Rend. Lincei Mat. Appl. 29 (2018), 635–659
DOI 10.4171/RLM/826

Partial Differential Equations — *A new approach to decay estimates – Application to a nonlinear and degenerate parabolic PDE*, by Maria Michaela Porzio, communicated on May 11, 2018.

Abstract. — In this paper we describe a new method to derive different type of decay estimates for solutions of evolution equations which allow to describe the asymptotic behavior of the solutions both in presence or absence of "immediate" regularizing properties. Moreover, we give various examples of applications some of which new and dealing with a class of nonlinear problems with degenerate coercivity.

Key words: Decay estimates, asymptotic behavior, regularity of solutions, nonlinear degenerate and singular parabolic equations, smoothing effect

Mathematics Subject Classification: 35K10, 35K55, 35K65

1. Introduction

In this paper we describe a new approach, developed in [17] and [19], to the study of the regularity, asymptotic behavior and decay estimates (of different type) for solutions of evolution equations. This new method allows to prove decay estimates of the type

$$\|u(t)\|_{L^r(\Omega)} \leq C \frac{\|u_0\|_{L^{r_0}(\Omega)}^{h_0}}{t^{h_1}} \quad \text{for every } t > 0.$$

where $1 \leq r \leq +\infty$, h_0 and h_1 are positive exponents and $u_0 \in L^{r_0}(\Omega)$ $(r_0 \geq 1)$ is the initial datum of the evolution problem satisfied by u.

We recall that if $r = +\infty$ the previous estimates are often called in literature as "ultracontractive estimates", while if $r_0 < r < +\infty$ they are referred as "supercontractive estimates".

Here both the cases of "immediate regularization phenomena" (supercontractive and ultracontractive estimates) $r_0 < r \leq +\infty$ are considered together with the case of decay without any improvement of regularity $1 \leq r \leq r_0$. This new approach will allow to derive all these decay estimates (of any type) simply by suitable integral estimates.

We will describe this method together with some possible applications in the following Section 2.

We point out that further examples and developments of the theory can be found in [15]–[23].

Then, in Section 3 we state new results concerning the asymptotic behavior of a class of nonlinear problems with degenerate coercivity whose prototype is the following equation

$$
(1.1)\qquad \begin{cases} u_t = \operatorname{div}\left(\frac{\nabla u}{(1+|u|)^\gamma}\right) & \text{in } \Omega \times (0, +\infty), \\ u = 0 & \text{on } \partial\Omega \times (0, +\infty) \\ u(x, 0) = u_0(x) & \text{on } \Omega, \end{cases}
$$

where $\gamma > 0$ and $u_0 \in L^{r_0}(\Omega)$, with $r_0 \geq 1$.

We will also investigate the influence in (1.1) of lower order terms of the type $\alpha_0|u|^{s-1}u$ $(s \geq 1)$ on the regularity and asymptotic behavior of these solutions proving, for example, that it is sufficient any power $s > 1$ to provoke the immediate boundedness of the solutions whatever is the summability of the initial datum and the value γ of the degeneracy of the equation.

Finally, in Section 4 we give the proofs of these results using the approach presented in Section 2.

2. A new approach to decay estimates

Let us consider the classical example of the heat equation

$$
\begin{cases} u_t = \triangle u & \text{in } \Omega \times (0, +\infty), \\ u = 0 & \text{on } \partial\Omega \times (0, \infty), \\ u(x, 0) = u_0(x) & \text{on } \Omega, \end{cases}
$$

where Ω is an open set of $\mathbb{R}^N$. Although the initial datum u_0 is only in $L^1(\Omega)$ the following regularity result (decay estimate) holds true

$$
\|u(t)\|_{L^\infty(\Omega)} \leq C\frac{\|u_0\|_{L^1(\Omega)}}{t^{\frac{N}{2}}} \quad \text{for every } t > 0.
$$

Moreover, if u_0 is more regular, for example if it belongs to $L^{r_0}(\Omega)$, $r_0 > 1$, then it results

$$
(2.1)\qquad \|u(t)\|_{L^\infty(\Omega)} \leq C\frac{\|u_0\|_{L^{r_0}(\Omega)}}{t^{\frac{N}{2r_0}}} \quad \text{for every } t > 0.
$$

In other words, it is sufficient to have summable initial data u_0 to obtain L^∞-estimates, but if the initial datum is more summable and belongs to $L^{r_0}(\Omega)$, then the exponent r_0 influences the L^∞-estimate satisfied by the solution.

Notice that estimate (2.1) remains true also doing a slight change in the equation. In detail, if we consider the following problem

$$
(2.2)\qquad \begin{cases} u_t = \operatorname{div}(A(x,t)\nabla u), & \text{in } \Omega \times (0, +\infty), \\ u = 0 & \text{on } \partial\Omega \times (0, +\infty) \\ u(x, 0) = u_0(x) & \text{on } \Omega, \end{cases}
$$

where A is a bounded matrix satisfying

$$(A\xi, \xi) \geq \alpha|\xi|^2 \quad \text{for every } \xi \in \mathbb{R}^N,\ \alpha > 0, \tag{2.3}$$

then there exists a solution u of (2.2) satisfying (2.1) (see [26], [9] and [17]).

Hence all these linear equations exhibit the following very strong "Regularizing effect":

$$u_0 \in L^1(\Omega) \quad \Rightarrow \quad u(t) \in L^\infty(\Omega) \quad \text{for every } t > 0$$

and L^∞ estimates of the type

$$\|u(t)\|_{L^\infty(\Omega)} \leq C \frac{\|u_0\|_{L^{r_0}(\Omega)}^{h_0}}{t^{h_1}} \quad \text{for every } t > 0$$

hold true with h_0 and h_1 positive exponents. As recalled in the introduction, estimates of the previous type are known in literature as "decay estimates" or "ultracontractive estimates".

The interest in studying these kind of estimates consists of the consequences and the applications that follow by them.

As a matter of fact, these estimates are the starting point of the "improvement process" of regularity: if you are interested in continuity (or more regularity of the solutions) the first step is to prove the boundedness of the solutions. Moreover, they allow to describe the "behavior of the solutions for t large" (i.e. how the solution decays when t tends to $+\infty$) together with the "behavior of the solutions for t small" (i.e. what happens to the solution for t that tends to zero). Indeed, the improvement of regularity is often strongly related also to the uniqueness of solutions. As a matter of fact, in many cases these solutions that regularize are the only regular solutions (see [8], [18], [20] and [22]).

Decay estimates have been proved not only in the linear framework but also for numerous nonlinear problems (or even doublynonlinear) which can be also degenerate or singular. A famous example is the porous medium equation

$$\begin{cases} u_t = \triangle(|u|^{m-1}u), & \text{in } \Omega_T \equiv \Omega \times (0, +\infty), \\ u = 0 & \text{on } \partial\Omega \times (0, +\infty) \\ u(x, 0) = u_0(x) & \text{on } \Omega, \end{cases}$$

where $m > 1$. It is known that there exists a solution of this degenerate equation satisfying

$$\|u(t)\|_{L^\infty(\Omega)} \leq c \frac{\|u_0\|_{L^{r_0}(\Omega)}^{\frac{2r_0}{N(m-1)+2r_0}}}{t^{\frac{N}{N(m-1)+2r_0}}} \quad \text{for every } t > 0. \tag{2.4}$$

(see [26], [4], [1], [5] and [25]).

It is worth to notice that if $m \to 1^+$ (formally) the porous medium equation becomes the heat equation and the decay estimate (2.4) becomes the decay estimate (2.1) satisfied by the solution of the heat equation.

Moreover, if Ω is a bounded set a faster decay (at infinity) appears and the following bound holds

$$\|u(t)\|_{L^\infty(\Omega)} \leq \frac{C}{t^{\frac{1}{m-1}}} \quad \text{for every } t > 0, \tag{2.5}$$

where the constant C does not depend of the initial datum u_0: for this reason these kind of inequalities are also called universal estimates (see [26]). Indeed, it is sufficient to assume that Ω has finite measure to get universal estimates (see [7] if Ω is a connected domain and [17] in the remaining case). Notice that in absence of this condition on Ω these estimates fail (see [25]).

We remark that if Ω has finite measure also the solution u of the heat equation has a faster decay at infinity but of a "different type" since the following exponential decay occurs

$$\|u(t)\|_{L^\infty(\Omega)} \leq C \frac{\|u_0\|_{L^{r_0}(\Omega)}}{t^{\frac{N}{2r_0}} e^{\sigma t}} \quad \text{for every } t > 0. \tag{2.6}$$

where σ depends on the measure of Ω (see [7] and [17]).

Another interesting case for which these estimates hold is the fast diffusion equation

$$\begin{cases} u_t = \triangle(|u|^{m-1}u), & \text{in } \Omega \times (0, +\infty), \\ u = 0 & \text{on } \partial\Omega \times (0, +\infty) \\ u(x, 0) = u_0(x) & \text{on } \Omega, \end{cases}$$

formally the same equation of the porous medium equation but now $0 < m < 1$ and hence the equation is singular. In this case, the behavior of the solutions changes: if Ω has finite measure it is not true that uniform estimates hold and restrictions on the coefficient m (in dependence of the summability r_0 of the initial datum u_0) are needed in order to have L^∞-regularization (see [25]). In detail, if

$$\frac{N - 2r_0}{N} < m < 1. \tag{2.7}$$

then there is a solution that becomes immediately bounded and satisfies the decay estimate (2.4) (see [3], [26], [13], [25]). Notice that once again this decay estimate becomes the decay estimate of the heat equation letting $m \to 1^-$.

Another nonlinear parabolic equation exhibiting regularizing phenomena is the p-Laplacian equation

$$\begin{cases} u_t - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0 & \text{in } \Omega \times (0, +\infty), \\ u = 0 & \text{on } \partial\Omega \times (0, +\infty) \\ u(x, 0) = u_0(x) & \text{on } \Omega. \end{cases} \tag{2.8}$$

In the degenerate case $p > 2$ the behavior is similar to the porous medium equation: we have the decay estimate

$$\|u(t)\|_{L^\infty(\Omega)} \le c_4 \frac{\|u_0\|_{L^{r_0}(\Omega)}^{\frac{pr_0}{N(p-2)+pr_0}}}{t^{\frac{N}{N(p-2)+pr_0}}} \quad \text{for every } t > 0, \tag{2.9}$$

that becomes the decay estimate (2.1) of the heat equation letting $p \to 2^+$ (i.e. when formally the p-Laplacian equation becomes the heat equation), while if Ω has finite measure the following universal bound holds

$$\|u(t)\|_{L^\infty(\Omega)} \le \frac{C}{t^{\frac{1}{p-2}}} \quad \text{for every } t > 0, \tag{2.10}$$

(see [26], [14], [10], [7] and [25]). In the singular case $1 < p < 2$, the behavior of the solutions of (2.8) is similar at all to the case of the fast diffusion: no universal bounds are satisfied and a restriction on the diffusion exponent p (once again depending on r_0) is needed to have a L^∞-decay estimate which is the same estimate of the degenerate case (see [26], [14], [11], and [25]).

The previous nonlinear problems are only possible examples of PDE problems exhibiting these L^∞-decay estimates and there is a wide literature on the subject.

The proofs of these decay estimates vary from one problem to the other and in general the main tool is to derive suitable families of logarithmic Sobolev inequalities which reflect the operator involved in the problem considered.

We recall a different approach developed in [17] that allows to derive L^∞-decay estimates simply by suitable integral estimates.

Let use define the following function which often appears in many regularity proofs

$$G_k(u) = (|u| - k)_+ \operatorname{sign}(u).$$

The method relies in applying one of the following results.

THEOREM 2.1 (Theorem 2.1 in [17]). *Assume that u is in $C((0,T); L^r(\Omega)) \cap L^b(0,T; L^q(\Omega)) \cap C([0,T); L^{r_0}(\Omega))$, where Ω is an open set of $\mathbb{R}^N$ (not necessary bounded) $N \ge 1$, $0 < T \le +\infty$ and*

$$1 \le r_0 < r < q \le +\infty, \quad b_0 < b < q, \quad b_0 = \frac{(r - r_0)}{1 - \frac{r_0}{q}}, \tag{2.11}$$

where here (and in the rest of the paper) we consider $\frac{1}{+\infty} = 0$. Assume that u satisfies the following integral estimates for every $k > 0$

$$\int_\Omega |G_k(u)|^r(t_2)\,dx - \int_\Omega |G_k(u)|^r(t_1)\,dx + c_1 \int_{t_1}^{t_2} \|G_k(u)(\tau)\|_{L^q(\Omega)}^b\,d\tau \le 0 \quad \textit{for every } 0 < t_1 < t_2 < T, \tag{2.12}$$

$$\|G_k(u)(t)\|_{L^{r_0}(\Omega)} \le c_2 \|G_k(u)(t_0)\|_{L^{r_0}(\Omega)} \quad \textit{for every } 0 \le t_0 < t < T, \tag{2.13}$$

where c_1 and c_2 are positive constants independent of k and

$$u_0 \equiv u(x,0) \in L^{r_0}(\Omega). \tag{2.14}$$

Then there exists a positive constant C_1 depending only on N, c_1, c_2, r, r_0, q and b such that

$$\|u(t)\|_{L^\infty(\Omega)} \leq C_1 \frac{\|u_0\|^{h_0}_{L^{r_0}(\Omega)}}{t^{h_1}} \quad \text{for every } t \in (0,T), \tag{2.15}$$

where

$$h_1 = \frac{1}{b-(r-r_0)-\frac{r_0 b}{q}}, \quad h_0 = h_1\Big(1-\frac{b}{q}\Big)r_0. \tag{2.16}$$

Moreover if Ω has finite measure we have an exponential decay if $b = r$ and universal bounds if $b > r$. More in details we have the following result.

Theorem 2.2 (Theorem 2.2 in [17]). *Let the assumptions of Theorem* 2.1 *hold true. If Ω has finite measure and $b = r$ the following exponential decay occurs*

$$\|u(t)\|_{L^\infty(\Omega)} \leq C_2 \frac{\|u_0\|_{L^{r_0}(\Omega)}}{t^{h_1}e^{\sigma t}} \quad \text{for every } t \in (0,T), \tag{2.17}$$

where C_2 is a positive constant depending only on N, c_1, c_2, r, r_0, b and q, h_1 is as in (2.16) *and*

$$\sigma = \frac{c_1\kappa}{4(r-r_0)|\Omega|^{1-\frac{b}{q}}}, \quad \kappa \text{ arbitrarily fixed in } \Big(0, 1-\frac{r_0}{r}\Big). \tag{2.18}$$

If otherwise Ω has finite measure and $b > r$ we have the following universal bound

$$\|u(t)\|_{L^\infty(\Omega)} \leq \frac{C_\#}{t^{h_2}} \quad \text{for every } t \in (0,T), \tag{2.19}$$

where

$$h_2 = h_1 + \frac{h_0}{b-r} = \frac{1}{b-r}, \tag{2.20}$$

and $C_\#$ is a constant depending only on r, r_0, q, b, c_1, c_2 and the measure of Ω.

We point out that the previous Theorems are "abstract results" where it is not assumed that u solves any partial differential equation. Moreover, it is possible to estimate the constants C_1, C_2 and $C_\#$ above (see [17]) and some further generalizations of these "abstract results" are allowed (see again [17]).

We show now how to apply the previous results by means of the following easy example. Consider the linear problem (2.2) above with $u_0 \in L^1(\Omega)$, i.e.

$$\begin{cases} u_t = \operatorname{div}(A(x,t)\nabla u) & \text{in } \Omega \times (0,+\infty), \\ u = 0 & \text{on } \partial\Omega \times (0,+\infty), \\ u(x,0) = u_0(x) & \text{on } \Omega, \end{cases}$$

where A is a bounded matrix satisfying (2.3) and Ω is an open set of $\mathbb{R}^N$, $N > 2$ (not necessary bounded or with finite measure). Notice that the case considered here is not included in the papers of [26] and [9]. To avoid technicality we do a "formal calculus" (but the rigorous one is very easy and can be found in [17]). Taking $G_k(u)$ as test function we deduce for every $0 < t_1 < t_2 < +\infty$

$$\frac{1}{2}\int_\Omega |G_k(u)|^2(t_2) - \frac{1}{2}\int_\Omega |G_k(u)|^2(t_1) + \int_{t_1}^{t_2}\int_\Omega A(x,t)\nabla u \nabla G_k(u) \le 0.$$

Using the coercivity condition (2.3) and the Sobolev inequality[1] (with $p = 2$) we obtain

$$\int_{t_1}^{t_2}\int_\Omega A(x,t)\nabla u\nabla G_k(u) = \int_{t_1}^{t_2}\int_\Omega A(x,t)\nabla G_k(u)\nabla G_k(u)$$
$$\ge \alpha\int_{t_1}^{t_2}\int_\Omega |\nabla G_k(u)|^2 \ge \alpha c_S \int_{t_1}^{t_2} \|G_k(u)\|^2_{L^{2^*}(\Omega)},$$

where c_S is the Sobolev immersion constant defined in (2.21). Putting together the previous estimates we deduce that for every $0 < t_1 < t_2 < +\infty$

$$\text{(2.22)} \qquad \int_\Omega |G_k(u)|^2(t_2) - \int_\Omega |G_k(u)|^2(t_1) + 2\alpha c_S \int_{t_1}^{t_2} \|G_k(u)\|^2_{L^{\frac{2N}{N-2}}(\Omega)} \le 0,$$

i.e. the integral estimate (2.12) holds true with $r = b = 2$, $c_1 = 2\alpha c_S$, $q = \frac{2N}{N-2}$, and $b_0 = \frac{2N}{N+2}$. Notice that now $r_0 = 1$ and this choice of exponents satisfies the algebraic conditions in (2.11). To show that also (2.13) holds true it is again sufficient to choose a suitable test function. In detail, take $\varphi = \left\{1 - \frac{1}{[1+|G_k(u)|]^\delta}\right\}\operatorname{sign}(u)$ $(\delta > 1)$ as test function in $\Omega \times (0,t)$, where $t > 0$ is arbitrarily fixed. Notice that φ is nonzero only on the set of finite measure $A_k \equiv \{(x,\tau) \in \Omega \times (0,t) : |u(x,\tau)| > k\}$. We deduce for every $0 \le t_0 \le t < +\infty$

[1] The Sobolev inequality: there exists a constant c_s depending only on N and p $(N > p)$ such that

$$\text{(2.21)} \qquad c_s\Big(\int_\Omega |v|^{p^*}\,dx\Big)^{\frac{p}{p^*}} \le \int_\Omega |\nabla v|^p\,dx \quad \text{for every } v \in W_0^{1,p}(\Omega), \quad p^* = \frac{pN}{N-p}.$$

$$\int_\Omega |G_k(u)|(t) + \frac{1}{\delta-1}\int_{A_k^{t_0}} \left\{1 - \frac{1}{[1+|G_k(u)|(t_0)]^{\delta-1}}\right\}$$
$$+\delta\int_{t_0}^t\int_\Omega a(x,t,u,\nabla u)\frac{\nabla G_k(u)}{[1+|G_k(u)|(t)]^{\delta+1}}$$
$$\leq \int_\Omega |G_k(u)|(t_0) + \frac{1}{\delta-1}\int_{A_k^{t}} \left\{1 - \frac{1}{[1+|G_k(u)|(t)]^{\delta-1}}\right\}. \tag{2.23}$$

where $A_k^\tau \equiv \{x \in \Omega : |u(x,\tau)| > k\}$. Notice that also A_k^τ has finite measure. Using again (2.3) (and that $\delta > 1$) from (2.23) we obtain

$$\int_\Omega |G_k(u)|(t) \leq \int_\Omega |G_k(u)|(t_0) + \frac{1}{\delta-1}\int_{A_k^{t}} \left\{1 - \frac{1}{[1+|G_k(u)|(t)]^{\delta-1}}\right\}$$
$$\leq \int_\Omega |G_k(u)|(t_0) + \frac{|A_k^t|}{\delta-1},$$

which implies that (2.13) holds true for every $0 \leq t_0 < t < +\infty$ thanks to the arbitrary choice of $\delta > 1$. Now applying Theorem 2.1 we obtain the decay estimate

$$\|u(t)\|_{L^\infty(\Omega)} \leq C_0\frac{\|u_0\|_{L^1(\Omega)}}{t^{\frac{N}{2}}} \quad \text{for every } t > 0, \tag{2.24}$$

with $C_0 = C_0(\alpha, c_S, N)$[2] which is, once again, the same decay estimate of the heat equation.

Moreover, if $|\Omega| < +\infty$, we can apply also Theorem 2.2 concluding that also the exponential decay estimate (2.6) is satisfied. Hence the behavior of the solution of the linear equation above is at all similar to that of the heat equation and the exponential estimates are not a peculiarity of the heat equation.

The previous method works also with the classical equations (porous medium etc.) recalled above and since the algebraic conditions become the sharp known bounds to have the L^∞ decay we think that these conditions on the exponents could be sharp. Indeed, also the formulas of the exponents give (in the known cases) the sharp exponents.

An obvious consequence of this new approach is the following result.

COROLLARY 2.1. *If the solutions of different evolution problems satisfy the same integral inequalities* (2.12) *and* (2.13) *then they satisfy the same L^∞-regularizing property and L^∞-decay estimates.*

[2] Here and everywhere in the paper we denote by $C(\cdot,\cdot,\cdot)$ a positive constant that depends only on the variables in brackets.

Hence, we can finally explain the reason why surprisingly the solutions of different problems like, for example, the heat equation, the linear problem (2.2) and the nonlinear Leray–Lions operator with growth 2 or the p-Laplacian and the Leray–Lions operators with growth p, satisfy exactly the same L^∞-decay estimates. The motivation is very easy, but not clear at all with different approaches, and relies (as stated in Corollary 2.1) in the fact that all these solutions that satisfy the same decay estimates verify the same integral estimates too and hence, by the results presented above, they have exactly the same regularizing and decay properties.

This new approach allows to prove a lot of new decay results in a very simple way: you just need to prove the integral estimates "of energy type" (2.12) and (2.13). Easy applications to evolution equations like for example anisotropic problems can be found in [17].

As just recalled above, in many "classical" evolution problems when this strong L^∞ regularizing property appears these "regular solutions" are unique, as for example, in the case of the porous medium equation with summable initial data (see [8]). Hence, now that you have an easy method to prove this improvement of regularity, you can also check if also other problems for which this improvement of regularity appears exhibit this uniqueness property too. We point out that this approach to uniqueness works also for other parabolic PDE like, for example, Leray–Lions problems (see [18]).

We have described above some interesting properties of the solutions that "immediately" regularize into $L^\infty(\Omega)$ and an easy method to recognize these solutions but there are many evolution problems for which this strong improvement of regularity do not appear. Hence it would be interesting to study also what is the behavior in these other cases.

Indeed, different behavior are allowed and the previous approach, that makes use only of integral inequalities, can be extended to study also this different framework. In particular, there are cases for which *"no smoothing effect"* appears:

$$u(t) \in L^1(\Omega) \quad \text{for every } t > 0, \text{ (with } u(t) \notin L^r(\Omega) \text{ for every } r > 1)$$

like for example when u is a solution of the singular p-Laplacian or of the fast diffusion equations for suitable choices of the exponents p and m. There are PDE problems exhibiting a *"very strong regularizing effect"*:

$$u(t) \in L^r(\Omega) \quad \text{for every } r \in (1, +\infty] \text{ and for every } t > 0$$

like the case of the solution of the heat equation or of the linear problem (2.2) discussed above. It is also possible that a "weak regularizing effect" appears:

$$u(t) \in L^r(\Omega) \quad \text{for every } r \in (r_0, +\infty) \text{ and } u(t) \notin L^\infty(\Omega)$$

as in the case of the solution u of the "porous media equation with two weights"

$$(2.25)\qquad \begin{cases} \rho_\nu u_t - \operatorname{div}(\rho_\mu \nabla(|u|^{m-1}u)) = 0 & \text{in } \Omega \times (0,\infty), \\ u = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x) & \text{on } \Omega, \end{cases}$$

where $m > 1$ and the weights $\nu = \nu(x)$ and $\mu = \mu(x)$ are suitable functions defined on Ω (see [12]). Another possibility is a "very weak regularizing effect":

$$u(t) \in L^r(\Omega) \quad r \in (r_0, r_1) \quad r_1 \neq +\infty$$

but

$$u(t) \notin L^r(\Omega) \quad \text{for every } r > r_1.$$

An example of problem for which this last phenomenon holds true is the heat equation with a singular potential term

$$(2.26)\qquad \begin{cases} u_t - \Delta u = \lambda \frac{u}{|x|^2} & \text{in } \Omega \times (0,\infty), \\ u = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x) & \text{on } \Omega, \end{cases}$$

where Ω is a bounded domain of $\mathbb{R}^N$ containing the origin and $\lambda > 0$ (see [19] and [23]).

It is also possible to prove that in all the different cases presented above of "strong", "weak" or "very weak" regularization phenomena a decay of the $L^r(\Omega)$-norms of the solutions (with r as above suitable chosen satisfying $r > r_0$) appears.

Indeed, it can also appear a decay of the $L^r(\Omega)$-norm of solutions that do not satisfy any regularization like for example the p-Laplacian equation with p suitable small; in these cases the values of r satisfy $r < r_0$ (see [18] and [19]).

All these results can be proved extending the previous approach to the L^∞-regularization phenomena. In other words, again if a solution satisfies suitable integral estimates (similar but weaker than (2.11) and (2.12) in Theorem 2.1) then it belongs to $L^r(\Omega)$ and decays in the $L^r(\Omega)$-norm, where the allowed values of r depend on the integral estimates that these solutions verify. This new approach, presented in [19] (together with some possible applications of this new method), allows to determine which kind of "regularization" appears together with the decay bounds satisfied by the solutions. Further developments and applications can be found in [15], [16], [18], [20]–[23] and in Section 3 of this paper.

For the convenience of the reader, we conclude this section stating two of the "abstract results" in [19] that we will use in Section 4 to prove new decay results on some nonlinear degenerate parabolic equations, that, as said before, we state with all the details in the following section.

THEOREM 2.3 (Theorem 2.1 in [19]). *Let u_0 be in $L^{r_0}(\Omega)$ and u in $C((0,T); L^r(\Omega)) \cap L^b(0,T; L^q(\Omega)) \cap C([0,T); L^{r_0}(\Omega))$, where Ω is an open set of $\mathbb{R}^N$ (not necessary bounded), $N \geq 1$ and $0 < T \leq +\infty$. Assume that for every*

$0 < t_1 < t_2 < T$

$$\int_\Omega |u|^r(t_2) - \int_\Omega |u|^r(t_1) + c_1 \int_{t_1}^{t_2} \|u(t)\|_{L^q(\Omega)}^b \, dt \le 0, \tag{2.27}$$

$$1 \le \min\{r_0, q\} < r < \max\{r_0, q\} \le +\infty, \quad b > b_0, \quad b_0 = \frac{r - r_0}{1 - \frac{r_0}{q}}, \tag{2.28}$$

and

$$\|u(t)\|_{L^{r_0}(\Omega)} \le c_2 \|u_0\|_{L^{r_0}(\Omega)} \quad \textit{a.e. } 0 < t < T. \tag{2.29}$$

Then the following estimate holds true

$$\|u(t)\|_{L^r(\Omega)} \le c_3 \frac{\|u_0\|_{L^{r_0}(\Omega)}^{\gamma_0}}{t^{\gamma_1}} \quad \textit{for every } t \in (0, T),$$

where

$$\gamma_1 = \frac{\frac{r}{r_0} - 1}{r\left[\frac{b}{r_0} - \left(\frac{r}{r_0} - 1\right) - \frac{b}{q}\right]} \quad \gamma_0 = \frac{b\left(1 - \frac{r}{q}\right)}{r\left[\frac{b}{r_0} - \left(\frac{r}{r_0} - 1\right) - \frac{b}{q}\right]}$$

and

$$c_3 = \left(\frac{r\gamma_1}{c_1}\right)^{\gamma_1} c_2^{\gamma_0}.$$

THEOREM 2.4 (Theorem 2.8 in [19]). *Let u be in $C((0,T); L^r(\Omega)) \cap L^\infty(0,T; L^{r_0}(\Omega))$ where $0 < r \le r_0 < \infty$. Suppose also that $|\Omega| < +\infty$ if $r \ne r_0$ (no assumption are needed on $|\Omega|$ if $r = r_0$). If u satisfies*

$$\int_\Omega |u|^r(t_2) - \int_\Omega |u|^r(t_1) + c_1 \int_{t_1}^{t_2} \|u(t)\|_{L^r(\Omega)}^r \, dt \le 0 \quad \textit{for every } 0 < t_1 < t_2 < T, \tag{2.30}$$

and there exists $u_0 \in L^{r_0}(\Omega)$ such that

$$\|u(t)\|_{L^{r_0}(\Omega)} \le c_2 \|u_0\|_{L^{r_0}(\Omega)} \quad \textit{for almost every } t \in (0, T), \tag{2.31}$$

where c_i, $i = 1, 2$ are real positive numbers, then the following estimate holds true

$$\|u(t)\|_{L^r(\Omega)} \le c_4 \frac{\|u_0\|_{L^{r_0}(\Omega)}}{e^{\sigma t}} \quad \textit{for every } 0 < t < T, \tag{2.32}$$

where

$$c_4 = \begin{cases} c_2 |\Omega|^{\frac{1}{r} - \frac{1}{r_0}} & \textit{if } r < r_0, \\ 1 & \textit{if } r = r_0, \end{cases} \qquad \sigma = \frac{c_1}{r}.$$

3. An application to nonlinear problems with degenerate coercivity

We show here some new applications of the method presented above. In detail, we consider a class of nonlinear, degenerate and non coercive parabolic equations

$$
(3.1)\qquad \begin{cases} u_t = \operatorname{div}(A(x,t,u)\nabla u) & \text{in } \Omega_T \equiv \Omega \times (0,T), \\ u = 0 & \text{on } \partial\Omega \times (0,T) \\ u(x,0) = u_0(x) & \text{on } \Omega, \end{cases}
$$

where Ω is a bounded domain of $\mathbb{R}^N$ $(N \geq 2)$ with a smooth boundary, $0 < T < +\infty$ and $A(x,t,s) : \Omega \times (0,T) \times \mathbb{R} \to \mathbb{R}^{N^2}$ is a bounded symmetric matrix function, continuous with respect to (x,s) and measurable with respect to t (for every $(x,s) \in \Omega \times \mathbb{R}$) such that for any $\xi \in \mathbb{R}^N$, $s \in \mathbb{R}$ and a.e. $(x,t) \in \Omega_T$

$$
(3.2)\qquad \frac{\alpha|\xi|^2}{(1+|s|)^\gamma} \leq \langle A(x,t,s)\xi, \xi\rangle \leq \frac{\beta|\xi|^2}{(1+|s|)^\gamma}
$$

where

$$
\alpha > 0 \quad \beta > 0 \quad \text{and} \quad \gamma > 0.
$$

The model case we have in mind is the following

$$
(3.3)\qquad \begin{cases} u_t = \operatorname{div}\left(\frac{\alpha\nabla u}{(1+|u|)^\gamma}\right) & \text{in } \Omega_T, \\ u = 0 & \text{on } \partial\Omega \times (0,T) \\ u(x,0) = u_0(x) & \text{on } \Omega, \end{cases}
$$

We observe that these parabolic problems degenerate as soon as the solutions are unbounded. Moreover, if the initial datum u_0 belongs to $L^{r_0}(\Omega)$, $r_0 \geq 1$, we cannot hope (without further assumptions) that ∇u always exists, even in a weak sense. Hence, to overcome this difficulty and to define a solution of this problem in [24] it was considered the gradient of an auxiliary function $G(u)$ defined as

$$
(3.4)\qquad G(s) \equiv \int_0^s \frac{1}{(1+|z|)^\gamma}\,dz \quad z \in \mathbb{R}
$$

and the following notion of solution is considered.

Definition 3.1. A measurable function u is a weak solution of (3.1) if $u \in L^\infty(0,T;L^1(\Omega))$, $G(u) \in L^1(0,T;W_0^1(\Omega))$, $A(x,t,u)(1+|u|)^\gamma \nabla G(u) \in (L^1(\Omega))^N$ and if it results

$$
(3.5)\qquad \int_0^T \int_\Omega \{u\varphi_t + \langle A(x,t,u)(1+|u|)^\gamma \nabla G(u), \nabla\varphi\rangle\}\,dx\,dt = \int_\Omega u_0\varphi(x,0)\,dx
$$

for every $\varphi \in W^{1,1}(0,T;L^\infty(\Omega)) \cap L^\infty(0,T;W_0^{1,\infty}(\Omega))$, with compact support in $[0,T) \times \overline{\Omega}$.

As noticed in Remark 2.2 in [24], if $\nabla u \in (L^1_{loc}(\Omega))^N$ then $A(x,t,u)(1+|u|)^\gamma \nabla G(u) = A(x,t,u)\nabla u$ and (3.5) becomes

$$\int_0^T \int_\Omega \{u\varphi_t + \langle A(x,t,u)\nabla u, \nabla\varphi\rangle\}\,dx\,dt = \int_\Omega u_0\varphi(x,0)\,dx$$

We recall that the previous assumptions guarantee the existence of a weak solution for every choice of $\gamma > 0$ and u_0 summable initial datum. As a matter of fact, it results

THEOREM 3.1 (Theorem 2.5 in [24]). *Let* (3.2) *hold true and* $u_0 \in L^{r_0}(\Omega)$ *with* $r_0 \geq 1$. *Then there exists a weak solution u of* (3.1) *in* $L^\infty(0,T;L^{r_0}(\Omega))$. *Moreover, it results*

$$G(u) \in L^q(0,T;W_0^{1,q}(\Omega)) \begin{cases} q = 2 & \text{if } r_0 > 2-\gamma, \\ q = \frac{2r_0}{2-\gamma} & \text{if } 1 < r_0 \leq 2-\gamma, \\ q \in \left[1, \frac{2}{2-\gamma}\right) & \text{if } 1 = r_0 \leq 2-\gamma. \end{cases} \tag{3.6}$$

Indeed, if (3.2) is satisfied for every $\xi \in \mathbb{R}^N$, $s \in \mathbb{R}$ and a.e. $(x,t) \in \Omega \times (0,\infty)$ (in such a case we say "shortly" that (3.2) is satisfied in Ω_∞) it is also possible to prove the existence of global weak solutions of

$$\begin{cases} u_t = \operatorname{div}(A(x,t,u)\nabla u) & \text{in } \Omega_T \equiv \Omega \times (0,+\infty), \\ u = 0 & \text{on } \partial\Omega \times (0,\infty) \\ u(x,0) = u_0(x) & \text{on } \Omega, \end{cases} \tag{3.7}$$

where by a global weak solution of (3.7) (or equivalently by a global weak solution of (3.1)) we mean a function u which is a weak solution of (3.1) for every $T > 0$. As a matter of fact, the following result holds.

THEOREM 3.2. *Assume that* (3.2) *is satisfied in* Ω_∞. *Let* u_0 *be in* $L^{r_0}(\Omega)$ *with* $r_0 \geq 1$. *Then there exists a global weak solution u of* (3.1) *in* $L^\infty_{loc}([0,+\infty);L^{r_0}(\Omega))$ *satisfying*

$$G(u) \in L^q_{loc}([0,+\infty);W_0^{1,q}(\Omega)) \tag{3.8}$$

where q is as in (3.6).

The proof of the previous result together with the following ones are given in Section 4.

REMARK 3.1. We point out that the previous theorem completes the existence result proved in [24] removing the restriction (imposed in Theorem 2.15 in [24]) that $r_0 > \frac{\gamma N}{2}$.

We recall that if u_0 is bounded than also u is bounded (see Theorem 2.6 in [24]) and the following estimate holds true

$$\|u\|_{L^\infty(\Omega_T)} \le \|u_0\|_{L^\infty(\Omega)}.$$

Moreover, even if the initial datum u_0 is not bounded, if r_0 and γ are suitable related, there exists a solution that becomes "immediately bounded". In details, if

$$r_0 > \frac{\gamma N}{2} \tag{3.9}$$

then $u(t) \in L^\infty(\Omega)$ and satisfies the following decay estimate

$$\|u(t)\|_{L^\infty(\Omega)} \le K_1 \frac{\|u_0\|_{L^{r_0}(\Omega)}}{t^\nu} e^{-K_2 t} \quad \text{for every } t > 0 \tag{3.10}$$

where K_1, K_2 and ν are positive constants depending on the data in the problem (see Theorem 2.15 in [24]). Notice that condition (3.9) is a sharp condition to have this "immediate boundedness" (see counterexamples in Section 6 in [24]).

To our knowledge, it is not known which is the behavior of global weak solutions when (3.9) is not satisfied. Hence we want to fill this gap here studying this lacking case.

We have the following result

THEOREM 3.3. *Assume that* (3.2) *is satisfied in* Ω_∞. *Let* u_0 *be in* $L^{r_0}(\Omega)$ *with* r_0 *verifying*

$$1 < r_0 \le \frac{\gamma N}{2}. \tag{3.11}$$

If it results

$$\gamma \le r_0, \tag{3.12}$$

then there exists a global weak solution u (*which is the same solution given by Theorem* 3.2) *which satisfies the following decay estimate*

$$\|u(t)\|_{L^r(\Omega)} \le C \frac{\|u_0\|_{L^{r_0}(\Omega)}}{t^{\frac{r_0 - r}{r\gamma}}} \quad a.e.\ t > 0, \tag{3.13}$$

for every $1 < r < r_0$, *where* C *is a positive constant independent of* u (*see formula* (4.17) *below*).

REMARK 3.2. We observe that if $\gamma \le 1$ then assumption (3.12) is ever satisfied. We do not know if it is possible to remove this assumption in Theorem 3.3. Any-way, we observe that (3.12) is ever satisfied in the regularizing case $r_0 > \frac{\gamma N}{2}$ when, as recalled above, the decay of the L^∞-norm of $u(t)$ (and hence of every L^r-norm) holds true.

Finally, in the borderline case $\gamma = 2 \le r_0$ further results and decay estimates in $(0, T)$ for weak solutions of (3.1) can be found in [6].

We show now that if we do a slight modification in problem (3.1) introducing a lower order term $\alpha_0 u$, with α_0 positive constant, then it is possible to show a decay of a global weak solution u also in $L^{r_0}(\Omega)$ and without assuming (3.12). Moreover, a faster decay in $L^r(\Omega)$ (as $t \to +\infty$) occurs for every $1 \le r \le r_0$. In detail, let us consider the following evolution problem

$$\begin{cases} u_t - \operatorname{div}(A(x,t,u)\nabla u) + \alpha_0 u = 0 & \text{in } \Omega_T \equiv \Omega \times (0, +\infty), \\ u = 0 & \text{on } \partial\Omega \times (0, \infty) \\ u(x, 0) = u_0(x) & \text{on } \Omega, \end{cases} \tag{3.14}$$

then we have the following result.

THEOREM 3.4. *Assume that* (3.2) *is satisfied in* Ω_∞ *and that* α_0 *is a positive constant. Let* u_0 *be in* $L^{r_0}(\Omega)$ *with* r_0 *verifying*

$$1 \le r_0 \le \frac{\gamma N}{2}. \tag{3.15}$$

Then there exists a global weak[3] *solution u of* (3.14) *in* $L^\infty_{loc}([0, +\infty); L^{r_0}(\Omega))$ *satisfying* (3.8) *and such that for every* $1 \le r \le r_0$ *the following decay estimate holds*

$$\|u(t)\|_{L^r(\Omega)} \le C \frac{\|u_0\|_{L^{r_0}(\Omega)}}{e^{\alpha_0 t}} \quad a.e.\ t > 0, \tag{3.17}$$

where

$$C = \begin{cases} |\Omega|^{\frac{1}{r} - \frac{1}{r_0}} & if\ r < r_0 \\ 1 & if\ r = r_0. \end{cases} \tag{3.18}$$

REMARK 3.3. Notice that the previous theorem allows to extend Theorem 3.3 to the case $r_0 = 1$ together with (as recalled above) the cases $r = r_0$.

Finally if we "increase" the power of the lower order term replacing $\alpha_0 u$ with $\alpha_0 |u|^{s-1} u$, with $s > 1$, then without any restriction on γ and $r_0 \ge 1$ (hence also when (3.9) is not satisfied) there exist global solutions that "immediately become

[3] Here a global solution of (3.14) is a weak solution in every set Ω_T, for every $T > 0$, where by a weak solution u in Ω_T we mean that u satisfies Definition 3.1 with (3.5) replaced by

$$\int_0^T \int_\Omega \{u\varphi_t + \langle A(x,t,u)(1 + |u|)^\gamma \nabla G(u), \nabla\varphi\rangle + \alpha_0 u\varphi\}\, dx\, dt = \int_\Omega u_0 \varphi(x, 0)\, dx. \tag{3.16}$$

bounded". Moreover, it is also possible to show that uniform L^∞-bounds hold true. In detail, let us consider the following evolution problem

$$(3.19)\qquad \begin{cases} u_t - \operatorname{div}(A(x,t,u)\nabla u) + \alpha_0 |u|^{s-1}u = 0 & \text{in } \Omega_T \equiv \Omega \times (0,+\infty), \\ u = 0 & \text{on } \partial\Omega \times (0,\infty) \\ u(x,0) = u_0(x) & \text{on } \Omega, \end{cases}$$

then we have the following result.

THEOREM 3.5. *Assume that* (3.2) *is satisfied in* Ω_∞. *Let* α_0 *and* s *be positive constants with* $s > 1$ *and let* u_0 *be in* $L^{r_0}(\Omega)$ *with* $r_0 \geq 1$. *Then there exists a global weak*[4] *solution* u *in* $L^\infty_{loc}([0,+\infty); L^{r_0}(\Omega)) \cap L^s_{loc}([0,+\infty); L^s(\Omega))$ *satisfying* (3.8). *Moreover,* u *belongs to* $L^\infty(\Omega \times (\varepsilon,+\infty))$ *for every* $\varepsilon > 0$ *and satisfies the following decay estimate*

$$(3.21)\qquad \|u(t)\|_{L^\infty(\Omega)} \leq \left[\frac{1}{\alpha_0(s-1)}\right]^{\frac{1}{s-1}} \frac{1}{t^{\frac{1}{s-1}}} \quad a.e.\ t > 0.$$

REMARK 4. We observe that (3.21) is an universal decay estimate, i.e. it is independent of the initial datum, that consequently, does not influence the decay of the solution.

We notice also that in many of the results above, the assumptions on A can be weakened. We have preferred to not consider the full generality to avoid further technicality.

4. PROOF OF THEOREMS 3.2–3.5

In this section we prove all the results stated in the Section 3.

4.1. Proof of Theorem 3.2

The proof proceed by steps.

Step 1. We show here that for every fixed $n \geq 1$ ($n \in \mathbf{N}$) there exists a global weak solution u_n of the following problem

$$(4.1)\qquad \begin{cases} (u_n)_t = \operatorname{div}(A_n(x,t,u_n)\nabla u_n) & \text{in } \Omega_T \equiv \Omega \times (0,+\infty), \\ u_n = 0 & \text{on } \partial\Omega \times (0,\infty) \\ u_n(x,0) = T_n(u_0(x)) & \text{on } \Omega, \end{cases}$$

[4] Here a global solution of (3.19) is a weak solution in every set Ω_T, for every $T > 0$, where by a weak solution u in Ω_T we mean that u is in $L^s(\Omega_T)$ and satisfies Definition 3.1 with (3.5) replaced by

$$(3.20)\qquad \int_0^T \int_\Omega \{u\varphi_t + \langle A(x,t,u)(1+|u|)^\gamma \nabla G(u), \nabla\varphi\rangle + \alpha_0 |u|^{s-1}u\varphi\}\,dx\,dt = \int_\Omega u_0\varphi(x,0)\,dx.$$

where $T_n(\sigma)$ is the usual truncated function

$$T_n(\sigma) = \min\{|\sigma|, n\}\, \text{sign}(\sigma). \tag{4.2}$$

and A_n is the same regularization introduced in the proof of Theorem 3.1 above (see Theorem 2.5 in [24]), i.e. we first extend $A(x, t, \sigma)$ in all $\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}$ defining $A(x, t, \sigma) \equiv \frac{\alpha I}{(1+|\sigma|)^\gamma}$ for any $(x, t) \in (\mathbb{R}^N \times \mathbb{R})/\Omega_\infty$ and then we consider a smooth approximation

$$A_n(x, t, \sigma) \equiv (A * j_n)(x, t, \sigma)$$

where $j_n \equiv n^{N+1} j(nx, n\sigma)$ (for any $n \in \mathbf{N}$) and $j \in C^\infty(\mathbb{R}^{N+1})$ is a nonnegative function with support in $B_1 \times (-1, 1)$ and such that $\int_{\mathbb{R}^{N+1}} j(x, \sigma)\, dx\, d\sigma = 1$. Notice that these assumptions imply that for a.e. $t \in (0, +\infty)$, $A_n(x, t, \sigma) \in C^\infty(\mathbb{R}^{N+1})$ and satisfies (3.2) in Ω_∞ with α and β replaced by $\alpha' = \alpha'(\alpha)$ and $\beta' = \beta'(\beta)$ positive constants independent of n. Hence by Theorem 2.6 in [24], for every fixed $T > 0$ there exists u_n in $C([0, T]; L^2(\Omega)) \cap L^\infty(\Omega_T) \cap L^2(0, T; H_0^1(\Omega)) \cap C^{\delta, \frac{\delta}{2}}(\Omega \times (0, T))$ solution in Ω_T of (4.1). Moreover, it results

$$\|u_n\|_{L^\infty(\Omega_T)} \le \|T_n(u_0)\|_{L^\infty(\Omega)} \le n \tag{4.3}$$

which implies

$$A_n(x, t, u_n) = A_n(x, t, T_n(u_n))$$

and

$$\alpha_n|\xi|^2 \le \langle A_n(x, t, T_n(\sigma))\xi, \xi\rangle \le \beta|\xi|^2 \quad \text{where } \alpha_n \equiv \frac{\alpha'}{(1 + |n|)^\gamma}$$

Hence we can extend every u_n into a global weak solution of (4.1) (which for sake of notation we denote again with u_n). Thus, we have constructed a sequence u_n which solves our approximating problem (4.1) in every set Ω_T (for every arbitrarily fixed $T > 0$).

By the regularity above and estimate (4.3) it follows that every u_n is in $C([0, +\infty); L^2(\Omega)) \cap L^\infty(\Omega_\infty) \cap L^2(0, \infty; H_0^1(\Omega)) \cap C^{\delta, \frac{\delta}{2}}(\Omega \times (0, +\infty))$.

Step 2. We complete the proof constructing a global weak solution u of our problem by means of the sequence u_n defined in the previous step. To this aim, we remark that by the proof of Theorem 2.5 in [24], for every fixed $T > 0$ there exists a subsequence of u_n that converges a.e. in Ω_T to a weak solution $u^{(T)}$ of (3.1) which belongs to $L^\infty(0, T; L^{r_0}(\Omega))$ and satisfies (3.6). Moreover, the previous property remain true for every subsequence of u_n[5]. Hence, let $T_0 > 0$ arbi-

[5] i.e. for every fixed $T > 0$ and for every subsequence u_{n_k} of u_n, there exists a subsequence of u_{n_k} that converges a.e. in Ω_T to a weak solution of (3.1) which belongs to $L^\infty(0, T; L^{r_0}(\Omega))$ and satisfies (3.6).

trarily fixed and consider the subsequence of u_n, that we denote $u_n^{(1)}$, such that

$$u_n^{(1)} \to u^{(T_0)} \quad \text{a.e. in } \Omega \times (0, T_0)$$

where $u^{(T_0)}$ is a weak solution of (3.1) in $\Omega \times (0, T_0)$. By construction every term of the subsequences $u_n^{(1)}$ is a global weak solution of (4.1). Consequently, every $u_n^{(1)}$ is a weak solution of (4.1) in $\Omega \times (0, 2T_0)$. Thus, it follows that there exists a subsequence of $u_n^{(1)}$, that we denote $u_n^{(2)}$ (whose element by construction are all global weak solution of (3.1)) such that

$$u_n^{(2)} \to u^{(2T_0)} \quad \text{a.e. in } \Omega \times (0, 2T_0)$$

where $u^{(2T_0)}$ is a weak solution of (3.1) in $\Omega \times (0, 2T_0)$. We point out that it results

$$u^{(T_0)} = u^{(2T_0)} \quad \text{a.e. in } \Omega \times (0, T_0).$$

We iterate this procedure and we define a function u in $\Omega \times (0, +\infty)$ as follows

$$\text{for every } T > 0 : u(x, t) = u^{(mT_0)}(x, t) \quad \text{a.e. in } \Omega \times (0, T)$$

where $m \in \mathbf{N}$ is such that $T \in (0, mT_0)$. We notice that the definition of u is well posed since by construction if $T \in (0, mT_0) \cap (0, hT_0)$ (with m and h in $\mathbf{N}$) then it results

$$u^{(mT_0)}(x, t) = u^{(hT_0)}(x, t) \quad \text{a.e. in } \Omega \times (0, T).$$

By construction, for every arbitrarily fixed $T > 0$ u solves (3.1) (hence is a global weak solution of (3.1)), belongs to $L^\infty_{loc}([0, +\infty); L^{r_0}(\Omega))$ and satisfies (3.8). □

4.2. Proof of Theorem 3.3

Let u be the global weak solution of (3.1) constructed in the proof of Theorem 3.2. The assertion will follow showing that for every arbitrarily fixed $T > 0$ it results

$$\|u(t)\|_{L^r(\Omega)} \le C \frac{\|u_0\|_{L^{r_0}(\Omega)}}{t^{\frac{r_0 - r}{r\gamma}}} \quad \text{a.e. } t \in (0, T), \tag{4.4}$$

where C (see formula (4.17) below) is a positive constant depending only on r_0, r, γ, Ω and u_0.

Hence, let $T > 0$ arbitrarily fixed. By construction, there exists a subsequence of u_n, that for sake of notation we denote again by u_n, such that

$$\begin{cases} (u_n)_t = \operatorname{div}(A_n(x, t, u_n)\nabla u_n) & \text{in } \Omega_T \equiv \Omega \times (0, T), \\ u_n = 0 & \text{on } \partial\Omega \times (0, T) \\ u_n(x, 0) = T_n(u_0(x)) & \text{on } \Omega, \end{cases} \tag{4.5}$$

and

$$u_n \to u \quad \text{a.e. in } \Omega_T. \tag{4.6}$$

Let us take $\varphi = \{[\varepsilon + |u_n|]^{r-1} - \varepsilon^{r-1}\}$ sign (u_n) as test function in (4.5) where ε is a positive constant arbitrarily fixed and $1 < r < r_0$ (if $2 < r < r_0$ we can choose directly $\varepsilon = 0$). We obtain for every $0 < t_1 < t_2 < T$

$$\begin{aligned} &\frac{1}{r}\int_\Omega \{[\varepsilon + |u_n(t_2)|]^r - \varepsilon^r\}\,dx - \frac{1}{r}\int_\Omega \{[\varepsilon + |u_n(t_1)|]^r - \varepsilon^r\}\,dx \\ &\quad - \varepsilon^{r-1}\int_\Omega |u_n(t_2)|\,dx + \varepsilon^{r-1}\int_\Omega |u_n(t_1)|\,dx \\ &\quad + (r-1)\int_{t_1}^{t_2}\int_\Omega \frac{|\nabla u_n|^2}{(1+|u_n|)^\gamma}(\varepsilon + |u_n|)^{r-2} \le 0 \end{aligned} \tag{4.7}$$

We estimate the last integral in (4.7). Let $1 \le p < 2$ arbitrarily fixed (it will be chosen below). It results

$$\begin{aligned} \int_\Omega |\nabla u_n|^p (\varepsilon + |u|)^{\frac{(r-2)p}{2}} &= \int_\Omega \frac{|\nabla u_n|^p (\varepsilon + |u_n|)^{\frac{(r-2)p}{2}}}{(1+|u_n|)^{\gamma\frac{p}{2}}}(1+|u|)^{\gamma\frac{p}{2}} \\ &\le \left[\int_\Omega \frac{|\nabla u_n|^2(\varepsilon + |u_n|)^{r-2}}{(1+|u|)^\gamma}\right]^{\frac{p}{2}} \left[\int_\Omega (1+|u_n|)^{\frac{p\gamma}{2-p}}\right]^{1-\frac{p}{2}}. \end{aligned} \tag{4.8}$$

Let us choose p satisfying

$$\frac{p\gamma}{2-p} = r_0 \quad \Leftrightarrow \quad p = \frac{2r_0}{\gamma + r_0}. \tag{4.9}$$

We observe that assumption (3.12) is equivalent to require $p \ge 1$. Hence, by (4.8) it follows

$$\int_\Omega |\nabla u_n|^p (\varepsilon + |u|)^{\frac{(r-2)p}{2}} \le \left[\int_\Omega \frac{|\nabla u_n|^2(\varepsilon + |u_n|)^{r-2}}{(1+|u|)^\gamma}\right]^{\frac{p}{2}} \left[\int_\Omega (1+|u_n|)^{r_0}\right]^{1-\frac{p}{2}}. \tag{4.10}$$

We observe that it results

$$\|u_n\|_{L^\infty(0,T;L^{r_0}(\Omega))} \le \|u_0\|_{L^{r_0}(\Omega)} \tag{4.11}$$

As a matter of fact, proceeding as in (4.7) but with $r = r_0$ and $t_1 = 0$ we deduce

$$\begin{aligned} &\frac{1}{r_0}\int_\Omega \{[\varepsilon + |u_n(t_2)|]^{r_0} - \varepsilon^r\}\,dx - \frac{1}{r}\int_\Omega \{[\varepsilon + |T_n(u_0)|]^{r_0} - \varepsilon^{r_0}\}\,dx + \\ &\quad - \varepsilon^{r_0-1}\int_\Omega |u_n(t_2)|\,dx + \varepsilon^{r_0-1}\int_\Omega |T_n(u_0)|\,dx \le 0 \end{aligned} \tag{4.12}$$

from which (4.11) follows letting $\varepsilon \to 0$ since it results

$$\|T_n(u_0)\|_{L^{r_0}(\Omega)} \leq \|u_0\|_{L^{r_0}(\Omega)}.$$

Hence, by (4.10) and (4.11) we deduce

$$\int_{t_1}^{t_2}\int_\Omega \frac{|\nabla u_n|^2(\varepsilon+|u_n|)^{r-2}}{(1+|u|)^\gamma} \geq C_0^{-\frac{2}{p}}\int_{t_1}^{t_2}\Big(\int_\Omega |\nabla u_n|^p(\varepsilon+|u|)^{\frac{(r-2)p}{2}}\Big)^{\frac{2}{p}} \tag{4.13}$$

where $C_0 = 2^{r_0}(|\Omega|^{1-\frac{p}{2}} + \|u_0\|_{L^{r_0}(\Omega)}^{r_0(1-\frac{p}{2})})$ is a constant independent of n. We estimate the right hand side of (4.13). It results (using Poincaré inequality[6])

$$\begin{aligned}\int_\Omega |\nabla u_n|^p(\varepsilon+|u|)^{\frac{(r-2)p}{2}} &= \Big(\frac{2}{r}\Big)^p \int_\Omega |\nabla[(\varepsilon+|u_n|)^{\frac{r}{2}} - \varepsilon^{\frac{r}{2}}]\operatorname{sign}(u_n)|^p \\ &\geq \Big(\frac{2}{r}\Big)^p c_P^p \int_\Omega |(\varepsilon+|u_n|)^{\frac{r}{2}} - \varepsilon^{\frac{r}{2}}|^p.\end{aligned}$$

By the previous estimates we deduce

$$\begin{aligned}&\int_\Omega \{[\varepsilon+|u_n(t_2)|]^r - \varepsilon^r\}\,dx - \int_\Omega \{[\varepsilon+|u_n(t_1)|]^r - \varepsilon^r\}\,dx \\ &\quad - r\varepsilon^{r-1}\int_\Omega |u_n(t_2)|\,dx + r\varepsilon^{r-1}\int_\Omega |u_n(t_1)|\,dx \\ &\quad + C_1\int_{t_1}^{t_2}\Big(\int_\Omega |(\varepsilon+|u_n|)^{\frac{r}{2}} - \varepsilon^{\frac{r}{2}}|^p\Big)^{\frac{2}{p}} \leq 0\end{aligned} \tag{4.15}$$

where $C_1 = r(r-1)C_0^{-\frac{2}{p}}\big(\frac{2}{r}\big)^2 c_P^2$ is a positive constant independent of n. Letting $\varepsilon \to 0$ we deduce for every $0 < t_1 < t_2 < T$

$$\int_\Omega |u_n(t_2)|^r - \int_\Omega |u_n(t_1)|^r + C_1\int_{t_1}^{t_2}\Big(\int_\Omega |u_n|^{\frac{rp}{2}}\Big)^{\frac{2}{p}} \leq 0 \tag{4.16}$$

that is the integral estimate (2.27) with $q = \frac{rp}{2}$ and $b = r$. Notice that these coefficients satisfy the algebraic condition (2.28) (with $q < r < r_0$). Moreover, by (4.11) it follows that also (2.29) is verified (with $c_2 = 1$). Thus, by Theorem 2.3 it follows that

$$\|u_n(t)\|_{L^r(\Omega)} \leq C\frac{\|T_n(u_0)\|_{L^{r_0}(\Omega)}}{t^{\frac{r_0-r}{r\gamma}}} \quad \text{for every } t \in (0,T),$$

[6] The Poincare inequality: there exists a constant c_P depending only on Ω, N and p such that

$$c_P\|v\|_{L^p(\Omega)} \leq \|\nabla v\|_{L^p(\Omega)} \quad \text{for every } v \in W_0^{1,p}(\Omega). \tag{4.14}$$

where

$$C = \left(\frac{r_0 - r}{\gamma C_1}\right)^{\frac{r_0-r}{r\gamma}} \tag{4.17}$$

from which the assert (4.4) follows. □

REMARK 4.1. We point out that since Ω is bounded, the decay estimates proved above imply the decay also of the L^1-norm of $u(t)$. As a matter of fact, for every $1 < r < r_0$ arbitrarily fixed it results

$$\|u(t)\|_{L^1(\Omega)} \leq C^* \frac{\|u_0\|_{L^1(\Omega)}}{t^{\frac{r_0-r}{r\gamma}}},$$

where $C^* = C|\Omega|^{1-\frac{1}{r}}$.

4.3. Proof of Theorem 3.4

The existence of a global weak solution of (3.14) follows proceeding as in the proof of Theorem 3.2 with the only change of replacing the approximating problem (4.1) with

$$\begin{cases} (u_n)_t - \operatorname{div}(A_n(x,t,u_n)\nabla u_n) + \alpha_0 u_n = 0 & \text{in } \Omega_T \equiv \Omega \times (0,+\infty), \\ u_n = 0 & \text{on } \partial\Omega \times (0,\infty) \\ u_n(x,0) = T_n(u_0(x)) & \text{on } \Omega. \end{cases} \tag{4.18}$$

and hence we omit it. To prove the assert, it is sufficient to show that for every arbitrarily fixed $T > 0$ it results

$$\|u_n(t)\|_{L^r(\Omega)} \leq C\frac{\|u_0\|_{L^{r_0}(\Omega)}}{e^{\alpha_0 t}} \quad \text{for every } t \in (0,T), \tag{4.19}$$

where C is as in (3.18). The proof proceeds distinguishing two cases: the case $r_0 > 1$ and the case $r_0 = 1$. If $r_0 > 1$, it is sufficient to prove

$$\|u_n(t)\|_{L^{r_0}(\Omega)} \leq \frac{\|u_0\|_{L^{r_0}(\Omega)}}{e^{\alpha_0 t}} \quad \text{for every } t \in (0,T), \tag{4.20}$$

(i.e. the assert for $r = r_0$) since if $1 \leq r < r_0$ it results

$$\|u_n(t)\|_{L^r(\Omega)} \leq \|u_n(t)\|_{L^{r_0}(\Omega)}|\Omega|^{\frac{1}{r}-\frac{1}{r_0}}$$

and hence the assertion follows by (4.20). Taking as test function $\varphi = \{[\varepsilon + |u_n|]^{r_0-1} - \varepsilon^{r_0-1}\}$ sign (u_n) we deduce that for every $0 < t_1 < t_2 < T$ it results

$$\frac{1}{r_0}\int_\Omega \{[\varepsilon + |u_n(t_2)|]^{r_0} - \varepsilon^{r_0}\}\,dx - \frac{1}{r_0}\int_\Omega \{[\varepsilon + |u_n(t_1)|]^{r_0} - \varepsilon^{r_0}\}\,dx$$
$$- \varepsilon^{r_0-1}\int_\Omega |u_n(t_2)|\,dx + \varepsilon^{r_0-1}\int_\Omega |u_n(t_1)|\,dx$$
$$+ (r_0-1)\int_{t_1}^{t_2}\int_\Omega \frac{|\nabla u_n|^2}{(1+|u_n|)^\gamma}(\varepsilon + |u_n|)^{r_0-2}$$
$$+ \alpha_0\int_{t_1}^{t_2}\int_\Omega \{[\varepsilon + |u_n|]^{r_0-1} - \varepsilon^{r_0-1}\}|u_n| \le 0$$

which implies

$$\frac{1}{r_0}\int_\Omega \{[\varepsilon + |u_n(t_2)|]^{r_0} - \varepsilon^{r_0}\}\,dx - \frac{1}{r_0}\int_\Omega \{[\varepsilon + |u_n(t_1)|]^{r_0} - \varepsilon^{r_0}\}\,dx$$
$$- \varepsilon^{r_0-1}\int_\Omega |u_n(t_2)|\,dx + \varepsilon^{r_0-1}\int_\Omega |u_n(t_1)|\,dx$$
$$+ \alpha_0\int_{t_1}^{t_2}\int_\Omega \{[\varepsilon + |u_n|]^{r_0-1} - \varepsilon^{r_0-1}\}|u_n| \le 0.$$

By the previous estimate we deduce (letting $\varepsilon \to 0$) for every $0 < t_1 < t_2 < T$

$$\int_\Omega |u_n(t_2)|^{r_0} - \int_\Omega |u_n(t_1)|^{r_0} + r_0\alpha_0\int_{t_1}^{t_2}\int_\Omega |u_n|^{r_0} \le 0. \tag{4.21}$$

Thus, we can apply Theorem 2.4 obtaining the following decay estimate

$$\|u_n(t)\|_{L^{r_0}(\Omega)} \le \frac{\|u_0\|_{L^{r_0}(\Omega)}}{e^{\alpha_0 t}} \quad \text{for every } t \in (0,T), \tag{4.22}$$

which concludes the proof if $r_0 > 1$.

If $r_0 = 1$, changing the test function in $\varphi = \left\{1 - \frac{1}{[1+|u_n|]^\delta}\right\}\operatorname{sign}(u_n)$, $\delta > 1$ (and using assumption (3.2)) we deduce for every $0 < t_1 < t_2 < T$

$$\int_\Omega |u_n(t_2)| + \frac{1}{\delta-1}\int_\Omega\left\{1 - \frac{1}{[1+|u_n(t_1)|]^{\delta-1}}\right\} + \alpha_0\int_{t_1}^{t_2}\int_\Omega |u_n|\left\{1 - \frac{1}{[1+|u_n|]^\delta}\right\}$$
$$\le \int_\Omega |u_n(t_1)| + \frac{1}{\delta-1}\int_\Omega\left\{1 - \frac{1}{[1+|u_n(t_2)|]^{\delta-1}}\right\}$$

from which, letting $\delta \to +\infty$ it follows

$$\int_\Omega |u_n(t_2)| - \int_\Omega |u_n(t_1)| + \alpha_0\int_{t_1}^{t_2}\int_\Omega |u_n| \le 0 \tag{4.23}$$

(which is (4.21) with $r_0 = 1$). Now the assertion follows, as in the previous case, applying Theorem 2.4. □

4.4. Proof of Theorem 3.5

As before, the existence of a global weak solution of (3.19) follows proceeding as in the proof of Theorem 3.2 with the only change of replacing the approximating problem (4.1) with

$$\text{(4.24)}\quad \begin{cases} (u_n)_t - \operatorname{div}(A_n(x,t,u_n)\nabla u_n) + \alpha_0|u_n|^{s-1}u_n = 0 & \text{in } \Omega_T \equiv \Omega \times (0,+\infty), \\ u_n = 0 & \text{on } \partial\Omega \times (0,\infty) \\ u_n(x,0) = T_n(u_0(x)) & \text{on } \Omega, \end{cases}$$

and hence we omit it. To prove the assert, it is sufficient to show that if $r_0 = 1$ for every arbitrarily fixed $T > 0$ it results

$$\text{(4.25)}\qquad \|u_n(t)\|_{L^\infty(\Omega)} \le \left[\frac{1}{\alpha_0(s-1)}\right]^{\frac{1}{s-1}} \frac{1}{t^{\frac{1}{s-1}}} \quad \text{for every } t \in (0,T).$$

Choosing $\varphi = |u_n|^{r-2}u_n$ as a test function, with $r > 2$ arbitrarily fixed we deduce for every $0 < t_1 < t_2 < T$

$$\text{(4.26)}\qquad \int_\Omega |u_n(t_2)|^r - \int_\Omega |u_n(t_1)|^r + r\alpha_0 \int_{t_1}^{t_2}\int_\Omega |u_n|^{s+r-1} \le 0$$

Hence assumption (2.27) of Theorem 2.3 is satisfied with $b = q = s + r - 1$. Notice that also the algebraic conditions in (2.28) are verified (with now $r_0 < r < q$). Moreover, proceeding exactly as in the proof of (4.23) we deduce that also (2.29) is satisfied. Hence, applying Theorem 2.3 we deduce the following decay estimate

$$\text{(4.27)}\qquad \|u_n(t)\|_{L^r(\Omega)} \le C(r)\frac{\|u_0\|_{L^1(\Omega)}^{\frac{1}{r}}}{t^{\frac{r-1}{r(s-1)}}} \quad \text{for every } t \in (0,T),$$

where

$$C(r) \equiv \left(\frac{r-1}{\alpha_0 r(s-1)}\right)^{\frac{r-1}{r(s-1)}}.$$

Notice that it results

$$\exists \lim_{r\to+\infty} C(r) = \left(\frac{1}{\alpha_0(s-1)}\right)^{\frac{1}{s-1}}.$$

Hence, (4.25) follows letting $r \to +\infty$ in (4.27). □

References

[1] D. G. Aronson - L. A. Peletier, *Large time behavior of solutions of the porous medium equation in bounded domains*, J. Diff. Eqns. 39 (1981), 378–412.

[2] P. Baras - J. Goldstein, *The heat equation with singular potential*, Trans. Amer. Math. Soc. 294 (1984), 121–139.

[3] Ph. Benilan, *Equations d'evolutions dans un espace de Banach quelconque et applications*, doctoral thesis, Orsay (1972).

[4] Ph. Benilan, *Opérateurs accrétifs et semi-groupes dans les espaces* L^p $(1 \leq p \leq \infty)$, France-Japan Seminar, Tokyo, 1976.

[5] Ph. Benilan - M. G. Crandall - M. Pierre, *Solutions of the porous medium in* $\mathbb{R}^N$ *under optimal conditions on initial values*, Indiana Univ. Math. J. 33 (1984), 51–87.

[6] L. Boccardo - M. M. Porzio, *Degenerate parabolic equations: existence and decay properties*, Discrete and Continuous Dynamical Systems Series S, vol. 7, n. 4, (2014), pp. 617–629.

[7] M. Bonforte - G. Grillo, *Super and ultracontractive bounds for doubly nonlinear evolution equations*, Rev. Mat. Iberoamericana 22 (2006), n. 1, 111–129.

[8] H. Brezis - M. G. Crandall, *Uniqueness of solutions of the initial-value problems for* $u_t - \triangle\varphi(u) = 0$, J. Math. Pures et Appl. 58 (1979), 153–163.

[9] F. Cipriani - G. Grillo, *Uniform bounds for solutions to quasilinear parabolic equations*, J. Differential Equations 177 (2001), 209–234.

[10] E. Di Benedetto - M. A. Herrero, *On the Cauchy problem and initial traces for a degenerate parabolic equation*, Trans. AMS 314 (1989), 187–224.

[11] E. Di Benedetto - M. A. Herrero, *Non negative solutions of the evolution p-Laplacian equation. Initial traces and Cauchy problem when* $1 < p < 2$, Arch. Rational Mech. Anal. 111, 3, (1990), 225–290.

[12] G. Grillo - M. Muratori - M. M. Porzio, *Porous media equations with two weights: existence, uniqueness, smoothing and decay properties of energy solutions via Poincaré inequalities*, Discrete and Continuous Dynamical Systems Series A, vol. 33, n. 8, (2013), 3599–3640.

[13] M. A. Herrero - M. Pierre, *The Cauchy problem for* $u_t = \triangle u^m$ *when* $0 < m < 1$, Trans. A.M.S. 291 (1985), 145–158.

[14] M. A. Herrero - J. L. Vazquez, *Asymptotic behavior of the solutions of a strongly nonlinear parabolic problem*, Ann. Fac. Sci., Toulose Math. (5) 3, n. 2 (1981), 113–127.

[15] L. Moreno - M. M. Porzio, *Existence and asymptotic behavior of a parabolic equation with* L^1 *data*, submitted.

[16] G. Moscariello - M. M. Porzio, *Quantitative asymptotic estimates for evolution problems*, Nonlinear Analysis TMA 154 (2017), 225–240.

[17] M. M. Porzio, *On decay estimates*, Journal of Evolution Equation, Vol. 9, Issue 3, (2009), 561–591.

[18] M. M. Porzio, *Existence, uniqueness and behavior of solutions for a class of nonlinear parabolic problems*, Nonlinear Analysis TMA, 74 (2011), 5359–5382.

[19] M. M. Porzio, *On uniform and decay estimates for unbounded solutions of partial differential equations*, Journal of Differential Equations 259 (2015), 6960–7011.

[20] M. M. Porzio, *Regularity and time behavior of the solutions of linear and quasilinear parabolic equations*, Advances in Differential Equations vol. 23, n. 5–6, (2018), 329–372.

[21] M. M. Porzio, *Regularity and time behavior of the solutions to weak monotone parabolic equations*, preprint.

[22] M. M. Porzio, *Asymptotic behavior and regularity properties of strongly nonlinear parabolic equations*, submitted.

[23] M. M. Porzio, *Quasilinear parabolic and elliptic equations with singular potentials*, 2017 MATRIX Annals, Editors: David R. Wood, Jan de Gier, Cheryl E. Praeger, Terence Tao. Matrix book series, Volume 2, Springer to appear.

[24] M. M. Porzio - M. A. Pozio, *Parabolic equations with non–linear, degenerate and space–time dependent operators*, Journal of Evolution Equations, (2008), vol. 8, 31–70.

[25] J. L. Vazquez, *Smoothing and decay estimates for nonlinear diffusion equations*, Oxford University press 2006.

[26] L. Veron, *Effects regularisants des semi-groupes non linéaires dans des espaces de Banach*, Ann. Fac. Sci., Toulose Math. (5) 1, n. 2 (1979), 171–200.

Received 18 December 2017,
and in revised form 23 April 2018.

Maria Michaela Porzio
Dipartimento di Matematica ‘‘Guido Castelnuovo’’
Sapienza Universitá di Roma
Piazzale A. Moro 2
00185 Roma, Italy
porzio@mat.uniroma1.it