Rend. Lincei Mat. Appl. 29 (2018), 679–687
DOI 10.4171/RLM/828

Functional Analysis — *Some progress on the polynomial Dunford–Pettis property*, by Raffaella Cilia and Joaquín M. Gutiérrez, communicated on May 11, 2018.

Abstract. — A Banach space E has the *Dunford–Pettis property* (*DPP*, for short) if every weakly compact (linear) operator on E is completely continuous. In 1979 R. A. Ryan proved that E has the DPP if and only if every weakly compact polynomial on E is completely continuous. Every k-homogeneous (continuous) polynomial $P \in \mathscr{P}(^kE, F)$ between Banach spaces E and F admits an extension $\tilde{P} \in \mathscr{P}(^kE^{**}, F^{**})$ to the biduals called the Aron–Berner extension. The Aron–Berner extension of every weakly compact polynomial $P \in \mathscr{P}(^kE, F)$ is F-valued, that is, $\tilde{P}(E^{**}) \subseteq F$, but there are non-weakly compact polynomials with F-valued Aron–Berner extension. For Banach spaces F with weak-star sequentially compact dual unit ball B_{F^*}, we strengthen Ryan's result by showing that E has the DPP if and only if every polynomial $P \in \mathscr{P}(^kE, F)$ with F-valued Aron–Berner extension is completely continuous. This gives a partial answer to a question raised in 2003 by I. Villanueva and the second named author. They proved the result for spaces E such that every operator from E into its dual E^* is weakly compact, but the question remained open for other spaces.

Key words: Aron–Berner extension of polynomials, Dunford–Pettis property, Dunford–Pettis set, completely continuous polynomials

Mathematics Subject Classification (primary; secondary): 47H60; 46G25, 46B03

1. Introduction

Throughout E, F, and G denote Banach spaces, E^* is the dual of E, and B_E stands for its closed unit ball. The closed unit ball B_{E^*} will always be endowed with the weak-star topology. By $\mathbb{N}$ we represent the set of all natural numbers and by $\mathbb{K}$ the scalar field (real or complex). We use the symbol $\mathscr{L}(E, F)$ for the space of all (linear bounded) operators from E into F endowed with the operator norm. For $T \in \mathscr{L}(E, F)$ we denote its adjoint by $T^* \in \mathscr{L}(F^*, E^*)$.

Given $k \in \mathbb{N}$, we use $\mathscr{P}(^kE, F)$ for the space of all k-homogeneous (continuous) polynomials from E into F endowed with the supremum norm. When $F = \mathbb{K}$, we omit the range space: hence, $\mathscr{P}(^kE)$ will stand for $\mathscr{P}(^kE, \mathbb{K})$. For the general theory of polynomials on Banach spaces, we refer the reader to [Din] and [Mu]. For unexplained notation and results in Banach space theory, the reader may see [Di, DJT, DU].

A polynomial $P \in \mathscr{P}(^kE, F)$ is (*weakly*) *compact* if $P(B_E)$ is relatively (weakly) compact in F.

Given a polynomial $P \in \mathcal{P}({}^k E, F)$, its *adjoint* P^* is the operator

$$P^* : F^* \to \mathcal{P}({}^k E)$$

given by $P^*(\psi) := \psi \circ P$ for every $\psi \in F^*$. It is well-known that P is (weakly) compact if and only if P^* is (weakly) compact (see [AS, Proposition 3.2] for the compact case and [R2, Proposition 2.1] for the weakly compact case).

We say that a polynomial $P \in \mathcal{P}({}^k E, F)$ is *completely continuous* if it takes weak Cauchy sequences into norm convergent sequences. We say that P is *unconditionally converging* if, for every weakly unconditionally Cauchy series $\sum x_n$ in E, the sequence $(P(s_m))_{m=1}^\infty$ is norm convergent, where $s_m := \sum_{n=1}^m x_n$.

We use the notation $\otimes^k E := E \otimes \overset{(k)}{\cdots} \otimes E$ for the k-fold tensor product of E, and $E \hat{\otimes}_\pi F$ for the completed projective tensor product of E and F (see [DU, DF] for the theory of tensor products). By $\otimes_s^k E := E \otimes_s \overset{(k)}{\cdots} \otimes_s E$ we denote the k-fold symmetric tensor product of E, that is, the set of all elements $u \in \otimes^k E$ of the form

$$u = \sum_{j=1}^n \lambda_j x_j \otimes \overset{(k)}{\cdots} \otimes x_j \quad (n \in \mathbb{N},\ \lambda_j \in \mathbb{K},\ x_j \in E,\ 1 \le j \le n). \tag{1}$$

On the space $\otimes_s^k E$ we can define the *projective symmetric tensor norm* π_s by

$$\pi_s(u) := \inf \sum_{j=1}^n |\lambda_j| \, \|x_j\|^k,$$

where the infimum is taken over all representations of $u \in \otimes_s^k E$ of the form (1). We write $\hat{\otimes}_{\pi_s,s}^k E$ for the completion of $\otimes_s^k E$ endowed with the π_s norm.

For symmetric tensor products, the reader is referred to [F].

For a polynomial $P \in \mathcal{P}({}^k E, F)$, its *linearization*

$$\bar{P} : \hat{\otimes}_{\pi_s,s}^k E \to F$$

is the operator given by

$$\bar{P}\Big(\sum_{j=1}^n \lambda_j x_j \otimes \overset{(k)}{\cdots} \otimes x_j\Big) := \sum_{j=1}^n \lambda_j P(x_j)$$

for all $x_j \in E$ and $\lambda_j \in \mathbb{K}$ $(1 \le j \le n)$.

Every polynomial $P \in \mathcal{P}({}^k E, F)$ between Banach spaces admits an extension $\tilde{P} \in \mathcal{P}({}^k E^{**}, F^{**})$ called the Aron–Berner extension. We recall its construction following [CGKM, §2]. Let A be the symmetric k-linear mapping associated with P. We can extend A to a k-linear mapping $\tilde{A}$ from E^{**} into F^{**} in such a way that for each fixed j $(1 \le j \le k)$ and for each fixed $x_1, \ldots, x_{j-1} \in E$ and $z_{j+1}, \ldots, z_k \in E^{**}$, the linear mapping

$$z \mapsto \tilde{A}(x_1, \ldots, x_{j-1}, z, z_{j+1}, \ldots, z_k) \quad (z \in E^{**}) \tag{2}$$

is weak*-to-weak* continuous. In other words, we define the image of the mapping in (2) to be the weak*-limit of the net

$$(\tilde{A}(x_1, \ldots, x_{j-1}, x_\alpha, z_{j+1}, \ldots, z_k))_\alpha$$

for a weak*-convergent net $(x_\alpha)_\alpha \subset E$. By this weak*-to-weak* continuity, A can be extended to a k-linear mapping $\tilde{A}$ from E^{**} into F^{**} beginning with the last variable and working backwards to the first. Then the restriction

$$\tilde{P}(z) := \tilde{A}(z, \ldots, z) \quad (z \in E^{**})$$

is called the *Aron–Berner extension* of P. Given $z \in E^{**}$ and $w \in F^*$, we have

$$\tilde{P}(z)(w) = \widetilde{w \circ P}(z). \tag{3}$$

Actually this equality is often used as the definition of the vector-valued Aron–Berner extension based upon the scalar-valued Aron–Berner extension. Recall that $\tilde{A}$ is not symmetric in general.

The Aron–Berner extension was introduced in [AB]. A survey of its properties may be seen in [Z]. It has been studied by many mathematicians. Some examples are [AB, ACG, Ca, CG, CL, CGKM, DG, DGG, GGMM, GV, PVWY].

The Aron–Berner extension of every weakly compact polynomial is F-valued, that is, $\tilde{P}(E^{**}) \subseteq F$ [Ca, Proposition 1.4], but there are polynomials with F-valued Aron–Berner extension which are not weakly compact. The most typical and basic example may be the polynomial $Q \in \mathscr{P}(^k\ell_2, \ell_1)$ given by $Q(x) := (x_n^k)_{n=1}^\infty$ for all $x = (x_n)_{n=1}^\infty \in \ell_2$. The polynomials with F-valued Aron–Berner extension are often more useful than the weakly compact polynomials when it comes to characterize isomorphic properties of Banach spaces: see for instance [GV]. In the polynomial setting they play somehow the role of the weakly compact operators in the linear setting.

It should be noted that the statement of [GV, Lemma 3.3] (given without proof) is wrong. This lemma is used in several places of [GV]. A corrected version of the lemma and subsequent results of [GV] is given in [PVWY, §2].

A Banach space has the *Dunford–Pettis property* (*DPP*, for short) if every weakly compact operator on E is completely continuous. Ryan proved [R1] that E has the DPP if and only if every weakly compact polynomial on E is completely continuous. An attempt to strengthen this result was made in [GV] where the question was raised whether the DPP of E implies the complete continuity of every polynomial from E into an arbitrary Banach space F with $\tilde{P}(E^{**}) \subseteq F$. A partial affirmative answer was given in [GV] for spaces E such that every operator from E into E^* is weakly compact (this happens in particular if E is an $\mathscr{L}_\infty$-space), but the question remained open in general and was unknown for instance for $\mathscr{L}_1$-spaces.

In the present paper we prove that E has the DPP if and only if whenever P is a polynomial from E into an arbitrary Banach space F with weak-star sequentially compact dual unit ball B_{F^*} so that $\tilde{P}(E^{**}) \subseteq F$, then P is completely continuous. We achieve this result by a careful study of the composition of Dunford–Pettis operators (see definition below) with polynomials having F-valued Aron–Berner extension.

Recall that a Banach space F is *weakly compactly generated* (*WCG*, for short) if F contains a weakly compact absolutely convex set whose linear span is dense in F. Separable Banach spaces are WCG. Reflexive Banach spaces are also WCG. A well-known result of D. Amir and J. Lindenstrauss states that every subspace of a WCG Banach space has a weak-star sequentially compact dual ball [Di, Theorem XIII.4]. If F^* contains no copy of ℓ_1, then B_{F^*} is weak-star sequentially compact [Di, page 226].

We summarize some characterizations of Banach space isomorphic properties that can be obtained using polynomials with F-valued Aron–Berner extension:

- The DPP as mentioned above, whenever B_{F^*} is weak-star sequentially compact.
- Recall that E has the *reciprocal Dunford–Pettis property* (*RDPP*, for short) if every completely continuous operator on E is weakly compact. A space E has the RDPP if and only if every completely continuous polynomial from E into an arbitrary Banach space F has F-valued Aron–Berner extension [GV, Corollary 3.5].
- E is said to have *property* (V) if every unconditionally converging operator on E is weakly compact. A space E has property (V) if and only if every unconditionally converging polynomial from E into an arbitrary Banach space F has F-valued Aron–Berner extension [GV, Corollary 4.3].
- E has the *Grothendieck property* if every operator from E into c_0 is weakly compact. A space E has the Grothendieck property if and only if every polynomial from E into c_0 has c_0-valued Aron–Berner extension [GG2, Corollary 15].

As far as we know, no other polynomial characterization of the RDPP and of property (V) has been found up to date. As for the Grothendieck property, other polynomial characterizations are given in [GG2, Theorem 14 and Corollary 16].

A subset A of a Banach space E is a *Dunford–Pettis set* (*DP set*, for short) [An, Theorem 1] if, for every weakly null sequence $(x_n^*) \subset E^*$, we have

$$\lim_n \sup_{x \in A} |\langle x, x_n^* \rangle| = 0.$$

An operator $S \in \mathscr{L}(G, E)$ is a *Dunford–Pettis operator* if $S(B_G)$ is a DP set in E. We denote by $\mathscr{DP}$ the ideal of Dunford–Pettis operators which has been studied with a different notation in [GG1].

A subset A of a Banach space E is said to be a *Rosenthal set* if every sequence in A contains a weak Cauchy subsequence.

If $A \subset E$ is a subset, $\mathrm{co}(A)$ denotes the convex hull of A.

2. The polynomial DPP

Given $k \in \mathbb{N}$ and an operator $S \in \mathscr{L}(G, E)$, we define the operator

$$S_k^* : \mathscr{P}(^kE) \to \mathscr{P}(^kG)$$

by $S_k^*(P)(g) := P(S(g))$ for all $P \in \mathscr{P}(^kE)$ and $g \in G$.

Note that, given an operator $S \in \mathscr{L}(G, E)$ and a polynomial $P \in \mathscr{P}(^kE, F)$, the adjoint of the polynomial $P \circ S \in \mathscr{P}(^kG, F)$ is the operator

$$S_k^* \circ P^* : F^* \to \mathscr{P}(^kE) \to \mathscr{P}(^kG). \tag{4}$$

Theorem 2.1. *Let $(P_n) \subset \mathscr{P}(^kE)$ be a sequence of scalar-valued polynomials such that, for every $x^{**} \in E^{**}$, we have $\widetilde{P_n}(x^{**}) \to 0$. Let $A \subset E$ be a DP set. Then,*

$$\lim_n \sup_{x \in A} |P_n(x)| = 0.$$

Proof. Assume the result fails. Then, passing to a subsequence if necessary, we can find a sequence $(x_n) \subset A$ and $\delta > 0$ such that $|P_n(x_n)| > \delta$ for all $n \in \mathbb{N}$. Let $A_n : E \times \overset{(k)}{\cdots} \times E \to \mathbb{K}$ be the unique symmetric k-linear form associated with P_n. Then,

$$|A_n(x_n, \overset{(k)}{\ldots}, x_n)| > \delta \quad (n \in \mathbb{N}).$$

Since A is a DP set in E, the sequence

$$(A_n(x_n, \overset{(k-1)}{\ldots}, x_n, \cdot))_{n=1}^\infty \subset E^*$$

is not weakly null. So, passing again to a subsequence if necessary, we can find $x_k^{**} \in E^{**}$ and $\delta_1 > 0$ such that

$$|\widetilde{A_n}(x_n, \overset{(k-1)}{\ldots}, x_n, x_k^{**})| > \delta_1 \quad (n \in \mathbb{N}).$$

In the same way, the sequence

$$(\widetilde{A_n}(x_n, \overset{(k-2)}{\ldots}, x_n, \cdot, x_k^{**}))_{n=1}^\infty \subset E^*$$

is not weakly null so, passing to a subsequence if necessary, we can find $x_{k-1}^{**} \in E^{**}$ and $\delta_2 > 0$ such that

$$|\widetilde{A_n}(x_n, \overset{(k-2)}{\ldots}, x_n, x_{k-1}^{**}, x_k^{**})| > \delta_2 \quad (n \in \mathbb{N}).$$

Proceeding up to the first variable, we can find $x_1^{**} \in E^{**}$ and $\delta_k > 0$ so that

$$|\widetilde{A_n}(x_1^{**}, x_2^{**}, \ldots, x_k^{**})| > \delta_k \quad (n \in \mathbb{N}).$$

Using the notation

$$\widetilde{A_n}(x^{**})^k := \widetilde{A_n}(x^{**}, \overset{(k)}{\ldots}, x^{**})$$

for $x^{**} \in E^{**}$ and the polarization formula [Mu, Theorem 1.10], we obtain

$$k!2^k\delta_k < \left|\sum_{\varepsilon_j=\pm 1} \varepsilon_1 \dots \varepsilon_k \widetilde{A_n}(\varepsilon_1 x_1^{**} + \dots + \varepsilon_k x_k^{**})^k\right|$$

$$\leq \sum_{\varepsilon_j=\pm 1} |\widetilde{A_n}(\varepsilon_1 x_1^{**} + \dots + \varepsilon_k x_k^{**})^k|$$

$$= \sum_{\varepsilon_j=\pm 1} |\widetilde{P_n}(\varepsilon_1 x_1^{**} + \dots + \varepsilon_k x_k^{**})| \quad (n \in N). \tag{5}$$

Since each summand of (5) tends to zero as n goes to ∞, we reach a contradiction. □

Lemma 2.2. *Given $P \in \mathscr{P}(^k E, F)$, suppose that the closed unit ball B_{F^*} is weak-star sequentially compact. If the polynomial P is non-compact, then there is a weak-star null sequence (f_n^*) in F^* such that $\|P^*(f_n^*)\| \not\to_n 0$.*

Proof. Since P is non-compact, its adjoint P^* is a non-compact operator, so there is a sequence $(f_n^*) \subset B_{F^*}$ such that the sequence $(P^*(f_n^*))_{n=1}^{\infty}$ does not have any convergent subsequence. We can assume that (f_n^*) is weak-star convergent. By linearity of P^*, we can assume that (f_n^*) is weak-star null. □

Theorem 2.3. *Given a Banach space F, suppose that its closed dual unit ball B_{F^*} is weak-star sequentially compact. Let $P \in \mathscr{P}(^k E, F)$ be a polynomial with $\tilde{P}(E^{**}) \subseteq F$. If $S \in \mathscr{DP}(G, E)$, the polynomial $P \circ S$ is compact.*

Proof. Assume $P \circ S$ is non-compact. From (4) we know that its adjoint is $(P \circ S)^* = S_k^* \circ P^*$. By Lemma 2.2, there is a weak-star null sequence (f_n^*) in F^* so that the sequence $(S_k^* \circ P^*(f_n^*))_{n=1}^{\infty}$ is weak-star null but is not norm null in $\mathscr{P}(^k G)$.

By passing to a subsequence, we can find a sequence $(g_n) \subset B_G$ and $\varepsilon > 0$ so that

$$|P^*(f_n^*)(S(g_n))| = |S_k^* \circ P^*(f_n^*)(g_n)| > \varepsilon \tag{6}$$

for all $n \in \mathbb{N}$.

The condition $\tilde{P}(E^{**}) \subseteq F$ implies that

$$\widetilde{f_n^* \circ P}(x^{**}) = f_n^* \circ \tilde{P}(x^{**}) \to 0$$

for all $x^{**} \in E^{**}$. By Theorem 2.1, we have

$$P^*(f_n^*)(S(g_n)) = f_n^* \circ P \circ S(g_n) \to 0,$$

in contradiction with (6). □

COROLLARY 2.4. *Given a Banach space F, suppose that its closed dual unit ball B_{F^*} is weak-star sequentially compact. Then every polynomial $P \in \mathcal{P}({}^k E, F)$ with $\tilde{P}(E^{**}) \subseteq F$ is weakly continuous on DP sets of E.*

PROOF. Apply Theorem 2.3 and [GG1, Proposition 3.6]. □

COROLLARY 2.5. *Given a Banach space E and $k \in \mathbb{N}$, the following assertions are equivalent:*

(a) *E has the DPP;*

(b) *for every Banach space F with weak-star sequentially compact dual unit ball B_{F^*}, every polynomial $P \in \mathcal{P}({}^k E, F)$ with $\tilde{P}(E^{**}) \subseteq F$ is completely continuous;*

(c) *every polynomial $P \in \mathcal{P}({}^k E, c_0)$ with $\tilde{P}(E^{**}) \subseteq c_0$ is completely continuous.*

PROOF. (a) ⇒ (b). If $\tilde{P}(E^{**}) \subseteq F$, Corollary 2.4 implies that P is weakly continuous on DP sets of E. Since E has the DPP, [GG1, Proposition 1.2] implies that P is weakly continuous on Rosenthal sets. By the comment preceding [GG1, Corollary 3.7], P is weakly *uniformly* continuous on Rosenthal sets and so P takes weak Cauchy sequences into norm convergent sequences.

(b) ⇒ (c) is obvious.

(c) ⇒ (a). Let $P \in \mathcal{P}({}^k E, c_0)$ be a weakly compact polynomial. Then $\tilde{P}(E^{**}) \subseteq c_0$ so P is completely continuous. By [GG1, Theorem 3.14], E has the DPP. □

ACKNOWLEDGMENTS. Both authors were supported in part by Dirección General de Investigación, MTM2015-65825-P (Spain). The first named author was partially supported by GNSAGA (Italy).

REFERENCES

[An] K. T. ANDREWS, *Dunford-Pettis sets in spaces of Bochner integrable functions*, Math. Ann. 241 (1979), 35–41.

[AB] R. M. ARON - P. D. BERNER, *A Hahn-Banach extension theorem for analytic mappings*, Bull. Soc. Math. France 106 (1978), 3–24.

[ACG] R. M. ARON - B. J. COLE - T. W. GAMELIN, *Weak-star continuous analytic functions*, Canad. J. Math. 47 (1995), 673–683.

[AS] R. M. ARON - M. SCHOTTENLOHER, *Compact holomorphic mappings on Banach spaces and the approximation property*, J. Funct. Anal. 21 (1976), 7–30.

[Ca] D. CARANDO, *Extendible polynomials on Banach spaces*, J. Math. Anal. Appl. 233 (1999), 359–372.

[CG] D. CARANDO - D. GALICER, *Extending polynomials in maximal and minimal ideals*, Publ. Res. Inst. Math. Sci. 46 (2010), 669–680.

[CL] D. CARANDO - S. LASSALLE, *Extension of vector-valued integral polynomials*, J. Math. Anal. Appl. 307 (2005), 77–85.

[CGKM] Y. S. CHOI - D. GARCÍA - S. G. KIM - M. MAESTRE, *Composition, numerical range and Aron-Berner extension*, Math. Scand. 103 (2008), 97–110.

[DG] A. M. Davie - T. W. Gamelin, *A theorem on polynomial-star approximation*, Proc. Amer. Math. Soc. 106 (1989), 351–356.

[DF] A. Defant - K. Floret, *Tensor Norms and Operator Ideals*, Math. Studies 176, North-Holland, Amsterdam 1993.

[Di] J. Diestel, *Sequences and Series in Banach Spaces*, Graduate Texts in Math. 92, Springer, Berlin 1984.

[DJT] J. Diestel - H. Jarchow - A. Tonge, *Absolutely Summing Operators*, Cambridge Stud. Adv. Math. 43, Cambridge University Press, Cambridge 1995.

[DU] J. Diestel - J. J. Uhl, Jr., *Vector Measures*, Math. Surveys Monographs 15, American Mathematical Society, Providence RI 1977.

[DGG] V. Dimant - D. Galicer - R. García, *Geometry of integral polynomials, M-ideals and unique norm preserving extensions*, J. Funct. Anal. 262 (2012), 1987–2012.

[Din] S. Dineen, *Complex Analysis on Infinite Dimensional Spaces*, Springer Monogr. Math., Springer, Berlin 1999.

[F] K. Floret, *Natural norms on symmetric tensor products of normed spaces*, Note Mat. 17 (1997), 153–188.

[GGMM] P. Galindo - D. García - M. Maestre - J. Mujica, *Extension of multilinear mappings on Banach spaces*, Studia Math. 108 (1994), 55–76.

[GG1] M. González - J. M. Gutiérrez, *Weakly continuous mappings on Banach spaces with the Dunford-Pettis property*, J. Math. Anal. Appl. 173 (1993), 470–482.

[GG2] M. González - J. M. Gutiérrez, *Polynomial Grothendieck properties*, Glasgow Math. J. 37 (1995), 211–219.

[GV] J. M. Gutiérrez - I. Villanueva, *Extensions of multilinear operators and Banach space properties*, Proc. Roy. Soc. Edinburgh 133A (2003), 549–566. See Corrigendum in [PVWY, §2].

[Mu] J. Mujica, *Complex Analysis in Banach Spaces*, Math. Studies 120, North-Holland, Amsterdam 1986.

[PVWY] A. M. Peralta - I. Villanueva - J. D. Maitland Wright - K. Ylinen, *Quasi-completely continuous multilinear operators*, Proc. Roy. Soc. Edinburgh 140A (2010), 635–649.

[R1] R. A. Ryan, *Dunford-Pettis properties*, Bull. Acad. Polon. Sci. Sér. Sci. Math. 27 (1979), 373–379.

[R2] R. A. Ryan, *Weakly compact holomorphic mappings on Banach spaces*, Pacific J. Math. 131 (1988), 179–190.

[Z] I. Zalduendo, *Extending polynomials on Banach spaces – A survey*, Rev. Un. Mat. Argentina 46 (2005), 45–72.

Received 5 February 2018,
and in revised form 8 May 2018.

Raffaella Cilia
Dipartimento di Matematica
Università di Catania
Viale Andrea Doria 6
95125 Catania, Italy
cilia@dmi.unict.it

Joaquín M. Gutiérrez
Departamento de Matemáticas
Escuela Técnica Superior de Ingenieros Industriales
Universidad Politécnica de Madrid
C. José Gutiérrez Abascal 2
28006 Madrid, Spain
joaquin.gutierrez.alamo@gmail.com