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Partial Differential Equations. – *Normalized solutions for the Klein–Gordon–Dirac system*, by Vittorio Coti Zelati and Margherita Nolasco, communicated on 10 November 2022.

Dedicated to Antonio Ambrosetti, maestro and friend.

Abstract. – We prove the existence of a stationary solution for the system describing the interaction between an electron coupled with a massless scalar field (a photon). We find a solution, with fixed L^2-norm, by variational methods, as a critical point of an energy functional.

Keywords. – Klein–Gordon–Dirac, critical point theory, min-max methods, nonlinear eigenvalue.

2020 Mathematics Subject Classification. – Primary 35Q40; Secondary 81Q05, 35P30, 47J10, 49J35.

1. Introduction

We study the interaction electron-photon analyzing the Euler–Lagrange equations for a system consisting of a spinor field coupled with a massless scalar field. More precisely, our system consists of the Dirac equation coupled with a massless Klein–Gordon equation, and looks for normalized and stationary solutions of the system

$$\begin{cases} (-i\gamma^\mu \partial_\mu + m - \sqrt{s}\varphi)\psi = 0 & \text{in } \mathbb{R}\times\mathbb{R}^3, \\ \partial^\mu \partial_\mu \varphi = 4\pi\sqrt{s}(\psi, \beta\psi) & \text{in } \mathbb{R}\times\mathbb{R}^3, \end{cases} \tag{1.1}$$

where $\psi\colon \mathbb{R}\times\mathbb{R}^3 \to \mathbb{C}^4$, $\varphi\colon \mathbb{R}\times\mathbb{R}^3 \to \mathbb{R}$, $m > 0$ is the mass of the electron, $\sqrt{s} > 0$ is the coupling constant, γ^μ are the 4×4 Dirac matrices

$$\gamma^0 = \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \gamma^k = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix}, \quad k = 1, \dots, 3,$$

σ^k are the 2×2 Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and $(z, w) = \sum_{i=1}^4 \bar{z}_i w_i$, the scalar product between $z, w \in \mathbb{C}^4$.

This problem is closely related to the one studied in [6], and we will prove the existence of a solution of (1.1) with essentially the same methods developed in that article (see also [2]).

More precisely, we prove the existence of stationary, normalized solutions of this system, that is solutions (ω, ψ) of the problem

$$
\begin{cases}
(-i\boldsymbol{\alpha}\cdot\nabla + m\beta - \sqrt{s}\varphi\beta)\psi = \omega\psi & \text{in } \mathbb{R}^3,\\
-\Delta\varphi = 4\pi\sqrt{s}(\psi, \beta\psi) & \text{in } \mathbb{R}^3,\\
|\psi|_2^2 = \int_{\mathbb{R}^3} \left|\psi(t,x)\right|^2 dx = 1,
\end{cases}
\tag{1.2}
$$

where $\alpha_i = \beta\gamma_i, i = 1, \ldots, 3$. From $-\Delta\varphi = 4\pi\sqrt{s}(\psi, \beta\psi)$, we deduce that

$$
\varphi = \sqrt{s}(\psi, \beta\psi) * \frac{1}{|x|}
$$

and hence our problem reduces to

$$
\begin{cases}
\left(-i\boldsymbol{\alpha}\cdot\nabla + m\beta - s(\psi, \beta\psi) * \frac{1}{|x|}\beta\right)\psi = \omega\psi & \text{in } \mathbb{R}^3,\\
|\psi|_2^2 = \int_{\mathbb{R}^3} \left|\psi(t,x)\right|^2 dx = 1.
\end{cases}
\tag{1.3}
$$

Our result is the following theorem.

THEOREM 1.4. *For all $s \in (0, \frac{1}{8\pi})$, there exists $\omega \in (0, m)$ and $\psi \in H^{1/2}(\mathbb{R}^3, \mathbb{C}^4)$ solutions of problem* (1.3).

In the article [3], the authors prove using critical point theory the existence of one stationary solution of equation (1.1) but do not prescribe its L^2-norm.

We will find such a solution as a critical point of the functional

$$
I(\psi) = \frac{1}{2}\int_{\mathbb{R}^3} (H\psi, \psi) - \frac{s}{4}\int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{(\psi, \beta\psi)(x)(\psi, \beta\psi)(y)}{|x-y|}
$$

restricted on the manifold $|\psi|_2^2 = 1$. Here

$$
H = -i\boldsymbol{\alpha}\cdot\nabla + m\beta.
$$

The functional I is strongly indefinite, and, following the method introduced in [2, 6], the solution will be found via a min-max procedure consisting in minimizing the supremum of I over subspaces of dimension 1 in the positive energy subspace of the linear operator H; see Proposition 2.13. Let us remark here that we know very few results on the existence of *normalized* solutions for Dirac's equation (and more generally for strongly indefinite problems – one of these is [1]).

1.1. *Notation and background results*

We let $|u|_p^p = \int_{\mathbb{R}^3} |u(x)|^p$, $(u \mid v) = \int_{\mathbb{R}^3} u(x)v(x)$.

Let us recall some well-known facts on the Dirac operator H (see [7] for more details): H is a first order, self-adjoint operator on $H^1(\mathbb{R}^3, \mathbb{C}^4)$ with purely absolutely continuous spectrum given by

$$\sigma(H) = (-\infty, -m] \cup [m, +\infty).$$

The orthogonal projectors $\Lambda_\pm$ on the positive and negative energies subspaces are such that

$$H\Lambda_\pm = \Lambda_\pm H = \pm\sqrt{-\Delta + m^2}\Lambda_\pm = \pm\Lambda_\pm\sqrt{-\Delta + m^2}$$

and hence

$$\int \big(\psi(x), H\psi(x)\big)\, dx = \big|(-\Delta + m^2)^{1/4}\Lambda_+\psi\big|_2^2 - \big|(-\Delta + m^2)^{1/4}\Lambda_-\psi\big|_2^2.$$

We will denote $X = H^{1/2}(\mathbb{R}^3, \mathbb{C}^4)$, $X_\pm = \Lambda_\pm X$, $\Sigma = \{\psi \in X \mid |\psi|_2 = 1\}$, and $\Sigma_\pm = \{\psi \in X_\pm \mid |\psi|_2 = 1\}$.

We have also that

$$\begin{aligned}
\hat{H} &= \mathcal{F} H \mathcal{F}^{-1} = \boldsymbol{\alpha} \cdot p + m\beta,\\
U\hat{H}U^{-1} &= \lambda(p)\beta,\\
\hat{\Lambda}_\pm &= \mathcal{F}\Lambda_\pm\mathcal{F}^{-1} = U^{-1}\left(\frac{\mathsf{I} \pm \beta}{2}\right)U = \frac{1}{2}\left(\mathsf{I} \pm \frac{m}{\lambda(p)}\beta \pm \frac{1}{\lambda(p)}\boldsymbol{\alpha} \cdot p\right),
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{F}\psi(p) &= \hat{\psi}(p)\left(= \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \mathrm{e}^{-ipx}\, \psi(x)\, dx \text{ for all } v \in \mathcal{S}(\mathbb{R}^3)\right),\\
\lambda(p) &= \sqrt{|p|^2 + m^2},\\
U &= u_+(p)\mathsf{I} + u_-(p)\beta\frac{\boldsymbol{\alpha} \cdot p}{|p|},\\
U^{-1} &= u_+(p)\mathsf{I} - u_-(p)\beta\frac{\boldsymbol{\alpha} \cdot p}{|p|},\\
u_\pm(p) &= \sqrt{\frac{1}{2}\left(1 \pm \frac{m}{\lambda(p)}\right)}.
\end{aligned}$$

Let, for ϕ and $\psi \in H^{1/2}(\mathbb{R}^3, C^4)$,

$$\langle \phi \mid \psi \rangle = \int \sqrt{|p|^2 + m^2}\big(\hat{\phi}(p), \hat{\psi}(p)\big)\, dp$$

and

$$\|\psi\|^2 = \langle \psi \mid \psi \rangle.$$

We have that

$$\langle \Lambda_+\phi \mid \Lambda_-\psi \rangle = (\Lambda_+\phi \mid \Lambda_-\psi) = 0.$$

Let us recall that, since $\mathcal{F}\frac{1}{|x|} = \sqrt{\frac{2}{\pi}}\frac{1}{|p|^2}$, for all $f \in L^1(\mathbb{R}^3) \cap L^{3/2}(\mathbb{R}^3)$

$$\int \frac{f(x)\bar{f}(y)}{|x-y|} = 4\pi \int \frac{\left|\hat{f}(p)\right|}{|p|^2} \geq 0 \tag{1.5}$$

and that for all $\rho \in L^1(\mathbb{R}^3)$, $\psi \in H^{1/2}(\mathbb{R}^3, \mathbb{C}^4)$

$$\int \frac{\rho(x)|\psi|^2(y)}{|x-y|} \leq \kappa|\rho|_1\left|(-\Delta)^{1/4}\psi\right|_2^2 \leq \kappa|\rho|_1\|\psi\|^2 \tag{1.6}$$

($\kappa = \frac{\pi}{2}$) and also that

$$\int \frac{|f_n|(x)|g_n|(x)|h_n|(y)|v|(y)}{|x-y|} \to 0 \tag{1.7}$$

when $f_n, g_n. h_n$ and $v \in H^{1/2}$, f_n, g_n bounded, $h_n \rightharpoonup 0$.

2. Maximization

Let $I\colon H^{1/2}(\mathbb{R}^3, \mathbb{C}^4) \to \mathbb{R}$

$$I(\psi) = \frac{1}{2}\|\Lambda_+\psi\|^2 - \frac{1}{2}\|\Lambda_-\psi\|^2 - \frac{s}{4}\int \frac{(\psi, \beta\psi)(x)(\psi, \beta\psi)(y)}{|x-y|}.$$

Let us fix $w \in \Sigma_+$ and let

$$B_1 = \left\{\eta \in X_- \mid |\eta|_2 < 1\right\}.$$

We will look, given w, for a maximizer of the functional J_w defined on B_1,

$$\begin{aligned} J_w(\eta) &= I\big(a(\eta)w + \eta\big) \\ &= \frac{1}{2}\left\|a(\eta)w\right\|^2 - \frac{1}{2}\|\eta\|^2 - \frac{s}{4}\int \frac{(\psi, \beta\psi)(x)(\psi, \beta\psi)(y)}{|x-y|}, \end{aligned}$$

where $a(\eta) = \sqrt{1 - |\eta|_2^2}$ and $\psi = a(\eta)w + \eta \in \Sigma$.

We have that $da(\eta)[\xi] = -a(\eta)^{-1}(\eta \mid \xi)$ and hence the derivative of J_w is given, for all $\xi \in X_-$, by

$$dJ_w(\eta)[\xi] = dI\big(a(\eta)w + \eta\big)\big[da(\eta)[\xi]w + \xi\big] = dI(\psi)[h] \tag{2.1}$$
$$= \big\langle a(\eta)w \mid da(\eta)[\xi]w\big\rangle - \langle \eta \mid \xi\rangle - s\int \frac{(\psi,\beta\psi)(x)(\psi,\beta h)(y)}{|x-y|}$$
$$= -(\eta \mid \xi)\|w\|^2 - \langle \eta \mid \xi\rangle - s\int \frac{(\psi,\beta\psi)(x)(\psi,\beta h)(y)}{|x-y|}$$

(here $h = da(\eta)[\xi]w + \xi$) and we have, in particular,

$$dJ_w(\eta)[\eta] = -|\eta|_2^2\|w\|^2 - \|\eta\|^2 - s\int \frac{(\psi,\beta\psi)(x)\big(\psi,\beta\big(da(\eta)[\eta]w + \eta\big)\big)(y)}{|x-y|}.$$

Lemma 2.2. *For all $w \in \Sigma_+$ and $\eta \in B_1$, we have*

$$\|\eta\|^2 \le a(\eta)^2\|w\|^2 - 2J_w(\eta), \tag{2.3}$$

and for all $\eta \in B_1$ such that $|\eta|_2^2 \ge \frac{1}{2}$ and $J_w(\eta) \ge 0$, we have that

$$dJ_w(\eta)[\eta] \le -\frac{1}{2}(1-4s\kappa)m < 0, \tag{2.4}$$

Proof. We have, thanks to (1.5), that for $\eta \in B_1$ and $\psi = a(\eta)w + \eta$,

$$\frac{1}{2}\|\eta\|^2 \le \frac{1}{2}\|\eta\|^2 + \frac{s}{4}\int \frac{(\psi,\beta\psi)(x)(\psi,\beta\psi)(y)}{|x-y|} = \frac{1}{2}\big\|a(\eta)w\big\|^2 - J_w(\eta)$$

and (2.3) follows.

From (2.3) it follows that $\|\eta\| \le a(\eta)\|w\|$ if $J_w(\eta) \ge 0$; hence we have, if $|\eta|_2^2 > \frac{1}{2}$,

$$\begin{aligned} dJ_w(\eta)[\eta] &= -|\eta|_2^2\|w\|^2 - \|\eta\|^2 - s\int \frac{(\psi,\beta\psi)(x)(\psi,\beta\psi)(y)}{|x-y|} \\ &\quad + s\int \frac{(\psi,\beta\psi)(x)(w,\beta w)(y)}{|x-y|} + sa(\eta)^{-1}\int \frac{(\psi,\beta\psi)(x)(\eta,\beta w)(y)}{|x-y|} \\ &\le -|\eta|_2^2\|w\|^2 - \|\eta\|^2 + s\kappa\big(\|w\|^2 + a(\eta)^{-1}\|\eta\|\|w\|\big) \\ &\le -\frac{1}{2}\|w\|^2 - \|\eta\|^2 + 2s\kappa\|w\|^2 < -\frac{1}{2}(1-4s\kappa)\|w\|^2 \\ &\le -\frac{1}{2}(1-4s\kappa)m|w|_2^2 = -\frac{1}{2}(1-4s\kappa)m, \end{aligned}$$

where we have used (1.5) and (1.6). ∎

Remark 2.5. It follows from Lemma 2.2 that if η_n is a Palais–Smale sequence for J_w such that $J_w(\eta_n) \ge 0$, then $|\eta_n|_2^2 < \frac{1}{2}$ for all $n \in \mathbb{N}$ large enough.

LEMMA 2.6. *Let $\eta_n \in B_1$ be a Palais–Smale sequence for J_w, that is*

$$J_w(\eta_n) \to c \geq 0, \quad dJ_w(\eta_n) \to 0.$$

Then η_n converges, up to a subsequence, to a critical point η of J_w.

PROOF. It follows from Lemma 2.2 and Remark 2.5 that $|\eta_n|_2^2 < \frac{1}{2}$ and that $\|\eta_n\|$ is bounded; hence $\eta_n \rightharpoonup \eta$ (up to a subsequence).

From

$$o(1) = dJ_w(\eta_n)[\eta_n - \eta] = -(\eta_n \mid \eta_n - \eta)\|w\|^2 - \langle \eta_n \mid \eta_n - \eta \rangle \\ - s\int \frac{(\psi_n, \beta\psi_n)(x)\big(\psi_n, \beta\big(-a(\eta_n)^{-1}(\eta_n \mid \eta_n - \eta)w + \eta_n - \eta\big)\big)(y)}{|x-y|},$$

we deduce that

$$\begin{aligned} &|\eta_n - \eta|_2^2\|w\|^2 + \|\eta_n - \eta\|^2 \\ &\quad = -(\eta \mid \eta_n - \eta)\|w\|^2 - \langle \eta \mid \eta_n - \eta\rangle \\ &\qquad + sa(\eta_n)^{-1}(\eta_n \mid \eta_n - \eta)\int \frac{(\psi_n, \beta\psi_n)(x)(\psi_n, \beta w)(y)}{|x-y|} \\ &\qquad - s\int \frac{(\psi_n, \beta\psi_n)(x)\big(\psi_n, \beta(\eta_n - \eta)\big)(y)}{|x-y|} + o(1). \end{aligned}$$

We have that

$$\begin{aligned} &\int \frac{(\psi_n, \beta\psi_n)(x)\big(\psi_n, \beta(\eta_n - \eta)\big)(y)}{|x-y|} \\ &= \int \frac{\big(\psi_n, \beta(\eta_n - \eta)\big)(x)\big(\psi_n, \beta(\eta_n - \eta)\big)(y)}{|x-y|} + \int \frac{(\psi_n, \beta\eta)(x)\big(\psi_n, \beta(\eta_n - \eta)\big)(y)}{|x-y|} \\ &\quad + a(\eta_n)\int \frac{(\psi_n, \beta w)(x)\big(\psi_n, \beta(\eta_n - \eta)\big)(y)}{|x-y|} \\ &= \int \frac{\big(\psi_n, \beta(\eta_n - \eta)\big)(x)\big(\psi_n, \beta(\eta_n - \eta)\big)(y)}{|x-y|} + o(1) \end{aligned}$$

and

$$\begin{aligned} &\left|\int \frac{(\psi_n, \beta\psi_n)(x)\big(\psi_n, \beta a(\eta_n)w\big)(y)}{|x-y|}\right| \\ &\quad \leq \kappa\|\psi_n\|\big\|a(\eta_n)w\big\| \leq \kappa\big(2\big\|a(\eta_n)w\big\|^2 + \|\eta_n\|^2\big) \leq 3\kappa a(\eta_n)^2\|w\|^2. \end{aligned}$$

Since $|\eta_n|_2^2 < \frac{1}{2}$, we have

$$a(\eta_n)^{-2}(\eta_n \mid \eta_n - \eta) = a(\eta_n)^{-2}|\eta_n - \eta|_2^2 + o(1)$$

and we deduce that

$$\begin{aligned}
&|\eta_n-\eta|_2^2\|w\|^2+\|\eta_n-\eta\|^2\\
&\quad\le 3s\kappa|\eta_n-\eta|_2^2\|w\|^2-s\int\frac{\big(\psi_n,\beta(\eta_n-\eta)\big)(x)\big(\psi_n,\beta(\eta_n-\eta)\big)(y)}{|x-y|}+o(1)\\
&\quad\le 3s\kappa|\eta_n-\eta|_2^2\|w\|^2+o(1)
\end{aligned}$$

and $\eta_n\to\eta$, with η critical point of J_w. ■

We now show that all the critical points of J_w at positive levels are strict local maxima. This lemma follows as in [2, 6].

Lemma 2.7. *Let $\eta\in X_-$ a critical point of J_w such that $J_w(\eta)\ge 0$.*
Then there exists $\delta>0$ such that

$$d^2J_w(\eta)[\xi,\xi]\le-\delta\|\xi\|^2\quad\textit{for all } \xi\in X_-.$$

Proof. In order to compute the second derivative we denote $\psi=a(\eta)w+\eta$ and $h=da(\eta)[\xi]w+\xi$ and observe that

$$d^2a(\eta)[\xi,\zeta]=-a(\eta)^{-1}\left((\xi\mid\zeta)+\frac{(\eta\mid\xi)(\eta,\zeta)}{1-|\eta|_2^2}\right).$$

Then

$$\begin{aligned}
&d^2J_w(\eta)[\xi,\xi]\\
&\quad=d^2I(\psi)[h,h]+dI(\psi)\big[d^2a(\eta)[\xi,\xi]w\big]\\
&\quad=\big\langle da(\eta)[\xi]w\mid da(\eta)[\xi]w\big\rangle-\langle\xi\mid\xi\rangle-s\int\frac{(\psi,\beta\psi)(x)(h,\beta h)(y)}{|x-y|}\\
&\qquad-2s\int\frac{(\psi,\beta h)(x)(\psi,\beta h)(y)}{|x-y|}+a(\eta)d^2a(\eta)[\xi,\xi]\langle w\mid w\rangle\\
&\qquad-s\int\frac{(\psi,\beta\psi)(x)\big(\psi,\beta d^2a(\eta)[\xi.\xi]w\big)(y)}{|x-y|}\\
&\quad=-|\xi|_2^2\|w\|^2+s|\xi|_2^2\int\frac{(\psi,\beta\psi)(x)(w,\beta w)(y)}{|x-y|}-\|\xi\|^2\\
&\qquad-s\int\frac{(\psi,\beta\psi)(x)(\xi,\beta\xi)(y)}{|x-y|}+2s\frac{(\eta\mid\xi)}{1-|\eta|_2^2}a(\eta)\Gamma(\xi)\\
&\qquad-2s\int\frac{(\psi,\beta h)(x)(\psi,\beta h)(y)}{|x-y|}+s\left(\frac{|\xi|_2^2}{1-|\eta|_2^2}+\frac{(\eta\mid\xi)^2}{\big(1-|\eta|_2^2\big)^2}\right)a(\eta)\Gamma(\eta),
\end{aligned}$$

where we have set

$$\Gamma(\zeta) = \int \frac{(\psi, \beta\psi)(x)(w, \beta\zeta)(y)}{|x-y|}.$$

Since η is a critical point for J_w, we have that

$$\begin{aligned} d^2 J_w(\eta)[\xi,\xi] &= d^2 J_w(\eta)[\xi,\xi] + 2\frac{(\eta \mid \xi)}{1-|\eta|_2^2} dJ_w(\eta)[\xi] \\ &\quad + \left(\frac{|\xi|_2^2}{1-|\eta|_2^2} + 3\frac{(\eta,\xi)^2}{\left(1-|\eta|_2^2\right)^2}\right) dJ_w(\eta)[\eta]. \end{aligned}$$

We have that

$$\begin{aligned} 0 = dJ_w(\eta)[\eta] &= -|\eta|_2^2 \|w\|^2 - \|\eta\|^2 + s\frac{|\eta|_2^2}{1-|\eta|_2^2} \int \frac{(\psi,\beta\psi)(x)(w,\beta w)(y)}{|x-y|} \\ &\quad - s\int \frac{(\psi,\beta\psi)(x)(\eta,\beta\eta)(y)}{|x-y|} - s\left(\frac{1-2|\eta|_2^2}{1-|\eta|_2^2}\right) a\Gamma(\eta) \\ &\le -|\eta|_2^2(1-\kappa)\|w\| - (1-\kappa)\|\eta\| - s\left(\frac{1-2|\eta|_2^2}{1-|\eta|_2^2}\right) a\Gamma(\eta) \end{aligned}$$

which implies that $\Gamma(\eta) < 0$. After some simplification, we get

$$\begin{aligned} d^2 J_w(\eta)[\xi,\xi] &\le -Q(1-s\kappa)\|w\|^2 - \frac{|\xi|_2^2}{1-|\eta|_2^2}(1-s\kappa)\|\eta\|^2 - (1-s\kappa)\|\xi + R\eta\|^2 \\ &\le -\frac{|\xi|_2^2}{1-|\eta|_2^2}(1-s\kappa)\|\eta\|^2 - (1-s\kappa)\left(\frac{1}{3}\|\xi\|^2 - \frac{1}{2}R^2\|\eta\|^2\right) \\ &\le -\frac{1}{3}(1-s\kappa)\|\xi\|^2, \end{aligned}$$

where

$$\begin{aligned} R &= \frac{(\eta,\xi)}{1-|\eta|_2^2}, \\ Q &= |\xi|_2^2 + 2R(\eta,\xi) - |\eta|_2^2\left(\frac{|\xi|_2^2}{1-|\eta|_2^2} + R^2\right). \end{aligned}$$

Remark that, since $|\eta|_2^2 < \frac{1}{2}$, we have

$$Q \ge \left(\frac{|\xi|_2}{1-|\eta|_2^2} + R^2\right)\left(1-2|\eta|_2^2\right) > 0$$

and

$$\frac{|\xi|_2^2}{1-|\eta|_2^2} - \frac{1}{2}\frac{(\eta \mid \xi)^2}{\left(1-|\eta|_2^2\right)^2} \ge \frac{|\xi|_2^2\left(2-3|\eta|_2^2\right)}{2\left(1-|\eta|_2^2\right)^2} > \frac{|\xi|_2^2}{4\left(1-|\eta|_2^2\right)^2} > 0. \qquad \blacksquare$$

We let, for all $w \in \Sigma_+$,

$$\mathcal{E}(w) = \sup_{\eta \in B_1} J_w(\eta).$$

LEMMA 2.8. *For all $w \in \Sigma_+$, we have*

$$0 < \frac{1}{4}(2 - s\kappa)m \le \frac{1}{4}(2 - s\kappa)\|w\|^2 \le \mathcal{E}(w) \le \frac{1}{2}\|w\|^2.$$

PROOF. We have that

$$\begin{aligned}\mathcal{E}(w) \ge J_w(0) &= \frac{1}{2}\|w\|^2 - \frac{s}{4}\int \frac{(w, \beta w)(x)(w, \beta w)(y)}{|x-y|} \\ &\ge \frac{1}{4}(2 - s\kappa)\|w\|^2 \ge \frac{1}{4}(2 - s\kappa)m|w|_2^2 = \frac{1}{4}(2 - s\kappa)m\end{aligned}$$

and, for all $\eta \in B_1$, we have

$$J_w(\eta) = \frac{1}{2}\big\|a(\eta)w\big\|^2 - \frac{1}{2}\|\eta\|^2 - \frac{s}{4}\int \frac{(\psi, \beta\psi)(x)(\psi, \beta\psi)(y)}{|x-y|} \le \frac{1}{2}\|w\|^2. \qquad \blacksquare$$

PROPOSITION 2.9. *For every $w \in \Sigma_+$, there is a unique $\eta(w) \in B_1$ such that*

$$J_w\big(\eta(w)\big) = \max_{\eta \in B_1} J_w(\eta) = \mathcal{E}(w).$$

$\eta(w)$ is a critical point of J_w on B_1 such that $|\eta(w)|_2 < \frac{1}{2}$ and

$$\big\|\eta(w)\big\|^2 + m \le \big\|a(\eta)w\big\|^2, \quad \big\|\eta(w)\big\|^2 \le \frac{s}{2}\kappa\|w\|^2. \tag{2.10}$$

Moreover, the map

$$w \in X_+ \setminus \{0\} \mapsto \gamma(w) = \eta\big(|w|_2^{-1}w\big) \in B_1$$

is smooth.

PROOF. We can find, by Lemma 2.8 and using Ekeland's variational principle, a maximizing Palais–Smale sequence η_n at a positive level.

Then, by Lemma 2.6, $\eta_n \to \eta$ (up to a subsequence), with

$$dJ_w(\eta) = 0, \quad J_w(\eta) = \mathcal{E}(w).$$

From

$$\begin{aligned}\mathcal{E}(w) &= \frac{1}{2}a(\eta)^2\|w\|^2 - \frac{1}{2}\|\eta\|^2 - \frac{s}{4}\int \frac{(\psi, \beta\psi)(x)(\psi, \beta\psi)(y)}{|x-y|} \\ &\ge \frac{1}{2}\|w\|^2 - \frac{s}{4}\int \frac{(w, \beta w)(x)(w, \beta w)(y)}{|x-y|},\end{aligned}$$

we deduce, using Lemma A.1 in the appendix, that

$$
\begin{aligned}
& a(\eta)^2\|w\|^2 - \|\eta\|^2 - m \\
& \geq \|w\|^2 - s\kappa|\eta|_2^2\|w\|^2 - 7sa(\eta)^2\kappa\big(\|w\|^2 - m|w|_2^2\big) - 9s\kappa\|\eta\|^2 - m|w|_2^2 \\
& \geq 9s\kappa\big(a(\eta)^2\|w\|^2 - \|\eta\|^2 - m|w|_2^2\big) + (1-16s\kappa)\big(\|w\|^2 - m|w|_2^2\big) + 8s\kappa|\eta|_2^2\|w\|^2
\end{aligned}
$$

and we immediately deduce that

$$
a(\eta)^2\|w\|^2 - \|\eta\|^2 - m|w|_2^2 \geq \frac{1-16s\kappa}{1-9s\kappa}\big(\|w\|^2 - m|w|_2^2\big) > 0.
$$

We also have that

$$
\begin{aligned}
|\eta|_2^2\|w\|^2 + \|\eta\|^2 & \leq \frac{s}{2}\int \frac{(w,\beta w)(x)(w,\beta w)(y)}{|x-y|} \\
& \leq \frac{s}{2}\kappa\big|(-\Delta)^{-1/4}w\big|_2^2 \leq \frac{s}{2}\kappa\|w\|^2
\end{aligned}
$$

from which we deduce that

$$
\|\eta\|^2 \leq \frac{s}{2}\kappa\|w\|^2.
$$

To prove the uniqueness of the maxima for $J_w(\eta)$, we assume, by contradiction, the existence of $\eta_1, \eta_2 \in B_1$ such that

$$
J_w(\eta_1) = J_w(\eta_2) = \mathcal{E}(w).
$$

It follows from Lemma 2.2 that $|\eta_1|_2^2 < \frac{1}{2}$ and $|\eta_2|_2^2 < \frac{1}{2}$. We will use the mountain pass lemma in order to reach a contradiction. Let

$$
\Gamma = \left\{ g \in C\big([0,1], B_1\big) \mid g(0) = \eta_1,\ g(1) = \eta_2,\ \big|g(t)\big|_2^2 < \frac{1}{2} \right\}
$$

and define the min-max level

$$
c = \sup_{g\in\Gamma} \min_{t\in[0,1]} J_w\big(g(t)\big).
$$

Let $g(t) = t\eta_1 + (1-t)\eta_2$. We have that $\big|g(t)\big|_2^2 < \frac{1}{2}$ and $a(g(t))^2 > \frac{1}{2}$ for all $t \in [0,1]$, so that we have, letting $\psi_t = a(g(t))w + g(t)$,

$$
\begin{aligned}
J_w\big(g(t)\big) & = \frac{1}{2}a\big(g(t)\big)^2\|w\|^2 - \frac{1}{2}\big\|g(t)\big\|^2 - \frac{s}{4}\int \frac{(\psi_t,\beta\psi_t)(x)(\psi_t,\beta\psi_t)(y)}{|x-y|} \\
& \geq \frac{1}{2}\left(1 - \frac{s\kappa}{2}\right)a\big(g(t)\big)^2\|w\|^2 - \frac{1}{2}\left(1 + \frac{s\kappa}{2}\right)\big\|g(t)\big\|^2
\end{aligned}
$$

$$\geq \frac{1}{2}\Big(1-\frac{s\kappa}{2}\Big)ta(\eta_1)^2\|w\|^2+\frac{1}{2}\Big(1-\frac{s\kappa}{2}\Big)(1-t)a(\eta_2)^2\|w\|^2$$
$$-\frac{1}{2}\Big(1+\frac{s\kappa}{2}\Big)t\|\eta_1\|^2-\frac{1}{2}\Big(1+\frac{s\kappa}{2}\Big)(1-t)\|\eta_2\|^2$$
$$\geq \frac{1}{4}\Big(1-\frac{s\kappa}{2}\Big)\|w\|^2-\frac{1}{2}\Big(1+\frac{s\kappa}{2}\Big)\frac{s}{2}\kappa\|w\|^2\geq\frac{1}{4}(1-2s\kappa)\|w\|^2,$$

where we have used the second inequality in (2.10) and $s\kappa < 2$. We deduce that $c > 0$. Since the set Γ is invariant for the gradient flow (see Lemma 2.2) and the Palais–Smale condition holds (see Lemma 2.6), we deduce from the mountain pass theorem that there exists a critical point at level c, which cannot be a strict local maximum. The contradiction then follows from Lemma 2.7.

To prove that the map $w \mapsto \gamma(w) = \eta(|w|_2^{-1}w)$ is smooth, we consider, since the map $w \mapsto P(w) = |w|_2^{-1}w$ is smooth, that $(w_0, \eta(w_0)) \in \Sigma_+ \times B_1$ and we let $V \subset X_+ \setminus \{0\}$ and $U \subset B_1$ be neighborhoods of w_0 and $\eta(w_0)$, respectively. Then we define the map $F\colon V \times U \to L(X_-)$ as

$$F(w,\eta)[\xi] = dJ_{P(w)}(\eta)[\xi] \quad \xi \in X_-.$$

Clearly, $P(w_0) = w_0$ and $F(w_0, \eta(w_0)) = 0$. We have that

$$d_\eta F\big(w_0,\eta(w_0)\big)[\xi][\zeta] = d^2J_{w_0}\big(\eta(w_0)\big)[\xi,\zeta], \quad \xi,\zeta \in X_-.$$

It follows from Lemma 2.7 that

$$-d_\eta F\big(w_0,\eta(w_0)\big)[\xi][\xi] = -d^2J_{w_0}\big(\eta(w_0)\big)[\xi,\xi] \geq \delta\|\xi\|^2 \quad \text{for all } \xi \in X_-$$

and hence we have from Lax–Milgram that for all linear functionals L on X_- there is a unique $\zeta \in X_-$ such that

$$-d_\eta F\big(w_0,\eta(w_0)\big)[\zeta][\xi] = L[\xi], \quad \text{for all } \xi \in X_-,$$

that is, $L = -d_\eta F(w_0, \eta(w_0))[\zeta]$. By the implicit function theorem, there exist $V_0 \subset V$ and $U_0 \subset U$, neighborhoods of w_0 and $\eta(w_0)$ and a smooth map $\gamma\colon V_0 \to U_0$ such that $F(w, \gamma(w)) = 0$ for all $w \in V_0$; that is, $\gamma(w)$ is a critical point of $J_{P(w)}$ on B_1 at a positive level. Then, by Proposition 2.7, $\gamma(w)$ is a strict local maximum of $J_{P(w)}$ on B_1. Again using the mountain pass theorem, we deduce that actually $\gamma(w) = \eta(P(w))$ is the unique (up to a phase factor) maximum of $J_{P(w)}$.

Finally, we have that

$$d\gamma(w)[v] = -d_\eta F\big(w,\gamma(w)\big)^{-1}\big[d_w F\big(w,\gamma(w)\big)[v]\big] \quad \text{for all } v \in X_+. \qquad ■$$

It follows from Proposition 2.9 that we can consider the smooth functional $\mathcal{E}\colon X_+ \setminus \{0\} \to \mathbb{R}$ defined as

$$\mathcal{E}(w) = J_{P(w)}\big(\gamma(w)\big) = \sup_{\eta \in B_1} J_{P(w)}(\eta).$$

Since

$$J_{P(w)}\big(\gamma(w)\big) = I\big(a\big(\gamma(w)\big)P(w) + \gamma(w)\big)$$

and recalling that

$$dJ_{P(w)}\big(\gamma(w)\big)[\xi] = dI(\psi_w)\big[da\big(\gamma(w)\big)[\xi]P(w) + \xi\big] = 0 \quad \text{for all } \xi \in X_-$$

(where $\psi_w = a(\gamma(w))P(w) + \gamma(w)$), we have that for all $v \in X_+$

$$\begin{aligned} d\mathcal{E}(w)[v] &= d_w J_{P(w)}\big(\gamma(w)\big)[v] \\ &= dI(\psi_w)\big[da\big(\gamma(w)\big)\big[d\gamma(w)[v]\big]P(w) + a\big(\gamma(w)\big)dP(w)[v] + d\gamma(w)[v]\big] \\ &= d_\eta J_{P(w)}\big(\gamma(w)\big)\big[d\gamma(w)[v]\big] + dI(\psi_w)\big[a\big(\gamma(w)\big)dP(w)[v]\big] \\ &= dI(\psi_w)\big[a\big(\gamma(w)\big)dP(w)[v]\big] \\ &= dI(\psi_w)\big[a\big(\gamma(w)\big)v\big] - dI(\psi_w)\big[a\big(\gamma(w)\big)(w \mid v)w\big] \end{aligned}$$

(we have used that $dP(w)[v] = v - (w \mid v)w$) and

$$d\mathcal{E}(w)[v] = a\big(\gamma(w)\big)dI(\psi_w)[v] - a\big(\gamma(w)\big)^2\omega(\psi_w)(w \mid v) \quad \text{for all } v \in X_+, \tag{2.11}$$

where

$$\omega(\psi_w) = a\big(\gamma(w)\big)^{-1} dI(\psi_w)[w]. \tag{2.12}$$

PROPOSITION 2.13. *Let $w_0 \in \Sigma_+$ be a critical point of $\mathcal{E}$ restricted on the manifold Σ_+. Then w_0 is a critical point for $\mathcal{E}$ on X_+ and the function*

$$\psi_0 = a\big(\eta(w_0)\big)w_0 + \eta(w_0) \in \Sigma$$

is a critical point for I on the manifold Σ and satisfies

$$dI(\psi_0)[h] = \omega(\psi_0 \mid h) \quad \textit{for all } h \in X, \tag{2.14}$$

where $\omega = \omega(\psi_0) \in \mathbb{R}$,

$$(1 - 3s\kappa)\|w_0\|^2 \le \omega(\psi_0) \le 2I(\psi_{w_0}) = 2\mathcal{E}(w_0). \tag{2.15}$$

Moreover, if $\psi_0 \in \Sigma$ satisfies $I(\psi_0) \ge 0$ and (2.14) for some $\omega \in \mathbb{R}$, then $w = |\Lambda_+\psi_0|_2^{-1}\Lambda_+\psi_0$ is a critical point for $\mathcal{E}(w)$.

PROOF. Let $w_0 \in \Sigma_+$ be a critical point for $\mathcal{E}$ on Σ_+, $\eta_0 = \eta(w_0) = \gamma(w_0)$, and $\psi_0 = a(\eta_0)w_0 + \eta_0$. Then

$$d\mathcal{E}(w_0)[h] = 0 \quad \text{for all } h \in T_{w_0}\Sigma_+ = \{h \in X_+ \mid (w_0 \mid h) = 0\}.$$

Since $dP(w_0)[w_0] = 0$, we immediately deduce that

$$d\mathcal{E}(w_0)[v] = 0 \quad \text{for all } v \in X_+.$$

From (2.1), we have that for all $\xi \in X_-$

$$0 = dJ_{w_0}(\eta_0)[\xi] = dI(\psi_0)[\xi] + dI(\psi_0)\big[\big(da(\eta_0)[\xi]\big)w_0\big]$$

while for all $v \in X_+$ we have

$$0 = d\mathcal{E}(w_0)[v] = a(\eta_0)dI(\psi_0)[v] - a(\eta_0)^2\omega(\psi_0)(w_0 \mid v)$$

and hence, for all $h = v + \xi$, $v \in X_+$, $\xi \in X_-$,

$$\begin{aligned} dI(\psi_0)[h] &= a(\eta_0)\omega(\psi_0)(w_0 \mid v) - dI(\psi_0)\big[da(\eta_0)[\xi]w_0\big] \\ &= a(\eta_0)\omega(\psi_0)(w_0 \mid v) + \omega(\psi_0)(\eta_0 \mid \xi) \\ &= \omega(\psi_0)(\psi_0 \mid h), \end{aligned}$$

that is

$$dI(\psi_0)[h] = \omega(\psi_0)(\psi_0 \mid h) \quad \text{for all } h \in X,$$

which shows that ψ_0 is a critical point for $I(\psi)$ under the constraint $|\psi|_2 = 1$. The Lagrange multiplier $\omega(\psi_0) = a(\eta_0)^{-1}dI(\psi_0)[w_0]$ is such that

$$\begin{aligned} \omega(\psi_0) &= a\big(\eta(w_0)\big)^{-1}dI(\psi_0)[w_0] \geq \|w_0\|^2 - s\kappa a\big(\eta(w_0)\big)^{-1}\|\psi_0\|\|w_0\| \\ &\geq \|w_0\|^2 - \frac{s\kappa}{2}\big(a\big(\eta(w_0)\big)^{-2}\|\psi_0\|^2 + \|w_0\|^2\big) \\ &\geq \|w_0\|^2 - \frac{s\kappa}{2}\big(\|w_0\|^2 + a\big(\eta(w_0)\big)^{-2}\|\eta_0\|^2 + \|w_0\|^2\big) \\ &\geq \|w_0\|^2 - \frac{3s\kappa}{2}\|w_0\|^2 \end{aligned}$$

and

$$\omega(\psi_0) = dI(\psi_0)[\psi_0] \leq 2I(\psi_0).$$

Suppose now that $\psi_0 \in \Sigma$ satisfies (2.14) for some $\tilde{\omega}$. Let $w_0 = |\Lambda_+\psi_0|_2^{-1}\Lambda_+\psi_0$ and $\eta_0 = \Lambda_-\psi_0$. Then we deduce from (2.1) that for all $\xi \in X_-$

$$dJ_{w_0}(\eta_0)[\xi] = dI(\psi_0)\big[da(\eta_0)[\xi]w_0 + \xi\big] = \tilde{\omega}\big(\psi_0 \mid da(\eta_0)[\xi]w_0 + \xi\big) = 0,$$

and η_0 is a critical point of J_{w_0}. From Lemma 2.8, we know that η_0 is a local maximum and, arguing as in the proof of Proposition 2.9, we deduce that $\eta_0 = \eta(w_0)$ and $\mathcal{E}(w_0) = J_{\omega_0}(\eta_0)$. We also have that

$$\tilde{\omega} = dI(\psi_0)[\psi_0] = \omega(\psi_0).$$

We then deduce from (2.11) that

$$\begin{aligned} d\mathcal{E}(w_0) &= a\big(\gamma(w_0)\big)dI(\psi_0)[v] - a\big(\gamma(w_0)\big)^2\omega(\psi_0)(w_0 \mid v) \\ &= a\big(\gamma(w_0)\big)\tilde{\omega}(\psi_0 \mid v) - a\big(\gamma(w_0)\big)^2\omega(\psi_{w_0})(w_0 \mid v) \\ &= a\big(\gamma(w_0)\big)\omega(\psi_0)\big(a\big(\gamma(w_0)\big)w_0 + \xi \mid v\big) - a\big(\gamma(w_0)\big)^2\omega(\psi_0)(w_0, v) = 0. \ \blacksquare \end{aligned}$$

From now on, we will make explicit the dependence of I, J, and $\mathcal{E}$ on $s > 0$ writing I_s, J_s, and $\mathcal{E}_s$, introduce the following minimization problem:

$$\begin{aligned} e(s) &= \inf_{w\in\Sigma_+} \mathcal{E}_s(w) \\ &= \inf_{w\in\Sigma_+} \Big\{\frac{1}{2}a\big(\eta_s(w)\big)^2\|w\|^2 - \frac{1}{2}\|\eta_s(w)\|^2 \\ &\qquad - \frac{s}{4}\int \frac{(\psi_w, \beta\psi_w)(x)(\psi_w, \beta\psi_w)(y)}{|x-y|}\Big\}, \end{aligned}$$

and let $E(s) = se(s)$.

The next lemma allows us to recover enough compactness (via the concentration-compactness lemma [4, 5]) in order to prove our main result; see also [6, Lemma 4.2].

LEMMA 2.16. *For all $s \in (0, \frac{1}{8\pi}]$ we have that $0 < e(s) < \frac{m}{2}$.*

PROOF. From Lemma 2.8, we have that $e(s) \ge \frac{1}{4}(2 - s\kappa)m \ge \frac{1}{4}(2-\kappa)m > 0$.

Using Lemma A.1, we deduce that

$$\begin{aligned} \mathcal{E}_s(w) &= I_s(\psi_w) \\ &\le \frac{m}{2} + \frac{1}{2}(1 + 8s\kappa)\big(\|w\|^2 - m|w|_2^2\big) - \frac{s}{4}\int \frac{(w, \beta w)(x)(w, \beta w)(y)}{|x-y|}. \end{aligned}$$

Fix $w_1 \in H^1(\mathbb{R}^3, \mathbb{C}^2)$ such that $|w_1|_2 = 1$, $w = \binom{w_1}{0}$ and let $w_\varepsilon(x) = \varepsilon^{3/2}w(\varepsilon x)$. We have that

$$|w_\varepsilon|_2^2 = |w|_2^2 = 1,$$

and

$$\hat{w}_\varepsilon(q) = \frac{1}{\varepsilon^{3/2}}w(\varepsilon^{-1}p),$$

so that

$$\|w_\varepsilon\|^2 - m|w_\varepsilon|_2^2 = \int \left(\sqrt{|q|^2+m^2}-m\right)\left|\hat{w}_\varepsilon(q)\right|^2$$
$$= \int \left(\sqrt{\varepsilon^2|q|^2+m^2}-m\right)\left|\hat{w}_1(q)\right|^2 \le \frac{\varepsilon^2}{2m}\int |q|^2\left|\hat{w}_1(q)\right|^2$$

and $\|w_\varepsilon\|^2 \le m + C\varepsilon^2$. We then observe that

$$\|w_\varepsilon - \Lambda_+ w_\varepsilon\|^2 = \|\Lambda_- w_\varepsilon\|^2$$
$$= \int \sqrt{\varepsilon^2|p|^2+m^2}\left|\frac{1}{2}\left[1 - \frac{m\beta}{\sqrt{\varepsilon^2|p|^2+m^2}} - \frac{\varepsilon\boldsymbol{\alpha}\cdot p}{\sqrt{\varepsilon^2|p|^2+m^2}}\right]\begin{pmatrix}\hat{w}_1(p)\\ 0\end{pmatrix}\right|^2$$
$$= \frac{1}{4}\int \sqrt{\varepsilon^2|p|^2+m^2}\left|\begin{pmatrix}\frac{\sqrt{\varepsilon|p|^2+m^2}-m}{\sqrt{\varepsilon|p|^2+m^2}}\hat{w}_1(p)\\ \frac{\varepsilon\boldsymbol{\sigma}\cdot p}{\sqrt{\varepsilon|p|^2+m^2}}\hat{w}_1(p)\end{pmatrix}\right|^2$$
$$= \frac{1}{2}\int \left(\sqrt{\varepsilon^2|p|^2+m^2}-m\right)\left|\hat{w}_1(p)\right|^2 \le \frac{\varepsilon^2}{4m}\int |p|^2\left|\hat{w}_1(p)\right|^2$$

and also

$$\left|1-|\Lambda_+ w_\varepsilon|_2\right| = \left||w_\varepsilon|_2 - |\Lambda_+ w_\varepsilon|_2\right| \le |w_\varepsilon - \Lambda_+ w_\varepsilon|_2$$
$$= |\Lambda_- w_\varepsilon|_2 \le \frac{\varepsilon}{2\sqrt{m}}\left(\int |p|^2\left|\hat{w}_1(p)\right|^2\right)^{1/2}.$$

We deduce from this that for $\varepsilon > 0$ small enough $|\Lambda_+ w_\varepsilon|_2 > \frac{1}{2}$.

Let

$$\varphi_\varepsilon(x) = |\Lambda_+ w_\varepsilon|_2^{-1}\Lambda_+ w_\varepsilon(x).$$

We have that

$$\|\varphi_\varepsilon\| \le |\Lambda_+ w_\varepsilon|_2^{-1}\|w_\varepsilon\| \le \sqrt{m} + C\varepsilon$$

and

$$\|w_\varepsilon - \varphi_\varepsilon\| \le \|w_\varepsilon - \Lambda_+ w_\varepsilon\| + \left\|\left(1-|\Lambda_+ w_\varepsilon|_2\right)\varphi_\varepsilon\right\|$$
$$\le \frac{\varepsilon}{2\sqrt{m}}\left(2+\|\varphi_\varepsilon\|\right)\left(\int |p|^2\left|\hat{w}_1(p)\right|^2\right)^{1/2}$$

and also

$$|\varphi_\varepsilon - w_\varepsilon|_2 = \frac{1}{|\Lambda_+ w_\varepsilon|_2}|w_\varepsilon - \Lambda_+ w_\varepsilon|_2 \le \frac{\varepsilon}{\sqrt{m}}\left(\int |p|^2\left|\hat{w}_1(p)\right|^2\right)^{1/2}$$

and we can estimate

$$\mathcal{E}_s(\varphi_\varepsilon) \le \frac{m}{2} + \frac{1}{2}(1+8s\kappa)\big(\|\varphi_\varepsilon\|^2 - m|\varphi_\varepsilon|_2^2\big) - \frac{s}{4}\int \frac{(\varphi_\varepsilon, \beta\varphi_\varepsilon)(x)(\varphi_\varepsilon, \beta\varphi_\varepsilon)(y)}{|x-y|}.$$

We have that

$$\begin{aligned}
&\|\varphi_\varepsilon\|^2 - m|\varphi_\varepsilon|_2^2 \\
&= \int \big(\sqrt{|q|^2+m^2} - m\big)\big|\hat{\varphi}_\varepsilon(q)\big|^2 \\
&\le 2\int \big(\sqrt{|q|^2+m^2} - m\big)\big|\hat{\varphi}_\varepsilon(q) - \hat{w}_\varepsilon(q)\big|^2 + 2\int \big(\sqrt{|q|^2+m^2} - m\big)\big|\hat{w}_\varepsilon(q)\big|^2 \\
&= 2\big(\|\varphi_\varepsilon - w_\varepsilon\|^2 - m|\varphi_\varepsilon - w_\varepsilon|_2^2\big) + 2\big(\|w_\varepsilon\|^2 - m|w_\varepsilon|_2^2\big) \\
&\le \frac{\varepsilon^2}{2m}\big(3 + \|\varphi_\varepsilon\|\big)^2 |\nabla w_1|_2^2.
\end{aligned}$$

We have that

$$\begin{aligned}
Q(\varphi_\varepsilon) - Q(w_\varepsilon) &= Q\big((\varphi_\varepsilon - w_\varepsilon) + w_\varepsilon\big) - Q(w_\varepsilon) \\
&= Q(\varphi_\varepsilon - w_\varepsilon) + 4\int \frac{(\varphi_\varepsilon - w_\varepsilon, \beta w_\varepsilon)(x)(w_\varepsilon, \beta w_\varepsilon)(x)}{|x-y|} \\
&\quad + 3\int \frac{(\varphi_\varepsilon - w_\varepsilon, \beta w_\varepsilon)(x)(\varphi_\varepsilon - w_\varepsilon, \beta w_\varepsilon)(x)}{|x-y|} \\
&\quad + 3\int \frac{\big(\varphi_\varepsilon - w_\varepsilon, \beta(\varphi_\varepsilon - w_\varepsilon)\big)(x)(w_\varepsilon, \beta w_\varepsilon)(x)}{|x-y|} \\
&\quad + 4\int \frac{\big(\varphi_\varepsilon - w_\varepsilon, \beta(\varphi_\varepsilon - w_\varepsilon)\big)(x)(\varphi_\varepsilon - w_\varepsilon, \beta w_\varepsilon)(x)}{|x-y|} \\
&\ge -4\kappa|\varphi_\varepsilon - w_\varepsilon|_2\big|(-\Delta)^{1/4}|w_\varepsilon|\big|_2^2 - 3\kappa|\varphi_\varepsilon - w_\varepsilon|_2^2\big|(-\Delta)^{1/4}|w_\varepsilon|\big|_2^2 \\
&\quad - 4\kappa|\varphi_\varepsilon - w_\varepsilon|_2\|\varphi_\varepsilon - w_\varepsilon\|^2.
\end{aligned}$$

Since

$$\big|(-\Delta)^{1/4}|w_\varepsilon|\big|_2^2 = \int |p||\hat{w}_\varepsilon|^2 = \varepsilon\int |p||w_1|^2,$$

we have

$$Q(\varphi_\varepsilon) \ge Q(w_\varepsilon) - c\varepsilon^2 \ge \varepsilon Q(w) - c\varepsilon^2;$$

we therefore deduce that

$$\mathcal{E}_s(w_\varepsilon) \le \frac{m}{2} + \frac{2\varepsilon^2}{m}(1+8s\kappa)|\nabla w|_2^2 - \varepsilon\frac{s}{4}\int \frac{(w, \beta w)(x)(w, \beta w)(y)}{|x-y|} + c\varepsilon^2.$$

Since $Q(w) > 0$, we deduce that

$$\mathcal{E}_s(w_\varepsilon) < \frac{m}{2}$$

for ε small enough and for all $s \in (0, \frac{1}{8\pi})$, and hence $e(s) < \frac{m}{2}$ for all $s \in (0, \frac{1}{8\pi})$. ■

PROPOSITION 2.17. *For all $\theta > 1$ and $s \in (0, \frac{1}{8\pi})$ such that $\theta s \in (0, \frac{1}{8\pi})$, we have that*

$$e(\theta s) < e(s).$$

PROOF. Let $\theta > 1$ and $s \in (0, \frac{1}{8\pi})$ such that $\theta s \in (0, \frac{1}{8\pi})$. Take $w \in \Sigma_+$ and let $\eta_s(w) \in B_1$ the function whose existence follows from Proposition 2.9. Since it follows from (2.10) that

$$\left\|a\big(\eta_{\theta s}(w)\big)w\right\|^2 - \left\|\eta_{\theta s}(w)\right\|^2 - m \geq 0,$$

we have that

$$\begin{aligned}
&\theta\bigg(\mathcal{E}_{\theta s}(w) - \frac{m}{2}\bigg)\\
&= \theta\bigg(\frac{1}{2}\left\|a\big(\eta_{\theta s}(w)\big)w\right\|^2 - \frac{1}{2}\left\|\eta_{\theta s}(w)\right\|^2 - \frac{m}{2} - \frac{\theta s}{4}\int \frac{(\psi_1, \beta\psi_1)(x)(\psi_1, \beta\psi_1)(y)}{|x-y|}\bigg)\\
&\leq \theta^2\bigg(\frac{1}{2}\left\|a\big(\eta_{\theta s}(w)\big)w\right\|^2 - \frac{1}{2}\left\|\eta_{\theta s}(w)\right\|^2 - \frac{m}{2} - \frac{s}{4}\int \frac{(\psi_1, \beta\psi_1)(x)(\psi_1, \beta\psi_1)(y)}{|x-y|}\bigg)\\
&\leq \theta^2\bigg(\frac{1}{2}\left\|a\big(\eta_{s}(w)\big)w\right\|^2 - \frac{1}{2}\left\|\eta_{s}(w)\right\|^2 - \frac{m}{2} - \frac{s}{4}\int \frac{(\psi_2, \beta\psi_2)(x)(\psi_2, \beta\psi_2)(y)}{|x-y|}\bigg)\\
&= \theta^2\bigg(\mathcal{E}_s(w) - \frac{m}{2}\bigg)
\end{aligned}$$

(here $\psi_1 = a(\eta_{\theta s}(w))w + \eta_{\theta s}(w))$ and $\psi_2 = a(\eta_s(w))w + \eta_s(w)$. We know that $e(s) < \frac{m}{2}$ and hence

$$\theta\bigg(e(\theta s) - \frac{m}{2}\bigg) \leq \theta^2\bigg(e(s) - \frac{m}{2}\bigg) < \theta\bigg(e(s) - \frac{m}{2}\bigg)$$

from which we deduce that $e(\theta s) < e(s)$. ■

PROOF OF THEOREM 1.4. By Ekeland's variational principle, there exists a sequence $w_n \in \Sigma_+$ such that

$$\mathcal{E}_s(w_n) \to e(s), \qquad \sup_{v \in \Sigma_+} \left|d\mathcal{E}_s(w_n)[v]\right| \to 0.$$

From $\mathcal{E}_s(w_n) \to e(s)$, we deduce from Lemma 2.8 that $\|w_n\| \leq \frac{4e(s)}{2 - s\kappa} + o(1)$ so that the sequence w_n is bounded. It follows from Proposition 2.9 that also $\eta_n = \eta(w_n)$ and

$\psi_n = a(\eta_n)w_n + \eta_n$ are bounded in X. Letting $\omega_n = a(\eta_n)^{-1}dI(\psi_n)[w_n]$, we have that

$$dI_s(\psi_n)[h] - \omega_n(\psi_n \mid h) = 0 \quad \text{for all } h \in X.$$

We can assume that (up to a subsequence) $\psi_n \rightharpoonup \psi$ in X and that $\omega_n \to \omega$. Then we have that for all $h \in X$

$$\begin{aligned} &dI_s(\psi_n)[h] - \omega_n(\psi_n \mid h) \\ &\quad = \langle \psi_n \mid \Lambda_+ h\rangle - \langle \psi_n \mid \Lambda_- h\rangle - s\int \frac{(\psi_n, \beta\psi_n)(x)(\psi_n, \beta h)(y)}{|x-y|} - \omega_n(\psi_n \mid h) \\ &\quad \to 0, \end{aligned}$$

since, by (1.7), we have that

$$\int \frac{(\psi_n, \beta\psi_n)(x)(\psi_n - \psi, \beta h)(y)}{|x-y|} \to 0.$$

As a consequence, we have that

$$dI_s(\psi)[h] - \omega(\psi \mid h) = 0 \quad \text{for all } h \in X.$$

The weak convergence does not, however, preserve the L^2 norm, so we only know that $|\psi|_2 \le |\psi_n|_2 = 1$ (it could even be that $\psi = 0$).

To conclude, we will now apply the concentration-compactness principle; see [4, 5]. First of all, let us show that no vanishing occurs. By contradiction, assume that

$$\limsup_{n\to+\infty} \sup_{y\in\mathbb{R}^3} \int_{B(y,1)} |\psi_n|^2 = 0.$$

Then we know, see [4] or [8, Lemma 1.21], that $\psi_n \to 0$ in $L^p(\mathbb{R}^3)$ for $2 < p < 3$.

Since

$$\int \frac{\big(\psi_n, \beta\psi_n(x)\big)\big(\psi_n, \beta\psi_n(y)\big)}{|x-y|} \le \int \frac{\left|\psi_n(x)\right|^2\left|\psi_n(y)\right|^2}{|x-y|} \le C\left|\psi_n\right|^4_{\frac{12}{5}} \to 0,$$

we deduce, using (2.10), (2.15), and Lemma 2.16, that

$$\begin{aligned} 0 &= dI_s(\psi_n)[\psi_n] - \omega_n|\psi_n|_2^2 \\ &= \left\|a(\eta_n)w_n\right\|^2 - \|\eta_n\|^2 - \omega_n|\psi_n|_2^2 - s\int \frac{\big(\psi_n, \beta\psi_n(x)\big)\big(\psi_n, \beta\psi_n(y)\big)}{|x-y|} \\ &= \left\|a(\eta_n)w_n\right\|^2 - \|\eta_n\|^2 - m|\psi_n|_2^2 + (m-\omega_n) + o(1) \ge (m-\omega_n) + o(1) > 0 \end{aligned}$$

for n large enough, a contradiction which shows that vanishing does not occur.

Then we know from the concentration-compactness principle, that there exist $p \geq 1$ functions $\phi_1, \dots, \phi_p \in X$, critical points for I_s under the constraint $|\psi|_2^2 = \mu_i$ (hence satisfying (2.14) with $\omega = \lim_n \omega_n > 0$), and p sequences of points $x_{i,n} \in \mathbb{R}^3$, $i = 1, \dots, p$ such that $|x_{i,n} - x_{j,n}| \to +\infty$ for all $i \neq j$ as $n \to +\infty$ and

$$\left\| \psi_n - \sum_{i=i}^{p} \phi_i(\cdot - x_{i,n}) \right\| \to 0 \quad \text{as } n \to +\infty.$$

From this it follows also that $|\psi_n|_2^2 = 1 = \sum_{i=1}^{p} \mu_i$.

We then observe that

$$\begin{aligned}
&\|\Lambda_+\psi_n\|^2 - \|\Lambda_-\psi_n\|^2 \\
&\quad = \langle \psi_n \mid \Lambda_+\psi_n - \Lambda_-\psi_n \rangle \\
&\quad = \left\langle \psi_n - \sum_{i=i}^{p} \phi_i(\cdot - x_{i,n}) \mid \Lambda_+\psi_n - \Lambda_-\psi_n \right\rangle \\
&\quad\quad + \sum_{i=i}^{p} \langle \phi_i(\cdot - x_{i,n}) \mid \Lambda_+\psi_n - \Lambda_-\psi_n \rangle \\
&\quad = \sum_{i=i}^{p} \big(\langle \Lambda_+\phi_i(\cdot - x_{i,n}) \mid \psi_n \rangle - \langle \Lambda_-\phi_i(\cdot - x_{i,n}) \mid \psi_n \rangle \big) + o(1) \\
&\quad = \sum_{i=i}^{p} \big(\|\Lambda_+\phi_i\|^2 - \|\Lambda_-\phi_i\|^2 \big) + o(1)
\end{aligned}$$

and also

$$\int \frac{\big(\psi_n, \beta\psi_n(x)\big)\big(\psi_n, \beta\psi_n(y)\big)}{|x-y|} = \sum_{i=1}^{p} \int \frac{\big(\phi_i, \beta\phi_i(x)\big)\big(\phi_i, \beta\phi_i(y)\big)}{|x-y|} + o(1).$$

Finally, we have that

$$e(s) = I_s(\psi_n) + o(1) = \sum_{i=1}^{p} I_s(\phi_i) + o(1). \tag{2.18}$$

Let, for $i = 1, \dots, n$, $\tilde{\psi}_i = |\phi_i|_2^{-1}\phi_i = \mu_i^{-1/2}\phi_i \in \Sigma$. We have that

$$I_s(\phi_i) = I_s(\sqrt{\mu_i}\tilde{\psi}_i) = \mu_i I_{s\mu_i}(\tilde{\psi}_i)$$

and

$$0 = dI_s(\phi_i)[h] - \omega(\phi_i \mid h) = \sqrt{\mu_i}\big(dI_{s\mu_i}(\tilde{\psi}_i)[h] - \omega(\tilde{\psi}_i \mid h)\big) \quad \text{for all } h \in X.$$

It follows from Proposition 2.13 that $\tilde{w}_i = |\Lambda_+\tilde{\psi}_i|_2^{-1}\Lambda_+\tilde{\psi}_i \in \Sigma_+$ is a critical point for $\mathcal{E}_{s\mu_i}$ and $\mathcal{E}_{s\mu_i}(\tilde{w}_i) = I_{s\mu_i}(\tilde{\psi}_i)$.

Since

$$\mathcal{E}_{s\mu_i}(\tilde{w}_i) \geq e(s\mu_i),$$

we deduce from Proposition 2.17 that

$$e(s) = \sum_{i=1}^{p} I_s(\phi_i) = \sum_{i=1}^{p} \mu_i I_{s\mu_i}(\tilde{\psi}_i) \geq \sum_{i=1}^{p} \mu_i e(s\mu_i) > \sum_{i=1}^{p} \mu_i e\Big(\frac{1}{\mu_i} s\mu_i\Big) = e(s) \sum_{i=1}^{p} \mu_i,$$

a contradiction if $p > 1$.

Since there is no vanishing or dichotomy, our sequence ψ_n converges strongly in X to a critical point $\psi \in X$ of (2.14) such that $|\psi|_2 = 1$ and the theorem follows. ■

A. A useful lemma

This lemma is similar to [6, Lemma 2.9]. We give here a slightly different proof.

LEMMA A.1. *For all* $\psi = \sqrt{1 - |w|_2^2}\, w + \eta$, $w \in \Sigma_+$, $\eta \in X_-$, *we have*

$$\int \frac{(\psi, \beta\psi)(x)(\psi, \beta\psi)(y)}{|x - y|} \geq \int \frac{(w, \beta w)(x)(w, \beta w)(y)}{|x - y|} - 2\kappa |\eta|_2^2 \|w\|^2 - 14 a(\eta)^2 \kappa \big(\|w\|^2 - m|w|_2^2\big) - 18\kappa \|\eta\|^2.$$

PROOF. We have

$$\begin{aligned}
&\int \frac{(\psi, \beta\psi)(x)(\psi, \beta\psi)(y)}{|x - y|} \\
&= a(\eta)^4 \int \frac{(w, \beta w)(x)(w, \beta w)(y)}{|x - y|} \\
&\quad + 4a(\eta)^3 \int \frac{(w, \beta w)(x)(w, \beta\eta)(y)}{|x - y|} + 3a(\eta)^2 \int \frac{(w, \beta w)(x)(\eta, \beta\eta)(y)}{|x - y|} \\
&\quad + 3a(\eta)^2 \int \frac{(w, \beta\eta)(x)(w, \beta\eta)(y)}{|x - y|} + 4a(\eta) \int \frac{(\eta, \beta\eta)(x)(w, \beta\eta)(y)}{|x - y|} \\
&\quad + \int \frac{(\eta, \beta\eta)(x)(\eta, \beta\eta)(y)}{|x - y|} \\
&\geq a(\eta)^4 \int \frac{(w, \beta w)(x)(w, \beta w)(y)}{|x - y|} + 4a(\eta)^3 \int \frac{(w, \beta w)(x)(w, \beta\eta)(y)}{|x - y|} \\
&\quad - 3a(\eta)^2 \kappa \|\eta\|^2 + 4a(\eta) \int \frac{(\eta, \beta\eta)(x)(w, \beta\eta)(y)}{|x - y|}.
\end{aligned}$$

We have

$$\begin{aligned}
&\left|\int \frac{(w,\beta w)(x)(w,\beta\eta)(y)}{|x-y|}\,dx\,dt\right| \\
&\quad = (2\pi)^{3/2}\sqrt{\frac{2}{\pi}}\left|\int \frac{\mathcal{F}\big[(w,\beta w)\big]\mathcal{F}\big[(w,\beta\eta)\big]}{|p|^2}\,dp\right| \\
&\quad \le (2\pi)^{3/2}\sqrt{\frac{2}{\pi}}\big|\mathcal{F}\big[(w,\beta w)\big]\big|_\infty\left|\int \frac{\big|\mathcal{F}\big[(w,\beta\eta)\big]\big|}{|p|^2}\,dp\right| \\
&\quad \le \sqrt{\frac{2}{\pi}}\big|(w,\beta w)\big|_1\left|\int \frac{\big|\mathcal{F}\big[(w,\beta\eta)\big]\big|}{|p|^2}\,dp\right| \le \sqrt{\frac{2}{\pi}}|w|_2^2\left|\int \frac{\big|\mathcal{F}\big[(w,\beta\eta)\big]\big|}{|p|^2}\,dp\right| \\
&\quad \le \sqrt{\frac{2}{\pi}}\int \frac{dp}{|p|^2}\left(\frac{1}{(2\pi)^{3/2}}\int \big|\big(\hat{w}(p-q),\beta\hat{\eta}(q)\big)\big|\,dq\right)
\end{aligned}$$

and

$$\begin{aligned}
&\left|\int \frac{(\eta,\beta\eta)(x)(w,\beta\eta)(y)}{|x-y|}\,dx\,dt\right| \\
&\qquad \le |\eta|_2^2\sqrt{\frac{2}{\pi}}\int \frac{dp}{|p|^2}\left(\frac{1}{(2\pi)^{3/2}}\int \big|\big(\hat{w}(p-q),\beta\hat{\eta}(q)\big)\big|\,dq\right)
\end{aligned}$$

so that

$$\begin{aligned}
&4a(\eta)^3\int \frac{(w,\beta w)(x)(w,\beta\eta)(y)}{|x-y|} + 4a(\eta)\int \frac{(\eta,\beta\eta)(x)(w,\beta\eta)(y)}{|x-y|} \\
&\qquad \le 4a(\eta)\sqrt{\frac{2}{\pi}}\int \frac{dp}{|p|^2}\left(\frac{1}{(2\pi)^{3/2}}\int \big|\big(\hat{w}(p-q),\beta\hat{\eta}(q)\big)\big|\,dq\right).
\end{aligned}$$

Since

$$\begin{aligned}
\big(\hat{w}(p-q),\beta\hat{\eta}(q)\big) &= \big(\hat{\Lambda}_+(p-q)\hat{w}(p-q),\beta\hat{\Lambda}_-(q)\hat{\eta}(q)\big) \\
&= \big(\hat{w}(p-q),\hat{\Lambda}_+(p-q)\beta\hat{\Lambda}_-(q)\hat{\eta}(q)\big),
\end{aligned}$$

we compute

$$\begin{aligned}
&4\hat{\Lambda}_+(p-q)\beta\hat{\Lambda}_-(q) \\
&\quad = \left(\mathsf{I} + \frac{m\beta}{\lambda(p-q)} + \frac{\boldsymbol{\alpha}\cdot(p-q)}{\lambda(p-q)}\right)\beta\left(\mathsf{I} - \frac{m\beta}{\lambda(q)} - \frac{\boldsymbol{\alpha}\cdot q}{\lambda(q)}\right) \\
&\quad = \beta\left(1 - \frac{m^2}{\lambda(q)\lambda(p-q)}\right) - \mathsf{I}\left(\frac{m}{\lambda(q)} - \frac{m}{\lambda(p-q)}\right) \\
&\qquad - \beta\boldsymbol{\alpha}\cdot\left(\frac{q}{\lambda(q)} + \frac{p-q}{\lambda(p-q)}\right) - \frac{m\boldsymbol{\alpha}\cdot\big(q+(p-q)\big)}{\lambda(q)\lambda(p-q)} \\
&\qquad + \frac{\beta}{\lambda(q)\lambda(p-q)}\big(\boldsymbol{\alpha}\cdot(p-q)\boldsymbol{\alpha}\cdot q\big)
\end{aligned}$$

$$= \beta\left(1 - \frac{m^2}{\lambda(q)\lambda(p-q)}\right) - \mathsf{I}\left(\frac{m}{\lambda(q)} - \frac{m}{\lambda(p-q)}\right)$$
$$- \beta\boldsymbol{\alpha}\cdot\left(\frac{q}{\lambda(q)} + \frac{p-q}{\lambda(p-q)}\right) - \frac{m\boldsymbol{\alpha}\cdot p}{\lambda(q)\lambda(p-q)} + \frac{\beta\boldsymbol{\Sigma}\cdot(p-q)\boldsymbol{\Sigma}\cdot q}{\lambda(q)\lambda(p-q)},$$

where

$$\Sigma_i = \begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix}.$$

We now estimate the different terms. First of all, from

$$m\big(|p-q| + |q|\big) \le \lambda(q)\lambda(p-q) \le |q||p-q| + m\big(|q| + |p-q|\big) + m^2$$

and

$$\big||q|\lambda(p-q) - |p-q|\lambda(q)\big| \le m|p|,$$

we deduce that

$$\left|\frac{|q|}{\lambda(q)} - \frac{|p-q|}{\lambda(p-q)}\right| = \left|\frac{|q|\lambda(p-q) - |p-q|\lambda(q)}{\lambda(q)\lambda(p-q)}\right| \le \frac{m|p|}{\lambda(q)\lambda(p-q)}$$

and

$$\left|\frac{m}{\lambda(q)} - \frac{m}{\lambda(p-q)}\right| = m\frac{\big|\lambda(p-q) - \lambda(q)\big|}{\lambda(q)\lambda(p-q)} = m\frac{\big|m^2 + |p-q|^2 - m^2 - |q|^2\big|}{\lambda(q)\lambda(p-q)\big(\lambda(q) + \lambda(p-q)\big)}$$
$$\le \frac{\big||p-q| - |q|\big|}{\lambda(q) + \lambda(p-q)} \le \frac{|p|}{\lambda(q) + \lambda(p-q)}$$
$$\le \frac{|p|}{\big(\lambda(q) + m\big)^{1/2}\big(\lambda(p-q) + m\big)^{1/2}}.$$

Then

$$\left|\frac{\lambda(q)\lambda(p-q) - m^2}{\lambda(q)\lambda(p-q)}\right| \le \frac{|q||p-q| + m\big(|q| + |p-q|\big)}{\lambda(q)\lambda(p-q)}$$
$$\le \frac{|q||p-q|}{\lambda(q)\lambda(p-q)} + \frac{m|p|}{\lambda(q)\lambda(p-q)} + 2\frac{m|p-q|}{\lambda(q)\lambda(p-q)}$$

and

$$\left|\frac{q}{\lambda(q)} + \frac{p-q}{\lambda(p-q)}\right| \le \left|\frac{|q|}{\lambda(q)} - \frac{|p-q|}{\lambda(p-q)}\right| + 2\frac{|p-q|}{\lambda(p-q)}$$
$$\le \frac{m|p|}{\lambda(q)\lambda(p-q)} + 2\frac{|p-q|}{\lambda(p-q)}.$$

Since

$$\left|\frac{\beta\boldsymbol{\Sigma}\cdot(p-q)\boldsymbol{\Sigma}\cdot q}{\lambda(q)\lambda(p-q)}\right| \leq \frac{|q||p-q|}{\lambda(q)\lambda(p-q)},$$

we finally have

$$\begin{aligned}
&4\big|\big(\hat{w}(p-q),\beta\hat{\eta}(q)\big)\big| \\
&\quad \leq \left(\frac{3|q||p-q|+3m|p|+2m|p-q|}{\lambda(q)\lambda(p-q)}+\frac{2|p-q|}{\lambda(p-q)}\right)\big|\hat{w}(p-q)\big|\big|\hat{\eta}(q)\big| \\
&\qquad +\frac{|p|}{\big(\lambda(q)+m\big)^{1/2}\big(\lambda(p-q)+m\big)^{1/2}}\big|\hat{w}(p-q)\big|\big|\hat{\eta}(q)\big|.
\end{aligned}$$

Let us analyze the different terms:

$$\begin{aligned}
&\sqrt{\frac{2}{\pi}}\int\frac{dp}{|p|^2}\left(\frac{1}{(2\pi)^{3/2}}\int\frac{|p-q|\big|\hat{w}(p-q)\big|}{\lambda(p-q)}\frac{|q|\big|\hat{\eta}(q)\big|}{\lambda(q)}\,dq\right) \\
&\quad =\sqrt{\frac{2}{\pi}}\int\frac{1}{|p|^2}\mathcal{F}\left[\mathcal{F}^{-1}\left[\frac{|p|\big|\hat{w}(p)\big|}{\lambda(p)}\right]\mathcal{F}^{-1}\left[\frac{|p|\big|\hat{\eta}(p)\big|}{\lambda(p)}\right]\right]dp \\
&\quad =\int\frac{1}{|x|}\mathcal{F}^{-1}\left[\frac{|p|\big|\hat{w}(p)\big|}{\lambda(p)}\right]\mathcal{F}^{-1}\left[\frac{|p|\big|\hat{\eta}(p)\big|}{\lambda(p)}\right]dx \\
&\quad \leq\left|\frac{1}{|x|^{1/2}}\mathcal{F}^{-1}\left[\frac{|p|\big|\hat{w}(p)\big|}{\lambda(p)}\right]\right|_2\left|\frac{1}{|x|^{1/2}}\mathcal{F}^{-1}\left[\frac{|p|\big|\hat{\eta}(p)\big|}{\lambda(p)}\right]\right|_2 \\
&\quad \leq\kappa\left|(-\Delta)^{1/4}\mathcal{F}^{-1}\left[\frac{|p|\big|\hat{w}(p)\big|}{\lambda(p)}\right]\right|_2\left|(-\Delta)^{1/4}\mathcal{F}^{-1}\left[\frac{|p|\big|\hat{\eta}(p)\big|}{\lambda(p)}\right]\right|_2 \\
&\quad \leq 2\kappa\sqrt{\|w\|^2-m|w|_2^2}\sqrt{\|\eta\|^2-m|\eta|_2^2}
\end{aligned}$$

since

$$\begin{aligned}
&\left|(-\Delta)^{1/4}\mathcal{F}^{-1}\left[\frac{|p|\big|\hat{w}(p)\big|}{\lambda(p)}\right]\right|_2^2=\int\frac{|p|^3\big|\hat{w}(p)\big|^2}{\lambda(p)^2}\,dp \\
&\qquad \leq 2\int\big(\sqrt{|p|^2+m^2}-m\big)\big|\hat{w}(p)\big|^2\,dp=2\big(\|w\|^2-m|w|_2^2\big)
\end{aligned}$$

and

$$\begin{aligned}
&\sqrt{\frac{2}{\pi}}\int\frac{dp}{|p|^2}\left(\frac{1}{(2\pi)^{3/2}}\int\frac{|p-q|\big|\hat{w}(p-q)\big|}{\lambda(p-q)}\frac{m\big|\hat{w}(q)\big|}{\lambda(q)}\,dq\right) \\
&\quad \leq\kappa\left|(-\Delta)^{1/4}\mathcal{F}^{-1}\left[\frac{|p|\big|\hat{w}(p)\big|}{\lambda(p)}\right]\right|_2\left|(-\Delta)^{1/4}\mathcal{F}^{-1}\left[\frac{m\big|\hat{\eta}(p)\big|}{\lambda(p)}\right]\right|_2 \\
&\quad \leq\sqrt{m}\kappa|\eta|_2\sqrt{\|w\|^2-m|w|_2^2}
\end{aligned}$$

since

$$\left|(-\Delta)^{1/4}\mathcal{F}^{-1}\left[\frac{m|\hat{\eta}(p)|}{\lambda(p)}\right]\right|_2^2 = \int \frac{m^2|p||\hat{\eta}(p)|^2}{\lambda(p)^2}\,dp \le \frac{m}{2}|\eta|_2^2.$$

We also have that

$$\begin{aligned}
&\sqrt{\frac{2}{\pi}}\int \frac{dp}{|p|^2}\left(\frac{1}{(2\pi)^{3/2}}\int |p|\frac{|\hat{w}(p-q)|}{\big(\lambda(p-q)+m\big)^{1/2}}\frac{|\hat{\eta}(q)|}{\big(\lambda(q)+m\big)^{1/2}}\,dq\right)\\
&\quad= \sqrt{\frac{2}{\pi}}\int \frac{1}{|p|}\mathcal{F}\left[\mathcal{F}^{-1}\left[\frac{|\hat{w}(p)|}{\big(\lambda(p)+m\big)^{1/2}}\right]\mathcal{F}^{-1}\left[\frac{|p||\hat{\eta}(p)|}{\big(\lambda(p)+m\big)^{1/2}}\right]\right]dp\\
&\quad= \frac{2}{\pi}\int \frac{1}{|x|^2}\mathcal{F}^{-1}\left[\frac{|\hat{w}(p)|}{\big(\lambda(p)+m\big)^{1/2}}\right]\mathcal{F}^{-1}\left[\frac{|\hat{\eta}(p)|}{\big(\lambda(p)+m\big)^{1/2}}\right]dx\\
&\quad\le \frac{2}{\pi}\left|\frac{1}{|x|}\mathcal{F}^{-1}\left[\frac{|\hat{w}(p)|}{\big(\lambda(p)+m\big)^{1/2}}\right]\right|_2\left|\frac{1}{|x|^{1/2}}\mathcal{F}^{-1}\left[\frac{|\hat{\eta}(p)|}{\big(\lambda(p)+m\big)^{1/2}}\right]\right|_2\\
&\quad\le \frac{8}{\pi}\left|\frac{|p||\hat{w}(p)|}{\big(\lambda(p)+m\big)^{1/2}}\right|_2\left|\frac{|p||\hat{\eta}(p)|}{\big(\lambda(p)+m\big)^{1/2}}\right|_2\\
&\quad\le 2\kappa\sqrt{\|w\|^2-m|w|_2^2}\sqrt{\|\eta\|^2-m|\eta|_2^2}
\end{aligned}$$

and

$$\begin{aligned}
&\sqrt{\frac{2}{\pi}}\int \frac{dp}{|p|^2}\left(\frac{1}{(2\pi)^{3/2}}\int m|p|\frac{|\hat{w}(p-q)|}{\lambda(p-q)}\frac{|\hat{\eta}(q)|}{\lambda(q)}\,dq\right)\\
&\quad\le \frac{8}{\pi}\left|\frac{\sqrt{m}|p||\hat{w}(p)|}{\lambda(p)}\right|_2\left|\frac{\sqrt{m}|p||\hat{\eta}(p)|}{\lambda(p)}\right|_2\\
&\quad\le 5\kappa\sqrt{\|w\|^2-m|w|_2^2}\sqrt{\|\eta\|^2-m|\eta|_2^2}
\end{aligned}$$

(we have used the fact that $\frac{mp^2}{p^2+m^2} \le \frac{5}{2}(\sqrt{p^2+m^2}-m)$).

Finally, we have

$$\begin{aligned}
&\sqrt{\frac{2}{\pi}}\int \frac{dp}{|p|^2}\left(\frac{1}{(2\pi)^{3/2}}\int \frac{|p-q||\hat{w}(p-q)||\hat{w}(q)|}{\lambda(p-q)}\,dq\right)\\
&\quad\le \kappa\left|(-\Delta)^{1/4}\mathcal{F}^{-1}\left[\frac{|p||\hat{w}(p)|}{\lambda(p)}\right]\right|_2\left|(-\Delta)^{1/4}\mathcal{F}^{-1}\big[|\hat{\eta}(p)|\big]\right|_2\\
&\quad\le \sqrt{2}\kappa\|\eta\|\sqrt{\|w\|^2-m|w|_2^2}.
\end{aligned}$$

We now collect the terms:

$$
\begin{aligned}
&4a(\eta)\sqrt{\frac{2}{\pi}}\int \frac{dp}{|p|^2}\Big(\frac{1}{(2\pi)^{3/2}}\int \big|\big(\hat{w}(p-q),\beta\hat{\eta}(q)\big)\big|\,dq\Big)\\
&\quad\le 23\kappa a(\eta)\sqrt{\|w\|^2-m|w|_2^2}\sqrt{\|\eta\|^2-m|\eta|_2^2}\\
&\qquad + 2\sqrt{m}\kappa a(\eta)|\eta|_2\sqrt{\|w\|^2-m|w|_2^2} + 2\sqrt{2}\kappa a(\eta)\|\eta\|\sqrt{\|w\|^2-m|w|_2^2}\\
&\quad\le 14a(\eta)^2\kappa\big(\|w\|^2-m|w|_2^2\big) + 15\kappa\|\eta\|^2
\end{aligned}
$$

to deduce that

$$
\begin{aligned}
&\int \frac{(\psi,\beta\psi)(x)(\psi,\beta\psi)(y)}{|x-y|} - \int \frac{(w,\beta w)(x)(w,\beta w)(y)}{|x-y|}\\
&\quad\ge -2\kappa|\eta|_2^2\|w\|^2 - 3a(\eta)^2\kappa\|\eta\|^2\\
&\qquad - 4a(\eta)\sqrt{\frac{2}{\pi}}\int \frac{dp}{|p|^2}\Big(\frac{1}{(2\pi)^{3/2}}\int \big|\big(\hat{w}(p-q),\beta\hat{\eta}(q)\big)\big|\,dq\Big)\\
&\quad\ge -14a(\eta)^2\kappa\big(\|w\|^2-m|w|_2^2\big) - 2\kappa|\eta|_2^2\|w\|^2 - 18\kappa\|\eta\|^2. \qquad \blacksquare
\end{aligned}
$$

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