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Mathematics. – *Pseudo-uniformities*, by TULLIO VALENT, communicated on 17 June 2022.

ABSTRACT. – The notions of pseudo-uniformity for a set X and of pseudo-metric presented in this paper are somehow connected. Unlike the uniformity, the elements of a pseudo-uniformity do not necessarily contain the diagonal Δ of $X \times X$ but only some points (at least one point) of it. Likewise, the pseudo-metrics considered and utilized here do not vanish on the whole of Δ, but only in some points of it. It will be showed how any family of pseudo-metrics defines a pseudo-uniformity. The main result achieved is the proof that every pseudo-uniformity is defined by a family of "pseudo-uniformly continuous" pseudo-metrics. In the final part of the paper, we prove that every topology is defined by a family of pseudo-metrics and, consequently, that every topology is pseudo-uniformizable: nay every topology defines a pseudo-uniformity which induces just the starting topology.

KEYWORDS. – General structure theory, uniform structures and generalizations.

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1. INTRODUCTION

With a view of making a systematic study of structures on a set X defined by a family of maps $d : X \times X \mapsto \mathbb{R}^+$, in this paper we will consider symmetric maps d that satisfy the "triangle inequality" $d(x_1, x_2) \le d(x_1, x) + d(x, x_2)$ $\forall x, x_1, x_2 \in X$, and vanish at least in a point of the diagonal Δ of $X \times X$. These maps will be called *pseudo-metrics*, although in the usual terminology a pseudo-metric is requested to vanish on the whole of Δ.

Recall that the elements of a uniformity for a set X contain Δ, and that the most important result of the classical theory of the uniformities is the fact that each uniformity is generated by a family of pseudo-metrics (the family of all pseudo-metrics which are uniformly continuous), where the pseudo-metrics have the usual meaning (they vanish on the whole of Δ).

In this paper, a theory of pseudo-uniformity is presented and developed. A pseudo-uniformity for X is defined as a particular Δ-local filter on $X \times X$, and its elements contain at least a point of Δ, but not necessarily the whole Δ. Each family $\mathscr{P}$ of pseudo-metrics with the property that for every $x \in X$ there is a $d \in \mathscr{P}$ vanishing at (x, x) defines a pseudo-uniformity.

In Section 4, it is shown that every pseudo-uniformity is defined by a family of "pseudo-uniformly continuous" pseudo-metrics. This result is somehow analogous to those above recalled in the context of the theory of uniformities, but in our case the treatment offers greater difficulties. In Section 5, after showing how every topology is defined by a family of pseudo-metrics, we can prove that every topology is pseudo-uniformizable, in the sense that every topology τ defines a pseudo-uniformity, whose induced topology is just τ.

2. Δ-LOCAL FILTERS AND PSEUDO-UNIFORMITIES. PRELIMINARY DEFINITIONS AND PROPERTIES

Throughout, X will denote any set. The diagonal of $X \times X$ shall be denoted by Δ. For any subset V of $X \times X$, the composition $V \circ V$ is defined by

$$V \circ V := \big\{(x_1, x_2) \in X \times X : (x_1, \xi), (\xi, x_2) \in V \text{ for some } \xi \in X\big\}.$$

DEFINITION 2.1. If, for every $x \in X$, $\mathscr{U}_x$ is a filter on $X \times X$ generated by a family of symmetric subsets of $X \times X$ which contain the point (x, x) of Δ, then the union

$$\mathscr{U} := \bigcup_{x \in X} \mathscr{U}_x$$

will be called a Δ-*local filter* on $X \times X$.

Evidently, $\mathscr{U}_x = \{U \in \mathscr{U} : (x, x) \in U\}$. Of course, $\mathscr{U}$ is a union of filters, but (in general) is not a filter.

DEFINITION 2.2. A Δ-local filter $\mathscr{U}$ on $X \times X$ will be said a *pseudo-uniformity* for X if for every $x \in X$ and $U_x \in \mathscr{U}_x$ there is $V_x \in \mathscr{U}_x$ such that $V_x \circ V_x \subseteq U_x$.

EXAMPLE 2.3 (Of Δ-local filter). Consider a family $\mathscr{F}$ of symmetric subsets of $X \times X$ such that, for every $x \in X$, the set

$$\mathscr{F}_x := \big\{V \in \mathscr{F} : (x, x) \in V\big\}$$

is non-empty, and so it generates a filter, say $\mathscr{U}_x$, on $X \times X$. The union of the filters $\mathscr{U}_x$, $x \in X$, is obviously a Δ-local filter and it will be called the Δ-*local filter on* $X \times X$ *generated by* $\mathscr{F}$.

DEFINITION 2.4. Let $\mathscr{P}$ be any family of symmetric maps $d : X \times X \mapsto \mathbb{R}^+$ which has the property that for every $x \in X$ there is $d \in \mathscr{P}$ such that $d(x, x) = 0$. The Δ-*local filter defined by* $\mathscr{P}$ is

$$\widehat{\mathscr{U}}(\mathscr{P}) := \bigcup_{x \in X} \widehat{\mathscr{U}}_x(\mathscr{P}),$$

where $\widehat{\mathscr{U}}_x(\mathscr{P})$ is the filter on $X \times X$ generated by the family of the subsets $d^{\leftarrow}([0, \varepsilon[)$ of $X \times X$, with $d \in \mathscr{P}$ such that $d(x, x) = 0$, ε is any number > 0, and $d^{\leftarrow}([0, \varepsilon[) := \{(x_1, x_2) \in X \times X : d(x_1, x_2) < \varepsilon\}$.

As precised in the introduction (Section 1), in this paper a *pseudo-metric* on X is a symmetric map $d : X \times X \mapsto \mathbb{R}^+$ that satisfies the inequality $d(x_1, x_2) \leq d(x_1, \xi) + d(\xi, x_2)$ for all $x_1, x_2, \xi \in X$, and vanishes in some points of the diagonal of $X \times X$, but not necessarily on the whole of Δ.

THEOREM 2.5. *Any Δ-local filter on $X \times X$ defined by a family $\mathscr{P}$ of pseudo-metrics, with the property that for every $x \in X$ there is $d \in \mathscr{P}$ such that $d(x, x) = 0$, is a pseudo-uniformity for X.*

PROOF. We must prove that if $\mathscr{P}$ is a family of pseudo-metrics on $X \times X$ such that for every $x \in X$ there is $d \in \mathscr{P}$ that vanishes at (x, x), and $\widehat{\mathscr{U}}(\mathscr{P})$ is the Δ-local filter on $X \times X$ defined by $\mathscr{P}$ (see Remark 2.4), then for every $x \in X$ and every $U_x \in \widehat{\mathscr{U}}_x(\mathscr{P})$ there is $V_x \in \widehat{\mathscr{U}}_x(\mathscr{P})$ such that $V_x \circ V_x \subseteq U_x$. From the definition of $\widehat{\mathscr{U}}_x(\mathscr{P})$ it follows that there are a finite subset $\{d_i : i \in I\}$ of $\mathscr{P}$, with $d_i(x, x) = 0$, and a positive number ε such that

$$U_x \supseteq \big\{(x_1, x_2) \in X \times X : d_i(x_1, x_2) < \varepsilon \ \forall i \in I\big\}.$$

Consider, for every $x \in X$, the element V_x of $\widehat{\mathscr{U}}_x(\mathscr{P})$ defined by

$$V_x := \left\{(x_1, x_2) \in X \times X : d_i(x_1, x_2) < \frac{\varepsilon}{2} \ \forall i \in I\right\}.$$

Since d_i is a pseudo-metric, we have

$$d_i(x_1, x_2) \leq d_i(x_1, \xi) + d_i(\xi, x_2) \quad \forall x_1, x_2, \xi \in X.$$

Therefore,

$$\begin{aligned} V_x \circ V_x &= \big\{(x_1, x_2) \in X \times X : (x_1, \xi), (\xi, x_2) \in V_x \ \exists \xi \in X\big\} \\ &= \left\{(x_1, x_2) \in X \times X : d_i(x_1, \xi) < \frac{\varepsilon}{2},\ d_i(\xi, x_2) < \frac{\varepsilon}{2} \ \forall i \in I \ \exists \xi \in X\right\} \\ &\subseteq \big\{(x_1, x_2) \in X \times X : d_i(x_1, x_2) < \varepsilon \ \forall i \in I\big\} \subseteq U_x. \end{aligned}$$ ∎

Obviously, if the pseudo-metrics were thought in the usual meaning, the property required to $\mathscr{P}$ in Theorem 2.5 would be unnecessary (because it would be satisfied by every family $\mathscr{P}$ of pseudo-metrics). The most important result of Section 4 will be the proof that every pseudo-uniformity can be defined by a family of pseudo-metrics.

THEOREM 2.6. *Any Δ-local filter on $X \times X$ induces a topology on X.*

PROOF. Let $\mathscr{U}$ be a Δ-local filter on $X \times X$. Recall that $\mathscr{U} = \bigcup_{x \in X} \mathscr{U}_x$, where $\mathscr{U}_x = \{U \in \mathscr{U} : (x,x) \in U\}$, and that each $U \in \mathscr{U}_x$ contains a symmetric element of $\mathscr{U}_x$. Set, for every $x \in X$ and every symmetric $U \in \mathscr{U}_x$,

$$U[x] = \{\xi \in X : (\xi, x) \in U\}.$$

Of course $x \in U[x]$. Now, put

$$\mathscr{U}[x] := \{U[x] : U \in \mathscr{U}_x\},$$
$$\hat{\tau}(\mathscr{U}) := \{A \subseteq X : A \in \mathscr{U}[a],\ \forall a \in A\},$$

and observe that $\hat{\tau}(\mathscr{U})$ *is closed under finite intersections*, namely, that if

$$A_1 \in \mathscr{U}[a_1]\ \forall a_1 \in A_1, \quad A_2 \in \mathscr{U}[a_2]\ \forall a_2 \in A_2, \tag{2.1}$$

then

$$A_1 \cap A_2 \in \mathscr{U}[a] \quad \forall a \in A_1 \cap A_2. \tag{2.2}$$

Indeed, from (2.1) it follows that there are $U_1, U_2 \in \mathscr{U}$ such that

$$A_1 \in U_1[a_1]\ \forall a_1 \in A_1, \quad A_2 \in U_2[a_2]\ \forall a_2 \in A_2$$

which implies that $A_1 \cap A_2 \in U_1[a_1] \cap U_2[a_2]$ for all $(a_1, a_2) \in A_1 \times A_2$, and so, putting, $U = U_1 \cap U_2$, it is easily seen that (2.2) is true. Of course, U may be empty; in this case $A_1 \cap A_2 = \emptyset$ and hence (2.2) holds trivially. Since, obviously, $\hat{\tau}(\mathscr{U})$ *is closed under each union*, we can conclude that $\hat{\tau}(\mathscr{U})$ is a topology on X. It will be said that *the topology is induced by* $\mathscr{U}$. ■

Let $\mathscr{U} = \bigcup_{x \in X} \mathscr{U}_x$ be a Δ-local filter on $X \times X$ (or, in particular, a pseudo-uniformity for X).

DEFINITION 2.7. A subset $\mathscr{V}$ of $\mathscr{U}$ is a *pre-base (resp. a base) of* $\mathscr{U}$ if, for each $x \in X$, the intersection $\mathscr{V} \cap \mathscr{U}_x$ is a pre-base (resp. a base) of the filter $\mathscr{U}_x$.

DEFINITION 2.8. A family $\mathscr{V}$ of symmetric subsets of $X \times X$ is a *pre-base (resp. a base) for a Δ-local filter* if, for each $x \in X$, the family

$$\mathscr{V}_x := \{V \in \mathscr{V} : (x,x) \in V\}$$

is a pre-base (resp. a base) for a filter on $X \times X$.

In the case of the pseudo-uniformities, Definition 2.8 becomes the following.

DEFINITION 2.9. A family $\mathscr{V}$ of symmetric subsets of $X \times X$ is a *pre-base (resp. a base) for a pseudo-uniformity* if, for each $x \in X$, the family $\mathscr{V}_x$ is a pre-base (resp. a base) for a filter $\mathscr{U}_x$ on $X \times X$ having the property that for every $U_x \in \mathscr{U}_x$ there is $V_x \in \mathscr{U}_x$ such that $V_x \circ V_x \subseteq U_x$.

PROPOSITION 2.10. *A family $\mathscr{V}$ of symmetric subsets of $X \times X$ is a pre-base for a pseudo-uniformity for X if and only if, for each $x \in X$, $\mathscr{V}_x$ is a non-empty family of non-empty sets which is closed under finite intersections and has the property that for every $U_x \in \mathscr{V}_x$ there is $V_x \in \mathscr{V}_x$ such that $V_x \circ V_x \subseteq U_x$.*

PROOF. It is easy to check that any finite intersection of elements of $\mathscr{V}_x$ with the property that for every $U_x \in \mathscr{V}_x$ there is $V_x \in \mathscr{V}_x$ such that $V_x \circ V_x \subseteq U_x$ has the same property. Proposition 2.10 follows readily from this fact. ■

PROPOSITION 2.11. *A family $\mathscr{V}$ of symmetric subsets of $X \times X$ is a base for a pseudo-uniformity for X if and only if, for each $x \in X$, $\mathscr{V}_x$ is a non-empty family of non-empty sets with the properties that the intersection of two elements of $\mathscr{V}_x$ contains an element of $\mathscr{V}_x$, and that for every $U_x \in \mathscr{V}_x$ there is $V_x \in \mathscr{V}_x$ such that $V_x \circ V_x \subseteq U_x$.*

The proof of this proposition is omitted, because it shall be like the proof of the previous proposition.

3. UNIFORM CONTINUITY WITH RESPECT TO Δ-LOCAL FILTERS. PRODUCT OF Δ-LOCAL FILTERS

Let X, Y be any sets, and let $\mathscr{U} := \bigcup_{x \in X} \mathscr{U}_x$ and $\mathscr{V} := \bigcup_{y \in Y} \mathscr{V}_y$ be Δ-local filters on X and Y, respectively.

DEFINITION 3.1. A map $f : X \mapsto Y$ will be said *uniformly continuous with respect to $\mathscr{U}_x$ and $\mathscr{V}_y$* if for every $V_y \in \mathscr{V}_y$ the set $\{(x_1, x_2) \in X \times X : (f(x_1), f(x_2)) \in V_y\}$ is an element of $\mathscr{U}_x$. Moreover, f will be said *uniformly continuous with respect to $\mathscr{U}$ and $\mathscr{V}$* when for every $V \in \mathscr{V}$ the set $\{(x_1, x_2) \in X \times X : (f(x_1), f(x_2)) \in V\}$ is an element of $\mathscr{U}$.

DEFINITION 3.2. The *product of the Δ-local filters $\mathscr{U}_x$ on $X \times X$ and $\mathscr{U}_y$ on $Y \times Y$* is the smallest Δ-local filter on $(X \times Y) \times (X \times Y)$ such that the projections of $X \times Y$ into X and into Y are uniformly continuous.

One can prove the following.

PROPOSITION 3.3. *The product of the Δ-local filters $\mathscr{U}_x$ on $X \times X$ and $\mathscr{U}_y$ on $Y \times Y$ is the Δ-local filter on $(X \times Y) \times (X \times Y)$ whose pre-base is the union of the two family of sets*

$$\big\{\big((x_1, y_1), (x_2, y_2)\big) \in (X \times Y) \times (X \times Y) : (x_1, x_2) \in U_x \text{ for some } U_x \in \mathscr{U}_x\big\}$$

and

$$\big\{\big((x_1, y_1), (x_2, y_2)\big) \in (X \times Y) \times (X \times Y) : (y_1, y_2) \in U_y \text{ for some } U_y \in \mathscr{U}_y\big\}.$$

In the case $X = Y$ from Proposition 3.3, the following corollary holds.

COROLLARY 3.4. *The product of the Δ-local filter $\mathscr{U}$ on $X \times X$ by itself is the Δ-local filter on $(X \times X) \times (X \times X)$ whose pre-base is the union of the two family of sets*

$$\big\{\big((x_1, y_1), (x_2, y_2)\big) \in (X \times X) \times (X \times X) : (x_1, x_2) \in U \text{ for some } U \in \mathscr{U}\big\}$$

and

$$\big\{\big((x_1, y_1), (x_2, y_2)\big) \in (X \times X) \times (X \times X) : (y_1, y_2) \in U \text{ for some } U \in \mathscr{U}\big\}.$$

THEOREM 3.5. *Let $\mathscr{U}$ be a Δ-local filter on $X \times X$ (in particular, a pseudo-uniformity for X). A pseudo-metric $d : X \times X \mapsto \mathbb{R}^+$ is uniformly continuous with respect to the product of $\mathscr{U}$ by itself if and only if for every $\varepsilon > 0$ there is $U \in \mathscr{U}$ such that*

$$\big|d(x_1, x_2) - d(x_2, x_2)\big| \le \varepsilon \quad \forall (x_1, x_2) \in U, \tag{3.1}$$

or, equivalently, for every $x \in X$ and every $\varepsilon > 0$ there is $U_x \in \mathscr{U}_x$ such that

$$\big|d(x_1, x_2) - d(x_2, x_2)\big| \le \varepsilon \quad \forall (x_1, x_2) \in U_x. \tag{3.2}$$

PROOF. We first observe that the equivalence of the two conditions appearing in the statement of this theorem follows from the fact that $\mathscr{U} = \bigcup_{x \in X} \mathscr{U}_x$. In view of Corollary 3.4, d is uniformly continuous with respect to the product of $\mathscr{U}$ by itself if and only if for each $\varepsilon > 0$ there is $U \in \mathscr{U}$ such that

$$\big|d(x_1, y_1) - d(x_2, y_2)\big| \le \varepsilon \quad \text{whenever } (x_1, x_2) \in U \text{ or } (y_1, y_2) \in U.$$

Observe that this condition is equivalent to each of the following (equivalent) conditions, where ξ is any element of X:

$$\begin{aligned} \big|d(x_1, \xi) - d(x_2, \xi)\big| &\le \varepsilon \quad \text{when } (x_1, x_2) \in U, \\ \big|d(\xi, y_1) - d(\xi, y_2)\big| &\le \varepsilon \quad \text{when } (y_1, y_2) \in U. \end{aligned}$$

These equivalences follow from the symmetry of d. Then the proof is concluded taking $\xi = x_2$ in the first equality, or $\xi = y_1$ in the second one. ∎

It will be useful, in the sequel, to consider the following corollary of Theorem 3.5.

Corollary 3.6. *Let* $\mathscr{U} := \bigcup_{x\in X} \mathscr{U}_x$ *be a* Δ*-local filter on* $X \times X$*. A pseudo-metric* $d : X \times X \mapsto \mathbb{R}^+$ *is uniformly continuous with respect to the product of* $\mathscr{U}_x$ *by itself if for every* $\varepsilon > 0$ *there is* $U_x \in \mathscr{U}_x$ *such that*

$$\big|d(x_1, x_2) - d(x, x)\big| \le \varepsilon \quad \forall (x_1, x_2) \in U_x. \tag{3.3}$$

Moreover, d *is uniformly continuous with respect to the product of* $\mathscr{U}$ *by itself if for every* $x \in X$ *it is uniformly continuous with respect to the product of* $\mathscr{U}_x$ *by itself, namely, if for every* $x \in X$ *and every* $\varepsilon > 0$ *there is* $U_x \in \mathscr{U}_x$ *such that* (3.3) *is true.*

Proof. It suffices to observe that (3.3) implies that

$$\big|d(x, x_2) - d(x, x)\big| \le \varepsilon \quad \forall (x, x_2) \in U_x,$$

and this can be also obtained by making in (3.1) the choice $x_2 = x$. ∎

Corollary 3.7. *If* $d(x, x) = 0$*, then the pseudo-metric* d *is uniformly continuous with respect to the product of the filter* $\mathscr{U}_x$ *by itself if for every* $\varepsilon > 0$ *there is* $U_x \in \mathscr{U}_x$ *such that*

$$\big|d(x_1, x_2)\big| \le \varepsilon \quad \forall (x_1, x_2) \in U_x.$$

Definition 3.8. A pseudo-metric $d : X \times X \mapsto \mathbb{R}^+$ will be said *pseudo–uniformly continuous* if it is uniformly continuous with respect to the product of $\mathscr{U}_x$ by itself for some $x \in X$.

4. Family of pseudo-metrics defining a pseudo-uniformity

This section is devoted to prove that every pseudo-uniformity is defined by a family of pseudo-metrics (where the word pseudo-metric has the meaning, weaker than the usual, precised in the introduction), which are pseudo-uniformly continuous. This is the main result of this paper.

We will follow the outline used, in the theory of uniform spaces, in order to prove that each uniformity for X is generated by the family of all pseudo-metrics (in the usual meaning) which are uniformly continuous on $X \times X$. In our case, the situation presents some further difficulties with respect to that of the uniform structures. They are mostly due to the fact that the elements of a pseudo-uniformity do not necessarily contain the diagonal Δ of $X \times X$.

We need a few preliminary lemmas. Let $\mathscr{U} := \bigcup_{x\in X} \mathscr{U}_x$ be a pseudo-uniformity for X.

LEMMA 4.1. *For each $x \in X$ and $U \in \mathcal{U}_x$, there is $W \in \mathcal{U}_x$ such that*

$$W \circ W \circ W \subseteq U.$$

PROOF. Let $U \in \mathcal{U}_x$. From the definition of pseudo-uniformity given in Section 2 it follows that there is $V \in \mathcal{U}_x$ such that $V \circ V \subseteq U$ and there is $W' \in \mathcal{U}_x$ such that $W' \circ W' \subseteq V$. The set $W := W' \cap V$ is an element of $\mathcal{U}_x$, because $\mathcal{U}_x$ is closed under finite intersections. Besides, $W \circ W \subseteq W' \circ W' \subseteq V$, and so $W \circ W \circ W \subseteq V \circ W \subseteq V \circ V \subseteq U$. ∎

Since each element of $\mathcal{U}_x$ contains a symmetric element of $\mathcal{U}_x$ (see Definitions 2.1 and 2.2), from Lemma 4.1, the following hold.

LEMMA 4.2. *For $x \in X$ and each symmetric element U of $\mathcal{U}_x$, there is a sequence $(U_n)_{n\in\mathbb{N}}$ of symmetric elements of $\mathcal{U}_x$ such that*

$$U_{n+1} \circ U_{n+1} \circ U_{n+1} \subseteq U_n \quad \textit{for all } n \geq 1, \tag{4.1}$$

with $U_0 = X \times X, U_1 = U$.

LEMMA 4.3. *Let $x \in X$, $U \in \mathcal{U}_x$. Let $(U_n)_{n\in\mathbb{N}}$ be a sequence of symmetric elements of $\mathcal{U}_x$ which satisfies (4.1). Then there is a symmetric map $d : X \times X \mapsto \mathbb{R}^+$ which satisfies the "triangle inequality" $d(x_1, x_2) \leq d(x_1, x_3) + d(x_3, x_2)$ for all $x_1, x_2, x_3 \in X$, and vanishes at the point (x, x), such that*

$$U_n \subseteq \left\{(x_1, x_2) \in X \times X : d(x_1, x_2) \leq \frac{1}{2^n}\right\} \subseteq U_{n-1} \quad \textit{for all } n \geq 1. \tag{4.2}$$

The map d is a pseudo-metric according to the definition of pseudo-metric adopted in this paper (see the introduction (Section 1)). The proof of Lemma 4.3 is omitted because it would be essentially analogous to the proof of the "Metrization Lemma" in [2, Ch. 6, n. 12], which provides a fundamental result in the theory of uniform structures (see also [1, Th. 8.1.10]). However, in the cited lemma there is the hypothesis (that we cannot use in this paper) that each U_n contains the diagonal of $X \times X$, and this allows in that case to find a map d which vanishes in all points of the diagonal, i.e., a pseudo-metric in the usual sense.

COROLLARY 4.4. *The pseudo metric d is uniformly continuous with respect to the product of $\mathcal{U}_x$ by itself.*

PROOF. Since $d(x, x) = 0$, it suffices to prove (see Corollary 3.7) that for each $\varepsilon > 0$ there is $U \in \mathcal{U}_x$ such that $|d(x_1, x_2)| \leq \varepsilon$ for all $(x_1, x_2) \in U$. In view of (4.2), this is true with $U = U_n$ and n such that $\frac{1}{2^n} \leq \varepsilon$. ∎

In Lemma 4.3, the pseudo-metric d depends on x and on a symmetric element $U \in \mathscr{U}_x$; hence it may be denoted by $d_{x,U}$. Then an obvious consequence of Lemma 4.3 can be expressed in the following.

Corollary 4.5. *For $x \in X$ and each symmetric element U of $\mathscr{U}_x$, there is a pseudo-metric $d_{x,U}$ such that $d_{x,U}(x,x) = 0$ and*

$$\left\{(x_1, x_2) \in X \times X : d_{x,U}(x_1, x_2) \le \frac{1}{4}\right\} \subseteq U, \quad (4.3)$$

$$U \subseteq \left\{(x_1, x_2) \in X \times X : d_{x,U}(x_1, x_2) \le \frac{1}{2}\right\}.$$

The pseudo-metric $d_{x,U}$ is uniformly continuous with respect to the product of $\mathscr{U}_x$ by itself; namely (by using Definition 3.8), *it is pseudo-uniformly continuous.*

Theorem 4.6. *Every pseudo uniformity for X is defined by a family of pseudo-uniformly continuous pseudo-metrics on $X \times X$.*

Proof. We will show how a pseudo-uniformity $\mathscr{U}$ $(:= \bigcup_{x \in X} \mathscr{U}_x)$ coincides with the Δ-local filter on $X \times X$ defined by the family $\mathscr{P} := \{d_{x,U} : x \in X, U \in \mathscr{U}_x\}$ of pseudo-metrics. Indeed, from Corollary 4.5 it follows that, for each $x \in X$, every element U of $\mathscr{U}_x$ belongs to $\widehat{\mathscr{U}_x}(\mathscr{P})$ (in concordance with the notation used in Remark 2.4). Then every element of $\mathscr{U}$ belongs to the Δ-local filter $\widehat{\mathscr{U}}(\mathscr{P})$ defined by $\mathscr{P}$, which, by Theorem 2.5, is a pseudo-uniformity.

Conversely, again from Corollary 4.5 (more precisely, from the second inclusion in (4.3)), it comes that the pseudo-uniformity $\widehat{\mathscr{U}}(\mathscr{P})$ defined by $\mathscr{P}$ is contained in $\mathscr{U}$. ∎

5. Topologies and pseudo-uniformities

If $\mathscr{P}$ is a family of maps $d : X \times X \mapsto \mathbb{R}^+$, *the topology defined by $\mathscr{P}$* is the topology generated by the set $\{U_{d,\varepsilon}(x) : d \in \mathscr{P}, x \in X, \varepsilon > 0\}$, where

$$U_{d,\varepsilon}(x) := \left\{\xi \in X : d(\xi, x) < \varepsilon,\ d(x, \xi) < \varepsilon\right\}.$$

Proposition 5.1. Let τ be a topology on X. For every $A \in \tau$, let d_A be the characteristic function of $\complement(A \times A)$, where $\complement(A \times A)$ is the complement of $A \times A$. Then the functions d_A are pseudo-metrics.

Proof. We must prove that the (symmetric) maps d_A (which vanish at any (x,x) with $x \in A$) have the property

$$d_A(x_1, x_2) \le d_A(x_1, x) + d_A(x, x_2) \quad \forall x, x_1, x_2 \in X. \quad (5.1)$$

Indeed, (5.1) holds for all $x \in X$ if $x_1, x_2 \in A$ because $d_A(x_1, x_2) = 0$ when $x_1, x_2 \in A$. It also holds for all $x \in X$ if $x_1 \notin A$ or $x_2 \notin A$, because in this case $d_A(x_1, x_2) = d_A(x_1, x) = d_A(x, x_2) = 1$, being $(x_1, x_2), (x_1, x), (x, x_2) \in \complement(A \times A)$. ■

THEOREM 5.2. *Every topology τ is defined by the family of the pseudo-metrics d_A, with $A \in \tau$.*

PROOF. Clearly, $d_A(x_1, x_2) = 0$ if $x_1, x_2 \in A$, while $d_A(x_1, x_2) = 1$ if $x_1 \notin A$ or $x_2 \notin A$. The topology defined by $\{d_A : A \in \tau\}$ is the one generated by $\{U_{d_A,\varepsilon}(x) : A \in \tau,\ \varepsilon > 0\}$, where $U_{d_A,\varepsilon}(x) = \{\xi \in X : d_A(\xi, x) < \varepsilon\}$.

Then the proof is readily concluded after observing that

$$U_{d_A,\varepsilon}(x) = \begin{cases} A & \text{if } x \in A \text{ and } \varepsilon \le 1, \\ X & \text{if } x \in A \text{ and } \varepsilon > 1, \end{cases}$$

while

$$U_{d_A,\varepsilon}(x) = \begin{cases} \emptyset & \text{if } x \notin A \text{ and } \varepsilon \le 1, \\ X & \text{if } x \notin A \text{ and } \varepsilon > 1. \end{cases}$$ ■

THEOREM 5.3. *All topologies τ on X are pseudo-uniformizable. Nay, τ defines a pseudo-uniformity $\widehat{\mathscr{U}}(\tau)$ for X such that*

$$\hat{\tau}\big(\widehat{\mathscr{U}}(\tau)\big) = \tau, \tag{5.2}$$

where $\hat{\tau}(\widehat{\mathscr{U}}(\tau))$ denotes the topology induced by $\widehat{\mathscr{U}}(\tau)$ (see Theorem 2.6).

PROOF. By Theorem 5.2, τ is defined by the family $\mathscr{P}_\tau := \{d_A : A \in \tau\}$ of the pseudo-metrics d_A, where d_A is the characteristic function of $\complement(A \times A)$. In view of Remark 2.4 and Theorem 2.5, the family $\mathscr{P}_\tau$ defines a pseudo-uniformity, say $\widehat{\mathscr{U}}(\tau)$, by setting

$$\widehat{\mathscr{U}}(\tau) = \bigcup_{x \in X} \widehat{\mathscr{U}_x}(\tau),$$

where $\widehat{\mathscr{U}_x}(\tau)$ denotes the filter on $X \times X$ generated by the family of the subsets $d_A^{\leftarrow}([0, \varepsilon[)$, with $A \in \tau$ such that $x \in A$ and ε is any number > 0.

As $d_A^{\leftarrow}([0, \varepsilon[) = A \times A$ if $\varepsilon \le 1$, and $d_A^{\leftarrow}([0, \varepsilon[) = X \times X$ if $\varepsilon > 1$, we have

$$\widehat{\mathscr{U}_x}(\tau) = \big\{U \subseteq X \times X : U \supseteq A \times A,\ x \in A \in \tau\big\} \supseteq \{A \times A : x \in A \in \tau\}. \tag{5.3}$$

Now, in order to prove (5.2), we start by observing that (with the notation used in the proof of Theorem 2.6)

$$(A \times A)[x] = \big\{\xi \in X : (\xi, x) \in A \times A\big\} = A \quad \text{for all } x \in A.$$

Hence, setting $\widehat{\mathscr{U}}(\tau)[x] = \{U[x] : U \in \widehat{\mathscr{U}_x}(\tau)\}$, and recalling (5.3), we obtain

$$\widehat{\mathscr{U}}(\tau)[x] \ni (A \times A)[x] = A$$

if A is an element of τ which contains x. This means that every neighborhood of x for the topology induced by $\widehat{\mathscr{U}}(\tau)$ is neighborhood of x also for the topology τ and vice-versa. Therefore, (5.2) is true. ∎

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