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Group Theory — *On mutually permutable products of finite groups*, by Adolfo Ballester-Bolinches, Yangming Li, Mari Carmen Pedraza-Aguilera and Ning Su, communicated on June 22, 2018.

Abstract. — The main purpose of this paper is to study mutually permutable products $G = AB$ in which the subgroups of prime order p and cyclic of order 4 (if $p = 2$) of the largest normal subgroup of G contained in $A \cap B$ are well situated in G. Our results confirm once again the important role of the intersection of the factors in the structural study of mutually permutable products.

Key words: Finite group, Sylow permutability, weakly s-supplementation, factorisation, saturated formation

Mathematics Subject Classification: 20D10, 20D20

1. Introduction and statement of results

All groups considered in this paper are finite.

We recall that two subgroups A and B of a group G are said to permute if AB is a subgroup of G. A and B are called *mutually permutable* if every subgroup of A permutes with B and every subgroup of B permutes with A. If every subgroup of A permutes with every subgroup of B we say that A and B are *totally permutable*. Obviously totally permutable subgroups are mutually permutable but the converse does not hold in general. However, if A and B are mutually permutable and $A \cap B = 1$, then they are totally permutable.

Products of mutually and totally permutable subgroups have been widely studied in the last twenty five years and receive a full discussion in [1]. The emphasis was on how the structure of the factors A and B affects the structure of the factorised group $G = AB$ and vice versa. One of the outcomes of these investigations is that the intersection $A \cap B$ determines how 'far away' the product of two totally (respectively mutually) permutable subgroups is from a central (respectively totally permutable) product. In this context, it is natural and interesting to ask how, for a mutually permutable product, the embedding of some distinguished families of subgroups of the intersection of the factors influences the structure of the product. Our main goal here is to prove local structural results on mutually permutable products in which the subgroups of order p (p a prime) and cyclic subgroups of order 4 (if $p = 2$) of the largest normal subgroup contained in the intersection of the factors are well situated in the group. Our starting point is the classical one: a mutually permutable product of two p-supersoluble groups is not p-supersoluble in general ([1, Example 4.1.32]).

We present now our relevant definition. A subgroup H of a group G is said to be *weakly s-supplemented* in G if there exist subgroups S and T of G such that the following conditions hold.

1. $G = HT$.
2. $H \cap T \subseteq S \subseteq H$.
3. $SQ = QS$ for every Sylow subgroup Q of G.

We mention that a subgroup S satisfying (3) is said to be S-*permutable* in G. According to [1, Theorem 1.2.2], the subgroup generated by two S-permutable subgroups of G is S-permutable so that every subgroup H of G contains a unique largest S-permutable subgroup H_{sG}. Hence (2) and (3) are often combined in a single condition: that $H \cap T$ is contained in H_{sG}.

Weakly s-supplementation is a subgroup embedding property introduced by Skiba in [8] as a common extension of S-permutability and complementation. It turns out to be very useful in establishing results about the group structure.

In the sequel, *p will be a fixed prime*.

A group G is said to be *p-supersoluble* if all chief factors of G whose order is divisible by p have order p. G is said to be *p-nilpotent* if G has a normal Hall p'-subgroup, or equivalently, every chief factor whose order is divisible by p is central in G.

Note if p is the smallest prime dividing the order of a group G, then G is p-supersoluble if and only if G is p-nilpotent. Moreover, a group G is supersoluble if and only if G is p-supersoluble for all primes p.

Our main first result concerns mutually permutable products $G = AB$ in which the cyclic subgroups of order p and 4 (if $p = 2$) of $\mathrm{Core}_G(A \cap B)$, the largest normal subgroup of G contained in $A \cap B$, are weakly s-supplemented in G.

Before we state our theorem, we recall that if $\mathscr{F}$ is a formation, the $\mathscr{F}$-residual $G^{\mathscr{F}}$ of a group G is the smallest normal subgroup of G with quotient in $\mathscr{F}$. In particular, we have that $G^{\mathscr{F}}$ is a p'-group if and only if $G/O_{p'}(G)$ is an $\mathscr{F}$-group.

Theorem 1. *Let $\mathscr{F}$ be a saturated formation containing the class of all p-supersoluble groups. Let the group $G = AB$ be the mutually permutable product of the $\mathscr{F}$-subgroups A and B such that $\mathrm{Core}_G(A \cap B)$ is p-soluble. Suppose that every cyclic subgroup of $\mathrm{Core}_G(A \cap B)$ of order p and order 4 (if $p = 2$) are weakly s-supplemented in G. Then $G^{\mathscr{F}}$ is a p'-group.*

The following statement is an immediate corollary of Theorem 1.

Corollary 1. *Let the group $G = AB$ be the mutually permutable product of the p-supersoluble subgroups A and B. Suppose that every cyclic subgroup of $\mathrm{Core}_G(A \cap B)$ of order p and order 4 (if $p = 2$) are weakly s-supplemented in G. Then G is p-supersoluble.*

If A and B are p-soluble and G is the mutually permutable product of A and B, then G is p-soluble ([1, Theorem 4.1.15]). However, if A and B are of p-length k, then G has not p-length k in general unless the product is totally permutable

([1, Theorem 4.2.10]). In a recent paper [3], Cossey and the second author proved that a group $G = AB$ which is a mutually permutable product of two p-soluble subgroups A and B of p-length at most k is a p-soluble group of p-length at mosk $k+1$. Since the class of all p-soluble groups of p-length at most k is a saturated formation containing all p-supersoluble groups which is extensible by p'-groups, we have:

COROLLARY 2. *Let the group $G = AB$ be the mutually permutable product of two p-soluble subgroups A and B of p-length at most k. If every cyclic subgroup of order p and order 4 (if $p = 2$) of $\mathrm{Core}_G(A \cap B)$ are weakly s-supplemented in G, then G is p-soluble of p-length at most k.*

By Corollary 1, the class of all supersoluble groups is closed under taking mutually permutable products G in which the cyclic subgroups of prime order and order 4 of the largest normal subgroup of G contained in the intersection of the factors are weakly s-supplemented. Our second main result shows that this result can also be obtained owing to a general completeness property of all saturated formations containing all supersoluble groups.

THEOREM 2. *Let $\mathcal{F}$ be a saturated formation containing the class of all supersoluble groups. Let the group $G = AB$ be the mutually permutable product of the $\mathcal{F}$-subgroups A and B. Suppose that every cyclic subgroup of prime order and order 4 of $\mathrm{Core}_G(A \cap B)$ are weakly s-supplemented in G. Then G is an $\mathcal{F}$-group.*

It is worth noting that Theorem 1 does not hold if we only assume that the subgroups of order p and order 4 of $\mathrm{Core}_G(A \cap B)$ are weakly s-supplemented in A and B. In [1, Example 4.1.32], a non-5-supersoluble group G with elementary abelian Sylow 5-subgroups which is a mutually permutable product of two 5-supersoluble subgroups A and B is presented. However, by [7, Lemma 4.5], every subgroup of order 5 of $\mathrm{Core}_G(A \cap B)$ is weakly s-supplemented in A and B. In the case $p = 2$, the following stronger version of Theorem 1 holds.

THEOREM 3. *Let $\mathcal{F}$ be a saturated formation containing the class of all 2-nilpotent groups. Let the group $G = AB$ be the mutually permutable product of the $\mathcal{F}$-subgroups A and B. Suppose that every subgroup of order 2 and every cyclic subgroup of order 4 of $\mathrm{Core}_G(A \cap B)$ are weakly s-supplemented in A and B. Then $G^{\mathcal{F}}$ is a $2'$-group.*

It is not difficult to prove that the saturated formation of all 2-nilpotent groups is closed under taking mutually permutable products. Hence Theorem 3 holds in this case without any restriction on the subgroups of order 2 and 4.

We have been unable to decide if the conclusion of Theorem 3 still holds for any saturated formation containing the class of all 2-nilpotent groups if the hypothesis on the subgroups of order 4 is removed.

However, if we demand permutability of the subgroups of order p with the Sylow q-subgroups of A and B for all primes $q \neq p$, we can confirm that p does not divide the order of $G^{\mathcal{F}}$.

THEOREM 4. *Let $\mathcal{F}$ be a saturated formation containing the class of all p-supersoluble groups. Let the group $G = AB$ be the mutually permutable product of the $\mathcal{F}$-subgroups A and B such that* $\mathrm{Core}_G(A \cap B)$ *is p-soluble. Suppose that every subgroup of order p and every cyclic subgroup of order 4 (if $p = 2$) of* $\mathrm{Core}_G(A \cap B)$ *permute with every Sylow q-subgroup of A and every Sylow q-subgroup of B, for each prime $q \neq p$. Then $G^{\mathcal{F}}$ is a p'-group.*

To show how the above results can be used, we present some applications. The first one can be considered as a global version of Theorem 4.

THEOREM 5. *Let $\mathcal{F}$ be a saturated formation containing the class of all supersoluble groups. Let the group $G = AB$ be the mutually permutable product of the $\mathcal{F}$-subgroups A and B. Suppose that for each prime p, every cyclic subgroup of* $\mathrm{Core}_G(A \cap B)$ *of order p and order 4 (if $p = 2$) are permutable with every Sylow q-subgroup of A and every Sylow q-subgroup of B, for each prime $q \neq p$. Then G is an $\mathcal{F}$-group.*

Finally, we present applications of Theorems 3 and 4 to the situation where $\mathcal{F}$ is the saturated formation of all p-soluble groups of p-length at most k.

COROLLARY 3. *Let the group $G = AB$ be the mutually permutable product of two soluble subgroups A and B of 2-length at most k. If every subgroup of order p and every cyclic subgroup of order 4 of* $\mathrm{Core}_G(A \cap B)$ *are weakly s-supplemented in A and B, then G is a soluble group of 2-length at most k.*

COROLLARY 4. *Let the group $G = AB$ be the mutually permutable product of two p-soluble subgroups A and B of p-length at most k, $p > 2$. Suppose that every subgroup of order p of* $\mathrm{Core}_G(A \cap B)$ *is permutable with every Sylow q-subgroup of A and every Sylow q-subgroup of B, for each prime $q \neq p$. Then G is a p-soluble group of p-length at most k.*

2. PROOFS

The proofs of our results depend heavily of the following three lemmas. The first two ones are particularly useful when an inductive argument involving subgroups or quotients is applied.

LEMMA 1 ([8, Lemma 2.10]). *Let G be a group and N a normal subgroup of G.*

(a) *If $H \le K \le G$ and H is weakly s-supplemented in G, then H is weakly s-supplemented in K.*
(b) *If $N \le H$ and H is weakly s-supplemented in G, then H/N is weakly s-supplemented in G/N.*
(c) *Suppose that E is a normal subgroup of G and $(|H|, |E|) = 1$. Then H is weakly s-supplemented in G if and only if HE/E is weakly s-supplemented in G/E.*

Lemma 2 ([1, Lemma 4.1.10]). *Let G be a group and let U and M be two subgroups of G and N a normal subgroup of G. If U and M are mutually permutable subgroups of G, then UN/N and MN/N are mutually permutable subgroups of G/N.*

Our next lemma is an important result confirming that the saturated formations containing the class of all supersoluble groups are closed under taking mutually permutable products with core-free intersection.

Lemma 3 ([1, Corollary 4.5.9]). *Let $\mathscr{F}$ be a saturated formation containing the class of all supersoluble groups. Let the group $G = AB$ be the mutually permutable product of the $\mathscr{F}$-subgroups A and B. If $\mathrm{Core}_G(A \cap B) = 1$, then $G \in \mathscr{F}$.*

The weakly s-supplementation of the subgroups of order p and cyclic subgroups of order 4 of a normal p-subgroup forces it to be contained in the supersoluble hypercentre of the group.

Lemma 4 ([6, Theorem 3.2 and Theorem 3.3]). *Let P be a normal p-subgroup of G. Assume that every subgroup of P of order p and every cyclic subgroups of order 4 (if $p = 2$) are weakly s-supplemented in G. Then every chief factor of G below P is of order p.*

Proof of Theorem 1. Suppose the result is false and G is a minimal counterexample. Let $N = \mathrm{Core}_G(A \cap B)$. By Lemma 2, G/N is a mutually permutable product of the subgroups A/N and B/N. Since A/N and B/N are both $\mathscr{F}$-subgroups and $\mathrm{Core}_{G/N}(A/N \cap B/N) = 1$, it follows from Lemma 3 that $G/N \in \mathscr{F}$. Hence $G^{\mathscr{F}} \subseteq N$.

Set $\bar{G} = G/O_{p'}(N)$ and let $\bar{A}$ and $\bar{B}$ be the images of A and B in $\bar{G}$. Then, by Lemma 2, $\bar{G}$ is the mutually permutable product of the $\mathscr{F}$-subgroups $\bar{A}$ and $\bar{B}$, $\mathrm{Core}_{\bar{G}}(\bar{A} \cap \bar{B})$ is p-soluble, and every subgroup of order p and every cyclic subgroup of order 4 (if $p = 2$) of $\mathrm{Core}_{\bar{G}}(\bar{A} \cap \bar{B})$ are weakly s-supplemented in $\bar{G}$ by Lemma 1. This shows that $\bar{G}$ satisfies the hypotheses of the theorem.

Now suppose that $O_{p'}(N) \neq 1$. We conclude by the minimality of G that $\bar{G}/O_{p'}(\bar{G}) \in \mathscr{F}$. Note that $\bar{G}/O_{p'}(\bar{G}) \cong G/O_{p'}(G)$, thus $G/O_{p'}(G) \in \mathscr{F}$, contrary to the choice of G. Therefore $O_{p'}(N) = 1$.

Let $D = O_p(N)$. Since N is p-soluble, it follows that $C_N(D) \subseteq D$ by [5, Chapter 6, Theorem 3.2]. Applying Lemma 4, we have that every G-chief factor below D is of order p. Let $1 = D_0 \leq D_1 \leq \cdots \leq D_m = D$ be part of a chief series of G below D, and let $E = \bigcap_{i=1}^{m}(C_N(D_i/D_{i-1}))$. Clearly E is a normal subgroup of N. Since $|D_i/D_{i-1}| = p$, $N/C_N(D_i/D_{i-1})$ is a cyclic group of exponent dividing $p-1$. It follows that N/E is an abelian group with exponent dividing $p-1$. By [5, Chapter 5, Theorem 3.2], $E/C_N(D)$ is a p-group. Since $C_N(D)$ is also a p-group, it follows that E is a p-group. Consequently, $E = D$ is the unique Sylow p-subgroup of N. This implies that every chief factor of G below N whose order is divisible by p is of order p. By [9, Lemma 2.11], it follows that $G/O_{p'}(G) \in \mathscr{F}$. This contradiction proves the theorem. □

Proof of Theorem 2. By Lemma 1, every subgroup of prime order and every cyclic subgroup of order 4 of $\mathrm{Core}_G(A \cap B)$ is weakly s-supplemented in $\mathrm{Core}_G(A \cap B)$. By [6, Theorem 3.5], $\mathrm{Core}_G(A \cap B)$ is soluble so that it is p-soluble for each prime p.

Fixed a prime p, let $\mathscr{G}$ be the class of all finite groups H such that $H^{\mathscr{F}}$ is a p'-group. Since the class of all p'-groups is a formation, we can apply [4, Chapter IV, Theorems 1.8 and 3.13] to conclude that $\mathscr{G}$ is a saturated formation. We argue that $\mathscr{G}$ contains all p-supersoluble groups. Let X be a p-supersoluble group. Set $\bar{X} = X/O_{p'}(X)$. Since $O_{p'}(\bar{X}) = 1$, we can apply [1, Lemma 2.1.6] to conclude that $\bar{X}$ has a normal Sylow subgroup, $\bar{P}$ say, such that $\bar{X}/\bar{P}$ is abelian. Therefore $\bar{X}$ is supersoluble and so $\bar{X}$ is an $\mathscr{F}$-group. In particular, $X \in \mathscr{G}$.

By Theorem 1, $G/O_{p'}(G) \in \mathscr{G}$. Hence $(G/O_{p'}(G))/(O_{p'}(G/O_{p'}(G))) \in \mathscr{F}$. Since $O_{p'}(G/O_{p'}(G)) = 1$, it follows that $G/O_{p'}(G) \in \mathscr{F}$.

Consequently, $G^{\mathscr{F}}$ is a p'-group for all primes p and thus $G \in \mathscr{F}$, as wanted. □

Proof of Theorem 3. We suppose that the theorem is false and derive a contradiction. Let G be a counterexample of minimal order. Then, arguing as in Theorem 1, $O_{2'}(N) = 1$, where $N = \mathrm{Core}_G(A \cap B)$. Moreover, since N is 2-nilpotent by [6, Theorem 3.1], we conclude that N is a 2-group. Therefore N is contained in the hypercentre of A and B by [6, Theorem 3.2]. In particular, N normalises all Sylow subgroups of A and B. Hence N is contained in the hypercentre of G. By [9, Lemma 2.11], it follows that $G/O_{2'}(G) \in \mathscr{F}$. This contradiction proves the theorem. □

Proof of Theorem 4. We suppose that the theorem is false and derive a contradiction. Let G be a counterexample of minimal order. Then, arguing as in Theorem 1, $O_{p'}(N) = 1$, where $N = \mathrm{Core}_G(A \cap B)$. Moreover, $G/N \in \mathscr{F}$ by Lemma 3 and if $D = O_p(N)$, we have that $C_N(D) \subseteq D$ by [5, Chapter 6, Theorem 3.2].

Assume that $p = 2$. Then N is a 2-group because N is 2-nilpotent by [6, Theorem 3.1]. In this case, every subgroup of order 2 and every cyclic subgroup of order 4 of N are S-permutable in A and B. In particular, they are weakly s-supplemented in A and B. By Theorem 3, $G^{\mathscr{F}}$ is a $2'$-group. Therefore we may assume that $p > 2$.

By [5, Chapter 5, Theorem 3.13], D possesses a characteristic subgroup X of exponent p such that every nontrivial p'-automorphism of D induces a nontrivial automorphism of X. Since D is normal in G and X is characteristic in D, we have that X is normal in G.

Let P be a Sylow p-subgroup of G, and let $1 = X_0 < X_1 < \cdots < X_m = X$ be a P-chief series of X. Clearly X_i/X_{i-1} is a cyclic group of order p for each $1 < i \leq m$. We use induction on i to show that X_i is normal in G, for each $1 < i \leq m$. Assume that X_{i-1} is normal in G and we want to prove that X_i is normal in G. Since $|X_i/X_{i-1}| = p$ and X is of exponent p, X_i has a cyclic subgroup of order p, L say, such that $X_i = LX_{i-1}$.

Fixed a prime $q \neq p$, and let Q_1 be a Sylow q-subgroup of A. Since A and B are mutually permutable, it follows that Q_1B is a subgroup of G. Let Q be a Sylow q-subgroup of Q_1B containing Q_1 and let Q_2 be a Sylow q-subgroup of B containing $Q \cap B$. Then $Q = Q_1Q_2$ is a Sylow q-subgroup of G. By hypothesis, LQ_1 and LQ_2 are subgroups of G. Since X is normal in G and $L = X \cap LQ_1 = X \cap LQ_2$, it follows that L is normalised by both Q_1 and Q_2. Hence L is normal in LQ. Since $X_i = LX_{i-1}$ and X_{i-1} is normal in G by induction, we have that X_i is normalized by Q.

Consequently, $|G : N_G(X_i)|$ is a power of p. Since X_i is also normal in P, it follows that X_i is normal in G, as desired.

Therefore, $1 = X_0 < X_1 < \cdots < X_m = X$ is part of a G-chief series below X. By [5, Chapter 5, Theorem 3.2], $(\bigcap_{i=1}^m (C_G(X_i/X_{i-1}))/C_G(X)$ is a p-group. Since every nontrivial p'-automorphism of D induces a nontrivial automorphism of X, we have that $C_G(X)/C_G(D)$ is a p-group. Hence $(\bigcap_{i=1}^m (C_G(X_i/X_{i-1}))/C_G(D)$ is a p-group. Since $|X_i/X_{i-1}| = p$ for each $1 < i \leq m$, $G/(\bigcap_{i=1}^m (C_G(X_i/X_{i-1})))$ is an abelian group of exponent dividing $p-1$. It follows that $(\bigcap_{i=1}^m (C_G(X_i/X_{i-1}))/C_G(D)$ is the unique Sylow p-subgroup of $G/C_G(D)$, and a Hall p'-subgroup of $G/C_G(D)$ is an abelian group of exponent dividing $p-1$.

Let H/K be any G-chief factor of G below D. Then $G/C_G(H/K)$ is a quotient of $G/C_G(D)$ and thus $G/C_G(H/K)$ has a normal Sylow p-subgroup, and a Hall p'-subgroup of $G/C_G(H/K)$ is an abelian group of exponent dividing $p-1$. But $O_p(H/K) = 1$ by [4, Chapter A, Lemma 13.6], and thus $G/C_G(H/K)$ is an abelian group with exponent dividing $p-1$. Since H/K is an irreducible and faithful $G/C_G(H/K)$-module, it follows that $|H/K| = p$ by [4, Chapter B, Theorem 9.8].

We have proved that any G-chief factor below D is of order p. Now arguing as in the proof of Theorem 1, we have that $G/O_{p'}(G) \in \mathscr{F}$. This contradicts the choice of G.

The proof of the theorem is complete. □

Proof of Theorem 5. We first show that $\mathrm{Core}_G(A \cap B)$ is 2-nilpotent. Assume this is not true and let X be a minimal non-2-nilpotent subgroup of $\mathrm{Core}_G(A \cap B)$. Then $X = P \rtimes Q$, where P is the normal Sylow 2-subgroup of X of exponent at most 4, and Q is a cyclic q-subgroup of X for some prime $q \neq 2$. Let x be any element of P and let $L = \langle x \rangle$. Then L is a cyclic subgroup of order 2 or 4. Let K be a Sylow q-subgroup of A containing Q. Then LK is a subgroup of G and thus $LK \cap X = L(K \cap X) = LQ$ is a subgroup of G. Since $L = P \cap LQ$ and P is normal in X, L is normalized by Q. Since the automorphism group of L is a 2-group, it follows that $Q \subseteq C_X(L) = C_X(x)$. Consequently $Q \subseteq C_G(P)$ and X is 2-nilpotent, contrary to the choice of X. Therefore, $\mathrm{Core}_G(A \cap B)$ is 2-nilpotent and hence it is soluble by the Feit–Thompson theorem.

Now we can apply the same arguments used in the proof of Theorem 2 together with Theorem 4 to conclude that $G \in \mathscr{F}$. □

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Adolfo Ballester-Bolinches
Department of Mathematics
Guangdong University of Education
351 Xingang Middle Road
510310 Guangzhou, China
and
Departament de Matemàtiques
Universitat de València
Dr. Moliner 50
Burjassot
46100 València, Spain
Adolfo.Ballester@uv.es

Yangming Li
Department of Mathematics
Guangdong University of Education
351 Xingang Middle Road
510310 Guangzhou, China
liyangming@gdei.edu.cn

Mari Carmen Pedraza-Aguilera
Instituto Universitario de Matemática Pura y Aplicada
Universitat Politècnica de València
Camino de Vera
46022 Valencia, Spain
mpedraza@mat.upv.es

Ning Su
Department of Mathematics
Sun Yat-sen University
135 Xingang West Road
510275 Guangzhou, China
suning3@mail.sysu.edu.cn