



**Partial Differential Equations** — *Nodal solutions for nonlinear nonhomogeneous Robin problems*, by NIKOLAOS S. PAPAGEORGIOU, VICENȚIU D. RĂDULESCU and DUŠAN D. REPOVŠ, communicated on June 22, 2018.

ABSTRACT. — We consider the nonlinear Robin problem driven by a nonhomogeneous differential operator plus an indefinite potential. The reaction term is a Carathéodory function satisfying certain conditions only near zero. Using suitable truncation, comparison, and cut-off techniques, we show that the problem has a sequence of nodal solutions converging to zero in the  $C^1(\bar{\Omega})$ -norm.

KEY WORDS: Nodal solutions, indefinite potential, nonhomogeneous differential operator, nonlinear regularity theory, truncation and cut-off techniques

MATHEMATICS SUBJECT CLASSIFICATION: 35J20, 35J60

## 1. INTRODUCTION

Let  $\Omega \subseteq \mathbb{R}^N$  be a bounded domain with a  $C^2$ -boundary  $\partial\Omega$ . We study the following nonlinear nonhomogeneous Robin problem:

$$(1) \quad \left\{ \begin{array}{l} -\operatorname{div} a(Du(z)) + \xi(z)|u(z)|^{p-2}u(z) = f(z, u(z)) \quad \text{in } \Omega, \\ \frac{\partial u}{\partial n_a} + \beta(z)|u|^{p-2}u = 0 \quad \text{on } \partial\Omega. \end{array} \right\}$$

In this problem,  $a : \mathbb{R}^N \rightarrow \mathbb{R}^N$  is a continuous and strictly monotone map (thus also maximal monotone), which satisfies certain regularity and growth conditions listed in hypotheses  $H(a)$  below. These conditions are general and they incorporate in our framework many differential operators of interest, such as the  $p$ -Laplacian and the  $(p, q)$ -Laplacian. We stress that  $a(\cdot)$  is not homogeneous and this is a source of difficulties in the study of problem (1). The potential function  $\xi \in L^\infty(\Omega)$  is indefinite (that is, sign changing). The reaction term (the right-hand side of (1)) is a Carathéodory function (that is, for all  $x \in \mathbb{R}$ , the function  $z \mapsto f(z, x)$  is measurable, and for almost all  $z \in \Omega$ , the function  $x \mapsto f(z, x)$  is continuous. We impose conditions on  $f(z, \cdot)$  only near zero. In the boundary condition,  $\frac{\partial u}{\partial n_a}$  denotes the conormal derivative corresponding to the differential operator  $u \mapsto \operatorname{div} a(Du)$  and is defined by extension of the map

$$C^1(\bar{\Omega}) \ni u \mapsto (a(Du), n)_{\mathbb{R}^N},$$

with  $n(\cdot)$  being the outward unit normal on  $\partial\Omega$ .

We are looking for nodal (that is, sign-changing) solutions for problem (1). Employing a symmetry condition on  $f(z, \cdot)$  near zero and using truncation, perturbation, comparison, and cut-off techniques, and a result of Kajikiya [7], we generate a whole sequence  $\{u_n\}_{n \geq 1} \subseteq C^1(\bar{\Omega})$  of distinct nodal solutions such that  $u_n \rightarrow 0$  in  $C^1(\bar{\Omega})$ .

The first result in this direction was produced by Wang [27], who used cut-off techniques to produce an infinity of solutions converging to zero in  $H_0^1(\Omega)$ . In Wang [27] the problem is semilinear driven by the Dirichlet Laplacian. There is no potential term (that is,  $\xi \equiv 0$ ). The sequence produced by Wang [27] does not consist of nodal solutions. More recently, Li & Wang [8] produced a sequence of nodal solutions for semilinear Schrödinger equations. For nonlinear equations we mention the recent works of He, Huang, Liang & Lei [5], and Papageorgiou & Rădulescu [19]. In He *et al.* [5], the problem is Neumann (that is,  $\beta \equiv 0$ ) and the differential operator is the  $p$ -Laplacian (that is,  $a(y) = |y|^{p-2}y$  for all  $y \in \mathbb{R}^N$ , with  $1 < p < \infty$ ). In Papageorgiou & Rădulescu [19], the differential operator is the same as in the present paper, but  $\xi \equiv 0$ . Also, the hypotheses on  $f(z, \cdot)$  near zero are more restrictive. In the present paper we extend the results of all aforementioned works.

## 2. PRELIMINARIES AND HYPOTHESES

In the study of problem (1) we will use the following spaces: the Sobolev space  $W^{1,p}(\Omega)$ , the Banach space  $C^1(\bar{\Omega})$ , and the boundary Lebesgue spaces  $L^r(\partial\Omega)$ ,  $1 \leq r \leq \infty$ .

We denote by  $\|\cdot\|$  the norm on the Sobolev space  $W^{1,p}(\Omega)$  defined by

$$\|u\| = [ \|u\|_p^p + \|Du\|_p^p ]^{1/p} \quad \text{for all } u \in W^{1,p}(\Omega).$$

The Banach space  $C^1(\bar{\Omega})$  is an ordered Banach space, with positive (order) cone

$$C_+ = \{u \in C^1(\bar{\Omega}) : u(z) \geq 0 \text{ for all } z \in \bar{\Omega}\}.$$

This cone has a nonempty interior which is the set

$$D_+ = \{u \in C_+ : u(z) > 0 \text{ for all } z \in \bar{\Omega}\}.$$

In fact,  $D_+$  is also the interior of  $C_+$  when furnished with the relative  $C(\bar{\Omega})$ -norm topology.

On  $\partial\Omega$  we consider the  $(N-1)$ -dimensional Hausdorff (surface) measure  $\sigma(\cdot)$ . Using this measure, we can define in the usual way the Lebesgue spaces  $L^r(\partial\Omega)$ ,  $1 \leq r \leq \infty$ . From the theory of Sobolev spaces we know that there exists a unique continuous linear map  $\gamma_0 : W^{1,p}(\Omega) \rightarrow L^p(\partial\Omega)$ , known as the “trace map”, such that

$$\gamma_0(u) = u|_{\partial\Omega} \quad \text{for all } u \in W^{1,p}(\Omega) \cap C(\bar{\Omega}).$$

So, the trace map assigns “boundary values” to all Sobolev functions. We know that the trace map is compact into  $L^r(\partial\Omega)$  for all  $1 \leq r < \frac{(N-1)p}{N-p}$  if  $p < N$ , and into  $L^r(\partial\Omega)$  for all  $1 \leq r < \infty$  if  $p \geq N$ . Furthermore, we have that

$$\ker \gamma_0 = W_0^{1,p}(\Omega) \quad \text{and} \quad \text{im } \gamma_0 = W^{\frac{1}{p'},p}(\partial\Omega) \quad \left(\frac{1}{p} + \frac{1}{p'} = 1\right).$$

In what follows, for the sake of notational simplicity, we will drop the use of the trace map  $\gamma_0(\cdot)$ . All restrictions of Sobolev functions on  $\partial\Omega$ , are understood in the sense of traces.

Let  $X$  be a Banach space and  $\varphi \in C^1(X, \mathbb{R})$ . We say that  $\varphi$  satisfies the “Palais-Smale condition” (the “PS-condition” for short), if the following property holds:

“Every sequence  $\{u_n\}_{n \geq 1} \subseteq X$  such that  $\{\varphi(u_n)\}_{n \geq 1} \subseteq \mathbb{R}$  is bounded and  $\varphi'(u_n) \rightarrow 0$  in  $X^*$  as  $n \rightarrow \infty$ , admits a strongly convergent subsequence.”

We shall need the following result of Kajikya [7].

**THEOREM 1.** *Assume that  $X$  is a Banach space,  $\varphi \in C^1(X, \mathbb{R})$  satisfies the PS-condition,  $\varphi$  is even and bounded below,  $\varphi(0) = 0$ , and for every  $n \in \mathbb{N}$ , there exists an  $n$ -dimensional subspace  $V_n$  of  $X$  and  $\rho_n > 0$  such that*

$$\sup\{\varphi(u) : u \in V_n \cap \partial B_{\rho_n}\} < 0,$$

where  $\partial B_{\rho_n} = \{u \in X : \|u\|_X = \rho_n\}$ . Then there exists a sequence  $\{u_n\}_{n \geq 1} \subseteq X \setminus \{0\}$  such that

- (i)  $\varphi'(u_n) = 0$  for all  $n \in \mathbb{N}$  (that is, each  $u_n$  is a critical point of  $\varphi$ );
- (ii)  $\varphi(u_n) \leq 0$  for all  $n \in \mathbb{N}$ ; and
- (iii)  $u_n \rightarrow 0$  in  $X$  as  $n \rightarrow \infty$ .

In the sequel, for any  $\varphi \in C^1(X, \mathbb{R})$ , we denote by  $K_\varphi$  the critical set of  $\varphi$ , that is,

$$K_\varphi = \{u \in X : \varphi'(u) = 0\}.$$

For  $X \in \mathbb{R}$ , we set  $x^\pm = \max\{\pm x, 0\}$ . Then for any  $u \in W^{1,p}(\Omega)$ , we define  $u^\pm(\cdot) = u(\cdot)^\pm$ . We know that

$$u^\pm \in W^{1,p}(\Omega), \quad u = u^+ - u^-, \quad |u| = u^+ + u^-.$$

Let  $\mathfrak{J} \in C^1(0, \infty)$  be such that  $\mathfrak{J}(t) > 0$  for all  $t > 0$  and

$$(2) \quad 0 < \hat{c} \leq \frac{\vartheta'(t)t}{\vartheta(t)} \leq c_0 \quad \text{and} \quad c_1 t^{p-1} \leq \vartheta(t) \leq c_2(t^{\tau-1} + t^{p-1})$$

for all  $t > 0$ , with  $c_1, c_2 > 0$ ,  $1 \leq \tau < p$ .

Then the hypotheses on the map  $a(\cdot)$  are the following:

$H(a) : a(y) = a_0(|y|)y$  for all  $y \in \mathbb{R}^N$  with  $a_0(t) > 0$  for all  $t > 0$  and

(i)  $a_0 \in C^1(0, \infty)$ ,  $t \mapsto a_0(t)t$  is strictly increasing on  $(0, \infty)$ ,  $a_0(t)t \rightarrow 0^+$  as  $t \rightarrow 0^+$  and

$$\lim_{t \rightarrow 0^+} \frac{a_0'(t)t}{a_0(t)} > -1;$$

(ii)  $|\nabla a(y)| \leq c_3 \frac{\vartheta(|y|)}{|y|}$  for all  $y \in \mathbb{R}^N \setminus \{0\}$ , and some  $c_3 > 0$ ;

(iii)  $(\nabla a(y)\xi, \xi)_{\mathbb{R}^N} \geq \frac{\vartheta(|y|)}{|y|} |\xi|^2$  for all  $y \in \mathbb{R}^N \setminus \{0\}$ ,  $\xi \in \mathbb{R}^N$ ; and

(iv) If  $G_0(t) = \int_0^t a_0(s)s \, ds$  for all  $t > 0$ , then there exists  $q \in (1, p]$  such that

$$t \mapsto G_0(t^{1/q}) \text{ is convex and } \limsup_{t \rightarrow 0^+} \frac{qG_0(t)}{t^q} < +\infty.$$

REMARK 1. Hypotheses  $H(a)(i)$ ,  $(ii)$ ,  $(iii)$  are dictated by the nonlinear global regularity theory of Lieberman [10] and the nonlinear maximum principle of Pucci & Serrin [24]. Hypothesis  $H(a)(iv)$  reflects the particular requirements of our problem. However,  $H(a)(iv)$  is not restrictive as the examples below illustrate.

Hypotheses  $H(a)$  imply that  $G_0(\cdot)$  is strictly convex and strictly increasing. We set  $G(y) = G_0(|y|)$  for all  $y \in \mathbb{R}^N$ . Evidently,  $G(\cdot)$  is convex and  $G(0) = 0$ . Also, we have

$$\nabla G(y) = G_0'(|y|) \frac{y}{|y|} = a_0(|y|)y = a(y) \quad \text{for all } y \in \mathbb{R}^N \setminus \{0\}, \quad \nabla G(0) = 0.$$

So,  $G(\cdot)$  is the primitive of  $a(\cdot)$ . Moreover, the convexity of  $G(\cdot)$  implies that

$$(3) \quad G(y) \leq (a(y), y)_{\mathbb{R}^N} \quad \text{for all } y \in \mathbb{R}^N.$$

The next lemma summarizes the main properties of the map  $a(\cdot)$  and it is an easy consequence of hypotheses  $H(a)$  and condition (2) above.

LEMMA 2. *If hypotheses  $H(a)(i)$ ,  $(ii)$ ,  $(iii)$  hold, then*

- (a)  $a(\cdot)$  is continuous, strictly monotone, hence maximal monotone, too;
- (b)  $|a(y)| \leq c_4(1 + |y|^{p-1})$  for all  $y \in \mathbb{R}^N$ , and some  $c_4 > 0$ ; and
- (c)  $(a(y), y)_{\mathbb{R}^N} \geq \frac{c_1}{p-1} |y|^p$  for all  $y \in \mathbb{R}^N$ .

This lemma and (3) lead to the following growth conditions on  $G(\cdot)$ .

**COROLLARY 3.** *If hypotheses  $H(a)(i), (ii), (iii)$  hold, then  $\frac{c_1}{p(p-1)}|y|^p \leq G(y) \leq c_5(1 + |y|^p)$  for all  $y \in \mathbb{R}^N$ , and some  $c_5 > 0$ .*

**EXAMPLE 1.** *The following maps  $a(y)$  satisfy hypotheses  $H(a)$ :*

(a)  $a(y) = |y|^{p-2}y, 1 < p < \infty$ .

*This map corresponds to the  $p$ -Laplace differential operator defined by*

$$\Delta_p u = \operatorname{div}(|Du|^{p-2}Du) \quad \text{for all } u \in W^{1,p}(\Omega).$$

(b)  $a(y) = |y|^{p-2}y + |y|^{q-2}y, 1 < q < p < \infty$ . *This map corresponds to the  $(p, q)$ -Laplace differential operator defined by*

$$\Delta_p u + \Delta_q u \quad \text{for all } u \in W^{1,p}(\Omega).$$

*Such operators arise in problems of mathematical physics. Recently  $(p, q)$ -equations have been studied by Bobkov & Tanaka [1], Li & Zhang [9], Marano & Mosconi [11], Marano, Mosconi & Papageorgiou [12, 13], Mugnai & Papageorgiou [16], Papageorgiou & Rădulescu [17], Sun, Zhang & Su [25], and Tanaka [26].*

(c)  $a(y) = (1 + |y|^2)^{\frac{p-2}{2}}y, 1 < p < \infty$ . *This map corresponds to the generalized  $p$ -mean curvature differential operator defined by*

$$\operatorname{div}((1 + |Du|^2)^{\frac{p-2}{2}}Du) \quad \text{for all } u \in W^{1,p}(\Omega).$$

(d)  $a(y) = |y|^{p-2}y \left(1 + \frac{1}{1 + |y|^p}\right), 1 < p < \infty$ .

We denote by  $\langle \cdot, \cdot \rangle$  the duality brackets for the pair

$$(W^{1,p}(\Omega)^*, W^{1,p}(\Omega)).$$

Let  $A : W^{1,p}(\Omega) \rightarrow W^{1,p}(\Omega)^*$  be the nonlinear map defined by

$$\langle A(u), h \rangle = \int_{\Omega} (a(Du), Dh)_{\mathbb{R}^N} dz \quad \text{for all } u, h \in W^{1,p}(\Omega).$$

From Gasinski & Papageorgiou [3], we have:

**PROPOSITION 4.** *The map  $A : W^{1,p}(\Omega) \rightarrow W^{1,p}(\Omega)^*$  is bounded (maps bounded sets to bounded sets), continuous, monotone (hence maximal monotone, too), and of type  $(S)_+$ , that is,*

$$“u_n \xrightarrow{w} u \text{ in } W^{1,p}(\Omega) \text{ and } \limsup_{n \rightarrow \infty} \langle A(u_n), u_n - u \rangle \Rightarrow u_n \rightarrow u”.$$

The hypotheses on the potential function  $\xi(\cdot)$  and on the boundary coefficient  $\beta(\cdot)$  are the following:

- $H(\xi) : \xi \in L^\infty(\Omega)$ .
- $H(\beta) : \beta \in C^{0,\alpha}(\partial\Omega)$  for some  $\alpha \in (0, 1)$  and  $\beta(z) \geq 0$  for all  $z \in \partial\Omega$ .

REMARK 2. If  $\beta \equiv 0$ , then we recover the Neumann problem.

Finally, we introduce our conditions on the reaction term  $f(z, x)$ :

$H(f)$ :  $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  is a Carathéodory function such that  $f(z, 0) = 0$  for almost all  $z \in \Omega$  and

- (i) there exists  $\eta > 0$  such that for almost all  $z \in \Omega$ ,  $f(z, \cdot)|_{[-\eta, \eta]}$  is odd;
- (ii)  $|f(z, x)| \leq a_\eta(z)$  for almost all  $z \in \Omega$ ,  $x \in [-\eta, \eta]$ , with  $a_\eta \in L^\infty(\Omega)$ ;
- (iii) with  $q \in (1, p]$  as in hypothesis  $H(a)(iv)$ , we have

$$\lim_{x \rightarrow 0} \frac{f(z, x)}{|x|^{q-2}x} = +\infty \quad \text{uniformly for almost all } z \in \Omega; \text{ and}$$

- (iv) there exists  $\hat{\xi} > 0$  such that for almost all  $z \in \Omega$

$$x \rightarrow f(z, x) + \hat{\xi}|x|^{p-2}x$$

is nondecreasing on  $[-\eta, \eta]$

REMARK 3. We point out that all the above hypotheses concern the behaviour of  $f(z, \cdot)$  only near zero.

Finally, we mention that nonlinear problems with an indefinite potential have recently been studied in the context of equations driven by the Neumann  $p$ -Laplacian by Gasinski & Papageorgiou [4] (resonant problems) and Fragnelli, Mugnai & Papageorgiou [2] (superlinear problems). Also, nodal solutions for nonlinear Robin problems with no potential term, were obtained by Papageorgiou & Rădulescu [21].

### 3. NODAL SOLUTIONS

Let  $\varepsilon \in (0, \eta)$  and consider an even function  $\gamma \in C^1(\mathbb{R})$  such that  $0 \leq \gamma \leq 1$ ,  $\gamma|_{[-\varepsilon, \varepsilon]} = 1$  and  $\text{supp } \gamma \subseteq [-\eta, \eta]$ .

We set

$$\hat{f}(z, x) = \gamma(x)f(z, x) + (1 - \gamma(x))\xi(z)|x|^{p-2}x.$$

Evidently,  $\hat{f}(z, x)$  is a Carathéodory function which is odd in  $x \in \mathbb{R}$  and has the following two additional properties:

- (4)  $\hat{f}(z, \cdot)|_{[-\varepsilon, \varepsilon]} = f(z, \cdot)|_{[-\varepsilon, \varepsilon]}$  for all  $z \in \Omega$ ;
- (5)  $\hat{f}(z, x) = \xi(z)|x|^{p-2}x$  for all  $z \in \Omega$ ,  $|x| \geq \eta$ .

It follows from (5) that

$$(6) \quad \hat{f}(z, \eta) - \xi(z)\eta^{p-1} = 0 \quad \text{for almost all } z \in \Omega.$$

Since  $\hat{f}(z, \cdot)$  is odd, we have

$$(7) \quad \hat{f}(z, -\eta) + \xi(z)\eta^{p-1} = 0 \quad \text{for almost all } z \in \Omega.$$

On account of hypothesis  $H(f)(iii)$ , given any  $\mu > 0$ , we can find  $\delta = \delta(\mu) \in (0, \varepsilon)$  such that

$$(8) \quad f(z, x)x = \hat{f}(z, x) \geq \mu|x|^q \quad \text{for almost all } z \in \Omega, \text{ and all } |x| \leq \delta \text{ (see (4)).}$$

Then (8) combined with hypothesis  $H(f)(ii)$  implies that given  $r > p$  we can find  $c_6 > 0$  such that

$$(9) \quad \hat{f}(z, x)x \geq \mu|x|^q - c_6|x|^r \quad \text{for almost all } z \in \Omega, \text{ and all } x \in \mathbb{R}.$$

We introduce the following function

$$(10) \quad k(z, x) = \mu|x|^{q-2}x - c_6|x|^{r-2}x.$$

This is a Carathéodory function which is odd in  $x \in \mathbb{R}$ .

We consider the following auxiliary nonlinear Robin problem:

$$(11) \quad \left\{ \begin{array}{l} -\operatorname{div} a(Du(z)) + |\xi(z)| |u(z)|^{p-2}u(z) = k(z, u(z)) \quad \text{in } \Omega, \\ \frac{\partial u}{\partial n_a} + \beta(z)|u|^{p-2}u = 0 \quad \text{on } \partial\Omega. \end{array} \right\}$$

**PROPOSITION 5.** *If hypotheses  $H(a)$ ,  $H(\xi)$ ,  $H(\beta)$  hold, then problem (11) admits a unique positive solution*

$$u^* \in D_+$$

and since  $k(z, \cdot)$  is odd,  $v^* = -u^* \in D_+$  is the unique negative solution of (11).

**PROOF.** We consider the Carathéodory function  $\hat{k}(z, x)$  defined by

$$(12) \quad \hat{k}(z, x) = \begin{cases} k(z, -\eta) - \eta^{p-1} & \text{if } x < -\eta \\ k(z, x) + |x|^{p-2}x & \text{if } -\eta \leq x \leq \eta \\ k(z, \eta) + \eta^{p-1} & \text{if } \eta < x. \end{cases}$$

We set  $\hat{K}(z, x) = \int_0^x \hat{k}(z, s) ds$  and consider the  $C^1$ -functional  $\hat{\phi}_+ : W^{1,p}(\Omega) \rightarrow \mathbb{R}$  defined by

$$\begin{aligned} \hat{\phi}_+(u) &= \int_{\Omega} G(Du) dz + \frac{1}{p} \int_{\Omega} [|\xi(z)| + 1]|u|^p dz + \frac{1}{p} \int_{\partial\Omega} \beta(z)|u|^p d\sigma \\ &\quad - \int_{\Omega} \hat{K}(z, u^+) dz \quad \text{for all } u \in W^{1,p}(\Omega). \end{aligned}$$

From (12) and Corollary 3 it is clear that

$\hat{\phi}_+(\cdot)$  is coercive.

Also, from the Sobolev embedding theorem and the compactness of the trace map, we deduce that

$\hat{\phi}_+(\cdot)$  is sequentially weakly lower semicontinuous.

So, by the Weierstrass–Tonelli theorem, we can find  $u^* \in W^{1,p}(\Omega)$  such that

$$(13) \quad \hat{\phi}_+(u^*) = \inf\{\hat{\phi}_+(u) : u \in W^{1,p}(\Omega)\}.$$

On account of hypothesis  $H(a)(iv)$ , we can find  $c_7 > 0$  such that

$$(14) \quad G(y) \leq \frac{c_7}{q}|y|^q \quad \text{for all } |y| \leq \delta,$$

with  $\delta > 0$  as in (8). Let  $u \in D_+$ . Then we can find  $t \in (0, 1)$  small such that

$$(15) \quad tu(z) \in (0, \delta] \quad \text{and} \quad |D(tu)(z)| \leq \delta \quad \text{for all } z \in \bar{\Omega}.$$

Using (10), (12), (14) and (15), we obtain

$$\begin{aligned} \hat{\phi}_+(tu) &\leq \frac{t^q c_7}{q} \|Du\|_q^q + \frac{t^q}{q} \int_{\Omega} |\xi(z)| |u|^q dz + \frac{t^q}{q} \int_{\partial\Omega} \beta(z) |u|^q d\sigma \\ &\quad + \frac{t^r}{r} \|u\|_r^r - \frac{t^q}{q} \mu \|u\|_q^q \\ &\quad \text{(since } t \in (0, 1), q \leq p < r) \\ &\leq [c_8 - \mu c_9] t^q \text{ for some } c_8, c_9 > 0 \text{ depending on } u. \end{aligned}$$

Choosing  $\mu > \frac{c_8}{c_9}$ , we infer that

$$\begin{aligned} &\hat{\phi}_+(tu) < 0, \\ \Rightarrow &\hat{\phi}_+(u^*) < 0 = \hat{\phi}_+(0) \quad \text{(see (13))}, \\ \Rightarrow &u^* \neq 0. \end{aligned}$$

From (13) we have

$$\begin{aligned} (16) \quad &\hat{\phi}'_+(u^*) = 0, \\ \Rightarrow &\langle A(u^*), h \rangle + \int_{\Omega} [|\xi(z)| + 1] |u^*|^{p-2} u^* h dz + \int_{\partial\Omega} \beta(z) |u^*|^{p-2} u^* h d\sigma \\ &= \int_{\Omega} \hat{k}(z, (u^*)^+) h dz \quad \text{for all } h \in W^{1,p}(\Omega). \end{aligned}$$



In (16) we choose  $h = -(u^*)^- \in W^{1,p}(\Omega)$ . Using Lemma 2(c), we obtain

$$\begin{aligned} & \frac{c_1}{p-1} \|D(u^*)^-\|_p^p + \|(u^*)^-\|_p^p \leq 0 \quad (\text{see hypothesis } H(B)), \\ \Rightarrow & \quad u^* \geq 0, u^* \neq 0. \end{aligned}$$

In (16) we choose  $h = (u^* - \eta)^+ \in W^{1,p}(\Omega)$ . Then

$$\begin{aligned} & \langle A(u^*), (u^* - \eta)^+ \rangle + \int_{\Omega} [|\xi(z)| + 1](u^*)^{p-1}(u^* - \eta)^+ dz \\ & \quad + \int_{\partial\Omega} \beta(z)(u^*)^{p-1}(u^* - \eta)^+ d\sigma \\ & = \int_{\Omega} [\mu\eta^{q-1} - c_6\eta^{r-1} + \eta^{p-1}](u^* - \eta)^+ dz \quad (\text{see (12) and (10)}) \\ & \leq \int_{\Omega} [\hat{f}(z, \eta) + \eta^{p-1}](u^* - \eta)^+ dz \quad (\text{see (9)}) \\ & = \int_{\Omega} [|\xi(z) + 1|\eta^{p-1}](u^* - \eta)^+ dz \quad (\text{see (6)}) \\ & \leq \langle A(\eta), (u^* - \eta)^+ \rangle + \int_{\Omega} [|\xi(z)| + 1]\eta^{p-1}(u^* - \eta)^+ dz \\ & \quad + \int_{\partial\Omega} \beta(z)\eta^{p-1}(u^* - \eta)^+ d\sigma \\ & \quad (\text{note that } A(\eta) = 0 \text{ and see hypothesis } H(\beta)), \\ \Rightarrow & \quad \langle A(u^*) - A(\eta), (u^* - \eta)^+ \rangle \\ & \quad + \int_{\Omega} [|\xi(z)| + 1]((u^*)^{p-1} - \eta^{p-1})(u^* - \eta)^+ dz \leq 0 \\ & \quad (\text{see hypothesis } H(\beta)), \\ \Rightarrow & \quad u^* \leq \eta. \end{aligned}$$

So, we have proved that

$$(17) \quad u^* \in [0, \eta] = \{u \in W^{1,p}(\Omega) : 0 \leq u(z) \leq \eta \text{ for almost all } z \in \Omega\}.$$

From (10), (12), (16) and (17), we infer that  $u^*$  is a positive solution of problem (11). From Papageorgiou & Rădulescu [20], we have

$$u^* \in L^\infty(\Omega).$$

Now the nonlinear regularity theory of Lieberman [10] implies that

$$u^* \in C_+ \setminus \{0\}.$$

From (16) and (17), we have

$$\left\{ \begin{array}{l} -\operatorname{div} a(Du^*(z)) + |\xi(z)|u^*(z)^{p-1} = k(z, u^*(z)) \quad \text{for almost all } z \in \Omega, \\ \frac{\partial u^*}{\partial n_a} + \beta(z)u^* = 0 \quad \text{on } \partial\Omega \end{array} \right\}$$

(see Papageorgiou & Rădulescu [18])

$$\begin{aligned} \Rightarrow & -\operatorname{div} a(Du^*(z)) + |\xi(z)|u^*(z)^{p-1} \\ & \geq -c_6 u^*(z)^{r-1} \quad \text{for almost all } z \in \Omega \text{ (see (10)),} \\ \Rightarrow & \operatorname{div} a(Du^*(z)) \leq [c_6 \|u^*\|_\infty^{r-p} + \|\xi\|_\infty] u^*(z)^{p-1} \quad \text{for almost all } z \in \Omega \\ & \text{(see hypothesis } H(\xi)), \\ \Rightarrow & u^* \in D_+ \quad \text{(see Pucci & Serrin [24, p. 120]).} \end{aligned}$$

Next, we show the uniqueness of this solution. To this end, let  $\hat{i}: L^1(\Omega) \rightarrow \bar{\mathbb{R}}$   $= \mathbb{R} \cup \{+\infty\}$  be the integral functional defined by

$$\hat{i}(u) = \begin{cases} \int_{\Omega} G(Du^{\frac{1}{q}}) dz + \frac{1}{p} \int_{\Omega} |\xi(z)|u^{\frac{p}{q}} dz \\ \quad + \frac{1}{p} \int_{\partial\Omega} \beta(z)u^{\frac{p}{q}} d\sigma & \text{if } u \geq 0, u^{\frac{1}{q}} \in W^{1,p}(\Omega) \\ +\infty & \text{otherwise.} \end{cases}$$

From Papageorgiou & Winkert [23] (see the proof of Proposition 3.3), we know that  $\hat{i}(\cdot)$  is convex and if  $u^*, v^* \in D_+$  are two positive solutions of (11), then

$$\begin{aligned} \hat{i}'((u^*)^q)(h) &= \frac{1}{q} \int_{\Omega} \frac{-\operatorname{div} a(Du^*) + |\xi(z)|(u^*)^{p-1}}{(u^*)^{q-1}} h dz \\ \hat{i}'((v^*)^q)(h) &= \frac{1}{q} \int_{\Omega} \frac{-\operatorname{div} a(Dv^*) + |\xi(z)|(v^*)^{p-1}}{(v^*)^{q-1}} h dz \quad \text{for } h = (u^*)^q - (v^*)^q. \end{aligned}$$

The convexity of  $\hat{i}(\cdot)$  implies the monotonicity of  $\hat{i}'(\cdot)$ . Hence

$$\begin{aligned} 0 &\leq \int_{\Omega} \left[ \frac{-\operatorname{div} a(Du^*) + |\xi(z)|(u^*)^{p-1}}{(u^*)^{q-1}} \right. \\ &\quad \left. - \frac{\operatorname{div} a(Dv^*) + |\xi(z)|(v^*)^{p-1}}{(v^*)^{q-1}} \right] ((u^*)^q - (v^*)^q) dz \\ &= \int_{\Omega} c_6 [(v^*)^{r-q} - (v^*)^{r-p}] ((u^*)^q - (v^*)^q) dz \quad \text{(see (10)),} \\ \Rightarrow & u^* = v^* \quad \text{(since } q \leq p < r). \end{aligned}$$

This proves the uniqueness of the positive solution  $u^* \in D_+$  of (11). Since problem (11) is odd, it follows that  $v^* = -u^* \in -D_+$  is the unique negative solution of problem (11).  $\square$

Consider the following Robin problem:

$$(18) \quad \left\{ \begin{array}{l} -\operatorname{div} a(Du(z)) + \xi(z)|u(z)|^{p-2}u(z) = \hat{f}(z, u(z)) \quad \text{in } \Omega, \\ \frac{\partial u}{\partial n_a} + \beta(z)|u|^{p-2}u = 0 \quad \text{on } \partial\Omega. \end{array} \right.$$

We denote by  $S^+$  (respectively  $S^-$ ) the set of positive (respectively negative) solutions of problem (18) which are in the order interval  $[0, \eta] = \{u \in W^{1,p}(\Omega) : 0 \leq u(z) \leq \eta \text{ for almost all } z \in \Omega\}$  (respectively in  $[-\eta, 0] = \{v \in W^{1,p}(\Omega) : -\eta \leq v(z) \leq 0 \text{ for almost all } z \in \Omega\}$ ). From Papageorgiou, Rădulescu & Repovš [22], we know that

- $S^+$  is downward directed (that is, if  $u_1, u_2 \in S^+$ , then we can find  $u \in S^+$  such that  $u \leq u_1, u \leq u_2$ ).
- $S^-$  is upward directed (that is, if  $v_1, v_2 \in S^-$ , then we can find  $v \in S^-$  such that  $v_1 \leq v, v_2 \leq v$ ).

Moreover, reasoning as in the proof of Proposition 5 (with  $k(z, x)$  replaced by  $\hat{f}(z, x)$ ), we show that

$$\emptyset \neq S^+ \subseteq D_+ \quad \text{and} \quad \emptyset \neq S^- \subseteq -D_+.$$

**PROPOSITION 6.** *If hypotheses  $H(a), H(\xi), H(\beta), H(f)$  hold, then  $u^* \leq u$  for all  $u \in S^+$  and  $v \leq v^*$  for all  $v \in S^-$ .*

**PROOF.** Let  $u \in S_+$  and let  $\hat{k}(z, x)$  be given by (12). We introduce the following truncation of  $\hat{k}(z, \cdot)$ :

$$(19) \quad e_+(z, x) = \begin{cases} 0 & \text{if } x < 0 \\ \hat{k}(z, x) & \text{if } 0 \leq x \leq u(z) \\ \hat{k}(z, u(z)) & \text{if } u(z) < x. \end{cases}$$

This is a Carathéodory function. We set  $E_+(z, x) = \int_0^x e_+(z, s) ds$  and consider the  $C^1$ -functional  $\Psi_+ : W^{1,p}(\Omega) \rightarrow \mathbb{R}$  defined by

$$\begin{aligned} \Psi_+(u) &= \int_{\Omega} G(Du) dz + \frac{1}{p} \int_{\Omega} [|\xi(z)| + 1]|u|^p dz + \frac{1}{p} \int_{\partial\Omega} \beta(z)|u|^p d\sigma \\ &\quad - \int_{\Omega} E_+(z, u) dz \quad \text{for all } u \in W^{1,p}(\Omega). \end{aligned}$$

Evidently,  $\Psi_+(\cdot)$  is coercive (see (19)) and sequentially weakly lower semi-continuous. So, we can find  $\hat{u}^* \in W^{1,p}(\Omega)$  such that

$$(20) \quad \Psi_+(\hat{u}^*) = \inf\{\Psi_+(u) : u \in W^{1,p}(\Omega)\}.$$

As in the proof of Proposition 5, using hypotheses  $H(a)(iv)$  and  $H(f)(iii)$ , we show that

$$\begin{aligned} \Psi_+(\hat{u}^*) &< 0 = \Psi_+(0), \\ \Rightarrow \hat{u}^* &\neq 0. \end{aligned}$$

From (20) we have

$$\begin{aligned} (21) \quad \Psi'_+(\hat{u}^*) &= 0, \\ \Rightarrow \langle A(\hat{u}^*), h \rangle &+ \int_{\Omega} [|\xi(z)| + 1] |\hat{u}^*|^{p-2} \hat{u}^* h \, dz + \int_{\partial\Omega} \beta(z) |\hat{u}^*|^{p-2} \hat{u}^* h \, d\sigma \\ &= \int_{\Omega} e_+(z, \hat{u}^*) h \, dz \quad \text{for all } h \in W^{1,p}(\Omega). \end{aligned}$$

In (21), we choose  $h = -(u^*)^- \in W^{1,p}(\Omega)$ . Then using Lemma 2(c), we have

$$\begin{aligned} &\frac{c_1}{p-1} \|D(\hat{u}^*)^-\|_p^p + \int_{\Omega} [|\xi(z) + 1|] ((\hat{u}^*)^-)^p \, dz \\ &\leq 0 \quad (\text{see hypothesis } H(\beta) \text{ and (19)}) \\ \Rightarrow \hat{u}^* &\geq 0, \hat{u}^* \neq 0. \end{aligned}$$

Next, in (21) we choose  $h = (\hat{u}^* - u)^+ \in W^{1,p}(\Omega)$ . We have

$$\begin{aligned} &\langle A(\hat{u}^*), (\hat{u}^* - u)^+ \rangle + \int_{\Omega} [|\xi(z) + 1|] (\hat{u}^*)^{p-1} (\hat{u}^* - u)^+ \, dz \\ &\quad + \int_{\partial\Omega} \beta(z) (u^*)^{p-1} (\hat{u}^* - u)^+ \, d\sigma \\ &= \int_{\Omega} [\mu u^{q-1} - c_6 u^{r-1} + u^{p-1}] (\hat{u}^* - u)^+ \, dz \\ &\quad (\text{see (19), (12), (10) and recall that } u \in S^+) \\ &\leq \int_{\Omega} [\hat{f}(z, u) + u^{p-1}] (\hat{u}^* - u)^+ \, dz \quad (\text{see (9)}) \\ &= \langle A(u), (\hat{u}^* - u)^+ \rangle + \int_{\Omega} [|\xi(z) + 1|] u^{p-1} (\hat{u}^* - u)^+ \, dz \\ &\quad + \int_{\partial\Omega} \beta(z) u^{p-1} (\hat{u}^* - u)^+ \, d\sigma \quad (\text{since } u \in S^+) \\ \Rightarrow \hat{u}^* &\leq u. \end{aligned}$$

So, we have proved that

$$\hat{u}^* \in [0, u] = \{y \in W^{1,p}(\Omega) : 0 \leq y(z) \leq u(z) \text{ for almost all } z \in \Omega\}.$$

This fact, together with (10), (12), (19), (21), imply that

$$-\operatorname{div} a(D\hat{u}^*z) + |\xi(z)|\hat{u}^*(z)^{p-1} = k(z, \hat{u}^*(z)) \quad \text{for almost all } z \in \Omega,$$

$$\frac{\partial \hat{u}^*}{\partial n_a} + \beta(z)(\hat{u}^*)^{p-1} = 0 \text{ on } \partial\Omega \quad (\text{see Papageorgiou \& R\u0105dulescu [18]}),$$

$$\Rightarrow \hat{u}^* = u^* \quad (\text{see Proposition 5}),$$

$$\Rightarrow u^* \leq u \quad \text{for all } u \in S^+.$$

Similarly, we show that

$$v \leq v^* \quad \text{for all } v \in S^-.$$

This completes the proof. □

Now we can establish the existence of extremal constant sign solutions for problem (18), that is, we show that problem (18) has a smallest positive solution and a biggest negative solution.

**PROPOSITION 7.** *If hypotheses  $H(a)$ ,  $H(\beta)$ ,  $H(\xi)$ ,  $H(f)$  hold, then there exists a smallest positive solution  $u_+ \in S^+ \subseteq D_+$  and a biggest negative solution  $v_+ \in S^- \subseteq -D_+$ .*

**PROOF.** Invoking Lemma 3.10 of Hu & Papageorgiou [6, p. 178], we can find a decreasing sequence  $\{u_n\}_{n \geq 1} \subseteq S^+$  such that

$$\inf S^+ = \inf_{n \geq 1} u_n.$$

Evidently,  $\{u_n\}_{n \geq 1} \subseteq W^{1,p}(\Omega)$  is bounded. So, we may assume that

$$(22) \quad u_n \xrightarrow{w} u_+ \text{ in } W^{1,p}(\Omega) \quad \text{and} \quad u_n \rightarrow u_+ \text{ in } L^p(\Omega) \text{ and } L^p(\partial\Omega).$$

We have

$$(23) \quad \begin{aligned} \langle A(u_n), h \rangle + \int_{\Omega} \xi(z)u_n^{p-1}h \, dz + \int_{\partial\Omega} \beta(z)u_n^{p-1}h \, d\sigma \\ = \int_{\Omega} \hat{f}(z, u_n)h \, dx \quad \text{for all } h \in W^{1,p}(\Omega), n \in \mathbb{N}. \end{aligned}$$

In (23) we choose  $h = u_n - u_+ \in W^{1,p}(\Omega)$ , pass to the limit as  $n \rightarrow \infty$  and use (22). Then

$$(24) \quad \begin{aligned} \lim_{n \rightarrow \infty} \langle A(u_n), u_n - u_+ \rangle = 0, \\ \Rightarrow u_n \rightarrow u_+ \text{ in } W^{1,p}(\Omega) \quad (\text{see Proposition 4}). \end{aligned}$$

In (23) we pass to the limit as  $n \rightarrow \infty$  and use (24). Then

$$(25) \quad \begin{aligned} \langle A(u_+), h \rangle + \int_{\Omega} \xi(z) u_+^{p-1} h \, dz + \int_{\partial\Omega} \beta(z) u_+^{p-1} h \, d\sigma \\ = \int_{\Omega} \hat{f}(z, u_+) h \, dz \quad \text{for all } h \in W^{1,p}(\Omega). \end{aligned}$$

From Proposition 6, we have

$$(26) \quad \begin{aligned} u^* \leq u_n \quad \text{for all } n \in \mathbb{N}, \\ \Rightarrow u^* \leq u_+ \quad (\text{see (24)}), \text{ hence } u_+ \neq 0. \end{aligned}$$

It follows from (25) and (26) that

$$u_+ \in S^+ \subseteq D_+, \quad u_+ = \inf S^+.$$

Similarly, we produce

$$v_- \in S^- \subseteq -D_+, \quad v_- = \sup S^-. \quad \square$$

Let  $\tau > \|\xi\|_{\infty}$  and consider the following truncation-perturbation of  $\hat{f}(z, \cdot)$ :

$$(27) \quad f_0(z, x) = \begin{cases} \hat{f}(z, v_-(z)) + \tau |v_-(z)|^{p-2} v_-(z) & \text{if } x < v_-(z) \\ \hat{f}(z, x) + \tau |x|^{p-2} x & \text{if } v_-(z) \leq x \leq u_+(z) \\ \hat{f}(z, u_+(z)) + \tau u_+(z)^{p-1} & \text{if } u_+(z) < x. \end{cases}$$

We set  $F_0(z, x) = \int_0^x f_0(z, s) \, ds$  and consider the  $C^1$ -functional  $\varphi_0 : W^{1,p}(\Omega) \rightarrow \mathbb{R}$  defined by

$$\begin{aligned} \varphi_0(u) = \int_{\Omega} G(Du) \, dz + \frac{1}{p} \int_{\Omega} [\xi(z) + \tau] |u|^p \, dz + \frac{1}{p} \int_{\Omega} \beta(z) |u|^p \, d\sigma \\ - \int_{\Omega} F_0(z, u) \, dz \quad \text{for all } u \in W^{1,p}(\Omega). \end{aligned}$$

Evidently,  $\varphi_0(\cdot)$  is coercive (see (27) and recall that  $\tau > \|\xi\|_{\infty}$ ). So,  $\varphi_0(\cdot)$  is bounded below and satisfies the PS-condition (see Marano & Papageorgiou [14, 15]).

**PROPOSITION 8.** *If hypotheses  $H(a)$ ,  $H(\xi)$ ,  $H(\beta)$ ,  $H(f)$  hold and  $V \subseteq W^{1,p}(\Omega)$  is a finite dimensional linear subspace, then there exists  $\rho_V > 0$  such that*

$$\sup\{\varphi_0(u) : u \in V, \|u\| = \rho_V\} < 0.$$

**PROOF.** Recall that  $u_+ \in D_+$  and  $v_- \in -D_+$ . So,  $m_0 = \min\left\{\min_{\Omega} u_+, -\max_{\Omega} v_-\right\} > 0$ . We set  $\epsilon_0 = \min\{\epsilon, m_0\}$  (where  $\epsilon > 0$  is from (4)). On account of hypothesis  $H(f)$ (iii), given any  $\mu > 0$ , we can find  $\delta = \delta(\mu) > 0 \in (0, \epsilon_0)$  such that

$$(28) \quad F_0(z, x) = \hat{F}(z, x) + \frac{\tau}{p}|x|^p = F(z, x) + \frac{\tau}{p}|x|^p$$

$$\geq \frac{\mu}{q}|x|^q + \frac{\tau}{p}|x|^p$$

(for almost all  $z \in \Omega$ , and all  $|x| \leq \delta$ , see (4) and (27)).

Moreover, on account of hypothesis  $H(a)(iv)$  and Corollary 3, we have

$$(29) \quad G(y) \leq c_{10}[|y|^q + |y|^p] \quad \text{for some } c_{10} > 0, \text{ and all } y \in \mathbb{R}^N.$$

Since the subspace  $V \subseteq W^{1,p}(\Omega)$  is finite dimensional, all norms are equivalent. So, we can find  $\rho_V \in (0, 1]$  such that

$$(30) \quad u \in V, \quad \|u\| \leq \rho_V \Rightarrow |u(z)| \leq \delta \quad \text{for all } z \in \bar{\Omega}.$$

Then for every  $u \in V$  with  $\|u\| \leq \rho_V$ , we have

$$\begin{aligned} \varphi_0(u) &\leq c_{11}\|u\|^q - \mu c_{12}\|u\|^q \quad \text{for some } c_{11}, c_{12} > 0 \\ &\text{(see (27), (28), (29), (30) and recall that } \rho_V \leq 1, q \leq p) \end{aligned}$$

Since  $\mu > 0$  is arbitrary, we choose  $\mu > \frac{c_{11}}{c_{12}}$  and conclude that

$$\varphi_0(u) < 0 \quad \text{for all } u \in V \text{ with } \|u\| = \rho_V.$$

The proof is now complete. □

We now obtain the following multiplicity theorem for the nodal solutions of problem (1).

**THEOREM 9.** *Assume that hypotheses  $H(a)$ ,  $H(\xi)$ ,  $H(\beta)$ ,  $H(f)$  hold. Then there exists a sequence  $\{u_n\}_{n \geq 1} \subseteq C^1(\bar{\Omega})$  of nodal solutions of problem (1) such that*

$$u_n \rightarrow 0 \quad \text{in } C^1(\bar{\Omega}).$$

**PROOF.** We know that  $\varphi_0(\cdot)$  is even, bounded below, satisfies the PS-condition, and  $\varphi_0(0) = 0$ . Moreover, using (27) as before, we can check that

$$(31) \quad K_{\varphi_0} \subseteq [v_-, u_+] \cap C^1(\bar{\Omega}).$$

The aforementioned properties of  $\varphi_0(\cdot)$  and Proposition 8 permit us to apply Theorem 1. So, we can find a sequence  $\{u_n\}_{n \geq 1} \subseteq W^{1,p}(\Omega)$  such that

$$(32) \quad u_n \in K_{\varphi_0} \subseteq [v_-, u_+] \cap C^1(\bar{\Omega}) \quad \text{(see (31))} \quad \text{and} \quad u_n \rightarrow 0 \text{ in } W^{1,p}(\Omega).$$

The nonlinear regularity theory of Lieberman [10] implies that we can find  $\gamma \in (0, 1)$  and  $c_{13} > 0$  such that

$$(33) \quad u_n \in C^{1,\gamma}(\bar{\Omega}), \quad \|u_n\|_{C^{1,\gamma}(\bar{\Omega})} \leq c_{13} \quad \text{for all } n \in \mathbb{N}.$$

We know that  $C^{1,\gamma}(\bar{\Omega})$  is compactly embedded in  $C^1(\bar{\Omega})$ . So, it follows from (32) and (33) that

$$\begin{aligned} u_n &\rightarrow 0 \quad \text{in } C^1(\bar{\Omega}), \\ \Rightarrow -\epsilon_0 &\leq u_n(z) \leq \epsilon_0 \quad \text{for all } z \in \bar{\Omega}, \text{ and all } n \geq n_0 \\ &\text{(recall that } \epsilon_0 = \min\{\epsilon, m_0\} > 0, \text{ see the proof of Proposition 8).} \end{aligned}$$

From (4), (32) and the extremality of  $u_+$ ,  $v_-$ , we get that  $\{u_n\}_{n \geq 1} \subseteq C^1(\bar{\Omega})$  are nodal solutions of (1) and we have  $u_n \rightarrow 0$  in  $C^1(\bar{\Omega})$ .  $\square$

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