_1, \dots, P_N\}$ is open and the energy densities $e_\varepsilon^i(u_i)$ are defined as in formula (3.1), and let

$$\mathscr{F}^0(U, B) \equiv \mathscr{F}^0(DU^\perp - \Gamma, B), \quad \mathscr{F}^\alpha(U, B) \equiv \mathscr{F}^\alpha(DU^\perp - \Gamma, B) \tag{3.6}$$

be defined as in (1.3) and (1.2). Let $c_0 > 0$ be defined as above. Then we have

$$\mathscr{F}_\varepsilon^0 \xrightarrow{\Gamma} c_0 \mathscr{F}^0 \quad \text{and} \quad \mathscr{F}_\varepsilon^\alpha \xrightarrow{\Gamma} c_0 \mathscr{F}^\alpha,$$

and Γ-convergence takes place with respect to the strong topology of $L^1(\mathbb{R}^2; \mathbb{R}^{N-1})$. In particular, up to subsequences, minimizers U_ε of $\mathscr{F}_\varepsilon^0$ converge, as $\varepsilon \to 0$, to $U \in SBV(\mathbb{R}^2; \mathbb{Z}^{N-1})$ with $DU^\perp - \Gamma = \tau \otimes g \cdot \mathscr{H}^1 \llcorner L$, and L a Steiner Minimal Tree with terminal points in A, while minimizers V_ε of $\mathscr{F}_\varepsilon^\alpha$ converge (up to subsequences), as $\varepsilon \to 0$, to $V \in SBV(\mathbb{R}^2; \mathbb{Z}^{N-1})$, where $DV^\perp - \Gamma = \tau' \otimes g' \cdot \mathscr{H}^1 \llcorner L_\alpha$ represents an optimal α-irrigation plan with sources $P_1, \dots, P_{N-1}$ and target point P_N.

4. Convex relaxation

Another possible way to address the minimization of $\mathscr{F}^\alpha$ is to provide suitable convex positively 1-homogeneous relaxations of it, which in this case maintain their validity in any dimension. We recall that the functional $\mathscr{F}^\alpha$ is defined as

$$\mathscr{F}^\alpha(\Lambda) = \|\Lambda\|_{\Psi_\alpha} = \int_{\mathbb{R}^d} |g|_{1/\alpha}\, d\mathscr{H}^1 \llcorner L$$

if the measure Λ is the canonical representation of an acyclic graph L with terminal points $\{P_1, \dots, P_N\} \subset \mathbb{R}^d$, so that in particular, according to Definition

1.1, we can write $\Lambda = \tau \otimes g \cdot \mathscr{H}^1 \llcorner L$ with $|\tau| = 1$, $g_i \in \{0, 1\}$. For any other $d \times (N-1)$ valued measure Λ on $\mathbb{R}^d$ we set $\mathscr{F}^\alpha(\Lambda) = +\infty$. We look at possible extensions of this type from two perspectives:

Extension to rank one tensor measures. Following [2], we first propose possible positively 1-homogeneous convex relaxations of $\mathscr{F}^\alpha$ on the class of rank one tensor valued Radon measures $\Lambda = \tau \otimes g \cdot |\vec{\mu}|_1$, where $|\tau| = 1$, $g \in \mathbb{R}^{N-1}$ and $\vec{\mu} = (\mu_1, \ldots, \mu_{N-1})$ is a family of positive Radon measures on $\mathbb{R}^d$ with $|\vec{\mu}|_1 = \sum_i \mu_i$. For such a Λ we consider extensions of the form

$$\mathscr{R}^\alpha(\Lambda) = \int_{\mathbb{R}^d} \Psi^\alpha(g)\, d|\vec{\mu}|_1$$

for a convex positively 1-homogeneous Ψ^α on $\mathbb{R}^{N-1}$ (i.e. a norm) verifying

$$|g|_{1/\alpha} \leq \Psi^\alpha(g) \leq \Phi^{**}_{\alpha 0}(g) \quad \text{for all } g \in \mathbb{R}^{N-1}, \tag{4.1}$$

with the convention $1/0 = \infty$. Here we have

$$\Phi^{**}_{\alpha 0}(g) = \Big(\sum_{1 \leq i \leq N-1} |g_i^+|^{1/\alpha}\Big)^\alpha + \Big(\sum_{1 \leq i \leq N-1} |g_i^-|^{1/\alpha}\Big)^\alpha \tag{4.2}$$

for $\alpha > 0$ and for $\alpha = 0$

$$\Phi^{**}_{00}(g) = \sup_{1 \leq i \leq N-1} g_i^+ - \inf_{1 \leq i \leq N-1} g_i^-, \tag{4.3}$$

with $g_i^+ = \max\{g_i, 0\}$ and $g_i^- = \min\{g_i, 0\}$. In practice $\Phi^{**}_{0\alpha}$ represents the convex positively 1-homogeneous envelope of the function $\Phi_{0\alpha}(g) := |g|_{1/\alpha}$ when $g_i \geq 0$ for all i, $+\infty$ otherwise. This relaxation depends on the choice of the function Ψ^α and in general it is not sharp, i.e. minimizers in suitable classes of weighted graphs with prescribed fluxes at their terminal points, or more generally in the class of rank one tensor valued measures having divergence prescribed by (1.1), do not represent minimal Steiner trees (or convex combination of them): for example, choosing $\Psi^0 = |\cdot|_\infty$ and minimizing $\mathscr{R}^0$ within the class of rank one tensor valued Radon measures $\Lambda = \tau \otimes g \cdot |\vec{\mu}|_1$ satisfying (1.1) leads to a minimizer which is not acyclic (see example 4.1 in [2]).

Extension to general matrix valued measures. We turn next to the convex relaxation of $\mathscr{F}^\alpha$ for generic $d \times (N-1)$ matrix valued measures $\Lambda = (\Lambda_1, \ldots, \Lambda_{N-1})$, where Λ_i, for $1 \leq i \leq N-1$, are the vector measures corresponding to the columns of Λ. Since we are looking for a positively 1-homogeneous extension, we observe that whenever $\Lambda = p \cdot \mathscr{H}^1 \llcorner L = \tau \otimes g \cdot \mathscr{H}^1 \llcorner L$, with $|\tau| = c \geq 0$ and $g_i \in \{0, 1\}$, we must have

$$\mathscr{R}^\alpha(\Lambda) = \int_{\mathbb{R}^d} |\tau|\, |g|_{1/\alpha}\, d\mathscr{H}^1 \llcorner L = \int_{\mathbb{R}^d} \Phi_\alpha(p)\, d\mathscr{H}^1 \llcorner L,$$

with $\Phi_\alpha(p) = |\tau|\,|g|_{1/\alpha}$ defined only for matrices $p \in K_0$ ($+\infty$ otherwise), where $K_0 = \{\tau \otimes g \in \mathbb{R}^{d\times(N-1)},\ g_i \in \{0,1\},\ |\tau| = c \geq 0\}$. Then, following [6], we look for a "local" convex envelope of this energy, i.e. we look for Φ_α^{**}, the positively 1-homogeneous convex envelope on $\mathbb{R}^{d\times(N-1)}$ of Φ_α, and use it to define a relaxation of our initial energy. Setting $q = (q_1, \ldots, q_{N-1})$, with $q_i \in \mathbb{R}^d$ its columns, it turns out that Φ_α^{**} is the support function of the convex set

$$K^\alpha = \left\{ q \in \mathbb{R}^{d\times(N-1)},\ \left| \sum_{i\in J} q_i \right| \leq |J|^\alpha\ \forall J \subset \{1, \ldots, N-1\} \right\},$$

and thus we are led to consider the relaxed functional

$$\mathscr{R}^\alpha(\Lambda) = \int_{\mathbb{R}^d} \Phi_\alpha^{**}(\Lambda) = \sup\left\{ \sum_{i=1}^{N-1} \int_{\mathbb{R}^d} \varphi_i\, d\Lambda_i,\ \varphi \in C_c^\infty(\mathbb{R}^d; K^\alpha) \right\},$$

where $\varphi = (\varphi_1, \ldots, \varphi_{N-1})$ is a matrix valued function. Observe that for Λ a rank one tensor valued measure the above expression coincides with the one obtained in the previous section choosing $\Psi^0 = \Phi_{00}^{**}$. In the planar case $d = 2$ we can now fix a measure Γ as for instance in (2.1) and define our relaxed energy on functions $U \in BV(\mathbb{R}^2; \mathbb{R}^{N-1})$ as $\mathscr{E}^\alpha(U) = \mathscr{R}^\alpha(DU^\perp - \Gamma)$ or equivalently

$$\mathscr{E}^\alpha(U) = \sup\left\{ \int_{\mathbb{R}^2} \sum_{i=1}^{N-1} (Du_i^\perp - \Gamma_i)\cdot\varphi_i,\ \varphi \in C_c^\infty(\mathbb{R}^2; K^\alpha) \right\}.$$

5. Numerical identification of optimal structures

The previous Γ-convergence result and the proposed relaxation $\mathscr{E}^\alpha$ can now be extensively used from a numerical point of view to identify optimal Steiner trees and α-irrigation networks.

The approximation of the functionals $\mathscr{F}_\varepsilon^\alpha$ is carried out through the use of ad-hoc finite element spaces which are designed to take into account the presence of the drift terms Γ_i (see [2] for details). In the experiments of figure 1 we approximate the optimal Steiner trees associated to the vertices of a triangle, a regular

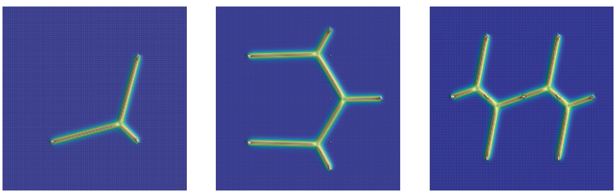

Figure 1. Local minimizers obtained by the Γ-convergence approach for 3, 5 and 7 points.

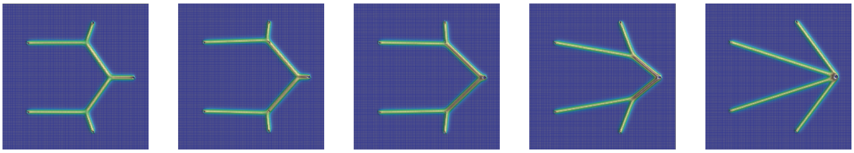

Figure 2. Gilbert–Steiner solutions for parameters $\alpha = 0.2, 0.4, 0.6, 0.8$ and 1 (left to right).

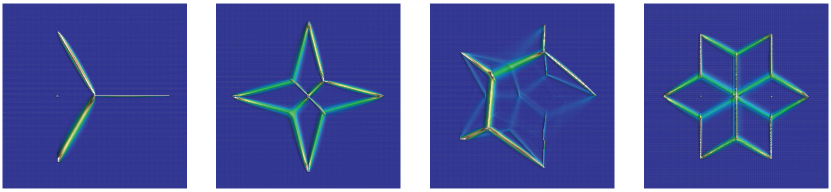

Figure 3. Results obtained by convex relaxation for 3, 4, 5 and 7 given points.

pentagon and a regular hexagon with its center. Observe that in the first two cases we are able to recover a global minimizer while the result obtained for the hexagon and its center is only a local minimizer. In figure 2 we focus on simple irrigation problems and we recover the solutions of Gilbert–Steiner problems for different values of α (for small α we can see that the irrigation network is close to an optimal Steiner tree).

Eventually the numerical optimization of the relaxed energy $\mathscr{E}^\alpha$ can be addressed within the same framework used for the approximation of $\mathscr{F}_\varepsilon^\alpha$, and in figure 3 we can see the results in four test cases. The convex formulation is able to approximate the (unique) optimal structure in the case of the triangle, while in the other three examples, where the solution is not unique, the result of the optimization is expected to be a convex combination of all solutions whenever the relaxation is sharp, as it can be observed on the second and fourth case of figure 3. However we do not expect this behaviour to hold for any configuration of points: for example in the third picture of figure 3 the numerical solution is not supported on a convex combination of global solutions since the density in the middle point is not 0.

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