**Partial Differential Equations** — *A variational approach to doubly nonlinear equations*, by Verena Bögelein, Frank Duzaar, Paolo Marcellini and Christoph Scheven, communicated on June 22, 2018.

Abstract. — This article presents a variational approach to the existence of solutions to equations of Porous Medium type. More generally, the method applies also to doubly nonlinear equations with a nonlinearity in $u$ and $Du$, whose prototype is given by

$$\partial_t u^m - \text{div}(|Du|^{p-2} Du) = 0,$$

where $m > 0$ and $p > 1$. The technique relies on a nonlinear version of the Minimizing Movement Method which has been introduced in [14] in the context of doubly nonlinear equations with general nonlinearities $\partial_t b(u)$ and more general operators with variational structure. The aim of this article is twofold. On the one hand it provides an introduction to variational solutions and outlines the method developed in [14]. In addition, we extend the results of [14] to initial data with potentially infinite energy. This requires a detailed discussion of the growth conditions of the variational energy integrand. The approach is flexible enough to treat various more general evolutionary problems, such as signed solutions, obstacle problems, time dependent boundary data or problems with linear growth.

Key words: Porous medium equation, doubly nonlinear equation, existence, minimizing movements

Mathematics Subject Classification:   35K86, 49J40, 49J45

## 1. Variational Solutions

A classical problem in the Calculus of Variations is to minimize integral functionals of the type

$$(1.1) \qquad \mathscr{F}[u] := \int_\Omega f(x, u, Du) \, dx$$

in a prescribed class of functions $u : \Omega \to \mathbb{R}$, where $\Omega$ is a domain in $\mathbb{R}^n$. A prominent model functional is the so-called *p-energy* $\mathscr{F}[u] = \frac{1}{p} \int_\Omega |Du|^p \, dx$, which corresponds to the integrand $f(x, u, \xi) = \frac{1}{p} |\xi|^p$. More generally, one can take any integrand $f : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ that is a Carathéodory-function, i.e. $\Omega \ni x \mapsto f(x, u, \xi)$ is measurable for any $(u, \xi) \in \mathbb{R} \times \mathbb{R}^n$ and $f(x, \cdot, \cdot)$ is

continuous for a.e. $x \in \Omega$, and satisfies the convexity and coercivity conditions

(1.2)     $\begin{cases} \mathbb{R} \times \mathbb{R}^n \ni (u, \xi) \to f(x, u, \xi) & \text{is convex for a.e. } x \in \Omega, \\ f(x, u, \xi) \geq v|\xi|^p & \text{for any } (x, u, \xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^n, \end{cases}$

for some $p > 1$ and $v > 0$. Then, for any prescribed Dirichlet-boundary datum $u_*$ in the Sobolev space $W^{1,p}(\Omega)$ with finite energy there exists a minimizer $u$ of the functional $\mathscr{F}$ in the class $W^{1,p}_{u_*}(\Omega) := u_* + W^{1,p}_0(\Omega)$, i.e. a function $u \in W^{1,p}_{u_*}(\Omega)$ satisfying

(1.3)                                    $\mathscr{F}[u] \leq \mathscr{F}[v]$

for any comparison function $v \in W^{1,p}_{u_*}(\Omega)$. The existence of the minimizer can be proven by the classical Direct Method of the Calculus of Variations. The underlying idea is to consider a minimizing sequence for the functional $\mathscr{F}$ and then pass to the limit by the use of the lower semicontinuity of the functional. In the time independent case of the classical Calculus of Variations various generalizations are possible. For instance, one can subtract lower order terms depending on $x$ and $u$, one can consider vector-valued minimizers and quasiconvex integrands instead of convex integrands.

   If the integrand $f$ is regular enough, subsequently it can be shown that any minimizer of (1.1) in $W^{1,p}_{u_*}(\Omega)$ is a (weak) solution to the Dirichlet problem of the associated Euler–Lagrange equation

(1.4)     $\begin{cases} \text{div } D_\xi f(x, u, Du) = D_u f(x, u, Du) & \text{in } \Omega, \\ u = u_* & \text{on } \partial\Omega. \end{cases}$

In the model case of the $p$-energy, $(1.4)_1$ is nothing more than the $p$-Laplace equation $\text{div}(|Du|^{p-2}Du) = 0$. Vice versa, since $f$ is convex any (weak) solution to (1.4) is also a minimizer of the functional (1.1). The Euler–Lagrange equation above is a quasi-linear elliptic partial differential equation. The domain $\Omega$ of the solution is considered as a spatial domain. If we additionally consider a time variable, then we end up with an $L^2$-gradient flow. The associated parabolic Cauchy–Dirichlet problem reads as

(1.5)     $\begin{cases} \partial_t u - \text{div } D_\xi f(x, u, Du) = -D_u f(x, u, Du) & \text{in } \Omega_T, \\ u = u_* & \text{on } \partial\Omega \times (0, T), \\ u(0) = u_o & \text{in } \Omega, \end{cases}$

where $u$ is now defined on the space-time cylinder $\Omega_T := \Omega \times (0, T)$, $T > 0$. Throughout this article we abbreviate $u(t) = u(\cdot, t)$ for $t \in [0, T]$. The initial values $u_o$ and lateral boundary values $u_*$ are prescribed on the parabolic boundary $\partial_\mathscr{P}\Omega_T := [\partial\Omega \times (0, T)] \cup [\overline{\Omega} \times \{0\}]$. As before, the derivatives in the diffusion part of the differential equation are only taken with respect to the spatial variable $x$. There are now several methods to prove the existence of (weak) solutions to the parabolic Cauchy–Dirichlet problem (1.5); for instance Galerkin type methods, Monotone Operators, Elliptic Regularization, or Minimizing Movements

are available. In general, the treatment of the diffusion term div $D_\xi f(x, u, Du)$ is difficult, in the sense that when considering an approximating sequence the passage to the limit has to be justified without knowledge of strong convergence properties of the gradients. Therefore, one usually has to assume a growth condition for the integrand $f$ of the form $f(\xi) \le L(1 + |\xi|^p)$ for some $L > 0$ and any $\xi \in \mathbb{R}^n$. On the other hand, such a condition is not necessary in order to prove the existence of a minimizer of the elliptic variational functional (1.1). This is due to the fact that the lower semicontinuity of the functional (1.1) can be exploited in limiting processes. With this respect, any variational approach is much more flexible than a PDE approach. For this reason there is a natural need for a variational approach to evolutionary problems. Such an approach promises a great potential for a more flexible existence theory and has recently been developed in [12, 16, 17]; for a related technique, the so-called method of elliptic regularization see also [6, 13, 20, 22, 24, 45].

To this aim, one first has to develop a variational formulation of the Cauchy–Dirichlet problem (1.5). The idea which has been performed in [13] goes back to Lichnewsky & Temam in [42], who introduced the notion of *variational solutions* (pseudo solutions) to the time dependent minimal surface equation. In our context, the idea is as follows. We multiply (1.5) by $v - u$, for some $v \in L^2(\Omega_T) \cap L^p(0, T; W^{1,p}_{u_*}(\Omega))$ with $\partial_t v \in L^2(\Omega_T)$, integrate the result over $\Omega_\tau$ for some $\tau \in (0, T]$ and integrate by parts. In this way, we obtain

$$\underbrace{\iint_{\Omega_\tau} \partial_t u (v - u) \, dx \, dt}_{=: \, \mathrm{I}}$$

$$+ \underbrace{\iint_{\Omega_\tau} [D_\xi f(x, u, Du) \cdot (Dv - Du) + D_u f(x, u, Du)(v - u)] \, dx \, dt}_{=: \, \mathrm{II}} = 0.$$

By the convexity of $\mathbb{R} \times \mathbb{R}^n \ni (u, \xi) \mapsto f(x, u, \xi)$ for a.e. $x \in \Omega$ we have

$$f(x, u, Du) + D_\xi f(x, u, Du) \cdot (Dv - Du) + D_u f(x, u, Du)(v - u) \le f(x, v, Dv)$$

and hence

(1.6) $$\mathrm{II} \le \iint_{\Omega_\tau} [f(x, v, Dv) - f(x, u, Du)] \, dx \, dt.$$

To treat I we would like to shift the time derivative "$\partial_t$" from $u$ to $v$. This is achieved by adding and subtracting $\partial_t v$ as follows

$$\mathrm{I} = \iint_{\Omega_\tau} \partial_t v (v - u) \, dx \, dt - \iint_{\Omega_\tau} \partial_t (v - u)(v - u) \, dx \, dt$$

$$= \iint_{\Omega_\tau} \partial_t v (v - u) \, dx \, dt - \frac{1}{2} \|(v - u)(\tau)\|^2_{L^2(\Omega)} + \frac{1}{2} \|v(0) - u_o\|^2_{L^2(\Omega)}.$$

We have thus shown that any (weak) solution to (1.5) satisfies the variational inequality

$$(1.7) \qquad \iint_{\Omega_\tau} f(x, u, Du)\, dx\, dt \le \iint_{\Omega_\tau} f(x, v, Dv)\, dx\, dt + \iint_{\Omega_\tau} \partial_t v(v - u)\, dx\, dt$$

$$- \frac{1}{2}\|(v - u)(\tau)\|_{L^2(\Omega)}^2 + \frac{1}{2}\|v(0) - u_o\|_{L^2(\Omega)}^2$$

for any $\tau \in (0, T]$ and any $v \in L^2(\Omega_T) \cap L^p(0, T; W_{u_*}^{1,p}(\Omega))$ with $\partial_t v \in L^2(\Omega_T)$. This is exactly the parabolic counterpart of (1.3) we are looking for.

For compatibility reasons we assume for the initial and lateral boundary data

$$(1.8) \qquad \begin{cases} u_o \in L^2(\Omega), \\ u_* \in L^2(\Omega) \cap W^{1,p}(\Omega) \quad \text{with} \\ \displaystyle\int_\Omega f(x, u_* + \varphi, D(u_* + \varphi))\, dx < \infty \quad \forall \varphi \in C_0^\infty(\Omega). \end{cases}$$

Later on in connection with the extension of the concept of the variational solution to doubly degenerate equations, we will discuss the above assumptions on the initial and boundary values in more detail; cf. the generalization (3.8) of (1.8) and the subsequent Remark 3.2. In particular, we will introduce in Remark 3.2 various sufficient (and easier to verify) conditions, that imply $(1.8)_3$. Now, we focus on the exact definition of the variation solution in the more classical case of standard evolutionary problems.

DEFINITION 1.1. Suppose that $f : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \overline{\mathbb{R}}$ is a variational integrand satisfying the convexity and coercivity assumption (1.2) and assume that the initial and lateral boundary data $u_o$ and $u_*$ fulfill (1.8). We identify

$$u \in C^0([0, T]; L^2(\Omega)) \cap L^p(0, T; W_{u_*}^{1,p}(\Omega)),$$

as a *variational solution* associated to the Cauchy–Dirichlet problem (1.5) if and only if the variational inequality (1.7) holds true.                                                    □

Note that the initial condition $u(0) = u_o$ is incorporated in the variational inequality (1.7). Indeed, one can show that the variational inequality (1.7) implies that $u(0) = u_o$ in the $C^0 - L^2$-sense, cf. [13, Lemma 2.1], [18, Lemma 3.4] or Lemma 4.2 (applied with $b(u) \equiv u$) below. Furthermore, if $f$ is regular enough, then the notions coincide, i.e. any variational solution is also a weak solution of (1.5). To obtain the equivalence of the two notions of solutions, we assume additionally to hypothesis (1.2), that $(u, \xi) \mapsto f(x, u, \xi)$ is $C^1$ for a.e. $x \in \Omega$ and that there exists a constant $L > 0$ such that

$$(1.9) \qquad \begin{cases} f(x, u, \xi) \le L(1 + |\xi|^p), \\ |D_u f(x, u, \xi)| \le L(1 + |\xi|^p), \\ |D_\xi f(x, u, \xi)| \le L(1 + |\xi|^{p-1}) \end{cases}$$

for a.e. $x \in \Omega$ and every $(u, \xi) \in \mathbb{R} \times \mathbb{R}^n$. The set (1.9) of growth conditions is commonly termed *natural p-growth condition*. They cover for example integrands of the form

$$f(x, u, Du) := a(x, u)|Du|^p + g(x, u).$$

Then, we have the following

LEMMA 1.2. *Assume that the integrand $f$ satisfies (1.2) and (1.9) and that the initial and lateral boundary values $u_o$ and $u_*$ satisfy (1.8). Then the notions of weak solution and variational solution coincide.*

PROOF. We have already sketched the proof that any weak solution is a variational solution. The formal computations above can be made rigorous by a standard approximation argument. Here, we present the proof of the opposite direction. To this aim, we consider a variational solution $u$ in the sense of Definition 1.1 and let $\varphi \in C_0^\infty(\Omega_T)$ be a generic testing function. Throughout the proof, we denote by $[w]_h$ for $h \in (0, T]$, $w_o \in L^1(\Omega)$ and $w \in L^1(\Omega_T) = L^1(0, T; L^1(\Omega))$ the time mollification

$$(1.10) \qquad [w]_h(t) := e^{-\frac{t}{h}} w_o + \frac{1}{h} \int_0^t e^{\frac{s-t}{h}} w(s) \, ds$$

for any $t \in [0, T]$. Note that the mollification $[w]_h$ is constructed in such a way that it satisfies

$$(1.11) \qquad \partial_t [w]_h = -\frac{1}{h}([w]_h - w).$$

For more information on this mollification we refer to [38, Lemma 2.2] and [13, Lemmas 2.2, 2.3]. We now define the comparison function $v_h := [u]_h + s[\varphi]_h$ with $s \in (0, 1)$. Here, the time mollification of $\varphi$ is defined according to (1.10) with initial values $w_o = 0 = \varphi(0)$, while for the mollification of $u$, we choose as initial values

$$u_o^{(\varepsilon)} := u_* + [(u_o - u_*)\chi_{2\varepsilon}] * \phi_\varepsilon$$

with $\varepsilon > 0$, where $\phi_\varepsilon(x) := \varepsilon^{-n} \phi(\frac{x}{\varepsilon})$ with $\phi \in C_0^\infty(B_1, \mathbb{R}_{\geq 0})$ denotes a standard mollifier and $\chi_{2\varepsilon}$ stands for the characteristic function of the inner parallel set $\Omega^{2\varepsilon} := \{x \in \Omega : \text{dist}(x, \partial\Omega) > 2\varepsilon\}$. Then, $u_o^{(\varepsilon)} - u_* \in C_0^\infty(\Omega)$ with $\text{spt}(u_o^{(\varepsilon)} - u_*) \subset \overline{\Omega^\varepsilon}$ and $u_o^{(\varepsilon)} \to u_o$ in $L^2(\Omega)$ as $\varepsilon \downarrow 0$. Moreover, on $\partial\Omega \times \{t\}$ we have $v_h(t) = [e^{-\frac{t}{h}} + \frac{1}{h} \int_0^t e^{\frac{s-t}{h}} ds] u_* = u_*$. Therefore, we are allowed to choose $v_h$ as comparison function in the variational inequality (1.7), with the result that

$$(1.12) \qquad \frac{1}{2} \|(v_h - u)(T)\|_{L^2(\Omega)}^2 + \iint_{\Omega_T} f(x, u, Du) \, dx \, dt$$

$$\leq \iint_{\Omega_T} f(x, v_h, Dv_h) \, dx \, dt + \iint_{\Omega_T} \partial_t v_h(v_h - u) \, dx \, dt$$

$$+\frac{1}{2}\|u_o^{(\varepsilon)} - u_o\|_{L^2(\Omega)}^2$$

$$=: \mathrm{I}_{h,\varepsilon} + \mathrm{II}_{h,\varepsilon} + \frac{1}{2}\|u_o^{(\varepsilon)} - u_o\|_{L^2(\Omega)}^2.$$

Note that the mollification (1.10) can be interpreted as the integral of $s \mapsto w(s)$ with respect to the measure $\mu = e^{-\frac{t}{h}}\delta_0 + \frac{1}{h}e^{\frac{s-t}{h}}\chi_{[0,t]}\,\mathrm{d}s$, which satisfies $\mu(\mathbb{R}) = 1$. Therefore, in view of the convexity of $(u, \xi) \mapsto f(x, u, \xi)$, Jensen's inequality implies

$$f(x, v_h, Dv_h) \le [f(x, u + s\varphi, Du + sD\varphi)]_h.$$

Here, the time mollification of $f(\cdot, u + s\varphi, Du + sD\varphi)$ must be taken with initial values $w_o = f(\cdot, u_o^{(\varepsilon)}, Du_o^{(\varepsilon)})$. Since $f(\cdot, u + s\varphi, Du + sD\varphi) \in L^1(\Omega_T)$ by assumption (1.9), [14, Lemma 6.2 (i)] yields

$$(1.13) \qquad \limsup_{h\downarrow 0} \mathrm{I}_{h,\varepsilon} \le \iint_{\Omega_T} f(x, u + s\varphi, Du + sD\varphi)\,\mathrm{d}x\,\mathrm{d}t.$$

Next we consider the term $\mathrm{II}_{h,\varepsilon}$. We use the definition of $v_h$, the identity (1.11), and afterwards perform an integration by parts (note that $\mathrm{spt}\,\varphi \Subset \Omega_T$), to obtain

$$\mathrm{II}_{h,\varepsilon} = \iint_{\Omega_T} \partial_t[u]_h([u]_h - u + s\varphi)\,\mathrm{d}x\,\mathrm{d}t + s\iint_{\Omega_T} \partial_t[\varphi]_h(v_h - u)\,\mathrm{d}x\,\mathrm{d}t$$

$$\le s\iint_{\Omega_T} \partial_t[u]_h\varphi\,\mathrm{d}x\,\mathrm{d}t + s\iint_{\Omega_T} \partial_t[\varphi]_h(v_h - u)\,\mathrm{d}x\,\mathrm{d}t$$

$$= -s\iint_{\Omega_T} [u]_h\partial_t\varphi\,\mathrm{d}x\,\mathrm{d}t + s\iint_{\Omega_T} \partial_t[\varphi]_h(v_h - u)\,\mathrm{d}x\,\mathrm{d}t.$$

Since $v_h - u \to s\varphi$ in $L^2(\Omega_T)$ as $h \downarrow 0$, we deduce

$$(1.14) \qquad \limsup_{h\downarrow 0} \mathrm{II}_{h,\varepsilon} \le -s\iint_{\Omega_T} u\partial_t\varphi\,\mathrm{d}x\,\mathrm{d}t + s^2\iint_{\Omega_T} \varphi\partial_t\varphi\,\mathrm{d}x\,\mathrm{d}t$$

$$= -s\iint_{\Omega_T} u\partial_t\varphi\,\mathrm{d}x\,\mathrm{d}t.$$

Here we used the fact that $\varphi(0) = 0 = \varphi(T)$. Now, we use (1.13) and (1.14) in (1.12) in order to pass to the limit $h \downarrow 0$ in the right-hand side. Since the first term on the left-hand side, i.e. the $L^2(\Omega)$-boundary term at $T$, is non-negative, we can discard this term from the left-hand side and infer, after passage to the limit $h \downarrow 0$, that

$$s \iint_{\Omega_T} u \partial_t \varphi \, dx \, dt \le \iint_{\Omega_T} [f(x, u + s\varphi, Du + sD\varphi) - f(x, u, Du)] \, dx \, dt$$

$$+ \frac{1}{2} \|u_o^{(\varepsilon)} - u_o\|_{L^2(\Omega)}^2$$

holds true. Here we pass to the limit $\varepsilon \downarrow 0$ by utilizing the $L^2(\Omega)$-convergence $u_o^{(\varepsilon)} \to u_o$. Subsequently, we divide both sides of the resulting inequality by $s > 0$ and obtain that

$$\iint_{\Omega_T} u \partial_t \varphi \, dx \, dt \le \frac{1}{s} \iint_{\Omega_T} [f(x, u + s\varphi, Du + sD\varphi) - f(x, u, Du)] \, dx \, dt$$

$$\le \iint_{\Omega_T} \int_0^1 [D_\xi f(x, u + s\sigma\varphi, Du + s\sigma D\varphi) \cdot D\varphi$$

$$+ D_u f(x, u + s\sigma\varphi, Du + s\sigma D\varphi)\varphi] \, d\sigma \, dx \, dt$$

holds true. Here we used the convexity of $(u, \xi) \mapsto f(x, u, \xi)$ and the $C^1$-assumption for the integrand $f$. At this stage the growth condition (1.9) allows the passage to the limit $s \downarrow 0$ on the right-hand side of the preceding inequality. In this way, we obtain

$$\iint_{\Omega_T} u \partial_t \varphi \, dx \, dt \le \iint_{\Omega_T} [D_\xi f(x, u, Du) \cdot D\varphi + D_u f(x, u, Du)\varphi] \, dx \, dt.$$

Here, we can replace $\varphi$ by $-\varphi$, and obtain that the opposite inequality holds as well. Together, this implies that $u$ is a weak solution as stated in the lemma. This finishes the proof. □

The advantage of the viewpoint, to interpret solutions of (1.5) as solutions of the associated variational inequality, is obvious: without any difficulty the concept can be introduced for general energies of the type $f(x, u, Du)$. As in the classical Calculus of Variations this approach could lead to variational solutions for energies, for which the corresponding Euler–Lagrange equation, i.e. the associated parabolic equation, might not hold, as in the case of functionals with non-standard $(p, q)$-growth. Prototype integrands we have in mind are

$$f(x, Du) := \alpha(x)|Du|^p + \beta(x)|Du|^q$$

with $1 < p < q < \infty$ and $\alpha(x) + \beta(x) > 0$ for $x \in \Omega$, or integrands with exponential growth such as

$$f(Du) := \exp\left[\frac{1}{2}|Du|^2\right].$$

Also functionals with linear growth as the area integrand

$$f(Du) := \sqrt{1 + |Du|^2}$$

or the total variation can be incorporated in the framework of variational solutions; cf. [12, 16, 20]. In fact, for boundary data $u_o = u_* \in W^{1,p}(\Omega)$ variational solutions to gradient flows have been constructed by the first three authors in [13] by the method of Elliptic Regularization (weighted energy dissipation). Even slightly more general integral functionals and the vectorial setting are considered; cf. [6, 19, 21, 22, 24, 45, 48] and the references therein for related existence results. The method of Elliptic Regularization has been suggested by De Giorgi [27, 28] in order to establish the existence of global solutions of inhomogeneous wave equations. The conjecture was recently solved by Serra & Tilli in [49]; see also [52] for a partial result. The method has also been applied in [2, 3, 43, 47, 51] for different types of parabolic partial differential equations. Our aim in this article is to present a different, and sometimes more flexible approach to the existence of variational solutions, the so-called Minimizing Movements Method and a certain nonlinear counterpart which allows also the treatment of porous medium type equations (and systems). In addition, we extend the proof to initial data with potentially infinite energy.

## 2. Minimizing Movements Method

By now, the Minimizing Movements Method, or Rothe's method is a standard tool in existence proofs and numerics for evolutionary problems. The overall strategy to construct a weak solution of the Cauchy–Dirichlet problem (1.5) is to perform a time discretization and to solve an elliptic (time independent) problem on each time slice. This yields a sequence of piecewise in time constant functions which, after passing to a subsequence, converges to a solution of the original problem. In the following we will outline the proof of the existence of variational solutions in the sense of Definition 1.1 for the case $u_o = u_* \in L^2(\Omega) \cap W^{1,p}(\Omega)$. We emphasize that the only assumptions on the integrand $f$ are (1.2), exactly the ones from the elliptic setting.

For an integer $k \in \mathbb{N}$ we consider a step-size $h_k := \frac{T}{k}$ and the times $t_i := ih_k$ with $i \in \{0, \ldots, k\}$. The idea now is to inductively select a sequence of minimizers $(u_{k,i})_{i=1}^k$ to certain elliptic variational problems. Therefore, suppose that for some $i \in \{1, \ldots, k\}$

$$u_{k,i-1} \in L^2(\Omega) \cap W_{u_o}^{1,p}(\Omega)$$

has already been selected. If $i = 1$, we let $u_{k,0} = u_o$. Then, we choose $u_{k,i}$ as the minimizer of the variational functional

$$(2.1) \qquad \mathbf{F}_{k,i}[v] := \int_\Omega f(x, v, Dv)\, \mathrm{d}x + \frac{1}{2h_k} \int_\Omega |u_{k,i-1} - v|^2 \, \mathrm{d}x$$

in the class of functions

$$v \in L^2(\Omega) \cap W_{u_o}^{1,p}(\Omega).$$

The existence of such a minimizer $u_{k,i}$ is guaranteed by the classical Direct Method of the Calculus of Variations. Then, we define

$$u^{(k)} : \Omega \times (-h_k, T] \to \mathbb{R}$$

as the piecewise in time constant function

$$u^{(k)}(t) := u_{k,i} \quad \text{for } t \in (t_{i-1}, t_i], \ i \in \{0, \dots, k\}.$$

Due to the minimizing property of the functions $u_{k,i}$ (note that $u_{k,i-1}$ is an admissible competitor for $u_{k,i}$), one can prove uniform energy bounds of the form

$$(2.2) \qquad \sup_{t \in [0,T]} \int_\Omega |u^{(k)}(t)|^2 \, \mathrm{d}x + \sup_{t \in [0,T]} \int_\Omega f(x, u^{(k)}(t), Du^{(k)}(t)) \, \mathrm{d}x \le C,$$

and

$$(2.3) \qquad \frac{1}{h_k^2} \iint_{\Omega_T} |u^{(k)}(t) - u^{(k)}(t - h_k)|^2 \, \mathrm{d}x \, \mathrm{d}t \le C,$$

where the constant $C$ is independent of $k$. In particular, due to the coercivity $(1.2)_2$ of $f$, (2.2) ensures that the sequence $\{u^{(k)}\}_{k=1}^\infty$ is uniformly bounded in the spaces $L^\infty(0, T; L^2(\Omega))$ and $L^\infty(0, T; W^{1,p}(\Omega))$. Therefore, by compactness we conclude that there exists a function

$$u \in L^\infty(0, T; L^2(\Omega)) \cap L^\infty(0, T; W^{1,p}(\Omega))$$

and a (not re-labelled) subsequence such that

$$u^{(k)} \overset{*}{\rightharpoonup} u \quad \text{weakly}^* \text{ in } L^\infty(0, T; L^2(\Omega)) \text{ and } L^\infty(0, T; W^{1,p}(\Omega)).$$

Furthermore, from the bound (2.3) for the difference quotient in time of $u^{(k)}$, we deduce for the time derivative of the limit function $u$ that

$$\partial_t u \in L^2(\Omega_T).$$

Finally, we have to ensure that the limit function is indeed a variational solution to the Cauchy–Dirichlet problem (1.5). We observe that the piecewise constant map $u^{(k)}$ is a minimizer of the functional

$$\mathbf{F}^{(k)}[w] := \iint_{\Omega_T} \left[ f(x, w, Dw) + \frac{1}{2h_k} |w(t) - u^{(k)}(t - h_k)|^2 \right] \mathrm{d}x \, \mathrm{d}t$$

in the class

$$w \in L^\infty(0, T; L^2(\Omega)) \cap L^\infty(0, T; W_{u_o}^{1,p}(\Omega)).$$

Testing with the admissible comparison map

$$w_s := u^{(k)} + s(v - u^{(k)}), \quad s \in (0, 1),$$

with $v$ as in Definition 1.1, and exploiting the convexity of $(u, \xi) \mapsto f(x, u, \xi)$, we end up – after passing to the limit $s \downarrow 0$ – with the inequality

$$\iint_{\Omega_T} f(x, u^{(k)}, Du^{(k)}) \, dx \, dt$$

$$\leq \iint_{\Omega_T} \left[ f(x, v, Dv) + \frac{u^{(k)}(t) - u^{(k)}(t - h_k)}{h_k} (v - u^{(k)}) \right] dx \, dt.$$

For the left-hand side we obtain by lower semi-continuity that

$$\iint_{\Omega_T} f(x, u, Du) \, dx \, dt \leq \liminf_{k \to \infty} \iint_{\Omega_T} f(x, u^{(k)}, Du^{(k)}) \, dx \, dt.$$

Furthermore, a compactness argument ensures that $u^{(k)} \to u$ strongly in $L^2(\Omega_T)$. Therefore, we are allowed to pass to the limit $k \to \infty$ in the right-hand side, with the result that

$$\lim_{k \to \infty} \iint_{\Omega_T} \frac{u^{(k)}(t) - u^{(k)}(t - h_k)}{h_k} (v - u^{(k)}) \, dx \, dt$$

$$= \iint_{\Omega_T} \partial_t u (v - u) \, dx \, dt$$

$$= \iint_{\Omega_T} \partial_t v (v - u) \, dx \, dt - \|(v - u)(T)\|_{L^2(\Omega)}^2 + \|v(0) - u_o\|_{L^2(\Omega)}^2.$$

This shows that the limit function is indeed a variational solution in the sense of Definition 1.1. The previous considerations roughly describe the main steps in the proof of the following existence result in the special case when $u_o = u_*$. The full statement with an initial datum $u_o \in L^2(\Omega)$ and lateral boundary values $u_*$ as in $(1.8)_2$ is proved as a special case in Theorem 5.1.

THEOREM 2.1. *Assume that* (1.2) *is in force and that the initial and boundary values $u_o$ and $u_*$ satisfy* (1.8). *Then, there exists a variational solution*

$$u \in C^0([0, T]; L^2(\Omega)) \cap L^p(0, T; W_{u_*}^{1,p}(\Omega))$$

*to the gradient flow* (1.5) *in the sense of Definition* 1.1.

*Furthermore, for the time derivative we have that $\partial_t u \in L^2(\Omega \times (\varepsilon, T))$ for any $\varepsilon > 0$. In the special case when $u_o = u_*$ there even holds $\partial_t u \in L^2(\Omega \times (0, T))$.*

Theorem 2.1 is the parabolic counterpart to the classical theory of the Calculus of Variations we are looking for. As already mentioned, for $u_o = u_*$ the result

has been proved in the vectorial setting in [13] by Elliptic Regularization. The method of Minimizing Movements as described here has been used in [17] to prove existence of variational solutions to time dependent obstacle problems. By more sophisticated arguments Theorem 2.1 can be generalized in several directions, concerning variable boundary data, obstacle problems, time dependent integrands, time dependent domains, or the total variation flow, cf. [15, 16, 17, 18, 48]. These generalizations are by far non-trivial, since the outlined Minimizing Movements Method is just the starting point in the proof. The difficulty stems from certain mollification procedures which have to be performed if the data (the boundary data or the obstacle) are not smooth enough.

## 3. Doubly nonlinear equations

In [26] De Giorgi posed the question what could be the effect if the second term in the functional $\mathbf{F}_{k,i}$ in (2.1) is modified. For instance, he suggested to replace the second term in $\mathbf{F}_{k,i}$ by the integral

$$\frac{1}{\beta h_k^{\beta-1}} \int_\Omega |u_{k,i-1} - v|^\beta \, dx$$

for some $\beta \in (1, 2]$. De Giorgi conjectured that the resulting modified Minimizing Movements Scheme leads to a solution of the differential equation

$$|\partial_t u|^{\beta-2} \partial_t u - \operatorname{div} D_\xi f(Du) = 0.$$

This is a certain kind of doubly nonlinear equation, since it is nonlinear in the diffusion part and also in the term containing the time derivative. For the integrand $f(\xi) = \frac{1}{2}|\xi|^2$ this conjecture has been verified by Gianazza & Savaré [32]. They obtained a solution to the differential equation

(3.1) $$|\partial_t u|^{\beta-2} \partial_t u - \Delta u = 0.$$

More general nonlinearities with respect to $\partial_t u$ have been treated by Mielke, Rossi & Savaré [44], who developed a minimizing movements scheme for equations of the type

$$b(\partial_t u) - \operatorname{div} D_\xi f(Du) = 0,$$

where $b = B'$ is the derivative of a convex function $B$ with superlinear growth. These equations are usually called doubly nonlinear evolution equations of second type, cf. [54]. Note that Akagi & Steffanelli [2, 3] proved an existence result for doubly nonlinear equations of second type by Elliptic Regularization.

Equation (3.1) looks somewhat similar to the Porous Medium Equation

$$\partial_t(|u|^{m-1} u) - \Delta u = 0, \quad m > 0.$$

However, both equations show a completely different behaviour, and it is not possible to transform one into the other. The Porous Medium Equation, how-

ever, is of particular interest since it possesses a wide spectrum of applications, for instance in fluid dynamics, soil science and filtration, cf. [7, 8, 9, 41, 50]. Note that in the applications usually non-negative solutions are relevant. Therefore, in the following we always assume that $u$ is non-negative, so that $|u|^{m-1}u = u^m$.

### 3.1. The prototype equation

More generally, in the following we will consider doubly nonlinear equations whose prototype is given by

$$(3.2) \qquad \partial_t u^m - \operatorname{div}(|Du|^{p-2}Du) = 0,$$

where $m > 0$ and $p > 1$. For $m = 1$ and $p = 2$, equation (3.2) reduces to the standard heat equation. For $m = 1$ and $p \in (1, \infty)$ the equation is known as the parabolic $p$-Laplace equation, while for $m \in (0, \infty)$ and $p = 2$ we are dealing with the Porous Medium Equation. For the doubly nonlinear equation the case $p - 1 > m$ is commonly known as a *slow diffusion equation*, while the case $p - 1 < m$ is named a *fast diffusion equation*, cf. [34]. The difference between these cases becomes apparent in the fact that slow diffusion equations allow solutions with compact support and perturbations propagate with finite speed, while in fast diffusion equations perturbations propagate with infinite speed, prohibiting compact support solutions. Moreover, doubly nonlinear equations can be subdivided into doubly degenerate parabolic equations ($p > 2$, $0 < m < 1$), singular-degenerate equations ($1 < p < 2$, $0 < m < 1$), degenerate-singular equations ($p > 2$, $m > 1$), and, finally, doubly singular equations ($1 < p < 2$, $m > 1$), cf. [34]. Despite their importance in applications and mathematics, a natural variational approach to the existence of solutions to the Porous Medium Equation or doubly nonlinear equations has not been invented before.

### 3.2. General doubly nonlinear equations

Before presenting the variational approach to doubly nonlinear equations, we will introduce the precise setting. In the following we consider the Cauchy–Dirichlet problem to the following quite general class of doubly nonlinear equations

$$(3.3) \qquad \begin{cases} \partial_t b(u) - \operatorname{div} D_\xi f(x, u, Du) = -D_u f(x, u, Du) & \text{in } \Omega_T, \\ u = u_* & \text{on } \partial\Omega \times (0, T), \\ u(0) = u_o & \text{in } \Omega, \end{cases}$$

where $u_*$ and $u_o$ are non-negative lateral and initial-boundary values and $f : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \bar{\mathbb{R}}$ denotes an integrand satisfying the convexity and coercivity conditions from (1.2). Note that with slightly more effort, one can subtract lower order terms on the right-hand side of $(1.2)_2$; see [14]. These type of equations clearly cover the model case (3.2) with the choices $b(u) = u^m$ and $f(\xi) = \frac{1}{p}|\xi|^p$. More generally, we consider a *nonlinearity* $b : [0, \infty) \to [0, \infty)$ which is continuous and piecewise $C^1$ on $(0, \infty)$. Without loss of generality, by replacing $b(u)$ with

$b(u) - b(0)$, we can reduce to the case $b(0) = 0$. Moreover, we assume that $b(u) > 0$ for $u > 0$ and that

$$(3.4) \qquad \ell \le \frac{ub'(u)}{b(u)} \le m,$$

holds true whenever $u > 0$ and $b'(u)$ exists, for given constants $m \ge \ell > 0$. We note that this implies in particular that $b' > 0$ whenever $b'$ exists. We define the primitive of $b$ by

$$\phi(u) := \int_0^u b(s)\, ds \quad \text{for all } u \ge 0.$$

Note that $\phi$ is a convex $C^1$-function with $\phi(0) = 0$. As usual, the convex conjugate of $\phi$ is defined by

$$\phi^*(x) := \sup_{v \ge 0}(xv - \phi(v)) \quad \text{for all } x \ge 0.$$

Using the convexity of $\phi$, it is straightforward to compute the convex conjugate at the point $x = b(u)$ as

$$(3.5) \qquad \phi^*(b(u)) = b(u)u - \phi(u) \quad \text{for all } u \ge 0.$$

An immediate consequence of the above definition is Fenchel's inequality

$$(3.6) \qquad uv \le \phi(u) + \phi^*(v) \quad \text{for all } u, v \ge 0.$$

Finally, we define the boundary term related to the nonlinearity $b$ by

$$(3.7) \qquad b[u, v] := \phi(v) - \phi(u) - b(u)(v - u)$$

for all numbers $u, v \ge 0$ and

$$\mathfrak{B}[u, v] := \int_\Omega b[u, v]\, dx,$$

for $u, v : \Omega \to \mathbb{R}_{\ge 0}$. For a domain $A \subset \mathbb{R}^k$, $k \in \mathbb{N}$, we define the Orlicz spaces related to the function $\phi$ by

$$L^\phi(A) := \left\{ v : A \to \mathbb{R}, \text{ measurable} : \int_A \phi(a|v|)\, dx < \infty, \text{ for some } a > 0 \right\}.$$

For more details on Orlicz spaces we refer to the monograph [46]. Since our assumptions on $b$ imply the $\Delta_2$-condition for $\phi$ (cf. [14, Lemma 2.1]), the previous definition is equivalent to

$$L^\phi(A) = \left\{ v : A \to \mathbb{R}, \text{ measurable} : \int_A \phi(|v|)\, dx < \infty \right\}.$$

The space $L^\phi(A)$ is equipped with the Orlicz norm

$$\|v\|_{L^\phi(A)} := \sup\left\{\left|\int_A vw\,dx\right| : \int_A \phi^*(w)\,dx \le 1\right\},$$

which is equivalent to the Luxemburg norm. Analogously, related to the convex conjugate $\phi^*$ we define the Orlicz space $L^{\phi^*}(A)$ equipped with the norm $\|\cdot\|_{L^{\phi^*}(A)}$. From [14, Lemma 2.3] we recall

LEMMA 3.1. *Under the assumption* (3.4) *on b, we have for any $u \ge 0$ that*

$$\frac{1}{m+1}ub(u) \le \phi(u) \le \frac{1}{\ell}\phi^*(b(u)) \le \frac{m}{\ell(m+1)}ub(u).$$

Instead of (1.8) we now assume the following compatibility condition for the non-negative initial and lateral boundary values $u_o$ and $u_*$:

$$(3.8) \qquad \begin{cases} u_o \in L^\phi(\Omega), \\ u_* \in L^\phi(\Omega) \cap W^{1,p}(\Omega) \quad \text{with} \\ \int_\Omega f(x, u_* + \varphi, D(u_* + \varphi))\,dx < \infty \quad \forall \varphi \in C_0^\infty(\Omega). \end{cases}$$

Note that the assumption $u_o \in L^\phi(\Omega)$ is equivalent to $b(u_o) \in L^{\phi^*}(\Omega)$; see Lemma 3.1. Moreover, the coercivity condition $(1.2)_2$ together with $(3.8)_2$ (for the choice $\varphi = 0$) already implies that $Du_* \in L^p(\Omega, \mathbb{R}^n)$.

REMARK 3.2. At this point some comments on assumption $(3.8)_3$ are in order. As can be seen from the proof of Lemma 4.2 below, this condition is only needed to guarantee the finiteness of the integral

$$\int_\Omega f(x, u_* + ((u_o - u_*)\chi_{2\varepsilon}) * \phi_\varepsilon, D[u_* + ((u_o - u_*)\chi_{2\varepsilon}) * \phi_\varepsilon])\,dx < \infty,$$

which of course is of the form

$$(3.9) \qquad \int_\Omega f(x, u_* + \varphi_\varepsilon, D[u_* + \varphi_\varepsilon])\,dx < \infty$$

with the smooth function $\varphi_\varepsilon := ((u_o - u_*)\chi_{2\varepsilon}) * \phi_\varepsilon \in C_0^\infty(\Omega)$. In a sense, the condition is tailor-made to enforce the finiteness of the integrals. Unfortunately, $(3.8)_3$ is relatively abstract and not easy to verify. Therefore, it is certainly useful to have some concrete conditions on the boundary data $u_*$ and the integrand $f$ at hand, that imply $(3.8)_3$. In the case of Lipschitz boundary values $u_* \in W^{1,\infty}(\Omega)$, it is enough to impose a condition ensuring the finiteness of the integrals of $f(x, \varphi, D\varphi)$ with $\varphi \in W^{1,\infty}(\Omega)$. However, this is guaranteed if there exists a measurable function $g : \Omega \times [0, \infty) \to [0, \infty)$ such that for any fixed $M \ge 0$ the par-

tial map $g(\cdot, M)$ is integrable, i.e. $g(\cdot, M) \in L^1(\Omega)$, and the pointwise bound

$$(3.10) \quad 0 \le f(x, u, \xi) \le g(x, M) \quad \text{for a.e. } x \in \Omega, \text{ provided } |u| \le M, |\xi| \le M,$$

holds true. Obviously, this condition includes any integrand $f$ satisfying a growth condition of the form

$$0 \le f(x, u, \xi) \le \alpha(x)\phi(|\xi|) + \beta(x)\psi(|u|) + \gamma(x)$$

with non-negative continuous functions $\phi, \psi : [0, \infty) \to [0, \infty)$ and integrable functions $\alpha, \beta, \gamma \in L^1(\Omega)$.

For general boundary data $u_* \notin W^{1,\infty}(\Omega)$ the situation is more complicated. Maybe the easiest way to ensure the validity of (3.9) in this case is to impose a standard $p$-growth condition for the integrand $f$ as in $(1.9)_1$. However, such a condition rules out interesting classes of integrands satisfying a non-standard $(p, q)$-growth condition. Another possibility to ensure the existence of integrals as in (3.9) is to exploit the convexity of the integrand $f$ with respect to the variables $(u, \xi)$ as follows:

$$f(x, u_* + \varphi_\varepsilon, D[u_* + \varphi_\varepsilon]) \le \frac{1}{2} f(x, 2u_*, 2Du_*) + \frac{1}{2} f(x, 2\varphi_\varepsilon, 2D\varphi_\varepsilon).$$

Therefore, it is natural to impose on the one hand a $\Delta_2$-condition for the integrand $f$ with respect to $(u, \xi)$ and on the other hand that the variational integral is finite for the given boundary values $u_*$ as well as for smooth functions $\varphi \in C_0^\infty(\Omega)$. More precisely, we require that the following three assumptions hold true:

(i) $f$ satisfies a $\Delta_2$-condition, i.e. there exists a constant $K > 0$ such that for a.e. $x \in \Omega$ and any $(u, \xi) \in \mathbb{R}_{\ge 0} \times \mathbb{R}^n$ there holds

$$f(x, 2u, 2\xi) \le Kf(x, u, \xi).$$

(ii) The boundary values have finite energy, i.e.

$$\int_\Omega f(x, u_*, Du_*)\, dx < \infty;$$

(iii) For any $\varphi \in C_0^\infty(\Omega)$ there holds

$$\int_\Omega f(x, \varphi, D\varphi)\, dx < \infty.$$

Under these assumptions, it is straightforward to deduce that the integral in (3.9) is finite. Indeed, the convexity allows to decompose the integral into two parts. The first one, i.e. the one with $f(x, 2u_*, 2Du_*)$ as integrand, is handled by the $\Delta_2$ condition and therefore it is finite. The second integral, i.e. the one with

integrand $f(x, \varphi_\varepsilon, D\varphi_\varepsilon)$ where $\varphi_\varepsilon := ((u_o - u_*)\chi_{2\varepsilon})) * \phi_\varepsilon$, is finite due to assumption (iii).

Unfortunately, the $\Delta_2$-condition is restrictive. Nevertheless, many interesting examples of integrands $f$ fulfill this condition; the most prominent examples are of course integrands satisfying a $p$-growth condition. Moreover, Orlicz-type functionals such as $|Du|^p \log(1 + |Du|)$ and more general variants of that are admissible. But also a large variety of functionals with $(p, q)$-growth falls into this class of functionals. For example, if the nonlinearity $b$ fulfills (3.4) with constants $0 < \ell < m < \infty$, then the primitive $\phi$ of $b$ is a convex function satisfying a $\Delta_2$-condition and $\phi$ satisfies a growth condition that is reminiscent of a $p, q$-growth condition with $p = \ell + 1$ and $q = m + 1$, more precisely

$$v \min\{|\xi|^{\ell+1}, |\xi|^{m+1}\} \le \phi(|\xi|) \le L \max\{|\xi|^{\ell+1}, |\xi|^{m+1}\}$$

for any $\xi \in \mathbb{R}^n$ and constants $0 < v \le L < \infty$. This can easily be converted into a $(\ell + 1, m + 1)$-growth condition of the form

$$v(|\xi|^{\ell+1} - 1) \le \phi(|\xi|) \le L(1 + |\xi|^{m+1})$$

for any $\xi \in \mathbb{R}^n$. For this reason, the structural requirements are satisfied by the integrand $\tilde{f}(\xi) := \phi(|\xi|) + v$, and thus lead to variational solutions associated to the integrand $f(\xi) := \phi(|\xi|)$. Therefore, integrands $f(\xi) = \phi(|\xi|)$ are possible examples. Other examples of non-standard $(p, q)$-growth integrands that satisfy the above conditions are the before mentioned double phase integrands $f(x, \xi) = \alpha(x)|\xi|^p + \beta(x)|\xi|^q$. After all we could also consider integrands $f(\xi)$ which do not behave like a power when $|\xi| \to \infty$. For instance, for $|\xi|$ large, the integrand could be of the type

$$f(\xi) = |\xi|^{a + b \sin(\log \log |\xi|)}.$$

A computation shows that such an integrand is a convex function for $|\xi| \ge e$ (and therefore it can be extended to all $\xi \in \mathbb{R}^n$ as a convex function on $\mathbb{R}^n$) if $a$, $b$ are positive real numbers such that $a > 1 + b\sqrt{2}$. In this case our integrand satisfies the bounds $|\xi|^p \le f(\xi) \le L(1 + |\xi|^q)$, with $p = a - b$ and $q = a + b$. Moreover, by a somewhat lengthy (but elementary) computation it can be shown that $f(\xi)$ fulfills a $\Delta_2$-condition.

At this point, however, it should be noted that not every integrand $f(\xi)$ satisfying a non-standard $p, q$-growth condition must necessarily fulfill a $\Delta_2$ condition. We refer to the following Remark 3.3, in which we present a counter-example.

Finally, also certain kind of functionals with exponential growth like $e^{|\xi|}$ and $e^{\sqrt{1+|\xi|^2}}$ are allowed in (3.8) although they do not satisfy a $\Delta_2$-condition. More general functionals of exponential growth are discussed in [14, Example 7.4]. However, one should mention that an exponential growth like $f(\xi) = e^{|\xi|^2}$ is not covered by the above assumptions.

In contrast to the $\Delta_2$-condition in (i), the assumption (iii) is not very restrictive. As already explained before, assumption (iii) for example holds if the integrand $f$ satisfies condition (3.10). $\qquad\square$

REMARK 3.3. Here, we give an example of a convex function $f : \mathbb{R}^n \to \mathbb{R}$ of $p, q$-growth which does not satisfy the $\Delta_2$-condition. This example, with exponents $p > 1$ and $q > p$ arbitrarily close to $p$, is inspired by [40, pp. 28–29], see also [31, pp. 342–343] and [25, Section 2.4]. We define $f = f(|\xi|)$ in integral form by

$$(3.11) \qquad f(|\xi|) = \int_0^{|\xi|} g(t)\, dt \quad \text{for } \xi \in \mathbb{R}^n,$$

where $g : [0, \infty) \to [0, \infty)$ is the piecewise polynomial (piecewise constant when $t \geq 1$) increasing function defined by

$$(3.12) \qquad g(t) = \begin{cases} t^{p-1} & \text{if } t \in [0, 1), \\ (k!)^{p-1} & \text{if } t \in [(k-1)!, k!),\ k \in \mathbb{N}_{\geq 2}. \end{cases}$$

Then $g(t) \geq t^{p-1}$ for every $t \in [0, \infty)$ and thus $f(|\xi|) \geq \frac{1}{p}|\xi|^p$ for all $\xi \in \mathbb{R}^n$. Next, we claim that $f$ satisfies a $q$-growth condition from above for an arbitrary $q > p$. For the proof of this claim, we first notice that, if $t \in [(k-1)!, k!)$ for some $k \in \mathbb{N}_{\geq 2}$ then $g(t) = (k!)^{p-1} = [(k-1)!k]^{p-1}$ and therefore $g(t) \leq k^{p-1}t^{p-1}$. Now we choose $\beta > 0$ in such a way that $q = p + \frac{1}{\beta}$. Since $\lim_{k \to \infty} k^{(p-1)\beta}/(k-1)! = 0$, there exists a positive constant $c = c(p, \beta) = c(p, q)$ with

$$\frac{k^{(p-1)\beta}}{(k-1)!} < c \quad \text{for every } k \in \mathbb{N}.$$

Hence, for every $t \in [(k-1)!, k!)$ we obtain the estimate

$$g(t) \leq k^{p-1}t^{p-1} \leq [c(k-1)!]^{\frac{1}{\beta}}t^{p-1} \leq [ct]^{\frac{1}{\beta}}t^{p-1} = c^{\frac{1}{\beta}}t^{q-1}.$$

Hence, for every $\xi \in \mathbb{R}^n$ we deduce

$$f(|\xi|) = \int_0^{|\xi|} g(t)\, dt \leq \frac{1}{p} + c^{\frac{1}{\beta}} \int_1^{|\xi|} t^{q-1}\, dt \leq \frac{1}{p} + \frac{1}{q}c^{\frac{1}{\beta}}|\xi|^q.$$

Combining this with the bound from below, we have shown that

$$\frac{1}{p}|\xi|^p \leq f(|\xi|) \leq L(1 + |\xi|^q) \quad \text{for every } \xi \in \mathbb{R}^n,$$

with $L := \max\{\frac{1}{p}, \frac{1}{q}c^{\frac{1}{\beta}}\}$. The fact that the function $f$ does not satisfy the $\Delta_2$-condition can be seen similarly to [40]. We consider a sequence $\xi_k \in \mathbb{R}^n$, $k \in \mathbb{N}$, with $|\xi_k| = k!$, and compare the values $f(|\xi_k|) = f(k!)$ and $f(|2\xi_k|) = f(2k!)$. Since $g(t) = (k!)^{p-1}$ for $t \in [(k-1)!, k!)$, and this value is also the maximum of

$g$ in the interval $[0, k!)$, we deduce

$$f(|\xi_k|) = f(k!) = \int_0^{k!} g(t)\,\mathrm{d}t \le k!^p,$$

while, since $g(k!)$ is the minimum value of $g$ in the interval $[k!, 2k!)$, we have

$$f(|2\xi_k|) = f(2k!) = \int_0^{2k!} g(t)\,\mathrm{d}t \ge \int_{k!}^{2k!} g(t)\,\mathrm{d}t \ge g(k!)k! = (k+1)!^{p-1}k!.$$

Therefore, we know

$$\lim_{k\to\infty} \frac{f(|2\xi_k|)}{f(|\xi_k|)} \ge \lim_{k\to\infty} \frac{(k+1)!^{p-1}k!}{k!^p} = \lim_{k\to\infty}(k+1)^{p-1} = \infty.$$

This implies that $f$ does not satisfy the $\Delta_2$-condition.                                    □

## 4. VARIATIONAL FORMULATION FOR DOUBLY NONLINEAR EQUATIONS

Our aim in [14] was to follow the conjecture of De Giorgi in [26] in order to construct variational solutions to Porous Medium type equations, or even more generally to doubly nonlinear equations by a modified nonlinear Minimizing Movements type approach. To this aim, one first has to find a suitable variational formulation of the Porous Medium Equation, respectively the doubly nonlinear equation.

Similar to our approach in Section 1, we multiply $(3.3)_1$ by $v - u$ with some non-negative function $v \in L^p(0, T; W_{u_o}^{1,p}(\Omega))$ with $\partial_t v \in L^{\phi^*}(\Omega_T)$ and $v(0) \in L^{\phi^*}(\Omega)$. Then, we integrate the result over $\Omega_\tau$ for some $\tau \in (0, T]$ and subsequently perform an integration by parts to obtain

$$\underbrace{\iint_{\Omega_\tau} \partial_t b(u)(v - u)\,\mathrm{d}x\,\mathrm{d}t}_{=:\,\mathrm{I}}$$

$$+ \underbrace{\iint_{\Omega_\tau} [D_\xi f(x, u, Du) \cdot (Dv - Du) + D_u f(x, u, Du)(v - u)]\,\mathrm{d}x\,\mathrm{d}t}_{=:\,\mathrm{II}} = 0.$$

The term II is treated exactly as in (1.6). The computation for I is more complicated than before. Nevertheless, taking into account that

$$u\partial_t b(u) = \partial_t[ub(u) - \phi(u)]$$

and that

$$b(v)\partial_t v = \partial_t \phi(v)$$

we are able to produce a boundary term by the following calculation

$$
\begin{aligned}
I &= -\iint_{\Omega_\tau} b(u)\partial_t v \,\mathrm{d}x\,\mathrm{d}t - \iint_{\Omega_\tau} u\partial_t b(u)\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega b(u)v\,\mathrm{d}x\Big|_0^\tau \\
&= \iint_{\Omega_\tau} \partial_t v(b(v)-b(u))\,\mathrm{d}x\,\mathrm{d}t - \iint_{\Omega_\tau} b(v)\partial_t v\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega [\phi(u)+b(u)(v-u)]\,\mathrm{d}x\Big|_0^\tau \\
&= \iint_{\Omega_\tau} \partial_t v(b(v)-b(u))\,\mathrm{d}x\,\mathrm{d}t - \int_\Omega \mathrm{b}[u,v]\,\mathrm{d}x\Big|_0^\tau \\
&= \iint_{\Omega_\tau} \partial_t v(b(v)-b(u))\,\mathrm{d}x\,\mathrm{d}t - \mathfrak{B}[u(\tau),v(\tau)] + \mathfrak{B}[u(0),v(0)].
\end{aligned}
$$

Here, we used the notation for the boundary term from (3.7). In the case $b(u)=u$ the boundary term simplifies to

$$
\mathrm{b}[u,v] = \frac{1}{2}|u-v|^2 \quad\text{and}\quad \mathfrak{B}[u,v] = \frac{1}{2}\|u-v\|_{L^2(\Omega)}^2,
$$

so that in the variational inequality the usual $L^2(\Omega)$-boundary terms appear. The above heuristic argument suggests the following definition.

DEFINITION 4.1. Assume that the initial and lateral boundary values $u_o, u_* : \Omega \to \mathbb{R}_{\geq 0}$ satisfy (3.8). We identify a non-negative map

$$
u \in C^0([0,T]; L^\phi(\Omega)) \cap L^p(0,T; W_{u_*}^{1,p}(\Omega))
$$

as a *variational solution* to the Cauchy–Dirichlet problem of the doubly nonlinear equation (3.3) if and only if the variational inequality

$$
\begin{aligned}
(4.1) \qquad & \mathfrak{B}[u(\tau),v(\tau)] + \iint_{\Omega_\tau} f(x,u,Du)\,\mathrm{d}x\,\mathrm{d}t \\
& \leq \mathfrak{B}[u_o,v(0)] + \iint_{\Omega_\tau} [f(x,v,Dv) + \partial_t v(b(v)-b(u))]\,\mathrm{d}x\,\mathrm{d}t
\end{aligned}
$$

holds true for any $\tau \in (0,T]$ and any non-negative map $v \in L^p(0,T; W_{u_*}^{1,p}(\Omega))$ with time derivative $\partial_t v \in L^\phi(\Omega_T)$ and initial value $v(0) \in L^\phi(\Omega)$. $\qquad\square$

This concept of solution can be viewed as the natural extension of the notion of pseudo solutions given by Lichnewsky & Temam in [42] to the framework of doubly nonlinear equations. Similarly to Definition 1.1, also the variational inequality (4.1) implies that the initial condition $u(0) = u_o$ is taken in the $C^0 - L^\phi$-sense. For the case $u_o = u_*$ this has been proved in [14, Lemma 2.9]. The general case is treated in the following lemma.

LEMMA 4.2. *Suppose that the variational integrand $f : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \overline{\mathbb{R}}$ and the nonlinearity $b : [0, \infty) \to [0, \infty)$ satisfy (1.2) and (3.4) and that the initial and lateral boundary values $u_o$ and $u_*$ satisfy (3.8). Then, any non-negative map*

$$u \in C^0((0, T]; L^\phi(\Omega)) \cap L^p(0, T; W^{1,p}_{u_*}(\Omega))$$

*solving the variational inequality (4.1) attains the initial values $u(0) = u_o$ in the $C^0 - L^\phi$-sense, that is*

$$\lim_{t \downarrow 0} \|u(t) - u_o\|_{L^\phi(\Omega)} = 0.$$

PROOF. For $\varepsilon > 0$ we define exactly as in the proof of Lemma 1.2

$$u_o^{(\varepsilon)} := u_* + ((u_o - u_*)\chi_{2\varepsilon}) * \phi_\varepsilon,$$

and recall that $u_o^{(\varepsilon)} - u_* \in C_0^\infty(\Omega)$ with $\mathrm{spt}(u_o^{(\varepsilon)} - u_*) \subset \overline{\Omega^\varepsilon}$ and $u_o^{(\varepsilon)} \to u_o$ in $L^\phi(\Omega)$ as $\varepsilon \downarrow 0$. Furthermore, assumption (3.8) ensures for any $\varepsilon > 0$ that

$$0 \le \int_\Omega f(x, u_o^{(\varepsilon)}, Du_o^{(\varepsilon)}) \, dx < \infty.$$

This allows to choose the time independent extension of $u_o^{(\varepsilon)}$ to $\Omega_\tau$ for $\tau \in (0, T]$ as comparison function in the variational inequality (4.1) on $\Omega_\tau$. Due to the coercivity assumption (1.2) we obtain for a.e. $\tau \in (0, T]$ that

$$\mathfrak{B}[u(\tau), u_o^{(\varepsilon)}] \le \iint_{\Omega_\tau} f(x, u_o^{(\varepsilon)}, Du_o^{(\varepsilon)}) \, dx \, dt + \mathfrak{B}[u_o, u_o^{(\varepsilon)}]$$

$$= \tau \int_\Omega f(x, u_o^{(\varepsilon)}, Du_o^{(\varepsilon)}) \, dx + \mathfrak{B}[u_o, u_o^{(\varepsilon)}].$$

In view of [14, Lemma 2.5], this implies

$$\limsup_{\tau \downarrow 0} \int_\Omega |\sqrt{\phi(u(\tau))} - \sqrt{\phi(u_o)}|^2 \, dx \le c \int_\Omega |\phi(u_o^{(\varepsilon)}) - \phi(u_o)| \, dx,$$

for a constant $c = c(\ell, m)$. Since the left-hand side does not depend on $\varepsilon$, we can pass to the limit $\varepsilon \downarrow 0$ in the right-hand side and obtain that

$$\lim_{\tau \downarrow 0} \int_\Omega |\sqrt{\phi(u(\tau))} - \sqrt{\phi(u_o)}|^2 \, dx = 0,$$

which immediately implies the bound

$$\limsup_{\tau \downarrow 0} \int_\Omega \phi(u(\tau)) \, dx \le 2 \int_\Omega \phi(u_o) \, dx < \infty.$$

In turn, we conclude that

$$\limsup_{\tau \downarrow 0} \int_\Omega |\phi(u(\tau)) - \phi(u_o)| \, \mathrm{d}x$$

$$\leq \left[ \limsup_{\tau \downarrow 0} \int_\Omega |\sqrt{\phi(u(\tau))} - \sqrt{\phi(u_o)}|^2 \, \mathrm{d}x \right]^{\frac{1}{2}}$$

$$\cdot \left[ \limsup_{\tau \downarrow 0} \int_\Omega [\sqrt{\phi(u(\tau))} + \sqrt{\phi(u_o)}]^2 \, \mathrm{d}x \right]^{\frac{1}{2}} = 0.$$

With [14, Lemma 2.6] we therefore obtain that $u(\tau) \to u_o$ in $L^\phi(\Omega)$ as $\tau \downarrow 0$. This proves the claim that $u(0) = u_o$ in the $C^0 - L^\phi$-sense. $\qquad \square$

Similarly to Lemma 1.2 we can also show in the setting of doubly non-linear evolution equations that the notion of variational solution and the notion of weak solution coincide, if the integrand $f$ is regular enough; see [14, Sections 6, 7].

## 5. VARIATIONAL APPROACH TO DOUBLY NONLINEAR EQUATIONS

Having the variational formulation from Definition 4.1 at hand, we are now in the position to explain our purely variational approach to the existence of doubly nonlinear equations by a modified nonlinear Minimizing Movements type approach. The key observation is that the second integral $\frac{1}{2} \int_\Omega |u - v|^2 \, \mathrm{d}x$ in (2.1) is exactly the boundary term in the variational formulation (1.7). Therefore, a natural choice for a Minimizing Movement scheme for doubly nonlinear equations could be the boundary term $\mathfrak{B}[u, v]$ from the variational inequality (4.1). Surprisingly enough, it turned out that the replacement of $\frac{1}{2}|u - v|^2$ by $\mathfrak{b}[u, v]$ in the Minimizing Movement scheme indeed yields, after passing to the limit $k \to \infty$, the existence of a variational solution to the doubly nonlinear equation (3.3). The final outcome of the whole procedure is the following existence result.

THEOREM 5.1. *Suppose that the nonlinearity $b : [0, \infty) \to [0, \infty)$ satisfies (3.4) and that $f : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \overline{\mathbb{R}}$ is a variational integrand satisfying the convexity and coercivity assumption (1.2). Then, for any initial and lateral boundary values $u_o$ and $u_*$ as in (3.8), there exists a variational solution $u$ in $\Omega_T$ to the doubly nonlinear evolutionary equation in the sense of Definition 4.1. Moreover, for any $\varepsilon > 0$ the solution satisfies $\partial_t \sqrt{\phi(u)} \in L^2(\Omega \times (\varepsilon, T))$.*

For initial data $u_o = u_*$ Theorem 5.1 has been obtained in [14] (see also [11] for a short summary), where also the case of unbounded domains is considered. Here, we will extend the result from [14] to the general case of initial data with possibly infinite energy. We emphasize that our method is flexible enough to treat time dependent lateral boundary data, problems with time-dependent obstacles,

or even a fast diffusion variant of the time dependent minimal surface equation. This will be investigated in forthcoming articles.

At this point, some words on the history of the problem are in order. Concerning existence of solutions to doubly nonlinear evolutionary equations one has to mention the pioneering papers of Grange & Mignot [33] and Alt & Luckhaus [5], in which – amongst other things – the existence of weak solutions to quasilinear parabolic equations of the form

$$(5.1) \qquad \partial_t b(u) - \operatorname{div} \mathbf{a}(b(u), Du) = \mathbf{f}(b(u))$$

was established. In [5] the continuous nonlinearity $b$ was assumed to satisfy $b(0) = 0$ and $b = \phi'$ for some convex $C^1$-function $\phi$. The coefficients $\mathbf{a}(b(u), \xi)$ were assumed to be continuous in $(u, \xi)$, to satisfy the ellipticity condition

$$(\mathbf{a}(b(u), \xi) - \mathbf{a}(b(u), \eta))(\xi - \eta) \geq c|\xi - \eta|^p$$

for some exponent $1 < p < \infty$ and some constant $c > 0$, and a standard growth condition from above, essentially ensuring $(p-1)$-growth in the gradient variable. The existence results of [5] and [33] have later been generalized to higher order doubly nonlinear equations on unbounded domains by Bernis [10].

A completely different approach to doubly nonlinear parabolic equations by methods from Convex Analysis has been suggested by Akagi & Stefanelli [4]. They consider the doubly nonlinear PDE

$$(5.2) \qquad \partial_t b(u) - \operatorname{div} \mathbf{a}(Du) \ni \mathbf{f},$$

where $b \subset \mathbb{R} \times \mathbb{R}$ and $\mathbf{a} \subset \mathbb{R}^n \times \mathbb{R}^n$ are maximal monotone graphs. The graphs are required to fulfill polynomial growth of power $m > 0$ and $p > 1$; for example the choices $b(u) = u^m$ and $\mathbf{a}(\xi) = |\xi|^{p-2}\xi$ cover the standard model (3.2). Instead of considering (5.2) the authors transform the equation to an equivalent dual problem for the unknown $v = b(u)$, which reads as $-\operatorname{div} \mathbf{a}(Db^{-1}(v)) \ni \mathbf{f} - \partial_t v$. For the porous medium equation one only changes the viewpoint in the sense that the nonlinearity is shifted from $\partial_t u^m - \Delta u = f$ to $\partial_t v - \Delta v^{\frac{1}{m}} = f$. The same transformation leads from the doubly nonlinear equation $\partial_t u^m - \operatorname{div}(|Du|^{p-2}Du) = f$ to the equation $\partial_t v - \operatorname{div}(|Dv^{\frac{1}{m}}|^{p-2}Dv^{\frac{1}{m}}) = f$. Now, let $B^* : W^{-1,p}(\Omega) \to W_0^{1,p}(\Omega)$ denote the solution operator to the nonlinear elliptic problem $h \mapsto w$, where $w$ solves the Dirichlet problem $-\operatorname{div} \mathbf{a}(Dw) \ni h$ in $\Omega$ with $w|_{\partial\Omega} = 0$. Then, (5.2) can be re-formulated as $-B^*(f - \partial_t v) + b^{-1}(v) \ni 0$. In the case $\mathbf{a} = \operatorname{id}$ one has $B^* = (-\Delta)^{-1}$ and the described strategy reduces to the classical dual formulation of (5.2) in $W^{-1,2}(\Omega)$, cf. [23]. Having arrived at this stage, the main idea of Akagi & Stefanelli was, to construct solutions to the latter problem by Elliptic Regularization (Weighted Energy-Dissipation Functional approach). As already mentioned, in this approach the nonlinearity $b$ and the coefficients $\mathbf{a}$ need to fulfill standard polynomial growth conditions.

Finally, we would like to mention that in [35, 36, 37] existence of solutions to doubly nonlinear equations has been proven by regularization and a priori esti-

mates, see also [53]. Moreover, in the special case of the porous medium equation, Kinnunen & Lindqvist & Lukkari [39] employed Perron's method for the construction of continuous solutions. For a discussion of regularity issues we refer to the monographs [29, 30].

Our approach to existence is completely different from all the methods mentioned above. Instead of using the equation itself, or to pass to a dual problem we work directly with the variational formulation of the equation introduced in Definition 4.1. As already explained at the beginning of this section, the idea is to replace in (2.1) the second integral $\frac{1}{2h} \int_\Omega |u - u_{i-1}|^2 \, dx$ by $\frac{1}{h} \mathfrak{B}[u_{i-1}, u]$, i.e. we consider the functionals

$$\mathbf{F}_i[u] := \int_\Omega f(x, u, Du) \, dx + \frac{1}{h} \mathfrak{B}[u_{i-1}, u],$$

where $\mathfrak{B}$ is defined according to (3.7). The precise proof will be given in the following section. It mostly follows the lines of [14] with the necessary changes in order to cope with the $L^\phi$-initial values.

The advantages of this new approach are obvious: on the one hand it is possible to deal with integrands satisfying a non-standard growth condition. In such cases it is not clear whether or not the associated parabolic equation makes sense (due to the non-standard growth condition). On the other hand, in this setup it is quite easy to incorporate side conditions, for example obstacle conditions. The technique could also be modified to treat time dependent Dirichlet boundary conditions on the lateral boundary, or a Neumann type boundary condition. As already mentioned, functionals with linear growth could also be treated.

## 6. Proof of the existence result from Theorem 5.1

For the proof of Theorem 5.1 we proceed in several steps.

### 6.1. A sequence of minimizers to elliptic variational functionals

We fix $k \in \mathbb{N}$ and the step size $h_k := \frac{T}{k}$. Our goal is to inductively construct a sequence $u_{k,i} \in L^\phi(\Omega) \cap W^{1,p}_{u_*}(\Omega)$ of non-negative minimizers to certain elliptic variational functionals whenever $i \in \{1, \ldots, k\}$. The construction is as follows. Suppose that for some $i \in \{1, \ldots, k-1\}$ the non-negative minimizer $0 \leq u_{k,i-1} \in L^\phi(\Omega) \cap W^{1,p}_{u_*}(\Omega)$ has already been defined. If $i = 1$, then $u_{k,0} = u_o \in L^\phi(\Omega)$ is the initial boundary datum. Then we choose $u_i$ as the minimizer of the functional

$$\mathbf{F}_{k,i}[u] := \int_\Omega f(x, u, Du) \, dx + \frac{1}{h_k} \mathfrak{B}[u_{k,i-1}, u],$$

in the class of functions $v \in L^\phi(\Omega) \cap W^{1,p}_{u_*}(\Omega)$ with $v \geq 0$ a.e. in $\Omega$. Note that this class is non-empty since $v = u_*$ is admissible. The existence of $u_{k,i}$ can be deduced by means of the Direct Method of the Calculus of Variation; see also [14, Proposition 4.1].

### 6.2. Reformulating the minimizing property

We fix a non-negative comparison map $w \in L^\phi(\Omega_T) \cap L^p(0, T; W^{1,p}_{u_*}(\Omega))$. For $i \in \{0, \ldots, k-1\}$ and $s \in (0, 1)$ the convex combination $w_s := u_{k,i+1} + s(w - u_{k,i+1})$ of $u_{k,i+1}$ and $w$ is an admissible comparison map for the functional $\mathbf{F}_{k,i+1}$, i.e. $0 \le w_s \in L^\phi(\Omega) \cap W^{1,p}_{u_*}(\Omega)$. Hence, the minimality of $u_{k,i+1}$ and the convexity of $f$ imply

$$\mathbf{F}_{k,i+1}[u_{k,i+1}] \le \mathbf{F}_{k,i+1}[w_s]$$

$$\le \int_\Omega \left[ (1-s)f(x, u_{k,i+1}, Du_{k,i+1}) + sf(x, w, Dw) + \frac{1}{h_k} \mathfrak{b}[u_{k,i}, w_s] \right] dx$$

for any $s \in (0, 1)$, with equality for $s = 0$. From this we deduce

$$\int_\Omega f(x, u_{k,i+1}, Du_{k,i+1}) \, dx$$

$$\le \int_\Omega \left[ f(x, w, Dw) + \frac{1}{sh_k} [\mathfrak{b}[u_{k,i}, w_s] - \mathfrak{b}[u_{k,i}, u_{k,i+1}]] \right] dx$$

$$= \int_\Omega \left[ f(x, w, Dw) + \frac{1}{h_k} \left[ \frac{1}{s} [\phi(w_s) - \phi(u_{k,i+1})] - b(u_{k,i})(w - u_{k,i+1}) \right] \right] dx.$$

Since $\phi$ is convex, the map $s \mapsto \frac{1}{s}[\phi(w_s) - \phi(u_{k,i+1})]$ is monotone and converges a.e. on $\Omega$ to the $L^1(\Omega)$-function $b(u_{k,i+1})(w - u_{k,i+1})$. Therefore, by the dominated convergence theorem we may pass to the limit $s \downarrow 0$ in the preceding inequality and obtain

$$(6.1) \qquad \int_\Omega f(x, u_{k,i+1}, Du_{k,i+1}) \, dx$$

$$\le \int_\Omega \left[ f(x, w, Dw) + \frac{1}{h_k} [b(u_{k,i+1}) - b(u_{k,i})](w - u_{k,i+1}) \right] dx,$$

for any non-negative comparison map $w \in L^\phi(\Omega) \cap W^{1,p}_{u_*}(\Omega)$ and any $i \in \{0, \ldots, k-1\}$.

### 6.3. Energy estimates and weak convergence

We set $t_{k,i} := ih_k$ for $i \in \{-1, \ldots, k\}$ and define $u^{(k)} : \Omega \times (-h_k, T] \to \mathbb{R}^N$ as a piecewise constant function with respect to the time variable by

$$u^{(k)}(t) := u_{k,i} \quad \text{for } t \in J_{k,i} := (t_{k,i-1}, t_{k,i}] \text{ with } i \in \{0, \ldots, k\}.$$

In the following, we shall derive energy bounds for the functions $u^{(k)}$. To this aim, we choose in (6.1) the comparison map $w = u_* \in L^\phi(\Omega) \cap W^{1,p}_{u_*}(\Omega)$ for $i \in$

$\{0, \ldots, k-1\}$ and obtain

$$\int_\Omega f(x, u_{k,i+1}, Du_{k,i+1}) \, dx + \frac{1}{h_k} \int_\Omega b(u_{k,i+1}) u_{k,i+1} \, dx$$

$$\leq \int_\Omega f(x, u_*, Du_*) \, dx + \frac{1}{h_k} \int_\Omega [b(u_{k,i}) u_{k,i+1} + (b(u_{k,i+1}) - b(u_{k,i})) u_*] \, dx.$$

By Fenchel's inequality (3.6), we have that

$$\int_\Omega b(u_{k,i}) u_{k,i+1} \, dx \leq \int_\Omega [\phi(u_{k,i+1}) + \phi^*(b(u_{k,i}))] \, dx,$$

so that due to identity (3.5), the preceding inequality can be re-written as

$$\iint_{\Omega \times J_{k,i+1}} f(x, u_{k,i+1}, Du_{k,i+1}) \, dx \, dt + \int_\Omega \phi^*(b(u_{k,i+1})) \, dx$$

$$\leq h_k \int_\Omega f(x, u_*, Du_*) \, dx + \int_\Omega [\phi^*(b(u_{k,i})) + (b(u_{k,i+1}) - b(u_{k,i})) u_*] \, dx.$$

Summing up the last inequalities from $i = 0, \ldots, j-1$ with $j \leq k$, we obtain

$$\iint_{\Omega \times (0, jh_k]} f(x, u^{(k)}, Du^{(k)}) \, dx \, dt + \int_\Omega \phi^*(b(u^{(k)}(jh_k))) \, dx$$

$$\leq T \int_\Omega f(x, u_*, Du_*) \, dx + \int_\Omega [\phi^*(b(u_o)) + (b(u^{(k)}(jh_k)) - b(u_o)) u_*] \, dx.$$

In view of Fenchel's inequality (3.6), the convexity of $\phi^*$ and [14, inequality (2.3)], we have that

$$b(u^{(k)}(jh_k)) u_* \leq \phi^*\left(\frac{1}{2} b(u^{(k)}(jh_k))\right) + \phi(2u_*)$$

$$\leq \frac{1}{2} \phi^*(b(u^{(k)}(jh_k))) + 2^{m+1} \phi(u_*).$$

Inserting this above and applying Lemma 3.1 therefore yields

$$(6.2) \qquad \iint_{\Omega \times (0, jh_k]} f(x, u^{(k)}, Du^{(k)}) \, dx \, dt + \frac{1}{2} \int_\Omega \phi^*(b(u^{(k)}(jh_k))) \, dx \leq M,$$

where

$$M := T \int_\Omega f(x, u_*, Du_*) \, dx + \int_\Omega [m\phi(u_o) + 2^{m+1} \phi(u_*)] \, dx.$$

On the one hand, in view of Lemma 3.1 this shows that

$$(6.3) \qquad \sup_{t \in [0,T)} \int_\Omega \phi(u^{(k)}(t)) \, dx \le \frac{2}{\ell} M,$$

while on the other hand we obtain due to hypothesis (1.2) for the choice $j = k$ that

$$(6.4) \qquad \iint_{\Omega_T} |Du^{(k)}|^p \, dx \, dt \le \frac{1}{\nu} M.$$

Together, (6.3) and (6.4) ensure that the sequence $(u^{(k)})_{k \in \mathbb{N}}$ is uniformly bounded in the spaces $L^\infty(0, T; L^\phi(\Omega))$ and $L^p(0, T; W^{1,p}(\Omega))$. Therefore, there exists a limit map

$$u \in L^p(0, T; W^{1,p}_{u_*}(\Omega))$$

and a subsequence $\mathfrak{K} \subset \mathbb{N}$ such that in the limit $\mathfrak{K} \ni k \to \infty$ we have

$$(6.5) \qquad u^{(k)} \rightharpoonup u \quad \text{weakly in } L^p(0, T; W^{1,p}(\Omega)).$$

Our next aim is to analyze the convergence of $\phi(u^{(k)})$. To this aim, we choose in (6.1) the comparison map $w = u_{k,i} \in L^\phi(\Omega) \cap W^{1,p}_{u_*}(\Omega)$ for $i \in \{1, \dots, k-1\}$ and apply [14, Lemma 2.5]. In this way, we obtain

$$\int_\Omega f(x, u_{k,i+1}, Du_{k,i+1}) \, dx + \frac{1}{ch_k} \int_\Omega \left| \sqrt{\phi(u_{k,i+1})} - \sqrt{\phi(u_{k,i})} \right|^2 dx$$

$$\le \int_\Omega f(x, u_{k,i}, Du_{k,i}) \, dx,$$

for a constant $c = c(m, \ell) \ge 1$. Iterating the preceding inequalities from $i = j_1, \dots, j_2 - 1$ for some $1 \le j_1 < j_2 \le k$, we obtain

$$\int_\Omega f(x, u_{k,j_2}, Du_{k,j_2}) \, dx + \frac{1}{ch_k} \sum_{i=j_1+1}^{j_2} \int_\Omega \left| \sqrt{\phi(u_{k,i})} - \sqrt{\phi(u_{k,i-1})} \right|^2 dx$$

$$\le \int_\Omega f(x, u_{k,j_1}, Du_{k,j_1}) \, dx.$$

Since the integrals on the left-hand side are non-negative, this implies for any index $j \in \{1, \dots, j_1\}$ that

$$\int_\Omega f(x, u_{k,j_2}, Du_{k,j_2}) \, dx + \frac{1}{ch_k} \sum_{i=j_1+1}^{j_2} \int_\Omega \left| \sqrt{\phi(u_{k,i})} - \sqrt{\phi(u_{k,i-1})} \right|^2 dx$$

$$\le \int_\Omega f(x, u_{k,j}, Du_{k,j}) \, dx,$$

which can be re-written as

$$\int_\Omega f(x, u^{(k)}(j_2 h_k), Du^{(k)}(j_2 h_k))\, dx + \frac{1}{c} \int_{j_1 h_k}^{j_2 h_k} \int_\Omega |\Delta_{-h_k} \sqrt{\phi(u^{(k)})}|^2\, dx\, dt$$

$$\leq \int_\Omega f(x, u^{(k)}(t), Du^{(k)}(t))\, dx,$$

for any $t \in [h_k, j_1 h_k]$. Here $\Delta_{-h_k}$ denotes the backward difference quotient in time. Now, we let $0 < \varepsilon < \tau \leq T$ and consider $k > \frac{4T}{\varepsilon}$. Choosing $j_1 = \lfloor \frac{\varepsilon}{h_k} \rfloor$ and $j_2 = \lceil \frac{\tau}{h_k} \rceil$, we find that

$$\int_\Omega f(x, u^{(k)}(\tau), Du^{(k)}(\tau))\, dx + \frac{1}{c} \int_\varepsilon^\tau \int_\Omega |\Delta_{-h_k} \sqrt{\phi(u^{(k)})}|^2\, dx\, dt$$

$$\leq \int_\Omega f(x, u^{(k)}(t), Du^{(k)}(t))\, dx,$$

for any $t \in [h_k, \varepsilon - h_k]$. Since the left-hand side is independent of $t$, we may take mean values on the right-hand side and obtain with the help of (6.2) that

$$(6.6) \qquad \int_\Omega f(x, u^{(k)}(\tau), Du^{(k)}(\tau))\, dx + \frac{1}{c} \int_\varepsilon^\tau \int_\Omega |\Delta_{-h_k} \sqrt{\phi(u^{(k)})}|^2\, dx\, dt$$

$$\leq \fint_{h_k}^{\varepsilon - h_k} \int_\Omega f(x, u^{(k)}, Du^{(k)})\, dx\, dt \leq \frac{M}{\varepsilon - 2h_k}.$$

Combining this estimate with assumption (1.2), we deduce

$$(6.7) \qquad \sup_{\tau \in (\varepsilon, T]} \int_\Omega |Du^{(k)}(\tau)|^p\, dx \leq \frac{1}{\varepsilon - 2h_k} \frac{M}{v}.$$

By virtue of the energy estimates (6.3), (6.6), and (6.7), the assumptions of [14, Proposition 3.1] are satisfied for the sequence $(u^{(k)})_{k \in \mathfrak{K}}$ on $\Omega \times (\varepsilon, T)$. Hence, after extraction of another subsequence, still denoted by $\mathfrak{K}$, we infer the convergence

$$(6.8) \qquad \sqrt{\phi(u^{(k)})} \to \sqrt{\phi(u)} \quad \text{strongly in } L^1(\Omega \times (\varepsilon, T)),$$

as $\mathfrak{K} \ni k \to \infty$. Since $\varepsilon > 0$ was arbitrary, this implies for another subsequence

$$(6.9) \qquad u^{(k)} \to u \quad \text{a.e. in } \Omega_T.$$

Furthermore, in view of (6.6) we can extract for any $\varepsilon > 0$ a subsequence such that $\Delta_{-h_k} \sqrt{\phi(u^{(k)})} \rightharpoonup w$ weakly in $L^2(\Omega \times (\varepsilon, T))$ for some function $w \in L^2(\Omega \times (\varepsilon, T))$. Using this fact together with (6.8), we obtain for any $\varphi \in C_0^\infty(\Omega \times (\varepsilon, T))$ that

$$\iint_{\Omega \times (\varepsilon, T)} \sqrt{\phi(u)} \partial_t \varphi \, dx \, dt = \lim_{\Re \ni k \to \infty} \iint_{\Omega \times (\varepsilon, T)} \sqrt{\phi(u^{(k)})} \Delta_{h_k} \varphi \, dx \, dt$$

$$= - \lim_{\Re \ni k \to \infty} \iint_{\Omega \times (\varepsilon, T)} \Delta_{-h_k} \sqrt{\phi(u^{(k)})} \varphi \, dx \, dt$$

$$= - \iint_{\Omega \times (\varepsilon, T)} w \varphi \, dx \, dt.$$

By a density argument this ensures that $w = \partial_t \sqrt{\phi(u)}$. Therefore, we have shown that $\partial_t \sqrt{\phi(u)} \in L^2(\Omega \times (\varepsilon, T))$ for any $\varepsilon > 0$. This, however, implies that $\partial_t \phi(u) = 2\sqrt{\phi(u)} \partial_t \sqrt{\phi(u)} \in L^1(\Omega \times (\varepsilon, T))$ for any $\varepsilon > 0$ and hence $\phi(u) \in C^0((0, T]; L^1(\Omega))$. From [14, Lemma 2.6] we therefore deduce that $u \in C^0((0, T]; L^\phi(\Omega))$.

### 6.4. Variational inequality

In this section we shall establish that $u$ satisfies the variational inequality (4.1). To this end, we consider a general non-negative comparison map $v \in L^p(0, T; W_{u_*}^{1,p}(\Omega))$ with $\partial_t v \in L^\phi(\Omega_T)$ and $v(0) \in L^\phi(\Omega)$ and let $\tau \in (0, T]$. Due to the definition of $u^{(k)}$ and inequality (6.1), we already know that

$$\iint_{\Omega_\tau} f(x, u^{(k)}, Du^{(k)}) \, dx \, dt \le \iint_{\Omega_\tau} f(x, v, Dv) + \Delta_{-h_k} b(u^{(k)})(v - u^{(k)}) \, dx \, dt$$

holds true. To proceed further, we apply the finite integration by parts formula from [14, Lemma 2.10] on the cylinder $\Omega_\tau$. This leads to

(6.10) $\quad \displaystyle\iint_{\Omega_\tau} f(x, u^{(k)}, Du^{(k)}) \, dx \, dt$

$$\le \iint_{\Omega_\tau} f(x, v, Dv) \, dx \, dt + \iint_{\Omega_\tau} \Delta_{h_k} v (b(v) - b(u^{(k)})) \, dx \, dt$$

$$- \mathrm{B}_\tau(h_k) + \mathrm{B}_0(h_k) + \boldsymbol{\delta}_1(h_k) + \boldsymbol{\delta}_2(h_k),$$

for every $\tau \in (0, T]$, where we have abbreviated

$$\mathrm{B}_\tau(h_k) := \frac{1}{h_k} \iint_{\Omega \times (\tau - h_k, \tau)} \mathfrak{b}[u^{(k)}(t), v(t + h_k)] \, dx \, dt,$$

$$\mathrm{B}_0(h_k) := \frac{1}{h_k} \iint_{\Omega \times (-h_k, 0)} \mathfrak{b}[u^{(k)}, v] \, dx \, dt.$$

The error terms $\boldsymbol{\delta}_1(h_k)$ and $\boldsymbol{\delta}_2(h_k)$ are given by

$$\delta_1(h_k) := \frac{1}{h_k} \iint_{\Omega_T} b[v(t), v(t + h_k)] \, dx \, dt,$$

$$\delta_2(h_k) := \iint_{\Omega \times (-h_k, 0)} \Delta_{h_k} v(b(v(t + h_k)) - b(u_k)) \, dx \, dt.$$

To be precise, in order to make all terms well defined, we extend $v$ to times $t > T$ by letting $v(t) = v(T)$. Since $\partial_t v \in L^\phi(\Omega_T)$ we know again from [14, Lemma 2.10] that

$$(6.11) \qquad \qquad \lim_{\mathfrak{K} \ni k \to \infty} \delta_1(h_k) = 0$$

and (note that $u^{(k)}(t) = u_o$ for $t \in (-h_k, 0)$)

$$(6.12) \qquad \lim_{\mathfrak{K} \ni k \to \infty} \delta_2(h_k) = \lim_{\mathfrak{K} \ni k \to \infty} \iint_{\Omega \times (-h_k, 0)} \Delta_{h_k} v(b(v(t + h_k)) - b(u_o)) \, dx \, dt = 0.$$

Our aim now is to pass to the limit $\mathfrak{K} \ni k \to \infty$ in (6.10). To this aim, we extend $v$ to negative times by $v(t) := v(0) \in L^\phi(\Omega)$ for $t < 0$. For the term involving the time derivative, i.e. the second integral on the right-hand side of (6.10), we have the strong convergence $\Delta_{h_k} v \to \partial_t v$ in $L^\phi(\Omega_T)$. Since $b(u^{(k)})$ is bounded in $L^{\phi^*}(\Omega_T)$ (see (6.3) and Lemma 3.1), $L^{\phi^*}(\Omega_T)$ is reflexive (cf. [1, Theorem 8.20]) and $b(u^{(k)}) \to b(u)$ a.e. in $\Omega_T$ (cf. (6.9)), we have for a subsequence (again denoted by $\mathfrak{K}$) that $b(u^{(k)}) \to b(u)$ weakly in $L^{\phi^*}(\Omega_T)$ as $\mathfrak{K} \ni k \to \infty$. As a consequence we get

$$(6.13) \qquad \lim_{\mathfrak{K} \ni k \to \infty} \iint_{\Omega_\tau} \Delta_{h_k} v(b(v) - b(u^{(k)})) \, dx \, dt = \iint_{\Omega_\tau} \partial_t v(b(v) - b(u)) \, dx \, dt.$$

Next, we turn our attention to the boundary terms $B_\tau$ and $B_0$ on the right-hand side of (6.10). Because of $u^{(k)}(t) = u_o$ and $v(t) = v(0)$ for $t \in (-h_k, 0)$, the term $B_0(h_k)$ in (6.10) takes the form

$$B_0(h_k) = \int_\Omega b[u_o, v(0)] \, dx.$$

Before passing to the limit $\mathfrak{K} \ni k \to \infty$ in (6.10), we integrate the inequality over $\tau \in [t_o, t_o + \delta]$ for some $\delta \in (0, T)$ and $t_o \in (0, T - \delta]$ and divide by $\delta$. Keeping in mind that $b \geq 0$, we deduce

$$(6.14) \qquad \fint_{t_o}^{t_o + \delta} \iint_{\Omega_\tau} f(x, u^{(k)}, Du^{(k)}) \, dx \, dt \, d\tau$$

$$\leq \fint_{t_o}^{t_o + \delta} \iint_{\Omega_\tau} f(x, v, Dv) \, dx \, dt \, d\tau$$

$$+ \fint_{t_o}^{t_o+\delta} \iint_{\Omega_\tau} \Delta_{h_k} v(b(v) - b(u^{(k)})) \, dx \, dt \, d\tau$$

$$- \frac{1}{\delta} \int_{t_o}^{t_o+\delta-h_k} \int_\Omega b[u^{(k)}, v(t + h_k)] \, dx \, dt + \int_\Omega b[u_o, v(0)] \, dx$$

$$+ \delta_1(h_k) + \delta_2(h_k).$$

As already mentioned in (6.9), we have $u^{(k)} \to u$ a.e. on $\Omega_T$. Moreover, since $\partial_t v \in L^\phi(\Omega_T)$ and $v(0) \in L^\phi(\Omega)$ imply $v \in C^0([0, T]; L^1(\Omega))$, we know that $v(t + h_k) \to v(t)$ a.e. on $\Omega_T$, in the limit $\mathfrak{K} \ni k \to \infty$; note that $h_k \downarrow 0$. Since $b$ is nonnegative, we can therefore apply Fatou's lemma, with the result that

$$(6.15) \qquad \frac{1}{\delta} \int_{t_o}^{t_o+\delta} \int_\Omega b[u(t), v(t)] \, dx \, dt$$

$$\leq \liminf_{\mathfrak{K} \ni k \to \infty} \frac{1}{\delta} \int_{t_o}^{t_o+\delta-h_k} \int_\Omega b[u^{(k)}, v(t + h_k)] \, dx \, dt$$

holds true. Using (6.11), (6.12), (6.13), (6.15) in combination with the lower semicontinuity of the integral $\iint_{\Omega_T} f(x, u^{(k)}, Du^{(k)}) \, dx \, dt$ with respect to the weak convergence of $(u^{(k)}, Du^{(k)})$ in $L^p(\Omega_T, \mathbb{R}^{n+1})$ and, finally, Fatou's lemma, we find, after passing to the limit $\mathfrak{K} \ni k \to \infty$ in (6.14), that

$$\fint_{t_o}^{t_o+\delta} \iint_{\Omega_\tau} f(x, u, Du) \, dx \, dt \, d\tau$$

$$\leq \fint_{t_o}^{t_o+\delta} \iint_{\Omega_\tau} f(x, v, Dv) \, dx \, dt \, d\tau + \fint_{t_o}^{t_o+\delta} \iint_{\Omega_\tau} \partial_t v(b(v) - b(u)) \, dx \, dt \, d\tau$$

$$- \frac{1}{\delta} \int_{t_o}^{t_o+\delta} \int_\Omega b[u, v] \, dx \, dt + \int_\Omega b[u_o, v(0)] \, dx$$

holds true, for any $\delta \in (0, T]$ and $t_o \in (0, T - \delta]$. Now, we claim that after passing to the limit $\delta \downarrow 0$, we infer that

$$(6.16) \qquad \iint_{\Omega_{t_o}} f(x, u, Du) \, dx \, dt$$

$$\leq \iint_{\Omega_{t_o}} f(x, v, Dv) \, dx \, dt + \iint_{\Omega_{t_o}} \partial_t v(b(v) - b(u)) \, dx \, dt$$

$$- \int_\Omega b[u(t_o), v(t_o)] \, dx + \int_\Omega b[u_o, v(0)] \, dx$$

holds true for any $t_o \in (0, T]$ and any comparison function $v \in L^p(0, T; W_{u_*}^{1,p}(\Omega))$ with $\partial_t v \in L^\phi(\Omega_T)$ and $v(0) \in L^\phi(\Omega)$. In fact, the convergence holds for every

$t_o \in (0, T]$ because each of the terms appearing in (6.16) depends continuously on time. This is clear for the first three integrals from the absolute continuity of the integral. For the continuity of the boundary term, we note that the integrand is majorized by

$$0 \le \mathrm{b}[u(t), v(t)] \le \phi(v(t)) + u(t)b(u(t)) \le \phi(v(t)) + (m + 1)\phi(u(t)),$$

because of $\phi(u) \ge 0$, $b(u)v \ge 0$ and by Lemma 3.1. Since $u \in C^0((0, T]; L^\phi(\Omega))$ and $v \in C^0([0, T]; L^\phi(\Omega))$ (cf. [14, Lemma 2.7]), the right-hand side of the preceding inequality depends continuously on time. Hence, the dominated convergence theorem ensures that

$$[0, T] \ni t \mapsto \int_\Omega \mathrm{b}[u(t), v(t)] \, \mathrm{d}x \quad \text{is continuous.}$$

This ensures that the variational inequality (6.16) holds for every $t_o \in (0, T]$. Therefore, we may apply Lemma 4.2 to infer that the initial datum $u_o$ is taken in the $C^0 - L^\phi$-sense, so that $u \in C^0([0, T]; L^\phi(\Omega))$ with $u(0) = u_o$. This finally shows that $u$ is a variational solution in the sense of Definition 4.1 satisfying $\partial_t \sqrt{\phi(u)} \in L^2(\Omega \times (\varepsilon, T))$ for any $\varepsilon > 0$. This finishes the proof of Theorem 5.1.

$\square$

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