



**Partial Differential Equations.** – *Normalized solutions for the Klein–Gordon–Dirac system*, by VITTORIO COTI ZELATI and MARGHERITA NOLASCO, communicated on 10 November 2022.

*Dedicated to Antonio Ambrosetti, maestro and friend.*

ABSTRACT. – We prove the existence of a stationary solution for the system describing the interaction between an electron coupled with a massless scalar field (a photon). We find a solution, with fixed  $L^2$ -norm, by variational methods, as a critical point of an energy functional.

KEYWORDS. – Klein–Gordon–Dirac, critical point theory, min-max methods, nonlinear eigenvalue.

2020 MATHEMATICS SUBJECT CLASSIFICATION. – Primary 35Q40; Secondary 81Q05, 35P30, 47J10, 49J35.

## 1. INTRODUCTION

We study the interaction electron-photon analyzing the Euler–Lagrange equations for a system consisting of a spinor field coupled with a massless scalar field. More precisely, our system consists of the Dirac equation coupled with a massless Klein–Gordon equation, and looks for normalized and stationary solutions of the system

$$(1.1) \quad \begin{cases} (-i\gamma^\mu \partial_\mu + m - \sqrt{s}\varphi)\psi = 0 & \text{in } \mathbb{R} \times \mathbb{R}^3, \\ \partial^\mu \partial_\mu \varphi = 4\pi \sqrt{s}(\psi, \beta\psi) & \text{in } \mathbb{R} \times \mathbb{R}^3, \end{cases}$$

where  $\psi: \mathbb{R} \times \mathbb{R}^3 \rightarrow \mathbb{C}^4$ ,  $\varphi: \mathbb{R} \times \mathbb{R}^3 \rightarrow \mathbb{R}$ ,  $m > 0$  is the mass of the electron,  $\sqrt{s} > 0$  is the coupling constant,  $\gamma^\mu$  are the  $4 \times 4$  Dirac matrices

$$\gamma^0 = \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \gamma^k = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix}, \quad k = 1, \dots, 3,$$

$\sigma^k$  are the  $2 \times 2$  Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and  $(z, w) = \sum_{i=1}^4 \bar{z}_i w_i$ , the scalar product between  $z, w \in \mathbb{C}^4$ .

This problem is closely related to the one studied in [6], and we will prove the existence of a solution of (1.1) with essentially the same methods developed in that article (see also [2]).

More precisely, we prove the existence of stationary, normalized solutions of this system, that is solutions  $(\omega, \psi)$  of the problem

$$(1.2) \quad \begin{cases} (-i\alpha \cdot \nabla + m\beta - \sqrt{s}\varphi\beta)\psi = \omega\psi & \text{in } \mathbb{R}^3, \\ -\Delta\varphi = 4\pi\sqrt{s}(\psi, \beta\psi) & \text{in } \mathbb{R}^3, \\ |\psi|_2^2 = \int_{\mathbb{R}^3} |\psi(t, x)|^2 dx = 1, \end{cases}$$

where  $\alpha_i = \beta\gamma_i$ ,  $i = 1, \dots, 3$ . From  $-\Delta\varphi = 4\pi\sqrt{s}(\psi, \beta\psi)$ , we deduce that

$$\varphi = \sqrt{s}(\psi, \beta\psi) * \frac{1}{|x|}$$

and hence our problem reduces to

$$(1.3) \quad \begin{cases} (-i\alpha \cdot \nabla + m\beta - s(\psi, \beta\psi) * \frac{1}{|x|}\beta)\psi = \omega\psi & \text{in } \mathbb{R}^3, \\ |\psi|_2^2 = \int_{\mathbb{R}^3} |\psi(t, x)|^2 dx = 1. \end{cases}$$

Our result is the following theorem.

**THEOREM 1.4.** *For all  $s \in (0, \frac{1}{8\pi})$ , there exists  $\omega \in (0, m)$  and  $\psi \in H^{1/2}(\mathbb{R}^3, \mathbb{C}^4)$  solutions of problem (1.3).*

In the article [3], the authors prove using critical point theory the existence of one stationary solution of equation (1.1) but do not prescribe its  $L^2$ -norm.

We will find such a solution as a critical point of the functional

$$I(\psi) = \frac{1}{2} \int_{\mathbb{R}^3} (H\psi, \psi) - \frac{s}{4} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{(\psi, \beta\psi)(x)(\psi, \beta\psi)(y)}{|x - y|}$$

restricted on the manifold  $|\psi|_2^2 = 1$ . Here

$$H = -i\alpha \cdot \nabla + m\beta.$$

The functional  $I$  is strongly indefinite, and, following the method introduced in [2, 6], the solution will be found via a min-max procedure consisting in minimizing the supremum of  $I$  over subspaces of dimension 1 in the positive energy subspace of the linear operator  $H$ ; see Proposition 2.13. Let us remark here that we know very few results on the existence of *normalized* solutions for Dirac's equation (and more generally for strongly indefinite problems – one of these is [1]).

### 1.1. Notation and background results

We let  $|u|_p^p = \int_{\mathbb{R}^3} |u(x)|^p$ ,  $(u | v) = \int_{\mathbb{R}^3} u(x)v(x)$ .

Let us recall some well-known facts on the Dirac operator  $H$  (see [7] for more details):  $H$  is a first order, self-adjoint operator on  $H^1(\mathbb{R}^3, \mathbb{C}^4)$  with purely absolutely continuous spectrum given by

$$\sigma(H) = (-\infty, -m] \cup [m, +\infty).$$

The orthogonal projectors  $\Lambda_{\pm}$  on the positive and negative energies subspaces are such that

$$H\Lambda_{\pm} = \Lambda_{\pm}H = \pm\sqrt{-\Delta + m^2}\Lambda_{\pm} = \pm\Lambda_{\pm}\sqrt{-\Delta + m^2}$$

and hence

$$\int (\psi(x), H\psi(x)) dx = |(-\Delta + m^2)^{1/4}\Lambda_+\psi|_2^2 - |(-\Delta + m^2)^{1/4}\Lambda_-\psi|_2^2.$$

We will denote  $X = H^{1/2}(\mathbb{R}^3, \mathbb{C}^4)$ ,  $X_{\pm} = \Lambda_{\pm}X$ ,  $\Sigma = \{\psi \in X \mid |\psi|_2 = 1\}$ , and  $\Sigma_{\pm} = \{\psi \in X_{\pm} \mid |\psi|_2 = 1\}$ .

We have also that

$$\hat{H} = \mathcal{F}H\mathcal{F}^{-1} = \boldsymbol{\alpha} \cdot p + m\beta,$$

$$U\hat{H}U^{-1} = \lambda(p)\beta,$$

$$\hat{\Lambda}_{\pm} = \mathcal{F}\Lambda_{\pm}\mathcal{F}^{-1} = U^{-1}\left(\frac{1 \pm \beta}{2}\right)U = \frac{1}{2}\left(1 \pm \frac{m}{\lambda(p)}\beta \pm \frac{1}{\lambda(p)}\boldsymbol{\alpha} \cdot p\right),$$

where

$$\mathcal{F}\psi(p) = \hat{\psi}(p)\left(= \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} e^{-ipx} \psi(x) dx \text{ for all } v \in \mathcal{S}(\mathbb{R}^3)\right),$$

$$\lambda(p) = \sqrt{|p|^2 + m^2},$$

$$U = u_+(p)1 + u_-(p)\beta \frac{\boldsymbol{\alpha} \cdot p}{|p|},$$

$$U^{-1} = u_+(p)1 - u_-(p)\beta \frac{\boldsymbol{\alpha} \cdot p}{|p|},$$

$$u_{\pm}(p) = \sqrt{\frac{1}{2}\left(1 \pm \frac{m}{\lambda(p)}\right)}.$$

Let, for  $\phi$  and  $\psi \in H^{1/2}(\mathbb{R}^3, \mathbb{C}^4)$ ,

$$\langle \phi | \psi \rangle = \int \sqrt{|p|^2 + m^2}(\hat{\phi}(p), \hat{\psi}(p)) dp$$

and

$$\|\psi\|^2 = \langle \psi \mid \psi \rangle.$$

We have that

$$\langle \Lambda_+ \phi \mid \Lambda_- \psi \rangle = (\Lambda_+ \phi \mid \Lambda_- \psi) = 0.$$

Let us recall that, since  $\mathcal{F} \frac{1}{|x|} = \sqrt{\frac{2}{\pi}} \frac{1}{|p|}$ , for all  $f \in L^1(\mathbb{R}^3) \cap L^{3/2}(\mathbb{R}^3)$

$$(1.5) \quad \int \frac{f(x)\bar{f}(y)}{|x-y|} = 4\pi \int \frac{|\hat{f}(p)|}{|p|^2} \geq 0$$

and that for all  $\rho \in L^1(\mathbb{R}^3)$ ,  $\psi \in H^{1/2}(\mathbb{R}^3, \mathbb{C}^4)$

$$(1.6) \quad \int \frac{\rho(x)|\psi|^2(y)}{|x-y|} \leq \kappa |\rho|_1 |(-\Delta)^{1/4} \psi|_2^2 \leq \kappa |\rho|_1 \|\psi\|^2$$

( $\kappa = \frac{\pi}{2}$ ) and also that

$$(1.7) \quad \int \frac{|f_n|(x)|g_n|(x)|h_n|(y)|v|(y)}{|x-y|} \rightarrow 0$$

when  $f_n, g_n, h_n$  and  $v \in H^{1/2}$ ,  $f_n, g_n$  bounded,  $h_n \rightarrow 0$ .

## 2. MAXIMIZATION

Let  $I: H^{1/2}(\mathbb{R}^3, \mathbb{C}^4) \rightarrow \mathbb{R}$

$$I(\psi) = \frac{1}{2} \|\Lambda_+ \psi\|^2 - \frac{1}{2} \|\Lambda_- \psi\|^2 - \frac{s}{4} \int \frac{(\psi, \beta \psi)(x)(\psi, \beta \psi)(y)}{|x-y|}.$$

Let us fix  $w \in \Sigma_+$  and let

$$B_1 = \{\eta \in X_- \mid |\eta|_2 < 1\}.$$

We will look, given  $w$ , for a maximizer of the functional  $J_w$  defined on  $B_1$ ,

$$\begin{aligned} J_w(\eta) &= I(a(\eta)w + \eta) \\ &= \frac{1}{2} \|a(\eta)w\|^2 - \frac{1}{2} \|\eta\|^2 - \frac{s}{4} \int \frac{(\psi, \beta \psi)(x)(\psi, \beta \psi)(y)}{|x-y|}, \end{aligned}$$

where  $a(\eta) = \sqrt{1 - |\eta|_2^2}$  and  $\psi = a(\eta)w + \eta \in \Sigma$ .

We have that  $da(\eta)[\xi] = -a(\eta)^{-1}(\eta \mid \xi)$  and hence the derivative of  $J_w$  is given, for all  $\xi \in X_-$ , by

$$\begin{aligned} (2.1) \quad dJ_w(\eta)[\xi] &= dI(a(\eta)w + \eta)[da(\eta)[\xi]w + \xi] = dI(\psi)[h] \\ &= \langle a(\eta)w \mid da(\eta)[\xi]w \rangle - \langle \eta \mid \xi \rangle - s \int \frac{(\psi, \beta\psi)(x)(\psi, \beta h)(y)}{|x-y|} \\ &= -(\eta \mid \xi)\|w\|^2 - \langle \eta \mid \xi \rangle - s \int \frac{(\psi, \beta\psi)(x)(\psi, \beta h)(y)}{|x-y|} \end{aligned}$$

(here  $h = da(\eta)[\xi]w + \xi$ ) and we have, in particular,

$$dJ_w(\eta)[\eta] = -|\eta|_2^2\|w\|^2 - \|\eta\|^2 - s \int \frac{(\psi, \beta\psi)(x)(\psi, \beta(da(\eta)[\eta]w + \eta))(y)}{|x-y|}.$$

LEMMA 2.2. *For all  $w \in \Sigma_+$  and  $\eta \in B_1$ , we have*

$$(2.3) \quad \|\eta\|^2 \leq a(\eta)^2\|w\|^2 - 2J_w(\eta),$$

and for all  $\eta \in B_1$  such that  $|\eta|_2^2 \geq \frac{1}{2}$  and  $J_w(\eta) \geq 0$ , we have that

$$(2.4) \quad dJ_w(\eta)[\eta] \leq -\frac{1}{2}(1 - 4s\kappa)m < 0,$$

PROOF. We have, thanks to (1.5), that for  $\eta \in B_1$  and  $\psi = a(\eta)w + \eta$ ,

$$\frac{1}{2}\|\eta\|^2 \leq \frac{1}{2}\|\eta\|^2 + \frac{s}{4} \int \frac{(\psi, \beta\psi)(x)(\psi, \beta\psi)(y)}{|x-y|} = \frac{1}{2}\|a(\eta)w\|^2 - J_w(\eta)$$

and (2.3) follows.

From (2.3) it follows that  $\|\eta\| \leq a(\eta)\|w\|$  if  $J_w(\eta) \geq 0$ ; hence we have, if  $|\eta|_2^2 > \frac{1}{2}$ ,

$$\begin{aligned} dJ_w(\eta)[\eta] &= -|\eta|_2^2\|w\|^2 - \|\eta\|^2 - s \int \frac{(\psi, \beta\psi)(x)(\psi, \beta\psi)(y)}{|x-y|} \\ &\quad + s \int \frac{(\psi, \beta\psi)(x)(w, \beta w)(y)}{|x-y|} + sa(\eta)^{-1} \int \frac{(\psi, \beta\psi)(x)(\eta, \beta w)(y)}{|x-y|} \\ &\leq -|\eta|_2^2\|w\|^2 - \|\eta\|^2 + s\kappa(\|w\|^2 + a(\eta)^{-1}\|\eta\|\|w\|) \\ &\leq -\frac{1}{2}\|w\|^2 - \|\eta\|^2 + 2s\kappa\|w\|^2 < -\frac{1}{2}(1 - 4s\kappa)\|w\|^2 \\ &\leq -\frac{1}{2}(1 - 4s\kappa)m|w|_2^2 = -\frac{1}{2}(1 - 4s\kappa)m, \end{aligned}$$

where we have used (1.5) and (1.6). ■

REMARK 2.5. It follows from Lemma 2.2 that if  $\eta_n$  is a Palais–Smale sequence for  $J_w$  such that  $J_w(\eta_n) \geq 0$ , then  $|\eta_n|_2^2 < \frac{1}{2}$  for all  $n \in \mathbb{N}$  large enough.

LEMMA 2.6. *Let  $\eta_n \in B_1$  be a Palais–Smale sequence for  $J_w$ , that is*

$$J_w(\eta_n) \rightarrow c \geq 0, \quad dJ_w(\eta_n) \rightarrow 0.$$

*Then  $\eta_n$  converges, up to a subsequence, to a critical point  $\eta$  of  $J_w$ .*

PROOF. It follows from Lemma 2.2 and Remark 2.5 that  $|\eta_n|_2^2 < \frac{1}{2}$  and that  $\|\eta_n\|$  is bounded; hence  $\eta_n \rightharpoonup \eta$  (up to a subsequence).

From

$$\begin{aligned} o(1) &= dJ_w(\eta_n)[\eta_n - \eta] = -(\eta_n | \eta_n - \eta)\|w\|^2 - \langle \eta_n | \eta_n - \eta \rangle \\ &\quad - s \int \frac{(\psi_n, \beta\psi_n)(x)(\psi_n, \beta(-a(\eta_n)^{-1}(\eta_n | \eta_n - \eta)w + \eta_n - \eta))(y)}{|x - y|}, \end{aligned}$$

we deduce that

$$\begin{aligned} &|\eta_n - \eta|_2^2\|w\|^2 + \|\eta_n - \eta\|^2 \\ &= -(\eta | \eta_n - \eta)\|w\|^2 - \langle \eta | \eta_n - \eta \rangle \\ &\quad + sa(\eta_n)^{-1}(\eta_n | \eta_n - \eta) \int \frac{(\psi_n, \beta\psi_n)(x)(\psi_n, \beta w)(y)}{|x - y|} \\ &\quad - s \int \frac{(\psi_n, \beta\psi_n)(x)(\psi_n, \beta(\eta_n - \eta))(y)}{|x - y|} + o(1). \end{aligned}$$

We have that

$$\begin{aligned} &\int \frac{(\psi_n, \beta\psi_n)(x)(\psi_n, \beta(\eta_n - \eta))(y)}{|x - y|} \\ &= \int \frac{(\psi_n, \beta(\eta_n - \eta))(x)(\psi_n, \beta(\eta_n - \eta))(y)}{|x - y|} + \int \frac{(\psi_n, \beta\eta)(x)(\psi_n, \beta(\eta_n - \eta))(y)}{|x - y|} \\ &\quad + a(\eta_n) \int \frac{(\psi_n, \beta w)(x)(\psi_n, \beta(\eta_n - \eta))(y)}{|x - y|} \\ &= \int \frac{(\psi_n, \beta(\eta_n - \eta))(x)(\psi_n, \beta(\eta_n - \eta))(y)}{|x - y|} + o(1) \end{aligned}$$

and

$$\begin{aligned} &\left| \int \frac{(\psi_n, \beta\psi_n)(x)(\psi_n, \beta a(\eta_n)w)(y)}{|x - y|} \right| \\ &\leq \kappa \|\psi_n\| \|a(\eta_n)w\| \leq \kappa(2\|a(\eta_n)w\|^2 + \|\eta_n\|^2) \leq 3\kappa a(\eta_n)^2 \|w\|^2. \end{aligned}$$

Since  $|\eta_n|_2^2 < \frac{1}{2}$ , we have

$$a(\eta_n)^{-2}(\eta_n | \eta_n - \eta) = a(\eta_n)^{-2}|\eta_n - \eta|_2^2 + o(1)$$

and we deduce that

$$\begin{aligned}
 & |\eta_n - \eta|_2^2 \|w\|^2 + \|\eta_n - \eta\|^2 \\
 & \leq 3s\kappa |\eta_n - \eta|_2^2 \|w\|^2 - s \int \frac{(\psi_n, \beta(\eta_n - \eta))(x)(\psi_n, \beta(\eta_n - \eta))(y)}{|x - y|} + o(1) \\
 & \leq 3s\kappa |\eta_n - \eta|_2^2 \|w\|^2 + o(1)
 \end{aligned}$$

and  $\eta_n \rightarrow \eta$ , with  $\eta$  critical point of  $J_w$ . ■

We now show that all the critical points of  $J_w$  at positive levels are strict local maxima. This lemma follows as in [2, 6].

LEMMA 2.7. *Let  $\eta \in X_-$  a critical point of  $J_w$  such that  $J_w(\eta) \geq 0$ .*

*Then there exists  $\delta > 0$  such that*

$$d^2 J_w(\eta)[\xi, \xi] \leq -\delta \|\xi\|^2 \quad \text{for all } \xi \in X_-.$$

PROOF. In order to compute the second derivative we denote  $\psi = a(\eta)w + \eta$  and  $h = da(\eta)[\xi]w + \xi$  and observe that

$$d^2 a(\eta)[\xi, \xi] = -a(\eta)^{-1} \left( \langle \xi | \xi \rangle + \frac{(\eta | \xi)(\eta, \xi)}{1 - |\eta|_2^2} \right).$$

Then

$$\begin{aligned}
 & d^2 J_w(\eta)[\xi, \xi] \\
 & = d^2 I(\psi)[h, h] + dI(\psi)[d^2 a(\eta)[\xi, \xi]w] \\
 & = \langle da(\eta)[\xi]w | da(\eta)[\xi]w \rangle - \langle \xi | \xi \rangle - s \int \frac{(\psi, \beta\psi)(x)(h, \beta h)(y)}{|x - y|} \\
 & \quad - 2s \int \frac{(\psi, \beta h)(x)(\psi, \beta h)(y)}{|x - y|} + a(\eta) d^2 a(\eta)[\xi, \xi] \langle w | w \rangle \\
 & \quad - s \int \frac{(\psi, \beta\psi)(x)(\psi, \beta d^2 a(\eta)[\xi, \xi]w)(y)}{|x - y|} \\
 & = -|\xi|_2^2 \|w\|^2 + s|\xi|_2^2 \int \frac{(\psi, \beta\psi)(x)(w, \beta w)(y)}{|x - y|} - \|\xi\|^2 \\
 & \quad - s \int \frac{(\psi, \beta\psi)(x)(\xi, \beta\xi)(y)}{|x - y|} + 2s \frac{(\eta | \xi)}{1 - |\eta|_2^2} a(\eta) \Gamma(\xi) \\
 & \quad - 2s \int \frac{(\psi, \beta h)(x)(\psi, \beta h)(y)}{|x - y|} + s \left( \frac{|\xi|_2^2}{1 - |\eta|_2^2} + \frac{(\eta | \xi)^2}{(1 - |\eta|_2^2)^2} \right) a(\eta) \Gamma(\eta),
 \end{aligned}$$

where we have set

$$\Gamma(\xi) = \int \frac{(\psi, \beta\psi)(x)(w, \beta\xi)(y)}{|x - y|}.$$

Since  $\eta$  is a critical point for  $J_w$ , we have that

$$\begin{aligned} d^2 J_w(\eta)[\xi, \xi] &= d^2 J_w(\eta)[\xi, \xi] + 2 \frac{(\eta | \xi)}{1 - |\eta|_2^2} dJ_w(\eta)[\xi] \\ &\quad + \left( \frac{|\xi|_2^2}{1 - |\eta|_2^2} + 3 \frac{(\eta, \xi)^2}{(1 - |\eta|_2^2)^2} \right) dJ_w(\eta)[\eta]. \end{aligned}$$

We have that

$$\begin{aligned} 0 &= dJ_w(\eta)[\eta] = -|\eta|_2^2 \|w\|^2 - \|\eta\|^2 + s \frac{|\eta|_2^2}{1 - |\eta|_2^2} \int \frac{(\psi, \beta\psi)(x)(w, \beta w)(y)}{|x - y|} \\ &\quad - s \int \frac{(\psi, \beta\psi)(x)(\eta, \beta\eta)(y)}{|x - y|} - s \left( \frac{1 - 2|\eta|_2^2}{1 - |\eta|_2^2} \right) a\Gamma(\eta) \\ &\leq -|\eta|_2^2(1 - \kappa) \|w\| - (1 - \kappa) \|\eta\| - s \left( \frac{1 - 2|\eta|_2^2}{1 - |\eta|_2^2} \right) a\Gamma(\eta) \end{aligned}$$

which implies that  $\Gamma(\eta) < 0$ . After some simplification, we get

$$\begin{aligned} d^2 J_w(\eta)[\xi, \xi] &\leq -Q(1 - s\kappa) \|w\|^2 - \frac{|\xi|_2^2}{1 - |\eta|_2^2} (1 - s\kappa) \|\eta\|^2 - (1 - s\kappa) \|\xi\| + R\eta\|^2 \\ &\leq -\frac{|\xi|_2^2}{1 - |\eta|_2^2} (1 - s\kappa) \|\eta\|^2 - (1 - s\kappa) \left( \frac{1}{3} \|\xi\|^2 - \frac{1}{2} R^2 \|\eta\|^2 \right) \\ &\leq -\frac{1}{3} (1 - s\kappa) \|\xi\|^2, \end{aligned}$$

where

$$\begin{aligned} R &= \frac{(\eta, \xi)}{1 - |\eta|_2^2}, \\ Q &= |\xi|_2^2 + 2R(\eta, \xi) - |\eta|_2^2 \left( \frac{|\xi|_2^2}{1 - |\eta|_2^2} + R^2 \right). \end{aligned}$$

Remark that, since  $|\eta|_2^2 < \frac{1}{2}$ , we have

$$Q \geq \left( \frac{|\xi|_2^2}{1 - |\eta|_2^2} + R^2 \right) (1 - 2|\eta|_2^2) > 0$$

and

$$\frac{|\xi|_2^2}{1 - |\eta|_2^2} - \frac{1}{2} \frac{(\eta | \xi)^2}{(1 - |\eta|_2^2)^2} \geq \frac{|\xi|_2^2(2 - 3|\eta|_2^2)}{2(1 - |\eta|_2^2)^2} > \frac{|\xi|_2^2}{4(1 - |\eta|_2^2)^2} > 0. \quad \blacksquare$$



We let, for all  $w \in \Sigma_+$ ,

$$\mathcal{E}(w) = \sup_{\eta \in B_1} J_w(\eta).$$

LEMMA 2.8. *For all  $w \in \Sigma_+$ , we have*

$$0 < \frac{1}{4}(2 - s\kappa)m \leq \frac{1}{4}(2 - s\kappa)\|w\|^2 \leq \mathcal{E}(w) \leq \frac{1}{2}\|w\|^2.$$

PROOF. We have that

$$\begin{aligned} \mathcal{E}(w) &\geq J_w(0) = \frac{1}{2}\|w\|^2 - \frac{s}{4} \int \frac{(w, \beta w)(x)(w, \beta w)(y)}{|x - y|} \\ &\geq \frac{1}{4}(2 - s\kappa)\|w\|^2 \geq \frac{1}{4}(2 - s\kappa)m|w|_2^2 = \frac{1}{4}(2 - s\kappa)m \end{aligned}$$

and, for all  $\eta \in B_1$ , we have

$$J_w(\eta) = \frac{1}{2}\|a(\eta)w\|^2 - \frac{1}{2}\|\eta\|^2 - \frac{s}{4} \int \frac{(\psi, \beta \psi)(x)(\psi, \beta \psi)(y)}{|x - y|} \leq \frac{1}{2}\|w\|^2. \quad \blacksquare$$

PROPOSITION 2.9. *For every  $w \in \Sigma_+$ , there is a unique  $\eta(w) \in B_1$  such that*

$$J_w(\eta(w)) = \max_{\eta \in B_1} J_w(\eta) = \mathcal{E}(w).$$

$\eta(w)$  is a critical point of  $J_w$  on  $B_1$  such that  $|\eta(w)|_2 < \frac{1}{2}$  and

$$(2.10) \quad \|\eta(w)\|^2 + m \leq \|a(\eta)w\|^2, \quad \|\eta(w)\|^2 \leq \frac{s}{2}\kappa\|w\|^2.$$

Moreover, the map

$$w \in X_+ \setminus \{0\} \mapsto \gamma(w) = \eta(|w|_2^{-1}w) \in B_1$$

is smooth.

PROOF. We can find, by Lemma 2.8 and using Ekeland's variational principle, a maximizing Palais–Smale sequence  $\eta_n$  at a positive level.

Then, by Lemma 2.6,  $\eta_n \rightarrow \eta$  (up to a subsequence), with

$$dJ_w(\eta) = 0, \quad J_w(\eta) = \mathcal{E}(w).$$

From

$$\begin{aligned} \mathcal{E}(w) &= \frac{1}{2}a(\eta)^2\|w\|^2 - \frac{1}{2}\|\eta\|^2 - \frac{s}{4} \int \frac{(\psi, \beta \psi)(x)(\psi, \beta \psi)(y)}{|x - y|} \\ &\geq \frac{1}{2}\|w\|^2 - \frac{s}{4} \int \frac{(w, \beta w)(x)(w, \beta w)(y)}{|x - y|}, \end{aligned}$$

we deduce, using Lemma A.1 in the appendix, that

$$\begin{aligned} & a(\eta)^2 \|w\|^2 - \|\eta\|^2 - m \\ & \geq \|w\|^2 - s\kappa |\eta|_2^2 \|w\|^2 - 7sa(\eta)^2 \kappa (\|w\|^2 - m|w|_2^2) - 9s\kappa \|\eta\|^2 - m|w|_2^2 \\ & \geq 9s\kappa (a(\eta)^2 \|w\|^2 - \|\eta\|^2 - m|w|_2^2) + (1 - 16s\kappa) (\|w\|^2 - m|w|_2^2) + 8s\kappa |\eta|_2^2 \|w\|^2 \end{aligned}$$

and we immediately deduce that

$$a(\eta)^2 \|w\|^2 - \|\eta\|^2 - m|w|_2^2 \geq \frac{1 - 16s\kappa}{1 - 9s\kappa} (\|w\|^2 - m|w|_2^2) > 0.$$

We also have that

$$\begin{aligned} |\eta|_2^2 \|w\|^2 + \|\eta\|^2 & \leq \frac{s}{2} \int \frac{(w, \beta w)(x)(w, \beta w)(y)}{|x - y|} \\ & \leq \frac{s}{2} \kappa |(-\Delta)^{-1/4} w|_2^2 \leq \frac{s}{2} \kappa \|w\|^2 \end{aligned}$$

from which we deduce that

$$\|\eta\|^2 \leq \frac{s}{2} \kappa \|w\|^2.$$

To prove the uniqueness of the maxima for  $J_w(\eta)$ , we assume, by contradiction, the existence of  $\eta_1, \eta_2 \in B_1$  such that

$$J_w(\eta_1) = J_w(\eta_2) = \mathcal{E}(w).$$

It follows from Lemma 2.2 that  $|\eta_1|_2^2 < \frac{1}{2}$  and  $|\eta_2|_2^2 < \frac{1}{2}$ . We will use the mountain pass lemma in order to reach a contradiction. Let

$$\Gamma = \left\{ g \in C([0, 1], B_1) \mid g(0) = \eta_1, g(1) = \eta_2, |g(t)|_2^2 < \frac{1}{2} \right\}$$

and define the min-max level

$$c = \sup_{g \in \Gamma} \min_{t \in [0, 1]} J_w(g(t)).$$

Let  $g(t) = t\eta_1 + (1 - t)\eta_2$ . We have that  $|g(t)|_2^2 < \frac{1}{2}$  and  $a(g(t))^2 > \frac{1}{2}$  for all  $t \in [0, 1]$ , so that we have, letting  $\psi_t = a(g(t))w + g(t)$ ,

$$\begin{aligned} J_w(g(t)) & = \frac{1}{2} a(g(t))^2 \|w\|^2 - \frac{1}{2} \|g(t)\|^2 - \frac{s}{4} \int \frac{(\psi_t, \beta \psi_t)(x)(\psi_t, \beta \psi_t)(y)}{|x - y|} \\ & \geq \frac{1}{2} \left(1 - \frac{s\kappa}{2}\right) a(g(t))^2 \|w\|^2 - \frac{1}{2} \left(1 + \frac{s\kappa}{2}\right) \|g(t)\|^2 \end{aligned}$$

$$\begin{aligned}
 &\geq \frac{1}{2} \left(1 - \frac{s\kappa}{2}\right) t a(\eta_1)^2 \|w\|^2 + \frac{1}{2} \left(1 - \frac{s\kappa}{2}\right) (1-t) a(\eta_2)^2 \|w\|^2 \\
 &\quad - \frac{1}{2} \left(1 + \frac{s\kappa}{2}\right) t \|\eta_1\|^2 - \frac{1}{2} \left(1 + \frac{s\kappa}{2}\right) (1-t) \|\eta_2\|^2 \\
 &\geq \frac{1}{4} \left(1 - \frac{s\kappa}{2}\right) \|w\|^2 - \frac{1}{2} \left(1 + \frac{s\kappa}{2}\right) \frac{s}{2} \kappa \|w\|^2 \geq \frac{1}{4} (1 - 2s\kappa) \|w\|^2,
 \end{aligned}$$

where we have used the second inequality in (2.10) and  $s\kappa < 2$ . We deduce that  $c > 0$ . Since the set  $\Gamma$  is invariant for the gradient flow (see Lemma 2.2) and the Palais–Smale condition holds (see Lemma 2.6), we deduce from the mountain pass theorem that there exists a critical point at level  $c$ , which cannot be a strict local maximum. The contradiction then follows from Lemma 2.7.

To prove that the map  $w \mapsto \gamma(w) = \eta(|w|_2^{-1}w)$  is smooth, we consider, since the map  $w \mapsto P(w) = |w|_2^{-1}w$  is smooth, that  $(w_0, \eta(w_0)) \in \Sigma_+ \times B_1$  and we let  $V \subset X_+ \setminus \{0\}$  and  $U \subset B_1$  be neighborhoods of  $w_0$  and  $\eta(w_0)$ , respectively. Then we define the map  $F: V \times U \rightarrow L(X_-)$  as

$$F(w, \eta)[\xi] = dJ_{P(w)}(\eta)[\xi] \quad \xi \in X_-.$$

Clearly,  $P(w_0) = w_0$  and  $F(w_0, \eta(w_0)) = 0$ . We have that

$$d_\eta F(w_0, \eta(w_0))[\xi][\zeta] = d^2 J_{w_0}(\eta(w_0))[\xi, \zeta], \quad \xi, \zeta \in X_-.$$

It follows from Lemma 2.7 that

$$-d_\eta F(w_0, \eta(w_0))[\xi][\xi] = -d^2 J_{w_0}(\eta(w_0))[\xi, \xi] \geq \delta \|\xi\|^2 \quad \text{for all } \xi \in X_-$$

and hence we have from Lax–Milgram that for all linear functionals  $L$  on  $X_-$  there is a unique  $\zeta \in X_-$  such that

$$-d_\eta F(w_0, \eta(w_0))[\zeta][\xi] = L[\xi], \quad \text{for all } \xi \in X_-,$$

that is,  $L = -d_\eta F(w_0, \eta(w_0))[\zeta]$ . By the implicit function theorem, there exist  $V_0 \subset V$  and  $U_0 \subset U$ , neighborhoods of  $w_0$  and  $\eta(w_0)$  and a smooth map  $\gamma: V_0 \rightarrow U_0$  such that  $F(w, \gamma(w)) = 0$  for all  $w \in V_0$ ; that is,  $\gamma(w)$  is a critical point of  $J_{P(w)}$  on  $B_1$  at a positive level. Then, by Proposition 2.7,  $\gamma(w)$  is a strict local maximum of  $J_{P(w)}$  on  $B_1$ . Again using the mountain pass theorem, we deduce that actually  $\gamma(w) = \eta(P(w))$  is the unique (up to a phase factor) maximum of  $J_{P(w)}$ .

Finally, we have that

$$d\gamma(w)[v] = -d_\eta F(w, \gamma(w))^{-1} [d_w F(w, \gamma(w))[v]] \quad \text{for all } v \in X_+. \quad \blacksquare$$

It follows from Proposition 2.9 that we can consider the smooth functional  $\mathcal{E}: X_+ \setminus \{0\} \rightarrow \mathbb{R}$  defined as

$$\mathcal{E}(w) = J_{P(w)}(\gamma(w)) = \sup_{\eta \in B_1} J_{P(w)}(\eta).$$

Since

$$J_{P(w)}(\gamma(w)) = I(a(\gamma(w))P(w) + \gamma(w))$$

and recalling that

$$dJ_{P(w)}(\gamma(w))[\xi] = dI(\psi_w)[da(\gamma(w))[\xi]P(w) + \xi] = 0 \quad \text{for all } \xi \in X_-$$

(where  $\psi_w = a(\gamma(w))P(w) + \gamma(w)$ ), we have that for all  $v \in X_+$

$$\begin{aligned} d\mathcal{E}(w)[v] &= d_w J_{P(w)}(\gamma(w))[v] \\ &= dI(\psi_w)[da(\gamma(w))[d\gamma(w)[v]]P(w) + a(\gamma(w))dP(w)[v] + d\gamma(w)[v]] \\ &= d_\eta J_{P(w)}(\gamma(w))[d\gamma(w)[v]] + dI(\psi_w)[a(\gamma(w))dP(w)[v]] \\ &= dI(\psi_w)[a(\gamma(w))dP(w)[v]] \\ &= dI(\psi_w)[a(\gamma(w))v] - dI(\psi_w)[a(\gamma(w))(w | v)w] \end{aligned}$$

(we have used that  $dP(w)[v] = v - (w | v)w$  and

$$(2.11) \quad d\mathcal{E}(w)[v] = a(\gamma(w))dI(\psi_w)[v] - a(\gamma(w))^2\omega(\psi_w)(w | v) \quad \text{for all } v \in X_+,$$

where

$$(2.12) \quad \omega(\psi_w) = a(\gamma(w))^{-1}dI(\psi_w)[w].$$

**PROPOSITION 2.13.** *Let  $w_0 \in \Sigma_+$  be a critical point of  $\mathcal{E}$  restricted on the manifold  $\Sigma_+$ . Then  $w_0$  is a critical point for  $\mathcal{E}$  on  $X_+$  and the function*

$$\psi_0 = a(\eta(w_0))w_0 + \eta(w_0) \in \Sigma$$

*is a critical point for  $I$  on the manifold  $\Sigma$  and satisfies*

$$(2.14) \quad dI(\psi_0)[h] = \omega(\psi_0 | h) \quad \text{for all } h \in X,$$

where  $\omega = \omega(\psi_0) \in \mathbb{R}$ ,

$$(2.15) \quad (1 - 3s\kappa)\|w_0\|^2 \leq \omega(\psi_0) \leq 2I(\psi_{w_0}) = 2\mathcal{E}(w_0).$$

*Moreover, if  $\psi_0 \in \Sigma$  satisfies  $I(\psi_0) \geq 0$  and (2.14) for some  $\omega \in \mathbb{R}$ , then  $w = |\Lambda_+\psi_0|_2^{-1}\Lambda_+\psi_0$  is a critical point for  $\mathcal{E}(w)$ .*

PROOF. Let  $w_0 \in \Sigma_+$  be a critical point for  $\mathcal{E}$  on  $\Sigma_+$ ,  $\eta_0 = \eta(w_0) = \gamma(w_0)$ , and  $\psi_0 = a(\eta_0)w_0 + \eta_0$ . Then

$$d\mathcal{E}(w_0)[h] = 0 \quad \text{for all } h \in T_{w_0}\Sigma_+ = \{h \in X_+ \mid (w_0 \mid h) = 0\}.$$

Since  $dP(w_0)[w_0] = 0$ , we immediately deduce that

$$d\mathcal{E}(w_0)[v] = 0 \quad \text{for all } v \in X_+.$$

From (2.1), we have that for all  $\xi \in X_-$

$$0 = dJ_{w_0}(\eta_0)[\xi] = dI(\psi_0)[\xi] + dI(\psi_0)[(da(\eta_0)[\xi])w_0]$$

while for all  $v \in X_+$  we have

$$0 = d\mathcal{E}(w_0)[v] = a(\eta_0)dI(\psi_0)[v] - a(\eta_0)^2\omega(\psi_0)(w_0 \mid v)$$

and hence, for all  $h = v + \xi$ ,  $v \in X_+$ ,  $\xi \in X_-$ ,

$$\begin{aligned} dI(\psi_0)[h] &= a(\eta_0)\omega(\psi_0)(w_0 \mid v) - dI(\psi_0)[da(\eta_0)[\xi]w_0] \\ &= a(\eta_0)\omega(\psi_0)(w_0 \mid v) + \omega(\psi_0)(\eta_0 \mid \xi) \\ &= \omega(\psi_0)(\psi_0 \mid h), \end{aligned}$$

that is

$$dI(\psi_0)[h] = \omega(\psi_0)(\psi_0 \mid h) \quad \text{for all } h \in X,$$

which shows that  $\psi_0$  is a critical point for  $I(\psi)$  under the constraint  $|\psi|_2 = 1$ . The Lagrange multiplier  $\omega(\psi_0) = a(\eta_0)^{-1}dI(\psi_0)[w_0]$  is such that

$$\begin{aligned} \omega(\psi_0) &= a(\eta(w_0))^{-1}dI(\psi_0)[w_0] \geq \|w_0\|^2 - s\kappa a(\eta(w_0))^{-1}\|\psi_0\|\|w_0\| \\ &\geq \|w_0\|^2 - \frac{s\kappa}{2}(a(\eta(w_0))^{-2}\|\psi_0\|^2 + \|w_0\|^2) \\ &\geq \|w_0\|^2 - \frac{s\kappa}{2}(\|w_0\|^2 + a(\eta(w_0))^{-2}\|\eta_0\|^2 + \|w_0\|^2) \\ &\geq \|w_0\|^2 - \frac{3s\kappa}{2}\|w_0\|^2 \end{aligned}$$

and

$$\omega(\psi_0) = dI(\psi_0)[\psi_0] \leq 2I(\psi_0).$$

Suppose now that  $\psi_0 \in \Sigma$  satisfies (2.14) for some  $\tilde{\omega}$ . Let  $w_0 = |\Lambda_+ \psi_0|_2^{-1} \Lambda_+ \psi_0$  and  $\eta_0 = \Lambda_- \psi_0$ . Then we deduce from (2.1) that for all  $\xi \in X_-$

$$dJ_{w_0}(\eta_0)[\xi] = dI(\psi_0)[da(\eta_0)[\xi]w_0 + \xi] = \tilde{\omega}(\psi_0 \mid da(\eta_0)[\xi]w_0 + \xi) = 0,$$

and  $\eta_0$  is a critical point of  $J_{w_0}$ . From Lemma 2.8, we know that  $\eta_0$  is a local maximum and, arguing as in the proof of Proposition 2.9, we deduce that  $\eta_0 = \eta(w_0)$  and  $\mathcal{E}(w_0) = J_{\omega_0}(\eta_0)$ . We also have that

$$\tilde{\omega} = dI(\psi_0)[\psi_0] = \omega(\psi_0).$$

We then deduce from (2.11) that

$$\begin{aligned} d\mathcal{E}(w_0) &= a(\gamma(w_0))dI(\psi_0)[v] - a(\gamma(w_0))^2\omega(\psi_0)(w_0 | v) \\ &= a(\gamma(w_0))\tilde{\omega}(\psi_0 | v) - a(\gamma(w_0))^2\omega(\psi_{w_0})(w_0 | v) \\ &= a(\gamma(w_0))\omega(\psi_0)(a(\gamma(w_0))w_0 + \xi | v) - a(\gamma(w_0))^2\omega(\psi_0)(w_0, v) = 0. \blacksquare \end{aligned}$$

From now on, we will make explicit the dependence of  $I$ ,  $J$ , and  $\mathcal{E}$  on  $s > 0$  writing  $I_s$ ,  $J_s$ , and  $\mathcal{E}_s$ , introduce the following minimization problem:

$$\begin{aligned} e(s) &= \inf_{w \in \Sigma_+} \mathcal{E}_s(w) \\ &= \inf_{w \in \Sigma_+} \left\{ \frac{1}{2}a(\eta_s(w))^2\|w\|^2 - \frac{1}{2}\|\eta_s(w)\|^2 \right. \\ &\quad \left. - \frac{s}{4} \int \frac{(\psi_w, \beta\psi_w)(x)(\psi_w, \beta\psi_w)(y)}{|x-y|} \right\}, \end{aligned}$$

and let  $E(s) = se(s)$ .

The next lemma allows us to recover enough compactness (via the concentration-compactness lemma [4, 5]) in order to prove our main result; see also [6, Lemma 4.2].

LEMMA 2.16. *For all  $s \in (0, \frac{1}{8\pi}]$  we have that  $0 < e(s) < \frac{m}{2}$ .*

PROOF. From Lemma 2.8, we have that  $e(s) \geq \frac{1}{4}(2 - s\kappa)m \geq \frac{1}{4}(2 - \kappa)m > 0$ .

Using Lemma A.1, we deduce that

$$\begin{aligned} \mathcal{E}_s(w) &= I_s(\psi_w) \\ &\leq \frac{m}{2} + \frac{1}{2}(1 + 8s\kappa)(\|w\|^2 - m|w|_2^2) - \frac{s}{4} \int \frac{(w, \beta w)(x)(w, \beta w)(y)}{|x-y|}. \end{aligned}$$

Fix  $w_1 \in H^1(\mathbb{R}^3, \mathbb{C}^2)$  such that  $|w_1|_2 = 1$ ,  $w = \begin{pmatrix} w_1 \\ 0 \end{pmatrix}$  and let  $w_\varepsilon(x) = \varepsilon^{3/2}w(\varepsilon x)$ . We have that

$$|w_\varepsilon|_2^2 = |w|_2^2 = 1,$$

and

$$\hat{w}_\varepsilon(q) = \frac{1}{\varepsilon^{3/2}}w(\varepsilon^{-1}p),$$

so that

$$\begin{aligned}\|w_\varepsilon\|^2 - m|w_\varepsilon|_2^2 &= \int (\sqrt{|q|^2 + m^2} - m) |\hat{w}_\varepsilon(q)|^2 \\ &= \int (\sqrt{\varepsilon^2|q|^2 + m^2} - m) |\hat{w}_1(q)|^2 \leq \frac{\varepsilon^2}{2m} \int |q|^2 |\hat{w}_1(q)|^2\end{aligned}$$

and  $\|w_\varepsilon\|^2 \leq m + C\varepsilon^2$ . We then observe that

$$\begin{aligned}\|w_\varepsilon - \Lambda_+ w_\varepsilon\|^2 &= \|\Lambda_- w_\varepsilon\|^2 \\ &= \int \sqrt{\varepsilon^2|p|^2 + m^2} \left| \frac{1}{2} \left[ 1 - \frac{m\beta}{\sqrt{\varepsilon^2|p|^2 + m^2}} - \frac{\varepsilon\alpha \cdot p}{\sqrt{\varepsilon^2|p|^2 + m^2}} \right] \begin{pmatrix} \hat{w}_1(p) \\ 0 \end{pmatrix} \right|^2 \\ &= \frac{1}{4} \int \sqrt{\varepsilon^2|p|^2 + m^2} \left| \begin{pmatrix} \frac{\sqrt{\varepsilon|p|^2 + m^2} - m}{\sqrt{\varepsilon|p|^2 + m^2}} \hat{w}_1(p) \\ \frac{\varepsilon\sigma \cdot p}{\sqrt{\varepsilon|p|^2 + m^2}} \hat{w}_1(p) \end{pmatrix} \right|^2 \\ &= \frac{1}{2} \int (\sqrt{\varepsilon^2|p|^2 + m^2} - m) |\hat{w}_1(p)|^2 \leq \frac{\varepsilon^2}{4m} \int |p|^2 |\hat{w}_1(p)|^2\end{aligned}$$

and also

$$\begin{aligned}|1 - |\Lambda_+ w_\varepsilon|_2| &= ||w_\varepsilon|_2 - |\Lambda_+ w_\varepsilon|_2| \leq |w_\varepsilon - \Lambda_+ w_\varepsilon|_2 \\ &= |\Lambda_- w_\varepsilon|_2 \leq \frac{\varepsilon}{2\sqrt{m}} \left( \int |p|^2 |\hat{w}_1(p)|^2 \right)^{1/2}.\end{aligned}$$

We deduce from this that for  $\varepsilon > 0$  small enough  $|\Lambda_+ w_\varepsilon|_2 > \frac{1}{2}$ .

Let

$$\varphi_\varepsilon(x) = |\Lambda_+ w_\varepsilon|_2^{-1} \Lambda_+ w_\varepsilon(x).$$

We have that

$$\|\varphi_\varepsilon\| \leq |\Lambda_+ w_\varepsilon|_2^{-1} \|w_\varepsilon\| \leq \sqrt{m} + C\varepsilon$$

and

$$\begin{aligned}\|w_\varepsilon - \varphi_\varepsilon\| &\leq \|w_\varepsilon - \Lambda_+ w_\varepsilon\| + \|(1 - |\Lambda_+ w_\varepsilon|_2)\varphi_\varepsilon\| \\ &\leq \frac{\varepsilon}{2\sqrt{m}} (2 + \|\varphi_\varepsilon\|) \left( \int |p|^2 |\hat{w}_1(p)|^2 \right)^{1/2}\end{aligned}$$

and also

$$|\varphi_\varepsilon - w_\varepsilon|_2 = \frac{1}{|\Lambda_+ w_\varepsilon|_2} |w_\varepsilon - \Lambda_+ w_\varepsilon|_2 \leq \frac{\varepsilon}{\sqrt{m}} \left( \int |p|^2 |\hat{w}_1(p)|^2 \right)^{1/2}$$

and we can estimate

$$\mathcal{E}_s(\varphi_\varepsilon) \leq \frac{m}{2} + \frac{1}{2}(1 + 8s\kappa)(\|\varphi_\varepsilon\|^2 - m|\varphi_\varepsilon|_2^2) - \frac{s}{4} \int \frac{(\varphi_\varepsilon, \beta\varphi_\varepsilon)(x)(\varphi_\varepsilon, \beta\varphi_\varepsilon)(y)}{|x-y|}.$$

We have that

$$\begin{aligned} & \|\varphi_\varepsilon\|^2 - m|\varphi_\varepsilon|_2^2 \\ &= \int (\sqrt{|q|^2 + m^2} - m)|\hat{\varphi}_\varepsilon(q)|^2 \\ &\leq 2 \int (\sqrt{|q|^2 + m^2} - m)|\hat{\varphi}_\varepsilon(q) - \hat{w}_\varepsilon(q)|^2 + 2 \int (\sqrt{|q|^2 + m^2} - m)|\hat{w}_\varepsilon(q)|^2 \\ &= 2(\|\varphi_\varepsilon - w_\varepsilon\|^2 - m|\varphi_\varepsilon - w_\varepsilon|_2^2) + 2(\|w_\varepsilon\|^2 - m|w_\varepsilon|_2^2) \\ &\leq \frac{\varepsilon^2}{2m}(3 + \|\varphi_\varepsilon\|)^2 |\nabla w_1|_2^2. \end{aligned}$$

We have that

$$\begin{aligned} Q(\varphi_\varepsilon) - Q(w_\varepsilon) &= Q((\varphi_\varepsilon - w_\varepsilon) + w_\varepsilon) - Q(w_\varepsilon) \\ &= Q(\varphi_\varepsilon - w_\varepsilon) + 4 \int \frac{(\varphi_\varepsilon - w_\varepsilon, \beta w_\varepsilon)(x)(w_\varepsilon, \beta w_\varepsilon)(x)}{|x-y|} \\ &\quad + 3 \int \frac{(\varphi_\varepsilon - w_\varepsilon, \beta w_\varepsilon)(x)(\varphi_\varepsilon - w_\varepsilon, \beta w_\varepsilon)(x)}{|x-y|} \\ &\quad + 3 \int \frac{(\varphi_\varepsilon - w_\varepsilon, \beta(\varphi_\varepsilon - w_\varepsilon))(x)(w_\varepsilon, \beta w_\varepsilon)(x)}{|x-y|} \\ &\quad + 4 \int \frac{(\varphi_\varepsilon - w_\varepsilon, \beta(\varphi_\varepsilon - w_\varepsilon))(x)(\varphi_\varepsilon - w_\varepsilon, \beta w_\varepsilon)(x)}{|x-y|} \\ &\geq -4\kappa|\varphi_\varepsilon - w_\varepsilon|_2|(-\Delta)^{1/4}|w_\varepsilon|_2^2 - 3\kappa|\varphi_\varepsilon - w_\varepsilon|_2^2|(-\Delta)^{1/4}|w_\varepsilon|_2^2 \\ &\quad - 4\kappa|\varphi_\varepsilon - w_\varepsilon|_2\|\varphi_\varepsilon - w_\varepsilon\|^2. \end{aligned}$$

Since

$$|(-\Delta)^{1/4}|w_\varepsilon|_2^2 = \int |p||\hat{w}_\varepsilon|^2 = \varepsilon \int |p||w_1|^2,$$

we have

$$Q(\varphi_\varepsilon) \geq Q(w_\varepsilon) - c\varepsilon^2 \geq \varepsilon Q(w) - c\varepsilon^2;$$

we therefore deduce that

$$\mathcal{E}_s(w_\varepsilon) \leq \frac{m}{2} + \frac{2\varepsilon^2}{m}(1 + 8s\kappa)|\nabla w|_2^2 - \varepsilon \frac{s}{4} \int \frac{(w, \beta w)(x)(w, \beta w)(y)}{|x-y|} + c\varepsilon^2.$$



Since  $Q(w) > 0$ , we deduce that

$$\mathcal{E}_s(w_\varepsilon) < \frac{m}{2}$$

for  $\varepsilon$  small enough and for all  $s \in (0, \frac{1}{8\pi})$ , and hence  $e(s) < \frac{m}{2}$  for all  $s \in (0, \frac{1}{8\pi})$ . ■

PROPOSITION 2.17. For all  $\theta > 1$  and  $s \in (0, \frac{1}{8\pi})$  such that  $\theta s \in (0, \frac{1}{8\pi})$ , we have that

$$e(\theta s) < e(s).$$

PROOF. Let  $\theta > 1$  and  $s \in (0, \frac{1}{8\pi})$  such that  $\theta s \in (0, \frac{1}{8\pi})$ . Take  $w \in \Sigma_+$  and let  $\eta_s(w) \in B_1$  the function whose existence follows from Proposition 2.9. Since it follows from (2.10) that

$$\|a(\eta_{\theta s}(w))w\|^2 - \|\eta_{\theta s}(w)\|^2 - m \geq 0,$$

we have that

$$\begin{aligned} & \theta \left( \mathcal{E}_{\theta s}(w) - \frac{m}{2} \right) \\ &= \theta \left( \frac{1}{2} \|a(\eta_{\theta s}(w))w\|^2 - \frac{1}{2} \|\eta_{\theta s}(w)\|^2 - \frac{m}{2} - \frac{\theta s}{4} \int \frac{(\psi_1, \beta \psi_1)(x)(\psi_1, \beta \psi_1)(y)}{|x-y|} \right) \\ &\leq \theta^2 \left( \frac{1}{2} \|a(\eta_{\theta s}(w))w\|^2 - \frac{1}{2} \|\eta_{\theta s}(w)\|^2 - \frac{m}{2} - \frac{s}{4} \int \frac{(\psi_1, \beta \psi_1)(x)(\psi_1, \beta \psi_1)(y)}{|x-y|} \right) \\ &\leq \theta^2 \left( \frac{1}{2} \|a(\eta_s(w))w\|^2 - \frac{1}{2} \|\eta_s(w)\|^2 - \frac{m}{2} - \frac{s}{4} \int \frac{(\psi_2, \beta \psi_2)(x)(\psi_2, \beta \psi_2)(y)}{|x-y|} \right) \\ &= \theta^2 \left( \mathcal{E}_s(w) - \frac{m}{2} \right) \end{aligned}$$

(here  $\psi_1 = a(\eta_{\theta s}(w))w + \eta_{\theta s}(w)$  and  $\psi_2 = a(\eta_s(w))w + \eta_s(w)$ ). We know that  $e(s) < \frac{m}{2}$  and hence

$$\theta \left( e(\theta s) - \frac{m}{2} \right) \leq \theta^2 \left( e(s) - \frac{m}{2} \right) < \theta \left( e(s) - \frac{m}{2} \right)$$

from which we deduce that  $e(\theta s) < e(s)$ . ■

PROOF OF THEOREM 1.4. By Ekeland's variational principle, there exists a sequence  $w_n \in \Sigma_+$  such that

$$\mathcal{E}_s(w_n) \rightarrow e(s), \quad \sup_{v \in \Sigma_+} |d\mathcal{E}_s(w_n)[v]| \rightarrow 0.$$

From  $\mathcal{E}_s(w_n) \rightarrow e(s)$ , we deduce from Lemma 2.8 that  $\|w_n\| \leq \frac{4e(s)}{2-s\kappa} + o(1)$  so that the sequence  $w_n$  is bounded. It follows from Proposition 2.9 that also  $\eta_n = \eta(w_n)$  and

$\psi_n = a(\eta_n)w_n + \eta_n$  are bounded in  $X$ . Letting  $\omega_n = a(\eta_n)^{-1}dI(\psi_n)[w_n]$ , we have that

$$dI_s(\psi_n)[h] - \omega_n(\psi_n | h) = 0 \quad \text{for all } h \in X.$$

We can assume that (up to a subsequence)  $\psi_n \rightharpoonup \psi$  in  $X$  and that  $\omega_n \rightarrow \omega$ . Then we have that for all  $h \in X$

$$\begin{aligned} & dI_s(\psi_n)[h] - \omega_n(\psi_n | h) \\ &= \langle \psi_n | \Lambda_+ h \rangle - \langle \psi_n | \Lambda_- h \rangle - s \int \frac{(\psi_n, \beta \psi_n)(x)(\psi_n, \beta h)(y)}{|x - y|} - \omega_n(\psi_n | h) \\ &\rightarrow 0, \end{aligned}$$

since, by (1.7), we have that

$$\int \frac{(\psi_n, \beta \psi_n)(x)(\psi_n - \psi, \beta h)(y)}{|x - y|} \rightarrow 0.$$

As a consequence, we have that

$$dI_s(\psi)[h] - \omega(\psi | h) = 0 \quad \text{for all } h \in X.$$

The weak convergence does not, however, preserve the  $L^2$  norm, so we only know that  $|\psi|_2 \leq |\psi_n|_2 = 1$  (it could even be that  $\psi = 0$ ).

To conclude, we will now apply the concentration-compactness principle; see [4, 5]. First of all, let us show that no vanishing occurs. By contradiction, assume that

$$\limsup_{n \rightarrow +\infty} \sup_{y \in \mathbb{R}^3} \int_{B(y, 1)} |\psi_n|^2 = 0.$$

Then we know, see [4] or [8, Lemma 1.21], that  $\psi_n \rightarrow 0$  in  $L^p(\mathbb{R}^3)$  for  $2 < p < 3$ .

Since

$$\int \frac{(\psi_n, \beta \psi_n(x))(\psi_n, \beta \psi_n(y))}{|x - y|} \leq \int \frac{|\psi_n(x)|^2 |\psi_n(y)|^2}{|x - y|} \leq C |\psi_n|_{\frac{12}{5}}^4 \rightarrow 0,$$

we deduce, using (2.10), (2.15), and Lemma 2.16, that

$$\begin{aligned} 0 &= dI_s(\psi_n)[\psi_n] - \omega_n |\psi_n|_2^2 \\ &= \|a(\eta_n)w_n\|^2 - \|\eta_n\|^2 - \omega_n |\psi_n|_2^2 - s \int \frac{(\psi_n, \beta \psi_n(x))(\psi_n, \beta \psi_n(y))}{|x - y|} \\ &= \|a(\eta_n)w_n\|^2 - \|\eta_n\|^2 - m |\psi_n|_2^2 + (m - \omega_n) + o(1) \geq (m - \omega_n) + o(1) > 0 \end{aligned}$$

for  $n$  large enough, a contradiction which shows that vanishing does not occur.

Then we know from the concentration-compactness principle, that there exist  $p \geq 1$  functions  $\phi_1, \dots, \phi_p \in X$ , critical points for  $I_s$  under the constraint  $|\psi|_2^2 = \mu_i$  (hence satisfying (2.14) with  $\omega = \lim_n \omega_n > 0$ ), and  $p$  sequences of points  $x_{i,n} \in \mathbb{R}^3$ ,  $i = 1, \dots, p$  such that  $|x_{i,n} - x_{j,n}| \rightarrow +\infty$  for all  $i \neq j$  as  $n \rightarrow +\infty$  and

$$\left\| \psi_n - \sum_{i=1}^p \phi_i(\cdot - x_{i,n}) \right\| \rightarrow 0 \quad \text{as } n \rightarrow +\infty.$$

From this it follows also that  $|\psi_n|_2^2 = 1 = \sum_{i=1}^p \mu_i$ .

We then observe that

$$\begin{aligned} & \|\Lambda_+ \psi_n\|^2 - \|\Lambda_- \psi_n\|^2 \\ &= \langle \psi_n \mid \Lambda_+ \psi_n - \Lambda_- \psi_n \rangle \\ &= \left\langle \psi_n - \sum_{i=1}^p \phi_i(\cdot - x_{i,n}) \mid \Lambda_+ \psi_n - \Lambda_- \psi_n \right\rangle \\ &\quad + \sum_{i=1}^p \langle \phi_i(\cdot - x_{i,n}) \mid \Lambda_+ \psi_n - \Lambda_- \psi_n \rangle \\ &= \sum_{i=1}^p (\langle \Lambda_+ \phi_i(\cdot - x_{i,n}) \mid \psi_n \rangle - \langle \Lambda_- \phi_i(\cdot - x_{i,n}) \mid \psi_n \rangle) + o(1) \\ &= \sum_{i=1}^p (\|\Lambda_+ \phi_i\|^2 - \|\Lambda_- \phi_i\|^2) + o(1) \end{aligned}$$

and also

$$\int \frac{(\psi_n, \beta \psi_n(x))(\psi_n, \beta \psi_n(y))}{|x - y|} = \sum_{i=1}^p \int \frac{(\phi_i, \beta \phi_i(x))(\phi_i, \beta \phi_i(y))}{|x - y|} + o(1).$$

Finally, we have that

$$(2.18) \quad e(s) = I_s(\psi_n) + o(1) = \sum_{i=1}^p I_s(\phi_i) + o(1).$$

Let, for  $i = 1, \dots, n$ ,  $\tilde{\psi}_i = |\phi_i|_2^{-1} \phi_i = \mu_i^{-1/2} \phi_i \in \Sigma$ . We have that

$$I_s(\phi_i) = I_s(\sqrt{\mu_i} \tilde{\psi}_i) = \mu_i I_{s\mu_i}(\tilde{\psi}_i)$$

and

$$0 = dI_s(\phi_i)[h] - \omega(\phi_i \mid h) = \sqrt{\mu_i} (dI_{s\mu_i}(\tilde{\psi}_i)[h] - \omega(\tilde{\psi}_i \mid h)) \quad \text{for all } h \in X.$$

It follows from Proposition 2.13 that  $\tilde{w}_i = |\Lambda_+ \tilde{\psi}_i|_2^{-1} \Lambda_+ \tilde{\psi}_i \in \Sigma_+$  is a critical point for  $\mathcal{E}_{s\mu_i}$  and  $\mathcal{E}_{s\mu_i}(\tilde{w}_i) = I_{s\mu_i}(\tilde{\psi}_i)$ .

Since

$$\mathcal{E}_{s\mu_i}(\tilde{w}_i) \geq e(s\mu_i),$$

we deduce from Proposition 2.17 that

$$e(s) = \sum_{i=1}^p I_s(\phi_i) = \sum_{i=1}^p \mu_i I_{s\mu_i}(\tilde{\psi}_i) \geq \sum_{i=1}^p \mu_i e(s\mu_i) > \sum_{i=1}^p \mu_i e\left(\frac{1}{\mu_i} s\mu_i\right) = e(s) \sum_{i=1}^p \mu_i,$$

a contradiction if  $p > 1$ .

Since there is no vanishing or dichotomy, our sequence  $\psi_n$  converges strongly in  $X$  to a critical point  $\psi \in X$  of (2.14) such that  $|\psi|_2 = 1$  and the theorem follows.  $\blacksquare$

#### A. A USEFUL LEMMA

This lemma is similar to [6, Lemma 2.9]. We give here a slightly different proof.

LEMMA A.1. *For all  $\psi = \sqrt{1 - |w|_2^2} w + \eta$ ,  $w \in \Sigma_+$ ,  $\eta \in X_-$ , we have*

$$\begin{aligned} \int \frac{(\psi, \beta\psi)(x)(\psi, \beta\psi)(y)}{|x-y|} &\geq \int \frac{(w, \beta w)(x)(w, \beta w)(y)}{|x-y|} - 2\kappa|\eta|_2^2\|w\|^2 \\ &\quad - 14a(\eta)^2\kappa(\|w\|^2 - m|w|_2^2) - 18\kappa\|\eta\|^2. \end{aligned}$$

PROOF. We have

$$\begin{aligned} &\int \frac{(\psi, \beta\psi)(x)(\psi, \beta\psi)(y)}{|x-y|} \\ &= a(\eta)^4 \int \frac{(w, \beta w)(x)(w, \beta w)(y)}{|x-y|} \\ &\quad + 4a(\eta)^3 \int \frac{(w, \beta w)(x)(w, \beta\eta)(y)}{|x-y|} + 3a(\eta)^2 \int \frac{(w, \beta w)(x)(\eta, \beta\eta)(y)}{|x-y|} \\ &\quad + 3a(\eta)^2 \int \frac{(w, \beta\eta)(x)(w, \beta\eta)(y)}{|x-y|} + 4a(\eta) \int \frac{(\eta, \beta\eta)(x)(w, \beta\eta)(y)}{|x-y|} \\ &\quad + \int \frac{(\eta, \beta\eta)(x)(\eta, \beta\eta)(y)}{|x-y|} \\ &\geq a(\eta)^4 \int \frac{(w, \beta w)(x)(w, \beta w)(y)}{|x-y|} + 4a(\eta)^3 \int \frac{(w, \beta w)(x)(w, \beta\eta)(y)}{|x-y|} \\ &\quad - 3a(\eta)^2\kappa\|\eta\|^2 + 4a(\eta) \int \frac{(\eta, \beta\eta)(x)(w, \beta\eta)(y)}{|x-y|}. \end{aligned}$$

We have

$$\begin{aligned}
 & \left| \int \frac{(w, \beta w)(x)(w, \beta \eta)(y)}{|x - y|} dx dt \right| \\
 &= (2\pi)^{3/2} \sqrt{\frac{2}{\pi}} \left| \int \frac{\mathcal{F}[(w, \beta w)] \mathcal{F}[(w, \beta \eta)]}{|p|^2} dp \right| \\
 &\leq (2\pi)^{3/2} \sqrt{\frac{2}{\pi}} |\mathcal{F}[(w, \beta w)]|_\infty \left| \int \frac{|\mathcal{F}[(w, \beta \eta)]|}{|p|^2} dp \right| \\
 &\leq \sqrt{\frac{2}{\pi}} |(w, \beta w)|_1 \left| \int \frac{|\mathcal{F}[(w, \beta \eta)]|}{|p|^2} dp \right| \leq \sqrt{\frac{2}{\pi}} |w|_2^2 \left| \int \frac{|\mathcal{F}[(w, \beta \eta)]|}{|p|^2} dp \right| \\
 &\leq \sqrt{\frac{2}{\pi}} \int \frac{dp}{|p|^2} \left( \frac{1}{(2\pi)^{3/2}} \int |(\hat{w}(p - q), \beta \hat{\eta}(q))| dq \right)
 \end{aligned}$$

and

$$\begin{aligned}
 & \left| \int \frac{(\eta, \beta \eta)(x)(w, \beta \eta)(y)}{|x - y|} dx dt \right| \\
 &\leq |\eta|_2^2 \sqrt{\frac{2}{\pi}} \int \frac{dp}{|p|^2} \left( \frac{1}{(2\pi)^{3/2}} \int |(\hat{w}(p - q), \beta \hat{\eta}(q))| dq \right)
 \end{aligned}$$

so that

$$\begin{aligned}
 & 4a(\eta)^3 \int \frac{(w, \beta w)(x)(w, \beta \eta)(y)}{|x - y|} + 4a(\eta) \int \frac{(\eta, \beta \eta)(x)(w, \beta \eta)(y)}{|x - y|} \\
 &\leq 4a(\eta) \sqrt{\frac{2}{\pi}} \int \frac{dp}{|p|^2} \left( \frac{1}{(2\pi)^{3/2}} \int |(\hat{w}(p - q), \beta \hat{\eta}(q))| dq \right).
 \end{aligned}$$

Since

$$\begin{aligned}
 (\hat{w}(p - q), \beta \hat{\eta}(q)) &= (\hat{\Lambda}_+(p - q) \hat{w}(p - q), \beta \hat{\Lambda}_-(q) \hat{\eta}(q)) \\
 &= (\hat{w}(p - q), \hat{\Lambda}_+(p - q) \beta \hat{\Lambda}_-(q) \hat{\eta}(q)),
 \end{aligned}$$

we compute

$$\begin{aligned}
 & 4\hat{\Lambda}_+(p - q) \beta \hat{\Lambda}_-(q) \\
 &= \left( 1 + \frac{m\beta}{\lambda(p - q)} + \frac{\alpha \cdot (p - q)}{\lambda(p - q)} \right) \beta \left( 1 - \frac{m\beta}{\lambda(q)} - \frac{\alpha \cdot q}{\lambda(q)} \right) \\
 &= \beta \left( 1 - \frac{m^2}{\lambda(q)\lambda(p - q)} \right) - i \left( \frac{m}{\lambda(q)} - \frac{m}{\lambda(p - q)} \right) \\
 &\quad - \beta \alpha \cdot \left( \frac{q}{\lambda(q)} + \frac{p - q}{\lambda(p - q)} \right) - \frac{m\alpha \cdot (q + (p - q))}{\lambda(q)\lambda(p - q)} \\
 &\quad + \frac{\beta}{\lambda(q)\lambda(p - q)} (\alpha \cdot (p - q) \alpha \cdot q)
 \end{aligned}$$

$$\begin{aligned}
&= \beta \left( 1 - \frac{m^2}{\lambda(q)\lambda(p-q)} \right) - \left| \frac{m}{\lambda(q)} - \frac{m}{\lambda(p-q)} \right| \\
&\quad - \beta \alpha \cdot \left( \frac{q}{\lambda(q)} + \frac{p-q}{\lambda(p-q)} \right) - \frac{m\alpha \cdot p}{\lambda(q)\lambda(p-q)} + \frac{\beta \Sigma \cdot (p-q) \Sigma \cdot q}{\lambda(q)\lambda(p-q)},
\end{aligned}$$

where

$$\Sigma_i = \begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix}.$$

We now estimate the different terms. First of all, from

$$m(|p-q| + |q|) \leq \lambda(q)\lambda(p-q) \leq |q||p-q| + m(|q| + |p-q|) + m^2$$

and

$$| |q|\lambda(p-q) - |p-q|\lambda(q) | \leq m|p|,$$

we deduce that

$$\left| \frac{|q|}{\lambda(q)} - \frac{|p-q|}{\lambda(p-q)} \right| = \left| \frac{|q|\lambda(p-q) - |p-q|\lambda(q)}{\lambda(q)\lambda(p-q)} \right| \leq \frac{m|p|}{\lambda(q)\lambda(p-q)}$$

and

$$\begin{aligned}
\left| \frac{m}{\lambda(q)} - \frac{m}{\lambda(p-q)} \right| &= m \frac{|\lambda(p-q) - \lambda(q)|}{\lambda(q)\lambda(p-q)} = m \frac{|m^2 + |p-q|^2 - m^2 - |q|^2|}{\lambda(q)\lambda(p-q)(\lambda(q) + \lambda(p-q))} \\
&\leq \frac{||p-q| - |q||}{\lambda(q) + \lambda(p-q)} \leq \frac{|p|}{\lambda(q) + \lambda(p-q)} \\
&\leq \frac{|p|}{(\lambda(q) + m)^{1/2} (\lambda(p-q) + m)^{1/2}}.
\end{aligned}$$

Then

$$\begin{aligned}
\left| \frac{\lambda(q)\lambda(p-q) - m^2}{\lambda(q)\lambda(p-q)} \right| &\leq \frac{|q||p-q| + m(|q| + |p-q|)}{\lambda(q)\lambda(p-q)} \\
&\leq \frac{|q||p-q|}{\lambda(q)\lambda(p-q)} + \frac{m|p|}{\lambda(q)\lambda(p-q)} + 2 \frac{m|p-q|}{\lambda(q)\lambda(p-q)}
\end{aligned}$$

and

$$\begin{aligned}
\left| \frac{q}{\lambda(q)} + \frac{p-q}{\lambda(p-q)} \right| &\leq \left| \frac{|q|}{\lambda(q)} - \frac{|p-q|}{\lambda(p-q)} \right| + 2 \frac{|p-q|}{\lambda(p-q)} \\
&\leq \frac{m|p|}{\lambda(q)\lambda(p-q)} + 2 \frac{|p-q|}{\lambda(p-q)}.
\end{aligned}$$

Since

$$\left| \frac{\beta \Sigma \cdot (p - q) \Sigma \cdot q}{\lambda(q) \lambda(p - q)} \right| \leq \frac{|q| |p - q|}{\lambda(q) \lambda(p - q)},$$

we finally have

$$\begin{aligned} & 4|(\hat{w}(p - q), \beta \hat{\eta}(q))| \\ & \leq \left( \frac{3|q| |p - q| + 3m|p| + 2m|p - q|}{\lambda(q) \lambda(p - q)} + \frac{2|p - q|}{\lambda(p - q)} \right) |\hat{w}(p - q)| |\hat{\eta}(q)| \\ & \quad + \frac{|p|}{(\lambda(q) + m)^{1/2} (\lambda(p - q) + m)^{1/2}} |\hat{w}(p - q)| |\hat{\eta}(q)|. \end{aligned}$$

Let us analyze the different terms:

$$\begin{aligned} & \sqrt{\frac{2}{\pi}} \int \frac{dp}{|p|^2} \left( \frac{1}{(2\pi)^{3/2}} \int \frac{|p - q| |\hat{w}(p - q)| |q| |\hat{\eta}(q)|}{\lambda(p - q) \lambda(q)} dq \right) \\ & = \sqrt{\frac{2}{\pi}} \int \frac{1}{|p|^2} \mathcal{F} \left[ \mathcal{F}^{-1} \left[ \frac{|p| |\hat{w}(p)|}{\lambda(p)} \right] \mathcal{F}^{-1} \left[ \frac{|p| |\hat{\eta}(p)|}{\lambda(p)} \right] \right] dp \\ & = \int \frac{1}{|x|} \mathcal{F}^{-1} \left[ \frac{|p| |\hat{w}(p)|}{\lambda(p)} \right] \mathcal{F}^{-1} \left[ \frac{|p| |\hat{\eta}(p)|}{\lambda(p)} \right] dx \\ & \leq \left\| \frac{1}{|x|^{1/2}} \mathcal{F}^{-1} \left[ \frac{|p| |\hat{w}(p)|}{\lambda(p)} \right] \right\|_2 \left\| \frac{1}{|x|^{1/2}} \mathcal{F}^{-1} \left[ \frac{|p| |\hat{\eta}(p)|}{\lambda(p)} \right] \right\|_2 \\ & \leq \kappa \left\| (-\Delta)^{1/4} \mathcal{F}^{-1} \left[ \frac{|p| |\hat{w}(p)|}{\lambda(p)} \right] \right\|_2 \left\| (-\Delta)^{1/4} \mathcal{F}^{-1} \left[ \frac{|p| |\hat{\eta}(p)|}{\lambda(p)} \right] \right\|_2 \\ & \leq 2\kappa \sqrt{\|w\|^2 - m|w|_2^2} \sqrt{\|\eta\|^2 - m|\eta|_2^2} \end{aligned}$$

since

$$\begin{aligned} & \left\| (-\Delta)^{1/4} \mathcal{F}^{-1} \left[ \frac{|p| |\hat{w}(p)|}{\lambda(p)} \right] \right\|_2^2 = \int \frac{|p|^3 |\hat{w}(p)|^2}{\lambda(p)^2} dp \\ & \leq 2 \int (\sqrt{|p|^2 + m^2} - m) |\hat{w}(p)|^2 dp = 2(\|w\|^2 - m|w|_2^2) \end{aligned}$$

and

$$\begin{aligned} & \sqrt{\frac{2}{\pi}} \int \frac{dp}{|p|^2} \left( \frac{1}{(2\pi)^{3/2}} \int \frac{|p - q| |\hat{w}(p - q)| |m| |\hat{w}(q)|}{\lambda(p - q) \lambda(q)} dq \right) \\ & \leq \kappa \left\| (-\Delta)^{1/4} \mathcal{F}^{-1} \left[ \frac{|p| |\hat{w}(p)|}{\lambda(p)} \right] \right\|_2 \left\| (-\Delta)^{1/4} \mathcal{F}^{-1} \left[ \frac{m |\hat{\eta}(p)|}{\lambda(p)} \right] \right\|_2 \\ & \leq \sqrt{m} \kappa |\eta|_2 \sqrt{\|w\|^2 - m|w|_2^2} \end{aligned}$$

since

$$\left| (-\Delta)^{1/4} \mathcal{F}^{-1} \left[ \frac{m|\hat{\eta}(p)|}{\lambda(p)} \right] \right|_2^2 = \int \frac{m^2 |p| |\hat{\eta}(p)|^2}{\lambda(p)^2} dp \leq \frac{m}{2} |\eta|_2^2.$$

We also have that

$$\begin{aligned} & \sqrt{\frac{2}{\pi}} \int \frac{dp}{|p|^2} \left( \frac{1}{(2\pi)^{3/2}} \int |p| \frac{|\hat{w}(p-q)|}{(\lambda(p-q)+m)^{1/2}} \frac{|\hat{\eta}(q)|}{(\lambda(q)+m)^{1/2}} dq \right) \\ &= \sqrt{\frac{2}{\pi}} \int \frac{1}{|p|} \mathcal{F} \left[ \mathcal{F}^{-1} \left[ \frac{|\hat{w}(p)|}{(\lambda(p)+m)^{1/2}} \right] \mathcal{F}^{-1} \left[ \frac{|p| |\hat{\eta}(p)|}{(\lambda(p)+m)^{1/2}} \right] \right] dp \\ &= \frac{2}{\pi} \int \frac{1}{|x|^2} \mathcal{F}^{-1} \left[ \frac{|\hat{w}(p)|}{(\lambda(p)+m)^{1/2}} \right] \mathcal{F}^{-1} \left[ \frac{|\hat{\eta}(p)|}{(\lambda(p)+m)^{1/2}} \right] dx \\ &\leq \frac{2}{\pi} \left| \frac{1}{|x|} \mathcal{F}^{-1} \left[ \frac{|\hat{w}(p)|}{(\lambda(p)+m)^{1/2}} \right] \right|_2 \left| \frac{1}{|x|^{1/2}} \mathcal{F}^{-1} \left[ \frac{|\hat{\eta}(p)|}{(\lambda(p)+m)^{1/2}} \right] \right|_2 \\ &\leq \frac{8}{\pi} \left| \frac{|p| |\hat{w}(p)|}{(\lambda(p)+m)^{1/2}} \right|_2 \left| \frac{|p| |\hat{\eta}(p)|}{(\lambda(p)+m)^{1/2}} \right|_2 \\ &\leq 2\kappa \sqrt{\|w\|^2 - m|w|_2^2} \sqrt{\|\eta\|^2 - m|\eta|_2^2} \end{aligned}$$

and

$$\begin{aligned} & \sqrt{\frac{2}{\pi}} \int \frac{dp}{|p|^2} \left( \frac{1}{(2\pi)^{3/2}} \int m|p| \frac{|\hat{w}(p-q)|}{\lambda(p-q)} \frac{|\hat{\eta}(q)|}{\lambda(q)} dq \right) \\ &\leq \frac{8}{\pi} \left| \frac{\sqrt{m}|p| |\hat{w}(p)|}{\lambda(p)} \right|_2 \left| \frac{\sqrt{m}|p| |\hat{\eta}(p)|}{\lambda(p)} \right|_2 \\ &\leq 5\kappa \sqrt{\|w\|^2 - m|w|_2^2} \sqrt{\|\eta\|^2 - m|\eta|_2^2} \end{aligned}$$

(we have used the fact that  $\frac{mp^2}{p^2+m^2} \leq \frac{5}{2}(\sqrt{p^2+m^2}-m)$ ).

Finally, we have

$$\begin{aligned} & \sqrt{\frac{2}{\pi}} \int \frac{dp}{|p|^2} \left( \frac{1}{(2\pi)^{3/2}} \int \frac{|p-q| |\hat{w}(p-q)| |\hat{w}(q)|}{\lambda(p-q)} dq \right) \\ &\leq \kappa \left| (-\Delta)^{1/4} \mathcal{F}^{-1} \left[ \frac{|p| |\hat{w}(p)|}{\lambda(p)} \right] \right|_2 \left| (-\Delta)^{1/4} \mathcal{F}^{-1} [|\hat{\eta}(p)|] \right|_2 \\ &\leq \sqrt{2}\kappa \|\eta\| \sqrt{\|w\|^2 - m|w|_2^2}. \end{aligned}$$



We now collect the terms:

$$\begin{aligned}
 & 4a(\eta) \sqrt{\frac{2}{\pi}} \int \frac{dp}{|p|^2} \left( \frac{1}{(2\pi)^{3/2}} \int |(\hat{w}(p-q), \beta \hat{\eta}(q))| dq \right) \\
 & \leq 23\kappa a(\eta) \sqrt{\|w\|^2 - m|w|_2^2} \sqrt{\|\eta\|^2 - m|\eta|_2^2} \\
 & \quad + 2\sqrt{m}\kappa a(\eta) |\eta|_2 \sqrt{\|w\|^2 - m|w|_2^2} + 2\sqrt{2}\kappa a(\eta) \|\eta\| \sqrt{\|w\|^2 - m|w|_2^2} \\
 & \leq 14a(\eta)^2 \kappa (\|w\|^2 - m|w|_2^2) + 15\kappa \|\eta\|^2
 \end{aligned}$$

to deduce that

$$\begin{aligned}
 & \int \frac{(\psi, \beta \psi)(x)(\psi, \beta \psi)(y)}{|x-y|} - \int \frac{(w, \beta w)(x)(w, \beta w)(y)}{|x-y|} \\
 & \geq -2\kappa |\eta|_2^2 \|w\|^2 - 3a(\eta)^2 \kappa \|\eta\|^2 \\
 & \quad - 4a(\eta) \sqrt{\frac{2}{\pi}} \int \frac{dp}{|p|^2} \left( \frac{1}{(2\pi)^{3/2}} \int |(\hat{w}(p-q), \beta \hat{\eta}(q))| dq \right) \\
 & \geq -14a(\eta)^2 \kappa (\|w\|^2 - m|w|_2^2) - 2\kappa |\eta|_2^2 \|w\|^2 - 18\kappa \|\eta\|^2. \quad \blacksquare
 \end{aligned}$$

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