Algebraic Geometry. – *On the singularities and on the projective normality of some Enriques–Fano threefolds*, by Vincenzo Martello, communicated on 13 May 2022.

Abstract. – In order to find useful information to complete the classification of Enriques–Fano threefolds, we will computationally study the singularities of some known Enriques–Fano threefolds of genus 6, 7, 8, 9, 10, 13, and 17. We will also deduce the projective normality of these threefolds.

Keywords. – Enriques–Fano threefolds, singularities, Macaulay2, projective normality, linear system of surfaces.

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1. Introduction

An *Enriques–Fano threefold* is a normal threefold W endowed with a complete linear system $\mathcal{L}$ of ample Cartier divisors such that the general element $S \in \mathcal{L}$ is an Enriques surface and such that W is not a *generalized cone* over S; i.e., W is not obtained by contraction of the negative section on the $\mathbb{P}^1$-bundle $\mathbb{P}(\mathcal{O}_S \oplus \mathcal{O}_S(S))$ over S. The linear system $\mathcal{L}$ defines a rational map $\phi_{\mathcal{L}} : W \dashrightarrow \mathbb{P}^p$, where $p := \frac{S^3}{2} + 1$ is called the *genus* of W and $2 \leq p \leq 17$ (see [16, 20]). If the elements of $\mathcal{L}$ are very ample divisors, then W is embedded in $\mathbb{P}^p$ via $\phi_{\mathcal{L}}$ as a non-degenerate threefold whose general hyperplane section is an Enriques surface. It is known that any Enriques–Fano threefold is singular with isolated canonical singularities (see [7, Lemma 3.2] and [3]). We will say that two distinct singular points of an Enriques–Fano threefold W are *associated* if the line joining them is contained in W. The way in which the singular points of an Enriques–Fano threefold are associated is called *configuration* and can be represented graphically: if two singular points are associated, one draws a segment joining them, otherwise not. In particular, if there is an m such that each singular point is associated with exactly m other singular points, we say that the configuration is *regular*. In Appendix B, we will graphically describe the configurations that we will find in this paper.

Definition 1.1. The singular points of an Enriques–Fano threefold W are said to be *similar* if they have the same multiplicity, they have biregular tangent cone, and their configuration is regular.

The first examples of Enriques–Fano threefolds were discovered by Fano (see [12]), under the assumption that their singularities were similar and that the blow-up at the singular points was sufficient to resolve them. It must be said that Fano's arguments contain many gaps, some of which have been solved by Conte and Murre (see [7], where the authors also re-proved a result of the paper [14] of Godeaux, useful for the arguments of Fano). By using a sort of inverse of [7, Theorem 7.2], Fano constructed four rational Enriques–Fano threefolds having eight similar singular points whose configurations are the ones in Table 1 of Appendix B (see also [18]). They are as follows:

(i) the Enriques–Fano threefold $W_F^6 \subset \mathbb{P}^6$ of genus 6 given by the image of the rational map defined by the linear system of the septic surfaces with double points along three twisted cubics having five points in common (see [12, Section 3]);

(ii) the Enriques–Fano threefold $W_F^7 \subset \mathbb{P}^7$ of genus 7 given by the image of the rational map defined by the linear system of the sextic surfaces having double points along the six edges of a tetrahedron and containing a plane cubic curve intersecting each edge at one point (see [12, Section 4]);

(iii) the Enriques–Fano threefold $W_F^9 \subset \mathbb{P}^9$ of genus 9 given by the image of the rational map defined by the linear system of the septic surfaces having double points along the six edges of two trihedra (see [12, Section 7]);

(iv) the Enriques–Fano threefold $W_F^{13} \subset \mathbb{P}^{13}$ of genus 13 given by the image of the rational map defined by the linear system of the sextic surfaces having double points along the six edges of a tetrahedron (see [12, Section 8]).

Fano also found an "exceptional" example of genus 4, which is a sextic hypersurface W_F^4 of $\mathbb{P}^4$ whose general hyperplane section is a sextic surface of $\mathbb{P}^3$ having double points along the six edges of a tetrahedron (see [12, Section 10]). The threefold W_F^4 has been proved to be *non-rational* (see [2]). Furthermore, as noted by Conte in [5, p. 225], there is also another exceptional example of genus 3, which is the threefold W_F^3 given by a quadruple $\mathbb{P}^3$ (see [12, Section 2]). We will refer to the above threefolds $W_F^{p=3,4,6,7,9,13}$ as *F-EF 3-folds*. However, Fano's classification is incomplete: indeed, it fails to include some other Enriques–Fano threefolds which have been discovered. Under the assumption that the singularities are terminal cyclic quotients, Enriques–Fano threefolds were classified by Bayle (see [1]) and, in a similar and independent way, by Sano (see [22]). We will refer to such Enriques–Fano threefolds as *BS-EF 3-folds*: they are fourteen and they have genus $2 \leq p \leq 10$ or $p = 13$. Six of them are endowed with a linear system of very ample divisors (see [1, Theorem A]): they have genus $p = 6, 7, 8, 9, 10, 13$ and we will denote them, respectively, by W_{BS}^6, W_{BS}^7, W_{BS}^8, W_{BS}^9, W_{BS}^{10}, W_{BS}^{13}. We recall that a fixed BS-EF 3-fold W is the quotient of a smooth Fano

threefold X via an involution σ having 8 fixed points, and that W has 8 quadruple points whose tangent cone is a cone over a Veronese surface. More generally, an Enriques–Fano threefold with only terminal singularities is a limit of some BS-EF 3-fold (see [19, Main Theorem 2]). Instead, only a few examples of Enriques–Fano threefolds with non-terminal canonical singularities are known: two of genus $p = 13, 17$ found by Prokhorov (see [20, Proposition 3.2 and Remark 3.3]) and one of genus $p = 9$ found by Knutsen, Lopez, and Muñoz (see [16, Section 13]). The Enriques–Fano threefolds of genus 13 and 17 found by Prokhorov (shortly, *P-EF 3-folds*) are obtained as the quotient of singular Fano threefolds V via an involution τ having five fixed points: we will denote them by W_P^{13} and W_P^{17}, respectively. The Enriques–Fano threefold found by Knutsen–Lopez–Muñoz (shortly, *KLM-EF 3-fold*) is a threefold $W_{KLM}^9 \subset \mathbb{P}^9$ given by the projection of the F-EF 3-fold $W_F^{13} \subset \mathbb{P}^{13}$ from the $\mathbb{P}^3$ spanned by a certain elliptic quartic curve $E_3 \subset W_F^{13}$.

In order to find useful information to complete the classification of Enriques–Fano threefolds, we will study the threefolds $W_{BS}^{p=6,7,8,9,10,13}$, $W_P^{p=13,17}$, W_{KLM}^9. We will do this from a computational point of view, with the help of the software Macaulay2. In Appendix A, we will collect the input codes used in Macaulay2.

Since the F-EF 3-folds $W_F^{p=6,7,9,13}$ have eight quadruple points whose tangent cone is a cone over a Veronese surface (see [12, p. 44]), then they only have terminal singularities (see [21, Example 1.3]) and therefore they are limits of the BS-EF 3-folds $W_{BS}^{p=6,7,9,13}$ (see [19, Main Theorem 2]). We can say something more about this link between F-EF 3-folds and BS-EF 3-folds. Indeed, in Sections 2, 3, 4, and 5 we will prove the following results.

THEOREM 1.2. *Let $p \in \{7, 13\}$. Then the eight quadruple points of the BS-EF 3-fold $W_{BS}^p \hookrightarrow \mathbb{P}^p$ are similar and they have the same configuration as the ones of the F-EF 3-fold $W_F^p \subset \mathbb{P}^p$. Furthermore, we have the following facts:*

(i) *the ideal of $W_{BS}^6 \subset \mathbb{P}^6$ is generated by cubics;*

(ii) *the ideal of $W_{BS}^7 \subset \mathbb{P}^7$ is generated by quadrics and cubics;*

(iii) *the ideal of $W_{BS}^9 \subset \mathbb{P}^9$ is generated by quadrics;*

(iv) *the ideal of $W_{BS}^{13} \subset \mathbb{P}^{13}$ is generated by quadrics.*

In Sections 2 and 4, we will see that the eight quadruple points of particular examples of BS-EF 3-folds of genus 6 (respectively, genus 9) are also similar and they have the same configuration as the ones of the F-EF 3-fold of genus 6 (respectively, genus 9). It would be interesting to show it for general W_{BS}^6 and W_{BS}^9.

THEOREM 1.3. *The embedding of the BS-EF 3-fold of genus 13 in $\mathbb{P}^{13}$ is a F-EF 3-fold of genus 13.*

In Section 4, we will also prove that particular BS-EF 3-folds of genus 9 are isomorphic to F-EF 3-folds of genus 9. It would be interesting to show it for general W_{BS}^9.

We have other links between F-EF 3-folds and BS-EF 3-folds. The BS-EF 3-fold of genus 4 described in [1, Section 6.33] (see also [22, Theorem 1.1 No. 5]) is endowed with a linear system defining a birational map onto the image, which is the F-EF 3-fold $W_F^4 \subset \mathbb{P}^4$. The BS-EF 3-fold of genus 3 described in [1, Section 6.1.5] (see also [22, Theorem 1.1 No. 2]) is endowed with a linear system defining a quadruple cover over $\mathbb{P}^3$, so it looks like the F-EF 3-fold W_F^3. This suggests that one could obtain the BS-EF 3-folds with ample (but not very ample) hyperplane sections by resuming the Fano–Conte–Murre techniques and by undermining some assumptions. Thus, reviewing the brilliant ideas of Fano with the techniques of Conte and Murre, would be very interesting, since new Enriques–Fano threefolds could be found, even if this problem seems not to have been studied yet. However, one must be careful of mistakes in resuming Fano's paper. For example, the BS-EF 3-folds W_{BS}^8 and W_{BS}^{10} do not appear in the description of Fano, although they behave like the other BS-EF 3-folds with very ample hyperplane sections. It is a situation that should be understood better. The problem could be the fact that Fano stated that if the genus is greater than 7, then there are no three mutually associated singular points (see [12, Section 5]). Indeed, we will see that the singularities of both threefolds W_{BS}^8 and W_{BS}^{10} have configurations which are in contradiction with this assertion. In Sections 6 and 7, we will prove the following result.

THEOREM 1.4. *Let $p \in \{8, 10\}$. The ideal of $W_{\mathrm{BS}}^p \subset \mathbb{P}^p$ is generated by quadrics and cubics. Furthermore, the eight singular points of W_{BS}^{10} are similar and they have the configuration in Table 2 of Appendix 2.*

In Section 6, we will show that the eight singular points of particular examples of B-EF 3-folds of genus 8 are also similar and they have the configuration in Table 2 of Appendix 2. It would be interesting to show it for a general W_{BS}^8.

Furthermore, we will identify W_{BS}^8 and W_{BS}^{10} as images of rational maps defined by linear systems of surfaces of $\mathbb{P}^3$, as happens for the rational F-EF 3-folds (see Theorems 6.1 and 7.1). We will also study the singularities of the P-EF 3-folds and the KLM-EF 3-fold. It is known that these threefolds have canonical non-terminal singularities, but so far there was no information about their multiplicities and tangent cones. This analysis is very interesting because it provides some clues about the link between the non-terminality of the singularities, the non-similarity of the singularities, and the fact that the blow-up at the singularities is not sufficient to resolve them. In Sections 8 and 9, we will prove the following result.

THEOREM 1.5. *Let $p \in \{13, 17\}$. Then the P-EF 3-fold of genus p can be embedded in $\mathbb{P}^p$ and its ideal is generated by quadrics. Furthermore, W_P^p has five non-similar singular points whose configuration is the one in Table 3 of Appendix B. Moreover, four of these five points are quadruple points, whose tangent cone is a cone over a Veronese surface. If $p = 13$, the last point is a quintuple point, whose tangent cone is a cone over the union of five planes; if $p = 17$, it is a sextuple point, whose tangent cone is a cone over the union of four planes and a quadric surface.*

For further details, see also Propositions 8.1, 9.1 and Theorems 8.2, 9.2. As for the KLM-EF 3-fold, we refer to Section 10. We will find that the ideal of W_{KLM}^9 in $\mathbb{P}^9$ is generated by quadrics and cubics. We will see that the images of the eight quadruple points of W_F^{13}, via the projection map, are five singular points of W_{KLM}^9 such that four of them are quadruple points, whose tangent cone is a cone over a Veronese surface (see Proposition 10.2), and one is a sextuple point, whose tangent cone is a cone over the union of four planes and a quadric surface (see Theorem 10.3).

Finally, in Section 11, we will study the projective normality of all the above Enriques–Fano threefolds, obtaining the following result.

THEOREM 1.6. *The following Enriques–Fano threefolds are projectively normal:*

$$W_{KLM}^9 \subset \mathbb{P}^9, \quad W_F^{p=7,9,13} \subset \mathbb{P}^p, \quad W_{BS}^{p=6,7,8,9,10,13} \xrightarrow{\phi_{\mathcal{L}}} \mathbb{P}^p, \quad W_P^{p=13,17} \subset \mathbb{P}^p.$$

We will work over the field $\mathbb{C}$ of the complex numbers. For the computational analysis we will work over a finite field (we will choose $\mathbb{F}_n := \mathbb{Z}/n\mathbb{Z}$ with $n = 10000019$).

2. THE BS-EF 3-FOLD OF GENUS 6

Let us study the Enriques–Fano threefold described in [1, Section 6.2.4] (see also [22, Theorem 1.1 No. 9]). We refer to Section A.1 of Appendix A for the computational techniques we will use. Let us take the smooth Fano threefold X given by the intersection of three divisors of bidegree $(1, 1)$ of $\mathbb{P}^3_{[x_0:\cdots:x_3]} \times \mathbb{P}^3_{[y_0:\cdots:y_3]}$ with equations

$$\sum_{i=0}^{3} \sum_{j=0}^{3} a_{ij} x_i y_j = 0, \quad \sum_{i=0}^{3} \sum_{j=0}^{3} b_{ij} x_i y_j = 0, \quad \sum_{i=0}^{3} \sum_{j=0}^{3} c_{ij} x_i y_j = 0,$$

where $a_{ij} = a_{ji}$, $b_{ij} = b_{ji}$, $c_{ij} = c_{ji}$, for $i, j \in \{0, 1, 2, 3\}$. Let $\sigma : X \to X$ be the involution of X defined by the restriction on X of the following map:

$$\sigma' : \mathbb{P}^3 \times \mathbb{P}^3 \to \mathbb{P}^3 \times \mathbb{P}^3,$$

$$[x_0 : \cdots : x_3] \times [y_0 : \cdots : y_3] \mapsto [y_0 : \cdots : y_3] \times [x_0 : \cdots : x_3].$$

We have that σ has eight fixed points p_1, p_2, p_3, p_4, p_5, p_6, p_7, and p_8 with coordinates $[x_0 : x_1 : x_2 : x_3] \times [x_0 : x_1 : x_2 : x_3]$ such that

$$\begin{cases} \sum_{i=0}^3 \sum_{j=0}^3 a_{ij} x_i x_j = 0, \\ \sum_{i=0}^3 \sum_{j=0}^3 b_{ij} x_i x_j = 0, \\ \sum_{i=0}^3 \sum_{j=0}^3 c_{ij} x_i x_j = 0. \end{cases}$$

The quotient map $\pi : X \to X/\sigma =: W_{\mathrm{BS}}^6$ is given by the restriction on X of the morphism $\varphi : \mathbb{P}^3 \times \mathbb{P}^3 \to \mathbb{P}^9$ defined by the σ'-invariant multihomogeneous polynomials of multidegree $(1, 1)$. Thus $\varphi : [x_0 : \cdots : x_3] \times [y_0 : \cdots : y_3] \mapsto [Z_0 : \cdots : Z_9]$, where $Z_0 = x_0 y_0$, $Z_1 = x_1 y_1$, $Z_2 = x_2 y_2$, $Z_3 = x_3 y_3$, $Z_4 = x_0 y_1 + x_1 y_0$, $Z_5 = x_0 y_2 + x_2 y_0$, $Z_6 = x_0 y_3 + x_3 y_0$, $Z_7 = x_1 y_2 + x_2 y_1$, $Z_8 = x_1 y_3 + x_3 y_1$, $Z_9 = x_2 y_3 + x_3 y_2$. By using Macaulay2, one can find that the image of $\mathbb{P}^3 \times \mathbb{P}^3$ via φ is a 6-dimensional algebraic variety F_6^{10} of degree 10, whose ideal is generated by the 10 polynomials

$$-2Z_1 Z_5 Z_6 + Z_4 Z_6 Z_7 + Z_4 Z_5 Z_8 - 2Z_0 Z_7 Z_8 + 4Z_0 Z_1 Z_9 - Z_4^2 Z_9,$$

$$-2Z_2 Z_4 Z_6 + Z_5 Z_6 Z_7 + 4Z_0 Z_2 Z_8 - Z_5^2 Z_8 + Z_4 Z_5 Z_9 - 2Z_0 Z_7 Z_9,$$

$$-4Z_1 Z_2 Z_6 + Z_6 Z_7^2 + 2Z_2 Z_4 Z_8 - Z_5 Z_7 Z_8 + 2Z_1 Z_5 Z_9 - Z_4 Z_7 Z_9,$$

$$-2Z_3 Z_4 Z_5 + 4Z_0 Z_3 Z_7 - Z_6^2 Z_7 + Z_5 Z_6 Z_8 + Z_4 Z_6 Z_9 - 2Z_0 Z_8 Z_9,$$

$$-4Z_1 Z_3 Z_5 + 2Z_3 Z_4 Z_7 - Z_6 Z_7 Z_8 + Z_5 Z_8^2 + 2Z_1 Z_6 Z_9 - Z_4 Z_8 Z_9,$$

$$-4Z_2 Z_3 Z_4 + 2Z_3 Z_5 Z_7 + 2Z_2 Z_6 Z_8 - Z_6 Z_7 Z_9 - Z_5 Z_8 Z_9 + Z_4 Z_9^2,$$

$$-4Z_1 Z_2 Z_3 + Z_3 Z_7^2 + Z_2 Z_8^2 - Z_7 Z_8 Z_9 + Z_1 Z_9^2,$$

$$-4Z_0 Z_2 Z_3 + Z_3 Z_5^2 + Z_2 Z_6^2 - Z_5 Z_6 Z_9 + Z_0 Z_9^2,$$

$$-4Z_0 Z_1 Z_3 + Z_3 Z_4^2 + Z_1 Z_6^2 - Z_4 Z_6 Z_8 + Z_0 Z_8^2,$$

$$-4Z_0 Z_1 Z_2 + Z_2 Z_4^2 + Z_1 Z_5^2 - Z_4 Z_5 Z_7 + Z_0 Z_7^2.$$

Let us observe that $W_{\mathrm{BS}}^6 = \varphi(X) = F_6^{10} \cap H_6$, where H_6 is the 6-dimensional projective subspace of $\mathbb{P}^9$ given by the zero locus of the three polynomials

$$a_{00} Z_0 + a_{11} Z_1 + a_{22} Z_2 + a_{33} Z_3 + 2a_{01} Z_4 + 2a_{02} Z_5 + 2a_{03} Z_6$$
$$+ 2a_{12} Z_7 + 2a_{13} Z_8 + 2a_{23} Z_9,$$

$$b_{00} Z_0 + b_{11} Z_1 + b_{22} Z_2 + b_{33} Z_3 + 2b_{01} Z_4 + 2b_{02} Z_5 + 2b_{03} Z_6$$
$$+ 2b_{12} Z_7 + 2b_{13} Z_8 + 2b_{23} Z_9,$$

$$c_{00} Z_0 + c_{11} Z_1 + c_{22} Z_2 + c_{33} Z_3 + 2c_{01} Z_4 + 2c_{02} Z_5$$
$$+ 2c_{03} Z_6 + 2c_{12} Z_7 + 2c_{13} Z_8 + 2c_{23} Z_9.$$

Therefore, we have $\pi = \varphi|_X : X \to W^6_{\mathrm{BS}} = \varphi(X) \subset H_6 \cong \mathbb{P}^6$. What follows has been proved for fixed values of a_{ij}, b_{ij}, and c_{ij}, in order to simplify the computational analysis.

2.1. Example

Let us take

$$
(a_{ij}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -7 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix},
$$

$$
(b_{ij}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -6 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix},
$$

$$
(c_{ij}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -7 & 0 \\ 0 & 0 & 0 & 7 \end{pmatrix}.
$$

Then the eight fixed points of $\sigma : X \to X$ are

$$
\begin{aligned}
p_1 &= [1 : 1 : 1 : 1] \times [1 : 1 : 1 : 1], \\
p_2 &= [-1 : 1 : 1 : 1] \times [-1 : 1 : 1 : 1], \\
p_3 &= [1 : -1 : 1 : 1] \times [1 : -1 : 1 : 1], \\
p_4 &= [-1 : -1 : 1 : 1] \times [-1 : -1 : 1 : 1], \\
p_5 &= [1 : 1 : -1 : 1] \times [1 : 1 : -1 : 1], \\
p_6 &= [-1 : 1 : -1 : 1] \times [-1 : 1 : -1 : 1], \\
p_7 &= [1 : -1 : -1 : 1] \times [1 : -1 : -1 : 1], \\
p_8 &= [-1 : -1 : -1 : 1] \times [-1 : -1 : -1 : 1].
\end{aligned}
$$

Furthermore, we have

$$
\begin{aligned}
H_6 &= \{Z_0 - 7Z_1 + 4Z_2 + 2Z_3 = 0, \ Z_0 - 6Z_1 + 2Z_2 + 3Z_3 = 0, \\
&\quad Z_0 - Z_1 - 7Z_2 + 7Z_3 = 0\} \\
&= \{Z_2 - Z_3 = 0, \ Z_1 - Z_3 = 0, \ Z_0 - Z_3 = 0\},
\end{aligned}
$$

which is the $\mathbb{P}^6_{[w_0:\cdots:w_6]}$ embedded in $\mathbb{P}^9_{[Z_0:\cdots:Z_9]}$ via the morphism such that

$$Z_i = w_0, \ i = 0, 1, 2, 3, \quad Z_j = w_{j-3}, \ j = 4, \ldots, 9.$$

By using Macaulay2, we find that the quotient map $\pi : X \to W^6_{\mathrm{BS}} \subset H_6 \cong \mathbb{P}^6$ is given by the restriction on X of the morphism $\varphi' : \mathbb{P}^3 \times \mathbb{P}^3 \to \mathbb{P}^6$ defined by $[x_0 : x_1 : x_2 : y_3 : y_4 : y_5] \mapsto [w_0 : \cdots : w_6]$, where

$$w_0 = x_3 y_3, \quad w_1 = x_0 y_1 + x_1 y_0, \quad w_2 = x_0 y_2 + x_2 y_0,$$
$$w_3 = x_0 y_3 + x_3 y_0, \quad w_4 = x_1 y_2 + x_2 y_1, \quad w_5 = x_1 y_3 + x_3 y_1,$$
$$w_6 = x_2 y_3 + x_3 y_2.$$

Thanks to Macaulay2, we obtain that this BS-EF 3-fold $W^6_{\mathrm{BS}} \subset \mathbb{P}^6$ has ideal generated by the 10 polynomials

$$-2w_0 w_2 w_3 + w_1 w_3 w_4 + w_1 w_2 w_5 - 2w_0 w_4 w_5 + 4w_0^2 w_6 - w_1^2 w_6,$$
$$-2w_0 w_1 w_3 + w_2 w_3 w_4 + 4w_0^2 w_5 - w_2^2 w_5 + w_1 w_2 w_6 - 2w_0 w_4 w_6,$$
$$-4w_0^2 w_3 + w_3 w_4^2 + 2w_0 w_1 w_5 - w_2 w_4 w_5 + 2w_0 w_2 w_6 - w_1 w_4 w_6,$$
$$-2w_0 w_1 w_2 + 4w_0^2 w_4 - w_3^2 w_4 + w_2 w_3 w_5 + w_1 w_3 w_6 - 2w_0 w_5 w_6,$$
$$-4w_0^2 w_2 + 2w_0 w_1 w_4 - w_3 w_4 w_5 + w_2 w_5^2 + 2w_0 w_3 w_6 - w_1 w_5 w_6,$$
$$-4w_0^2 w_1 + 2w_0 w_2 w_4 + 2w_0 w_3 w_5 - w_3 w_4 w_6 - w_2 w_5 w_6 + w_1 w_6^2,$$
$$-4w_0^3 + w_0 w_4^2 + w_0 w_5^2 - w_4 w_5 w_6 + w_0 w_6^2,$$
$$-4w_0^3 + w_0 w_2^2 + w_0 w_3^2 - w_2 w_3 w_6 + w_0 w_6^2,$$
$$-4w_0^3 + w_0 w_1^2 + w_0 w_3^2 - w_1 w_3 w_5 + w_0 w_5^2,$$
$$-4w_0^3 + w_0 w_1^2 + w_0 w_2^2 - w_1 w_2 w_4 + w_0 w_4^2.$$

Furthermore, W^6_{BS} has eight singular points $P_i := \pi(p_i)$, for $1 \leq i \leq 8$; they are

$$P_1 = [1 : 2 : 2 : 2 : 2 : 2 : 2], \qquad P_2 = [1 : -2 : -2 : -2 : 2 : 2 : 2],$$
$$P_3 = [1 : -2 : 2 : 2 : -2 : -2 : 2], \quad P_4 = [1 : 2 : -2 : -2 : -2 : -2 : 2],$$
$$P_5 = [1 : 2 : -2 : 2 : -2 : 2 : -2], \quad P_6 = [1 : -2 : 2 : -2 : -2 : 2 : -2],$$
$$P_7 = [1 : -2 : -2 : 2 : 2 : -2 : -2], \quad P_8 = [1 : 2 : 2 : -2 : 2 : -2 : -2].$$

One can verify that all the lines joining the points P_i and P_j, for $1 \leq i < j \leq 8$, are contained in W^6_{BS}. So we can say that each one of the eight singular points of W^6_{BS} is associated with all the other $m = 7$ points, as in Table 1 of Appendix B. Thus, the singularities of the BS-EF 3-fold W^6_{BS} have the same configuration as the ones of the F-EF 3-fold W^6_F.

3. THE BS-EF 3-FOLD OF GENUS 7

Let us study the Enriques–Fano threefold described in [1, Section 6.4.1] (see also [22, Theorem 1.1 No. 11]). We refer to Section A.2 of Appendix A for the computational techniques we will use. Let X be the smooth Fano threefold given by a divisor of

$$\mathbb{P}^1_{[x_0:x_1]} \times \mathbb{P}^1_{[y_0:y_1]} \times \mathbb{P}^1_{[z_0:z_1]} \times \mathbb{P}^1_{[t_0:t_1]}$$

of type

$$\sum_{i+j+k+l \text{ odd}} a_{ijkl} x_i y_j z_k t_l = 0.$$

Let $\sigma : X \to X$ be the involution of X defined by the restriction on X of the map

$$\sigma' : \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$$

given by

$$[x_0 : x_1] \times [y_0 : y_1] \times [z_0 : z_1] \times [t_0 : t_1]$$
$$\mapsto [x_0 : -x_1] \times [y_0 : -y_1] \times [z_0 : -z_1] \times [t_0 : -t_1].$$

Then $\sigma : X \to X$ has the eight fixed points

$$p_1 = [0:1] \times [0:1] \times [0:1] \times [0:1], \quad p_1' = [1:0] \times [1:0] \times [1:0] \times [1:0],$$
$$p_2 = [0:1] \times [1:0] \times [1:0] \times [0:1], \quad p_2' = [1:0] \times [0:1] \times [0:1] \times [1:0],$$
$$p_3 = [1:0] \times [1:0] \times [0:1] \times [0:1], \quad p_3' = [0:1] \times [0:1] \times [1:0] \times [1:0],$$
$$p_4 = [1:0] \times [0:1] \times [1:0] \times [0:1], \quad p_4' = [0:1] \times [1:0] \times [0:1] \times [1:0].$$

The quotient map $\pi : X \to X/\sigma =: W_{\mathrm{BS}}^7$ is given by the restriction on X of the morphism $\varphi : \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^7$, defined by the σ'-invariant multihomogeneous polynomials of multidegree $(1, 1, 1, 1)$. In particular, we have

$$\varphi : [x_0 : x_1] \times [y_0 : y_1] \times [z_0 : z_1] \times [t_0 : t_1]$$
$$\mapsto [w_0 : w_1 : w_2 : w_3 : w_4 : w_5 : w_6 : w_7],$$

where

$$w_0 = x_1 y_1 z_1 t_1, \quad w_1 = x_1 y_0 z_0 t_1, \quad w_2 = x_0 y_0 z_1 t_1, \quad w_3 = x_1 y_0 z_1 t_0,$$
$$w_4 = x_0 y_0 z_0 t_0, \quad w_5 = x_0 y_1 z_1 t_0, \quad w_6 = x_1 y_1 z_0 t_0, \quad w_7 = x_0 y_1 z_0 t_1.$$

By fixing (random) values for $a_{0001}, a_{0010}, a_{0100}, a_{1000}, a_{1110}, a_{1101}, a_{1011}$, and a_{0111}, one can verify, with Macaulay2, that the ideal of the BS-EF 3-fold W_{BS}^7 is

generated by the following 11 polynomials of degree 2 or 3:

$$w_2 w_6 - w_3 w_7, \quad w_1 w_5 - w_3 w_7, \quad w_0 w_4 - w_3 w_7,$$

$$a_{1110} w_0 w_5 w_6 + a_{1011} w_0 w_3 w_7 + a_{0111} w_0 w_5 w_7 + a_{0010} w_3 w_5 w_7$$
$$+ a_{1101} w_0 w_6 w_7 + a_{1000} w_3 w_6 w_7 + a_{0100} w_5 w_6 w_7 + a_{0001} w_3 w_7^2,$$

$$a_{1000} w_1 w_4 w_6 + a_{1011} w_1 w_3 w_7 + a_{0001} w_1 w_4 w_7 + a_{0010} w_3 w_4 w_7$$
$$+ a_{1101} w_1 w_6 w_7 + a_{1110} w_3 w_6 w_7 + a_{0100} w_4 w_6 w_7 + a_{0111} w_3 w_7^2,$$

$$a_{0010} w_3 w_4 w_5 + a_{1000} w_3 w_4 w_6 + a_{1110} w_3 w_5 w_6 + a_{0100} w_4 w_5 w_6$$
$$+ a_{1011} w_3^2 w_7 + a_{0001} w_3 w_4 w_7 + a_{0111} w_3 w_5 w_7 + a_{1101} w_3 w_6 w_7,$$

$$a_{0010} w_2 w_4 w_5 + a_{1011} w_2 w_3 w_7 + a_{0001} w_2 w_4 w_7 + a_{1000} w_3 w_4 w_7$$
$$+ a_{0111} w_2 w_5 w_7 + a_{1110} w_3 w_5 w_7 + a_{0100} w_4 w_5 w_7 + a_{1101} w_3 w_7^2,$$

$$a_{1011} w_1 w_2 w_3 + a_{0001} w_1 w_2 w_4 + a_{1000} w_1 w_3 w_4 + a_{0010} w_2 w_3 w_4$$
$$+ a_{1101} w_1 w_3 w_7 + a_{0111} w_2 w_3 w_7 + a_{1110} w_3^2 w_7 + a_{0100} w_3 w_4 w_7,$$

$$a_{1011} w_0 w_2 w_3 + a_{0111} w_0 w_2 w_5 + a_{1110} w_0 w_3 w_5 + a_{0010} w_2 w_3 w_5$$
$$+ a_{1101} w_0 w_3 w_7 + a_{0001} w_2 w_3 w_7 + a_{1000} w_3^2 w_7 + a_{0100} w_3 w_5 w_7,$$

$$a_{1011} w_0 w_1 w_3 + a_{1101} w_0 w_1 w_6 + a_{1110} w_0 w_3 w_6 + a_{1000} w_1 w_3 w_6$$
$$+ a_{0111} w_0 w_3 w_7 + a_{0001} w_1 w_3 w_7 + a_{0010} w_3^2 w_7 + a_{0100} w_3 w_6 w_7,$$

$$a_{1011} w_0 w_1 w_2 + a_{1101} w_0 w_1 w_7 + a_{0111} w_0 w_2 w_7 + a_{0001} w_1 w_2 w_7$$
$$+ a_{1110} w_0 w_3 w_7 + a_{1000} w_1 w_3 w_7 + a_{0010} w_2 w_3 w_7 + a_{0100} w_3 w_7^2.$$

The eight singular points of W_{BS}^7 are $P_i := \pi(p_i) = \{w_k = 0 \mid k \neq i - 1\}$ and $P_i' := \pi(p_i') = \{w_k = 0 \mid k \neq 3 + i\}$, for $1 \leq i \leq 4$. Let $l_{i,j}$ be the line joining P_i and P_j with $i, j \in \{1, 2, 3, 4, 1', 2', 3', 4'\}$ and $i \neq j$. We have that W_{BS}^7 does not contain the lines $l_{1,1'}$, $l_{2,2'}$, $l_{3,3'}$, and $l_{4,4'}$, while it contains the others. So each one of the eight singular points of W_{BS}^7 is associated with $m = 6$ of the other singular points, as in Table 1 of Appendix B. Thus, the singularities of the BS-EF 3-fold W_{BS}^7 have the same configuration as the ones of the F-EF 3-fold W_F^7.

4. The BS-EF 3-fold of genus 9

Let us study the Enriques–Fano threefold described in [1, Section 6.1.4] (see also [22, Theorem 1.1 No. 12]). We refer to Section A.3 of Appendix A for the computational techniques we will use. Let us take two quadric hypersurfaces of $\mathbb{P}^5_{[x_0:x_1:x_2:y_3:y_4:y_5]}$,

$$Q_1 := \{s_1(x_0, x_1, x_2) + r_1(y_3, y_4, y_5) = 0\},$$
$$Q_2 := \{s_2(x_0, x_1, x_2) + r_2(y_3, y_4, y_5) = 0\},$$

where s_1, s_2, r_1, r_2 are the quadratic homogeneous forms

$$s_1(x_0, x_1, x_2) := \sum_{i,j \in \{0,1,2\}} a_{i,j} x_i x_j, \quad s_2(x_0, x_1, x_2) := \sum_{i,j \in \{0,1,2\}} a'_{i,j} x_i x_j,$$

$$r_1(y_3, y_4, y_5) := \sum_{i,j \in \{3,4,5\}} b_{i,j} y_i y_j, \quad r_2(y_3, y_4, y_5) := \sum_{i,j \in \{3,4,5\}} b'_{i,j} y_i y_j.$$

Let us consider the smooth Fano threefold $X := Q_1 \cap Q_2$ and the involution σ of X defined by the restriction on X of the morphism

$$\sigma' : \mathbb{P}^5 \to \mathbb{P}^5, \quad [x_0 : x_1 : x_2 : y_3 : y_4 : y_5] \mapsto [x_0 : x_1 : x_2 : -y_3 : -y_4 : -y_5].$$

Then $\sigma : X \to X$ has eight fixed points p_1, p_2, p_3, p_4, p'_1, p'_2, p'_3, p'_4 such that

$$\{p_1, p_2, p_3, p_4\} = X \cap \{y_3 = y_4 = y_5 = 0\},$$
$$\{p'_1, p'_2, p'_3, p'_4\} = X \cap \{x_0 = x_1 = x_2 = 0\}.$$

The quotient map $\pi : X \to X/\sigma =: W_{BS}^9$ is given by the restriction on X of the morphism defined by the linear system of the σ-invariant quadric hypersurfaces of $\mathbb{P}^5$, that is the morphism $\varphi : \mathbb{P}^5 \to \mathbb{P}^{11}_{[Z_0:\cdots:Z_{11}]}$ such that

$$\left[x_0 : x_1 : x_2 : y_3 : y_4 : y_5 \right]$$

$$\downarrow$$

$$\left[x_0^2 : x_1^2 : x_2^2 : x_0 x_1 : x_0 x_2 : x_1 x_2 : y_3^2 : y_4^2 : y_5^2 : y_3 y_4 : y_3 y_5 : y_4 y_5 \right].$$

By using Macaulay2, one can find that the image of $\mathbb{P}^5$ via φ is a 5-dimensional algebraic variety F_5^{16} of degree 16, whose ideal is generated by the 12 polynomials

$$Z_9 Z_{10} - Z_6 Z_{11}, \quad Z_7 Z_{10} - Z_9 Z_{11}, \quad Z_8 Z_9 - Z_{10} Z_{11}, \quad Z_7 Z_8 - Z_{11}^2,$$
$$Z_6 Z_8 - Z_{10}^2, \quad Z_6 Z_7 - Z_9^2, \quad Z_3 Z_4 - Z_0 Z_5, \quad Z_1 Z_4 - Z_3 Z_5,$$
$$Z_2 Z_3 - Z_4 Z_5, \quad Z_1 Z_2 - Z_5^2, \quad Z_0 Z_2 - Z_4^2, \quad Z_0 Z_1 - Z_3^2.$$

We observe that $W_{BS}^9 = \varphi(X) = F_5^{16} \cap H_9$, where H_9 is the following 9-dimensional projective subspace of $\mathbb{P}^{11}$:

$$\begin{aligned}
H_9 := \{ &a_{00} Z_0 + a_{11} Z_1 + a_{22} Z_2 + (a_{01} + a_{10}) Z_3 + (a_{02} + a_{20}) Z_4 \\
&+ (a_{12} + a_{21}) Z_5 + b_{33} Z_6 + b_{44} Z_7 + b_{55} Z_8 + (b_{34} + b_{43}) Z_9 \\
&+ (b_{35} + b_{53}) Z_{10} + (b_{45} + b_{54}) Z_{11} = 0, \\
&a'_{00} Z_0 + a'_{11} Z_1 + a'_{22} Z_2 + (a'_{01} + a'_{10}) Z_3 + (a'_{02} + a'_{20}) Z_4 \\
&+ (a'_{12} + a'_{21}) Z_5 + b'_{33} Z_6 + b'_{44} Z_7 + b'_{55} Z_8 + (b'_{34} + b'_{43}) Z_9 \\
&+ (a'_{35} + b'_{53}) Z_{10} + (b'_{45} + b'_{54}) Z_{11} = 0 \}.
\end{aligned}$$

Therefore, we have $\pi = \varphi|_X : X \to W_{\mathrm{BS}}^9 = \varphi(X) \subset H_9 \cong \mathbb{P}^9$. What follows has been proved for fixed values of $a_{ij}, b_{ij}, a'_{ij},$ and b'_{ij}, in order to simplify the computational analysis.

4.1. Example

If $(a_{ij}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 2 \end{pmatrix} = (b'_{ij})$ and $(b_{ij}) = \begin{pmatrix} 3 & 0 & 0 \\ 0 & -8 & 0 \\ 0 & 0 & 5 \end{pmatrix} = (a'_{ij})$, then we obtain

$$p_1 = [1 : 1 : 1 : 0 : 0 : 0], \quad p'_1 = [0 : 0 : 0 : 1 : 1 : 1],$$
$$p_2 = [-1 : 1 : 1 : 0 : 0 : 0], \quad p'_2 = [0 : 0 : 0 : -1 : 1 : 1],$$
$$p_3 = [1 : -1 : 1 : 0 : 0 : 0], \quad p'_3 = [0 : 0 : 0 : 1 : -1 : 1],$$
$$p_4 = [1 : 1 : -1 : 0 : 0 : 0], \quad p'_4 = [0 : 0 : 0 : 1 : 1 : -1].$$

Furthermore, we have

$$H_9 = \{Z_0 - 3Z_1 + 2Z_2 + 3Z_6 - 8Z_7 + 5Z_8 = 0,$$
$$3Z_0 - 8Z_1 + 5Z_2 + Z_6 - 3Z_7 + 2Z_8 = 0\}$$
$$= \{Z_1 - Z_2 - 8Z_6 + 21Z_7 - 13Z_8 = 0,$$
$$Z_0 - Z_2 - 21Z_6 + 55Z_7 - 34Z_8 = 0\},$$

which is the $\mathbb{P}_{[w_0 : \cdots : w_9]}^9$ embedded in $\mathbb{P}_{[Z_0 : \cdots : Z_{11}]}^{11}$ via the morphism such that

$$Z_0 = w_0 + 21w_4 - 55w_5 + 34w_6,$$
$$Z_1 = w_0 + 8w_4 - 21w_5 + 13w_6,$$
$$Z_{i+2} = w_i, \quad i = 0, \dots, 9.$$

By using Macaulay2, we find that the quotient map $\pi : X \to W_{\mathrm{BS}}^9 \subset H_9 \cong \mathbb{P}^9$ is given by the restriction on X of the morphism $\varphi' : \mathbb{P}^5 \to \mathbb{P}^9$ such that

$$\left[x_0 : x_1 : x_2 : y_3 : y_4 : y_5\right]$$
$$\downarrow$$
$$\left[x_2^2 : x_0 x_1 : x_0 x_2 : x_1 x_2 : y_3^2 : y_4^2 : y_5^2 : y_3 y_4 : y_3 y_5 : y_4 y_5\right].$$

In particular, we obtain a BS-EF 3-fold $W_{\mathrm{BS}}^9 \subset \mathbb{P}^9$ whose ideal is generated by the 12 quadratic polynomials

$$w_7 w_8 - w_4 w_9, \quad w_5 w_8 - w_7 w_9, \quad w_6 w_7 - w_8 w_9,$$
$$w_5 w_6 - w_9^2, \quad w_4 w_6 - w_8^2, \quad w_4 w_5 - w_7^2,$$
$$w_2^2 - w_3^2 - 13 w_0 w_4 + 34 w_0 w_5 - 21 w_0 w_6,$$
$$w_1 w_2 - w_0 w_3 - 21 w_3 w_4 + 55 w_3 w_5 - 34 w_3 w_6,$$

$$w_0w_2 - w_1w_3 + 8w_2w_4 - 21w_2w_5 + 13w_2w_6,$$
$$w_1^2 - w_3^2 - 21w_0w_4 - 168w_4^2 + 55w_0w_5 - 1155w_5^2 - 34w_0w_6$$
$$- 442w_6^2 + 881w_7^2 - 545w_8^2 + 1429w_9^2, \quad w_0w_1 - w_2w_3,$$
$$w_0^2 - w_3^2 + 8w_0w_4 - 21w_0w_5 + 13w_0w_6.$$

This threefold W_{BS}^9 has singular points at $P_i := \pi(p_i)$ and $P_i' := \pi(p_i')$, for $1 \leq i \leq 4$, that is

$$P_1 = [1 : 1 : 1 : 1 : 0 : 0 : 0 : 0 : 0 : 0],$$
$$P_1' = [0 : 0 : 0 : 0 : 1 : 1 : 1 : 1 : 1 : 1],$$
$$P_2 = [1 : -1 : -1 : 1 : 0 : 0 : 0 : 0 : 0 : 0],$$
$$P_2' = [0 : 0 : 0 : 0 : 1 : 1 : 1 : -1 : -1 : 1],$$
$$P_3 = [1 : -1 : 1 : -1 : 0 : 0 : 0 : 0 : 0 : 0],$$
$$P_3' = [0 : 0 : 0 : 0 : 1 : 1 : 1 : -1 : 1 : -1],$$
$$P_4 = [1 : 1 : -1 : -1 : 0 : 0 : 0 : 0 : 0 : 0],$$
$$P_4' = [0 : 0 : 0 : 0 : 1 : 1 : 1 : 1 : -1 : -1].$$

Let $l_{i,j}$ be the line joining P_i and P_j for $i, j \in \{1, 2, 3, 4, 1', 2', 3', 4'\}$ and $i \neq j$. It follows that W_{BS}^9 contains the lines $l_{1,1'}, l_{1,2'}, l_{1,3'}, l_{1,4'}, l_{2,1'}, l_{2,3'}, l_{2,3'}, l_{2,4'}, l_{3,1'}, l_{3,2'}, l_{3,3'}, l_{3,4'}, l_{4,1'}, l_{4,3'}, l_{4,3'}, l_{4,4'}$, but it does not contain the others. So each one of the eight singular points of W_{BS}^9 is associated with $m = 4$ of the other singular points, as in Table 1 of Appendix B. Thus, the singularities of the BS-EF 3-fold W_{BS}^9 have the same configuration as the ones of the F-EF 3-fold W_F^9. We can say something more, namely that $W_{\mathrm{BS}}^9 = W_F^9$. Let us see how. Let us project $\mathbb{P}^9$ from the $\mathbb{P}^5$ spanned by the singular points $P_2, P_3, P_4, P_2', P_3', P_4'$ of W_{BS}^9. By using Macaulay2, we obtain the rational map $\rho : \mathbb{P}^9 \dashrightarrow \mathbb{P}^3_{[z_0:\cdots:z_3]}$ such that

$$[w_0 : \cdots : w_9]$$
$$\mapsto [w_0 + w_1 + w_2 + w_3 : -w_4 + w_5 : -w_4 + w_6 : w_4 + w_7 + w_8 + w_9].$$

The restriction $\rho|_{W_{\mathrm{BS}}^9} : W_{\mathrm{BS}}^9 \dashrightarrow \mathbb{P}^3$ is a birational map (it can be verified through Macaulay2), whose inverse map is the rational map $\nu : \mathbb{P}^3 \dashrightarrow W_{\mathrm{BS}}^9 \subset \mathbb{P}^9$ defined by the linear system of the septic surfaces of $\mathbb{P}^3$ double along the six edges of the two trihedra

$$T := \{(z_0 - 21z_1 + 13z_2)z_0(z_0 - 55z_1 + 34z_2) = 0\},$$
$$T' := \{(z_2 + z_3)(z_1 + z_3)z_3 = 0\},$$

and containing the lines given by the intersection of a face of T and one of T'.

5. THE BS-EF 3-FOLD OF GENUS 13

Let us study the Enriques–Fano threefold described in [1, Section 6.3.2] (see also [22, Theorem 1.1 No. 14]). We refer to Section A.4 of Appendix A for the computational techniques we will use. Let us take the smooth Fano threefold $X := \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ and the map $\sigma : X \to X$ defined by

$$[x_0 : x_1] \times [y_0 : y_1] \times [z_0 : z_1] \mapsto [x_0 : -x_1] \times [y_0 : -y_1] \times [z_0 : -z_1].$$

Then σ is an involution of X having the eight fixed points

$$\begin{aligned}
p_1' &= [0 : 1] \times [1 : 0] \times [1 : 0], & p_1 &= [1 : 0] \times [0 : 1] \times [0 : 1], \\
p_2' &= [0 : 1] \times [0 : 1] \times [0 : 1], & p_2 &= [1 : 0] \times [1 : 0] \times [1 : 0], \\
p_3 &= [0 : 1] \times [1 : 0] \times [0 : 1], & p_3' &= [1 : 0] \times [0 : 1] \times [1 : 0], \\
p_4 &= [0 : 1] \times [0 : 1] \times [1 : 0], & p_4' &= [1 : 0] \times [1 : 0] \times [0 : 1].
\end{aligned}$$

The σ-invariant multihomogeneous polynomials of multidegree $(2, 2, 2)$ define the coordinates of the quotient map $\pi : X \to X/\sigma =: W_{\mathrm{BS}}^{13} \subset \mathbb{P}^{13}$; i.e.,

$$\pi : [x_0 : x_1] \times [y_0 : y_1] \times [z_0 : z_1] \mapsto [w_0 : \cdots : w_{13}],$$

where

$$\begin{aligned}
w_0 &= x_0^2 y_0^2 z_0^2, & w_1 &= x_0^2 y_0^2 z_1^2, & w_2 &= x_0^2 y_0 y_1 z_0 z_1, \\
w_3 &= x_0^2 y_1^2 z_0^2, & w_4 &= x_0^2 y_1^2 z_1^2, & w_5 &= x_0 x_1 y_0^2 z_0 z_1, \\
w_6 &= x_0 x_1 y_0 y_1 z_0^2, & w_7 &= x_0 x_1 y_0 y_1 z_1^2, & w_8 &= x_0 x_1 y_1^2 z_0 z_1, \\
w_9 &= x_1^2 y_0^2 z_0^2, & w_{10} &= x_1^2 y_0^2 z_1^2, & w_{11} &= x_1^2 y_0 y_1 z_0 z_1, \\
w_{12} &= x_1^2 y_1^2 z_0^2, & w_{13} &= x_1^2 y_1^2 z_1^2.
\end{aligned}$$

The use of Macaulay2 enables us to find that the BS-EF 3-fold W_{BS}^{13} has ideal generated by the 42 quadratic polynomials

$$\begin{array}{llll}
w_{10}w_{12} - w_9 w_{13}, & w_7 w_{12} - w_6 w_{13}, & w_4 w_{12} - w_3 w_{13}, & w_1 w_{12} - w_0 w_{13}, \\
w_{11}^2 - w_9 w_{13}, & w_8 w_{11} - w_6 w_{13}, & w_7 w_{11} - w_5 w_{13}, & w_6 w_{11} - w_5 w_{12}, \\
w_4 w_{11} - w_2 w_{13}, & w_3 w_{11} - w_2 w_{12}, & w_2 w_{11} - w_0 w_{13}, & w_8 w_{10} - w_5 w_{13}, \\
w_6 w_{10} - w_5 w_{11}, & w_4 w_{10} - w_1 w_{13}, & w_3 w_{10} - w_0 w_{13}, & w_2 w_{10} - w_1 w_{11}, \\
w_8 w_9 - w_5 w_{12}, & w_7 w_9 - w_5 w_{11}, & w_4 w_9 - w_0 w_{13}, & w_3 w_9 - w_0 w_{12}, \\
w_2 w_9 - w_0 w_{11}, & w_1 w_9 - w_0 w_{10}, & w_8^2 - w_3 w_{13}, & w_7 w_8 - w_2 w_{13}, \\
w_6 w_8 - w_2 w_{12}, & w_5 w_8 - w_0 w_{13}, & w_7^2 - w_1 w_{13}, & w_6 w_7 - w_0 w_{13}, \\
w_5 w_7 - w_1 w_{11}, & w_3 w_7 - w_2 w_8, & w_2 w_7 - w_1 w_8, & w_6^2 - w_0 w_{12},
\end{array}$$

$$w_5w_6 - w_0w_{11}, \qquad w_4w_6 - w_2w_8, \qquad w_2w_6 - w_0w_8, \qquad w_1w_6 - w_0w_7,$$

$$w_5^2 - w_0w_{10}, \qquad w_4w_5 - w_1w_8, \qquad w_3w_5 - w_0w_8, \qquad w_2w_5 - w_0w_7,$$

$$w_1w_3 - w_0w_4, \qquad w_2^2 - w_0w_4.$$

The above threefold W_{BS}^{13} has the eight singular points

$$P_1 := \pi(p_1) = \{w_i = 0 \mid i \neq 4\}, \quad P_1' := \pi(p_1') = \{w_i = 0 \mid i \neq 9\},$$

$$P_2 := \pi(p_2) = \{w_i = 0 \mid i \neq 0\}, \quad P_2' := \pi(p_2') = \{w_i = 0 \mid i \neq 13\},$$

$$P_3 := \pi(p_3) = \{w_i = 0 \mid i \neq 10\}, \quad P_3' := \pi(p_3') = \{w_i = 0 \mid i \neq 3\},$$

$$P_4 := \pi(p_4) = \{w_i = 0 \mid i \neq 12\}, \quad P_4' := \pi(p_4') = \{w_i = 0 \mid i \neq 1\}.$$

Let $l_{i,j}$ be the line joining P_i and P_j with $i, j \in \{1, 2, 3, 4, 1', 2', 3', 4'\}$ and $i \neq j$. We see that W_{BS}^{13} contains the lines $l_{1,2'}, l_{1,3'}, l_{1,4'}, l_{2,1'}, l_{2,3'}, l_{2,4'}, l_{3,1'}, l_{3,2'}, l_{3,4'}, l_{4,1'}, l_{4,2'}, l_{4,3'}$, while it does not contain the others. So each one of the eight singular points of W_{BS}^{13} is associated with $m = 3$ of the other singular points, as in Table 1 of Appendix B. Thus, the singularities of the BS-EF 3-fold W_{BS}^{13} have the same configuration as the ones of the F-EF 3-fold W_F^{13}. We can say something more, namely that $W_{\mathrm{BS}}^{13} = W_F^{13}$. Let us see how. Let $\mathcal{S}$ be the linear system of the sextic surfaces of $\mathbb{P}^3$ having double points along the six edges of a fixed tetrahedron $T \subset \mathbb{P}^3$. Up to a change of coordinates, we can take the tetrahedron $T := \{t_0t_1t_2t_3 = 0\} \subset \mathbb{P}^3_{[t_0:\cdots:t_3]}$. Then $\mathcal{S}$ defines a rational map $\nu_{\mathcal{S}} : \mathbb{P}^3 \dashrightarrow \mathbb{P}^{13}$ given by $[t_0 : t_1 : t_2 : t_3] \mapsto [w_0 : \cdots : w_{13}]$, where

$$w_0 = t_0t_1^3t_2t_3, \qquad w_1 = t_0^2t_1^2t_2^2, \qquad w_2 = t_0^2t_1^2t_2t_3,$$

$$w_3 = t_0^2t_1^2t_3^2, \qquad w_4 = t_0^3t_1t_2t_3, \qquad w_5 = t_0t_1^2t_2^2t_3,$$

$$w_6 = t_0t_1^2t_2t_3^2, \qquad w_7 = t_0^2t_1t_2^2t_3, \qquad w_8 = t_0^2t_1t_2t_3^2,$$

$$w_9 = t_1^2t_2^2t_3^2, \qquad w_{10} = t_0t_1t_2^3t_3, \qquad w_{11} = t_0t_1t_2^2t_3^2,$$

$$w_{12} = t_0t_1t_2t_3^3, \qquad w_{13} = t_0^2t_2^2t_3^2$$

(see [15, p. 635]). The above map is birational onto the image, which is the F-EF 3-fold $W_F^{13} \subset \mathbb{P}^{13}$ of genus 13 (see [12, Section 8]). Thanks to Macaulay2, we see that the threefold $W_F^{13} \subset \mathbb{P}^{13}$ coincides with the threefold $W_{\mathrm{BS}}^{13} \subset \mathbb{P}^{13}$. Thus, we have the assertion of Theorem 1.3 for $p = 13$.

6. THE BS-EF 3-FOLD OF GENUS 8

Let us study the Enriques–Fano threefold described in [1, Section 6.4.2]. Sano erroneously omits it (see [22, p. 378]). We refer to Section A.5 of Appendix A for the

computational techniques we will use. Let us take the hyperplane

$$\{x_4 = 0\} \subset \mathbb{P}^4_{[x_0:x_1:x_2:x_3:x_4]}$$

and two quadric surfaces

$$Q, R \subset \{x_4 = 0\} \cong \mathbb{P}^3_{[x_0:x_1:x_2:x_3]},$$

respectively with equations

$$Q(x_0, x_1, x_2, x_3) := q_{00}x_0^2 + q_{11}x_1^2 + q_{22}x_2^2 + q_{33}x_3^2 + q_{01}x_0x_1 + q_{23}x_2x_3 = 0,$$

$$R(x_0, x_1, x_2, x_3) := r_{00}x_0^2 + r_{11}x_1^2 + r_{22}x_2^2 + r_{33}x_3^2 + r_{01}x_0x_1 + r_{23}x_2x_3 = 0.$$

Let $C := Q \cap R$ be the elliptic quartic curve given by the complete intersection of the above quadrics, and let $Y \subset \mathbb{P}^4$ be the cone over Q with vertex at the point $v := [0:0:0:0:1]$. Let $bl : X \to Y$ be the blow-up of Y at the point v and along the curve C. We have that X is a smooth Fano threefold. Let us explain this. Let us consider the blow-up of $\mathbb{P}^4$ at v and along C, that is the map $bl' : \mathrm{Bl}_{v \cup C} \mathbb{P}^4 \to \mathbb{P}^4$ with exceptional divisors $E_v := bl'^{-1}(v)$ and $E_C := bl'^{-1}(C)$. By definition, we have that X is the strict transform of Y on $\mathrm{Bl}_{v \cup C} \mathbb{P}^4$ and that $bl = bl'|_X$. If H denotes the pullback of the hyperplane class h of $\mathbb{P}^4$, we have that $X \sim 2H - 2E_v - E_C$. By the adjunction formula, we have that $-K_X = -(K_{\mathrm{Bl}_{v \cup C} \mathbb{P}^4} + X)|_X \sim (3H - E_v - E_C)|_X$. We want to show that $-K_X$ is ample. Let $\mathcal{C}$ be the linear system of the cubic hypersurfaces of $\mathbb{P}^4$ containing the curve C and passing through the point v, and let us fix a general hyperplane $h_v \subset \mathbb{P}^4$ passing through v. Thus, $\mathcal{C}$ contains a sublinear system $\overline{\mathcal{C}} \subset \mathcal{C}$ whose fixed part is given by $h_v \cup \{x_4 = 0\}$. Since the movable part of $\overline{\mathcal{C}}$ is given by the hyperplanes of $\mathbb{P}^4$, then we obtain the ampleness of $\mathcal{C}$ at least outside $v \cup C$. So we have the ampleness of $-K_X$ at least outside $E_v \cap X$ and $E_C \cap X$, since $|-K_X|$ coincides with the restriction on X of the strict transform of $\mathcal{C}$. Furthermore, the movable part of $\overline{\mathcal{C}}$ also contains the hyperplanes of $\mathbb{P}^4$ through v, whose strict transforms are very ample on E_v: indeed, we have $|\mathcal{O}_{E_v}(H - E_v)| = |\mathcal{O}_{E_v}(-E_v^2)| \cong |\mathcal{O}_{\mathbb{P}^3}(1)|$ (see [15, Chapter 4, Section 6]). Therefore, the ampleness of $-K_X$ along $E_v \cap X$ follows by the fact that $E_v \cap X$ is a smooth quadric surface in $E_v \cong \mathbb{P}^3$. It remains to show the ampleness of $-K_X$ along $S' := E_C \cap X$, which is a $\mathbb{P}^1$-bundle over C, identified with the projectification $\mathbb{P}(\mathcal{N}_{C|Y})$ of the normal bundle of C in Y (see [15, Chapter 4, Section 6]). Since C is the complete intersection of a hyperplane section and of a quadric section of Y, then $S' = \mathbb{P}(\mathcal{N}_{C|Y}) \cong \mathbb{P}(\mathcal{O}_C(h) \oplus \mathcal{O}_C(2h))$ (see [4, Example 10.2]). In particular, we have that the class $S'|_{S'}$ is the class of the tautological bundle on S' (see [15, Chapter 4, Section 6]). Thus, $-E_C|_{S'} = -S'|_{S'}$ is ample on S', and so $(-K_X)|_{S'} = (3H - E_C)|_{S'}$ is ample too.

Let $\sigma : X \to X$ be now the morphism defined by the birational map $\sigma' : Y \dashrightarrow Y$ given by $[x_0 : x_1 : x_2 : x_3 : x_4] \mapsto [x_4 x_0 : x_4 x_1 : -x_4 x_2 : -x_4 x_3 : R(x_0, x_1, x_2, x_3)]$. The map σ is an involution of X with eight fixed points, which are the preimages via $bl : X \to Y$ of the eight points $p_1, p_2, p_3, p_4, p_1', p_2', p_3', p_4' \in Y$ such that

$$\{p_1, p_1', p_2, p_2'\} = Y \cap \{x_2 = 0, \ x_3 = 0, \ x_4^2 - R(x_0, x_1, x_2, x_3) = 0\},$$

$$\{p_3, p_3', p_4, p_4'\} = Y \cap \{x_0 = 0, \ x_1 = 0, \ x_4^2 + R(x_0, x_1, x_2, x_3) = 0\}.$$

The σ'-invariant elements of $\mathcal{C}$ define the rational map $\varphi : Y \dashrightarrow \mathbb{P}^9$ given by $[x_0 : \cdots : x_4] \mapsto [Z_0 : \cdots : Z_9]$, where

$$Z_0 = x_4^2 x_0 + x_0 R(x_0, x_1, x_2, x_3), \quad Z_1 = x_4^2 x_1 + x_1 R(x_0, x_1, x_2, x_3),$$
$$Z_2 = x_4^2 x_2 - x_2 R(x_0, x_1, x_2, x_3), \quad Z_3 = x_4^2 x_3 - x_3 R(x_0, x_1, x_2, x_3),$$
$$Z_4 = x_4 x_0^2, \quad Z_5 = x_4 x_1^2, \quad Z_6 = x_4 x_2^2, \quad Z_7 = x_4 x_3^2,$$
$$Z_8 = x_4 x_0 x_1, \quad Z_9 = x_4 x_2 x_3.$$

Let us observe that $\varphi(Y)$ is contained in the hyperplane

$$H_8 := \{q_{00} Z_4 + q_{11} Z_5 + q_{22} Z_6 + q_{33} Z_7 + q_{01} Z_8 + q_{23} Z_9 = 0\} \cong \mathbb{P}^8 \subset \mathbb{P}^9.$$

Therefore, the rational map φ defines the quotient map $\pi : X \to X/\sigma =: W_{\mathrm{BS}}^8$, thanks to the commutative diagram

$$
\begin{array}{ccc}
X & & \\
\downarrow {\scriptstyle bl} & \searrow {\scriptstyle \pi} & \\
Y & \overset{\varphi}{\dashrightarrow} & \varphi(Y) = \pi(X) = W_{\mathrm{BS}}^8 \subset H_8 \cong \mathbb{P}^8.
\end{array}
$$

What follows has been proved for fixed values of q_{ij} and r_{ij}, in order to simplify the computational analysis.

6.1. Example

Let us take

$$Q(x_0, x_1, x_2, x_3) = x_0^2 - x_1^2 - x_2^2 + x_3^2$$

and

$$R(x_0, x_1, x_2, x_3) = 2x_0^2 - x_1^2 - 3x_2^2 + 2x_3^2.$$

Then $\varphi(Y)$ is contained in the hyperplane $H_8 = \{Z_4 - Z_5 - Z_6 + Z_7\}$, which we can see as the image of the morphism $i : \mathbb{P}^8 \hookrightarrow \mathbb{P}^9$ such that

$$[w_0 : \cdots : w_8] \overset{i}{\mapsto} [w_0 : w_1 : w_2 : w_3 : w_4 + w_5 - w_6 : w_4 : w_5 : w_6 : w_7 : w_8].$$

Thanks to Macaulay2, one can verify that we obtain a BS-EF 3-fold $W_{BS}^8 \subset H_8 \cong \mathbb{P}^8$ whose ideal is generated by the following 11 polynomials of degree 2 or 3:

$$w_5w_6 - w_8^2, \quad w_2w_6 - w_3w_8, \quad w_3w_5 - w_2w_8, \quad w_4^2 + w_4w_5 - w_4w_6 - w_7^2,$$

$$w_1w_4 + w_1w_5 - w_1w_6 - w_0w_7, \quad w_0w_4 - w_1w_7,$$

$$w_0^2 - w_1^2 - w_2^2 + w_3^2 - 4w_4w_5 + 4w_5^2 + 4w_4w_6 - 4w_8^2,$$

$$w_2w_3w_7 - w_0w_1w_8 + 4w_4w_7w_8 - 4w_5w_7w_8,$$

$$w_0w_1w_6 - w_3^2w_7 - 4w_4w_6w_7 + 4w_7w_8^2,$$

$$w_3^2w_4 - w_1^2w_6 + 4w_4w_6^2 + 4w_6w_7^2 - 8w_4w_8^2,$$

$$w_2w_3w_4 - w_1^2w_8 - 8w_4w_5w_8 + 4w_4w_6w_8 + 4w_7^2w_8.$$

This threefold W_{BS}^8 has the eight singular points

$$P_1 := i^{-1}\big(\varphi(p_1)\big) = \varphi\big([1:1:0:0:1]\big) = [2:2:0:0:1:0:0:1:0],$$

$$P_2 := i^{-1}\big(\varphi(p_2)\big) = \varphi\big([-1:-1:0:0:1]\big) = [-2:-2:0:0:1:0:0:1:0],$$

$$P_3 := i^{-1}\big(\varphi(p_3)\big) = \varphi\big([0:0:1:1:1]\big) = [0:0:2:2:0:1:1:0:1],$$

$$P_4 := i^{-1}\big(\varphi(p_4)\big) = \varphi\big([0:0:-1:1:1]\big) = [0:0:-2:-2:0:1:1:0:1],$$

$$P_1' := i^{-1}\big(\varphi(p_1')\big) = \varphi\big([-1:1:0:0:1]\big) = [2:-2:0:0:-1:0:0:1:0],$$

$$P_2' := i^{-1}\big(\varphi(p_2')\big) = \varphi\big([1:-1:0:0:1]\big) = [-2:2:0:0:-1:0:0:1:0],$$

$$P_3' := i^{-1}\big(\varphi(p_3')\big) = \varphi\big([0:0:-1:-1:1]\big) = [0:0:2:-2:0:-1:-1:0:1],$$

$$P_4' := i^{-1}\big(\varphi(p_4')\big) = \varphi\big([0:0:1:-1:1]\big) = [0:0:-2:2:0:-1:-1:0:1].$$

Let $l_{i,j}$ be the line joining P_i and P_j for $i, j \in \{1, 2, 3, 4, 1', 2', 3', 4'\}$ and $i \neq j$. We have that W_{BS}^8 does not contain the lines $l_{1,1'}, l_{1,2'}, l_{2,1'}, l_{2,2'}, l_{3,3'}, l_{3,4'}, l_{4,3'}, l_{4,4'}$, while it contains the others. So each one of the eight singular points of W_{BS}^8 is associated with $m = 5$ of the other singular points, as in Table 2 of Appendix B. Hence there exist three mutually associated points (for example P_1, P_2, and P_3). This case had been excluded by Fano for $p > 7$ (see [12, Section 5]). So this suggests that in Fano's paper there are other gaps to be discovered.

THEOREM 6.1. *Let T be a trihedron with edges l_0, l_1, l_2 and vertex v as in Figure 1. Let us choose a general point $q_1 \in l_1$, a general point $q_2 \in l_2$, three distinct points $a_r, a_s, a_t \in l_0$, a general point $b_1 \in r_1 := \langle q_1, a_r \rangle$, and a general point $b_2 \in r_2 := \langle q_2, a_r \rangle$. Let us take a general conic C through the points q_1, q_2, b_1, b_2, in the plane spanned by the three points a_r, q_1, q_2. Finally, let us consider the lines $s_1 := \langle q_1, a_s \rangle$, $s_2 := \langle q_2, a_s \rangle$, $t_1 := \langle b_1, a_t \rangle$, $t_2 := \langle b_2, a_t \rangle$ and the lines $l_1' := \langle q_1', q_2 \rangle$ and $l_2' := \langle q_2', q_1 \rangle$, where q_1' is a general point on t_1 and q_2' a general point on t_2. Then the BS-EF 3-fold W_{BS}^8 can be obtained as the image of $\mathbb{P}^3$ via the rational map $v_{\mathcal{N}} : \mathbb{P}^3 \dashrightarrow \mathbb{P}^8$*
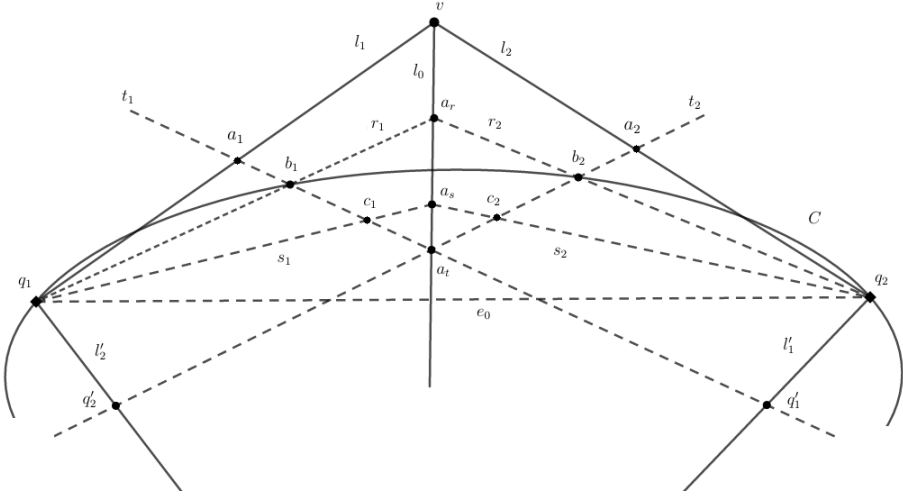
FIGURE 1. Base locus of the linear system $\mathcal{N}$.

defined by the linear system $\mathcal{N}$ of the septic surfaces of $\mathbb{P}^3$ which are quadruple at the points q_1 and q_2, triple at the vertex v, and double along the lines l_0, l_1, l_2, l_1', l_2', along the conic C and at the points $c_1 := t_1 \cap s_1$ and $c_2 := t_2 \cap s_2$. Furthermore, a general element of $\mathcal{N}$ contains the lines t_1, t_2, r_1, r_2, s_1, s_2 and $e_0 := \langle q_1, q_2 \rangle$.

PROOF. Let us project $\mathbb{P}^8$ from the $\mathbb{P}^4$ spanned by the singular points P_1, P_1', P_2, P_3, and P_3' of the BS-EF 3-fold W_{BS}^8 of Example 6.1. By using Macaulay2, we obtain the rational map $\rho : \mathbb{P}^8 \dashrightarrow \mathbb{P}^3_{[z_0:\cdots:z_3]}$ such that

$$[w_0 : \cdots : w_{13}] \mapsto [w_2 - 2w_8 : w_5 - w_6 : w_3 - 2w_6 : w_0 - w_1 + 2w_4 - 2w_7].$$

One can verify, via Macaulay2, that the restriction $\rho|_{W_{\mathrm{BS}}^8} : W_{\mathrm{BS}}^8 \dashrightarrow \mathbb{P}^3$ is birational and that its inverse map is given by the rational map $\nu_{\mathcal{N}} : \mathbb{P}^3 \dashrightarrow W_{\mathrm{BS}}^8 \subset \mathbb{P}^8$ defined by the linear system $\mathcal{N}$ of the septic surfaces

(i)   quadruple at $q_1 = [1 : 0 : -2 : 0]$ and $q_2 = [1 : 0 : 2 : 0]$;

(ii)   triple at the vertex $v = [0 : 0 : 0 : 1]$ of the trihedron

$$T = \{z_1(2z_0 + z_2)(2z_0 - z_2) = 0\};$$

(iii)   double at the points $c_1 = [1 : -2 : -2 : 0]$ and $c_2 = [1 : 2 : 2 : 0]$; double along the line $l_1' = \{z_3 = 2z_0 + 2z_1 - z_2 = 0\} \ni q_1' = [1 : -2 : -2 : 0]$ and along the line $l_2' = \{z_3 = 2z_0 - 2z_1 + z_2 = 0\} \ni q_2' = [1 : 2 : 2 : 0]$; double along the edges $l_0 = \{z_0 = z_2 = 0\}, l_1 = \{z_1 = 2z_0 + z_2 = 0\}, l_2 = \{z_1 = 2z_0 - z_2 = 0\}$

of the trihedron T; double along the conic

$$C = \{2z_1 + z_3 = 4z_0^2 - z_2^2 - 2z_2z_3 - 2z_3^2 = 0\}$$

passing through $q_1, q_2, b_1 = [1 : -1 : -2 : 2]$ and $b_2 = [1 : 1 : 2 : -2]$;

(iv) containing the lines

$$r_1 = \{2z_1 + z_3 = 2z_0 + z_2 = 0\}, \quad r_2 = \{2z_1 + z_3 = 2z_0 - z_2 = 0\},$$
$$s_1 = \{z_3 = 2z_0 + z_2 = 0\}, \quad s_2 = \{z_3 = 2z_0 - z_2 = 0\},$$
$$t_1 = \{2z_1 - 2z_2 - z_3 = 2z_0 + z_2 = 0\},$$
$$t_2 = \{2z_1 - 2z_2 - z_3 = 2z_0 - z_2 = 0\}. \qquad \blacksquare$$

7. THE BS-EF 3-FOLD OF GENUS 10

Let us study the Enriques–Fano threefold described in [1, Section 6.5.1] (see also [22, Theorem 1.1 No. 13]). We refer to Section A.6 of Appendix A for the computational techniques we will use. Let us take the smooth Fano threefold $X := \mathbb{P}^1 \times S_6$, where S_6 is a smooth sextic del Pezzo surface. We recall that S_6 is the image of $\mathbb{P}^2$ via the rational map $\lambda : \mathbb{P}^2 \dashrightarrow \mathbb{P}^6$ defined by the linear system of the plane cubic curves passing through three fixed points a_1, a_2, a_3 in general position. Up to a change of coordinates, we may assume that $a_1 = [1 : 0 : 0], a_2 = [0 : 1 : 0], a_3 = [0 : 0 : 1]$, and

$$\lambda : [u_0 : u_1 : u_2] \mapsto [u_1^2 u_2 : u_1 u_2^2 : u_0^2 u_2 : u_0 u_2^2 : u_0^2 u_1 : u_0 u_1^2 : u_0 u_1 u_2].$$

Thanks to Macaulay2, we can say that $S_6 = \lambda(\mathbb{P}^2) \subset \mathbb{P}^6_{[x_0 : x_1 : x_2 : x_3 : x_4 : x_5 : x_6]}$ has ideal generated by the polynomials

$$x_3 x_5 - x_6^2, \quad x_2 x_5 - x_4 x_6, \quad x_1 x_5 - x_0 x_6, \quad x_3 x_4 - x_2 x_6, \quad x_1 x_4 - x_6^2,$$
$$x_0 x_4 - x_5 x_6, \quad x_0 x_3 - x_1 x_6, \quad x_0 x_2 - x_3 x_6, \quad x_0 x_2 - x_6^2.$$

The quadratic transformation $q_{a_1, a_2, a_3} : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$, given by the linear system of the conics passing through a_1, a_2, and a_3, defines an involution of the above sextic del Pezzo surface. Indeed, we have

$$
\begin{array}{ccc}
[u_0 : u_1 : u_2] & \xrightarrow{\quad q_{a_1, a_2, a_3} \quad} & [\frac{1}{u_0} : \frac{1}{u_1} : \frac{1}{u_2}] \\
\Big\downarrow{\scriptstyle\lambda} & & \Big\downarrow{\scriptstyle\lambda} \\
[u_1^2 u_2 : u_1 u_2^2 : u_0^2 u_2 : u_0 u_2^2 : u_0^2 u_1 : u_0 u_1^2 : u_0 u_1 u_2] & & \\
& & [u_0^2 u_2 : u_1 u_0^2 : u_1^2 u_2 : u_0 u_1^2 : u_1 u_2^2 : u_0 u_2^2 : u_0 u_1 u_2],
\end{array}
$$

and then we obtain the involution t' of $\mathbb{P}^6$ given by

$$[x_0 : x_1 : x_2 : x_3 : x_4 : x_5 : x_6] \overset{t'}{\longmapsto} [x_2 : x_4 : x_0 : x_5 : x_1 : x_3 : x_6].$$

The locus of t'-fixed points of $\mathbb{P}^6$ consists of two projective subspaces

$$F_1 := \{x_0 + x_2 = x_1 + x_4 = x_3 + x_5 = x_6 = 0\} \cong \mathbb{P}^2,$$
$$F_2 := \{x_0 - x_2 = x_1 - x_4 = x_3 - x_5 = 0\} \cong \mathbb{P}^3.$$

In particular, we have $F_1 \cap S_6 = \emptyset$ and $F_2 \cap S_6 = \{d_1, d_2, d_3, d_4\}$, where

$$d_1 := [1 : 1 : 1 : 1 : 1 : 1 : 1], \qquad d_2 := [1 : -1 : 1 : -1 : -1 : -1 : 1],$$
$$d_3 := [-1 : 1 : -1 : -1 : 1 : -1 : 1], \quad d_4 := [-1 : -1 : -1 : 1 : -1 : 1 : 1].$$

Thus, $\sigma_2 := t'|_{S_6}$ is an involution of S_6 with four fixed points. We also consider the involution of $\mathbb{P}^1$ with two fixed points $[0 : 1]$ and $[1 : 0]$, that is the map $\sigma_1 : \mathbb{P}^1 \to \mathbb{P}^1$ given by $[y_0 : y_1] \mapsto [y_0 : -y_1]$. Therefore, the map $\sigma := (\sigma_1 \times \sigma_2) : X \to X$ is an involution of X having eight fixed points $p_1, p_2, p_3, p_4, p_1', p_2', p_3', p_4'$, where

$$p_i := [0 : 1] \times d_i, \quad p_i' = [1 : 0] \times d_i, \quad i = 1, 2, 3, 4.$$

The quotient map $\pi : X \to X/\sigma =: W_{\mathrm{BS}}^{10}$ is given by the restriction on X of the morphism $\varphi : \mathbb{P}^1 \times \mathbb{P}^6 \to \mathbb{P}_{[w_0 : \cdots : w_{10}]}^{10}$ defined by the $(\sigma_1 \times t')$-invariant multihomogeneous polynomials of multidegree $(2, 1)$; i.e., $\varphi : [y_0 : y_1] \times [x_0 : \cdots : x_6] \mapsto [w_0 : \cdots : w_{10}]$ where

$$w_0 = y_0^2 x_6, \quad w_1 = y_0^2(x_0 + x_2), \quad w_2 = y_0^2(x_1 + x_4), \quad w_3 = y_0^2(x_3 + x_5),$$
$$w_4 = y_1^2 x_6, \quad w_5 = y_1^2(x_0 + x_2), \quad w_6 = y_1^2(x_1 + x_4), \quad w_7 = y_1^2(x_3 + x_5),$$
$$w_8 = y_0 y_1(x_0 - x_2), \quad w_9 = y_0 y_1(x_1 - x_4), \quad w_{10} = y_0 y_1(x_3 - x_5).$$

Thanks to Macaulay2, one can find that the BS-EF 3-fold W_{BS}^{10} has ideal generated by the following 20 polynomials of degree 2 or 3:

$$w_7 w_8 - 2w_4 w_9 + w_5 w_{10}, \quad w_6 w_8 - w_5 w_9 + 2w_4 w_{10},$$
$$2w_4 w_8 - w_7 w_9 + w_6 w_{10}, \quad w_3 w_8 - 2w_0 w_9 + w_1 w_{10},$$
$$w_2 w_8 - w_1 w_9 + 2w_0 w_{10}, \quad 2w_0 w_8 - w_3 w_9 + w_2 w_{10},$$
$$w_3 w_6 - w_2 w_7, \quad w_2 w_6 - w_3 w_7 - w_9^2 + w_{10}^2, \quad w_1 w_6 - 2w_0 w_7 - w_8 w_9,$$
$$2w_0 w_6 - w_1 w_7 - w_8 w_{10}, \quad w_3 w_5 - w_1 w_7, \quad w_2 w_5 - 2w_0 w_7 - w_8 w_9,$$
$$w_1 w_5 - w_3 w_7 - w_8^2 + w_{10}^2, \quad 2w_0 w_5 - w_2 w_7 + w_9 w_{10}, \quad w_3 w_4 - w_0 w_7,$$
$$2w_2 w_4 - w_1 w_7 - w_8 w_{10}, \quad 2w_1 w_4 - w_2 w_7 + w_9 w_{10}, \quad 4w_0 w_4 - w_3 w_7 + w_{10}^2,$$
$$4w_4^3 - w_4 w_5^2 - w_4 w_6^2 + w_5 w_6 w_7 - w_4 w_7^2, \quad 4w_0^3 - w_0 w_1^2 - w_0 w_2^2 + w_1 w_2 w_3 - w_0 w_3^2.$$
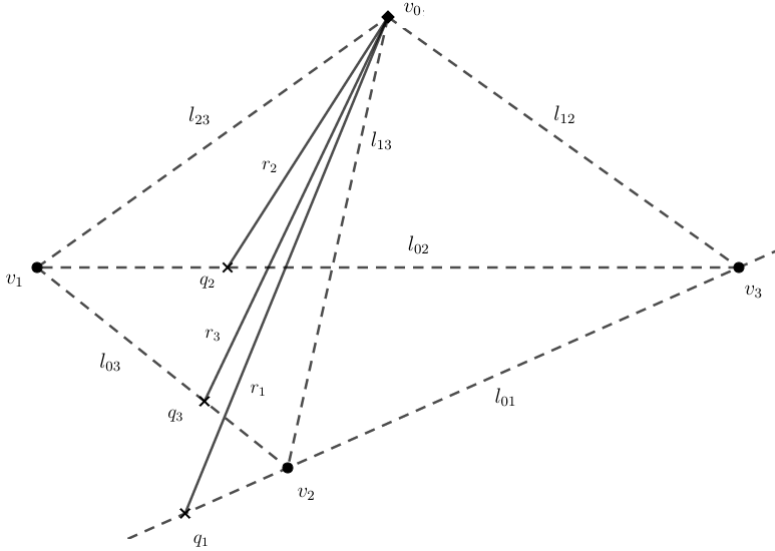
FIGURE 2. Base locus of the linear system $\mathcal{M}$.

The eight singular points of W_{BS}^{10} are $P_i := \pi(p_i)$ and $P'_i := \pi(p'_i)$, for $1 \le i \le 4$, that is

$$P_1 = [0:0:0:0:1:2:2:2:0:0:0], \qquad P'_1 = [1:2:2:2:0:\cdots:0],$$

$$P_2 = [0:0:0:0:1:2:-2:-2:0:0:0], \quad P'_2 = [1:2:-2:-2:0:\cdots:0],$$

$$P_3 = [0:0:0:0:1:-2:2:-2:0:0:0], \quad P'_3 = [1:-2:2:-2:0:\cdots:0],$$

$$P_4 = [0:0:0:0:1:-2:-2:2:0:0:0], \quad P'_4 = [1:-2:-2:2:0:\cdots:0].$$

Let $l_{i,j}$ be the line joining P_i and P_j for $i, j \in \{1, 2, 3, 4, 1', 2', 3', 4'\}$ and $i \ne j$. We have that W_{BS}^{10} contains the lines $l_{1,2}, l_{1,3}, l_{1,4}, l_{1,1'}, l_{2,3}, l_{2,4}, l_{2,2'}, l_{3,4}, l_{3,3'}, l_{4,4'}, l_{1',2'}, l_{1',3'}, l_{1',4'}, l_{2',3'}, l_{2',4'}, l_{3',4'}$, while it does not contain the others. So each one of the eight singular points of W_{BS}^{10} is associated with $m = 4$ of the other singular points, as in Table 2 of Appendix B. Hence there exist three mutually associated points (for example P_1, P_2, and P_3). This case had been excluded by Fano for $p > 7$ (see [12, Section 5]). So this suggests that in Fano's paper there are other gaps to be discovered.

THEOREM 7.1. *Let $T \subset \mathbb{P}^3$ be a tetrahedron with faces f_i and edges $l_{ij} := f_i \cap f_j$ for $0 \le i < j \le 3$. Let v_i be the vertex opposite to the face f_i, for $0 \le i \le 3$. Let π be a plane through the vertex v_0, which intersects the face f_i along a line r_i, for $1 \le i \le 3$, and let us define the point $q_i := r_i \cap l_{0i}$ (see Figure 2). Then W_{BS}^{10} can be obtained as the image of $\mathbb{P}^3$ via the rational map $v_{\mathcal{M}} : \mathbb{P}^3 \dashrightarrow \mathbb{P}^{10}$ defined by the linear system $\mathcal{M}$*

of the sextic surfaces of $\mathbb{P}^3$ which are quadruple at the vertex v_0, triple at the other three vertices v_1, v_2, v_3, and double along the three lines r_1, r_2, r_3. Furthermore, a general element of $\mathcal{M}$ also contains the six edges of T.

Proof. Let us project $\mathbb{P}^{10}$ from the $\mathbb{P}^6$ spanned by the singular points P_1, P_2, P_3, P_4, P_1', P_2', and P_3' of W_{BS}^{10}. By using Macaulay2, we obtain the rational map

$$\rho : \mathbb{P}^{10} \dashrightarrow \mathbb{P}^3_{[z_0 : \cdots : z_3]}, \quad [w_0 : \cdots : w_{13}] \mapsto [-2w_0 + w_1 + w_2 - w_3 : w_8 : w_9 : w_{10}].$$

Thanks to Macaulay2, we see that the restriction $\rho|_{W_{\mathrm{BS}}^{10}} : W_{\mathrm{BS}}^{10} \dashrightarrow \mathbb{P}^3$ is birational. We also compute its inverse map, which is the rational map $\nu_{\mathcal{M}} : \mathbb{P}^3 \dashrightarrow W_{\mathrm{BS}}^{10} \subset \mathbb{P}^{10}$ defined by the linear system $\mathcal{M}$ of the sextic surfaces

(i)   containing the six edges $l_{23} = \{z_1 = z_2 - z_3 = 0\}$, $l_{13} = \{z_3 = z_1 + z_2 = 0\}$, $l_{12} = \{z_2 = z_1 + z_3 = 0\}$, $l_{01} = \{z_0 = z_1 + z_2 + z_3 = 0\}$, $l_{03} = \{z_0 = z_1 - z_2 + z_3 = 0\}$, and $l_{02} = \{z_0 = z_1 + z_2 - z_3 = 0\}$ of the tetrahedron T with faces $f_0 = \{z_0 = 0\}$, $f_1 = \{z_1 + z_2 + z_3 = 0\}$, $f_2 = \{z_1 - z_2 + z_3 = 0\}$, and $f_3 = \{z_1 + z_2 - z_3 = 0\}$;

(ii)   double along the lines $r_1 = \{z_1 = z_2 + z_3 = 0\}$, $r_2 = \{z_3 = z_1 - z_2 = 0\}$ and $r_3 = \{z_2 = z_1 - z_3 = 0\}$ contained in the plane $\pi = \{z_1 - z_2 - z_3 = 0\}$, and obviously double at the points $q_1 = [0 : 0 : -1 : 1]$, $q_2 = [0 : 1 : 1 : 0]$, and $q_3 = [0 : 1 : 0 : 1]$;

(iii)   triple at the following vertices of T:

$$v_1 = [0 : 0 : 1 : 1], \quad v_2 = [0 : 1 : -1 : 0], \quad v_3 = [0 : 1 : 0 : -1];$$

(iv)   and quadruple at the vertex $v_0 = [1 : 0 : 0 : 0]$.   ∎

8. The P-EF 3-fold of genus 17

Let us study the Enriques–Fano threefold described in [20, Section 3]. We refer to Section A.7 of Appendix A for the computational techniques we will use. Let P be the octic del Pezzo surface given by the image of the anticanonical embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ in $\mathbb{P}^8$, which is defined by the linear system of the divisors of bidegree $(2, 2)$, i.e.,

$$[u_0 : u_1] \times [v_0 : v_1] \mapsto [y_{0,0} : y_{0,1} : y_{0,2}, y_{1,0} : y_{1,1} : y_{1,2} : y_{2,0} : y_{2,1} : y_{2,2}],$$

where $y_{i,j} := u_0^i u_1^{2-i} v_0^j v_1^{2-j}$. Let us consider $\mathbb{P}^8$ as the hyperplane $\{x = 0\}$ in $\mathbb{P}^9_{[y_{0,0} : y_{0,1} : y_{0,2} : y_{1,0} : y_{1,1} : y_{1,2} : y_{2,0} : y_{2,1} : y_{2,2} : x]}$. Let V be the cone over P with vertex at the point

$$v := [0 : 0 : 0 : 0 : 0 : 0 : 0 : 0 : 0 : 1];$$

then V is a singular Fano threefold (see [20, Lemma 3.1]). By using Macaulay2, we see that the ideal of V is generated by the polynomials

$$y_{2,1}^2 - y_{2,0}y_{2,2}, \qquad y_{1,2}y_{2,1} - y_{1,1}y_{2,2}, \quad y_{1,1}y_{2,1} - y_{1,0}y_{2,2},$$

$$y_{0,2}y_{2,1} - y_{0,1}y_{2,2}, \quad y_{0,1}y_{2,1} - y_{0,0}y_{2,2}, \quad y_{1,2}y_{2,0} - y_{1,0}y_{2,2},$$

$$y_{1,1}y_{2,0} - y_{1,0}y_{2,1}, \quad y_{0,2}y_{2,0} - y_{0,0}y_{2,2}, \quad y_{0,1}y_{2,0} - y_{0,0}y_{2,1},$$

$$y_{1,2}^2 - y_{0,2}y_{2,2}, \qquad y_{1,1}y_{1,2} - y_{0,1}y_{2,2}, \quad y_{1,0}y_{1,2} - y_{0,0}y_{2,2},$$

$$y_{1,1}^2 - y_{0,0}y_{2,2}, \qquad y_{1,0}y_{1,1} - y_{0,0}y_{2,1}, \quad y_{0,2}y_{1,1} - y_{0,1}y_{1,2},$$

$$y_{0,1}y_{1,1} - y_{0,0}y_{1,2}, \quad y_{1,0}^2 - y_{0,0}y_{2,0}, \qquad y_{0,2}y_{1,0} - y_{0,0}y_{1,2},$$

$$y_{0,1}y_{1,0} - y_{0,0}y_{1,1}, \quad y_{0,1}^2 - y_{0,0}y_{0,2}.$$

Let us consider the involution t of $\mathbb{P}^9$ given by $t(x) = -x$ and $t(y_{i,j}) = (-1)^{i+j}y_{i,j}$. Let $v_{i,j} := \{x = 0, \ y_{k,l} = 0 \mid (k,l) \neq (i,j)\}$. The locus of t-fixed points in $\mathbb{P}^9$ consists of two projective subspaces

$$F_1 = \{y_{0,0} = y_{0,2} = y_{1,1} = y_{2,0} = y_{2,2} = 0\} \cong \mathbb{P}^4$$

and

$$F_2 = \{y_{0,1} = y_{1,0} = y_{1,2} = y_{2,1} = x = 0\} \cong \mathbb{P}^4$$

such that $F_1 \cap V = \{v\}$ and $F_2 \cap V = \{v_{0,0}, v_{0,2}, v_{2,0}, v_{2,2}\}$. Thus, t defines an involution $\tau := t|_V : V \to V$ of V with five fixed points. The quotient of V via the involution τ is an Enriques–Fano threefold of genus 17 (see [20, Proposition 3.2]). In particular, the quotient map $\pi : V \to V/\tau =: W_P^{17}$ is defined by the restriction on V of the linear system $\mathcal{Q}$ of the quadric hypersurfaces of $\mathbb{P}^9$ of type

$$q_1(y_{0,0}, y_{0,2}, y_{1,1}, y_{2,0}, y_{2,2}) + q_2(y_{0,1}, y_{1,0}, y_{1,2}, y_{2,1}, x) = 0,$$

where q_1 and q_2 are quadratic homogeneous forms. Let us observe that the linear system $\mathcal{Q}$ defines a morphism $\varphi : \mathbb{P}^9 \to \mathbb{P}^{29}$, $[y_{0,0} : \cdots : y_{2,2} : x] \mapsto [Z_0 : \cdots : Z_{29}]$, where

$$\begin{aligned}
&Z_0 = y_{1,1}^2, & &Z_1 = y_{0,0}^2, & &Z_2 = y_{0,2}^2, & &Z_3 = y_{2,0}^2, \\
&Z_4 = y_{2,2}^2, & &Z_5 = x^2, & &Z_6 = y_{0,1}^2, & &Z_7 = y_{1,0}^2, \\
&Z_8 = y_{1,2}^2, & &Z_9 = y_{2,1}^2, & &Z_{10} = y_{0,1}x, & &Z_{11} = y_{1,0}x, \\
&Z_{12} = y_{1,2}x, & &Z_{13} = y_{2,1}x, & &Z_{14} = y_{0,0}y_{1,1}, & &Z_{15} = y_{0,2}y_{1,1}, \\
&Z_{16} = y_{2,0}y_{1,1}, & &Z_{17} = y_{2,2}y_{1,1}, & &Z_{18} = y_{0,1}y_{1,0}, & &Z_{19} = y_{0,1}y_{1,2}, \\
&Z_{20} = y_{1,0}y_{2,1}, & &Z_{21} = y_{1,2}y_{2,1}, & &Z_{22} = y_{0,0}y_{0,2}, & &Z_{23} = y_{0,0}y_{2,0}, \\
&Z_{24} = y_{0,2}y_{2,2}, & &Z_{25} = y_{2,0}y_{2,2}, & &Z_{26} = y_{0,1}y_{2,1}, & &Z_{27} = y_{0,0}y_{2,2}, \\
&Z_{28} = y_{0,2}y_{2,0}, & &Z_{29} = y_{1,0}y_{1,2}.
\end{aligned}$$

By looking at the ideal of $V \subset \mathbb{P}^9$, we deduce that $\varphi(V)$ is contained in a 17-dimensional projective subspace of $\mathbb{P}^{29}$ given by

$$H_{17} := \{Z_{18} = Z_{14}, \, Z_{19} = Z_{15}, \, Z_{20} = Z_{16}, \, Z_{21} = Z_{17}, \, Z_{22} = Z_6, \, Z_{23} = Z_7,$$
$$Z_{24} = Z_8, \, Z_{25} = Z_9, \, Z_{26} = Z_0, \, Z_{27} = Z_0, \, Z_{28} = Z_0, \, Z_{29} = Z_0\}.$$

Hence we obtain the morphism $\pi : V \to W_P^{17} = \varphi(V) \subset H_{17} \cong \mathbb{P}^{17}$ defined by

$$[y_{0,0} : y_{0,1} : y_{0,2} : y_{1,0} : y_{1,1} : y_{1,2} : y_{2,0} : y_{2,1} : y_{2,2} : x] \mapsto [z_0 : z_1 : \cdots : z_{16} : z_{17}],$$

where

$$
\begin{array}{llll}
z_0 = y_{1,1}^2, & z_1 = y_{0,0}^2, & z_2 = y_{0,2}^2, & z_3 = y_{2,0}^2, \\
z_4 = y_{2,2}^2, & z_5 = x^2, & z_6 = y_{0,1}^2, & z_7 = y_{1,0}^2, \\
z_8 = y_{1,2}^2, & z_9 = y_{2,1}^2, & z_{10} = y_{0,1}x, & z_{11} = y_{1,0}x, \\
z_{12} = y_{1,2}x, & z_{13} = y_{2,1}x, & z_{14} = y_{0,0}y_{1,1}, & z_{15} = y_{0,2}y_{1,1}, \\
z_{16} = y_{2,0}y_{1,1}, & z_{17} = y_{2,2}y_{1,1}. &
\end{array}
$$

Thanks to Macaulay2, we find that the threefold W_P^{17} has ideal generated by the 88 quadratic polynomials

$$
\begin{array}{llll}
z_{15}z_{16} - z_{14}z_{17}, & z_{12}z_{16} - z_{11}z_{17}, & z_9z_{16} - z_3z_{17}, & z_8z_{16} - z_0z_{17}, \\
z_6z_{16} - z_1z_{17}, & z_4z_{16} - z_9z_{17}, & z_2z_{16} - z_6z_{17}, & z_0z_{16} - z_7z_{17}, \\
z_{13}z_{15} - z_{10}z_{17}, & z_9z_{15} - z_0z_{17}, & z_8z_{15} - z_2z_{17}, & z_7z_{15} - z_1z_{17}, \\
z_4z_{15} - z_8z_{17}, & z_3z_{15} - z_7z_{17}, & z_0z_{15} - z_6z_{17}, & z_{13}z_{14} - z_{10}z_{16}, \\
z_{12}z_{14} - z_{11}z_{15}, & z_9z_{14} - z_7z_{17}, & z_8z_{14} - z_6z_{17}, & z_7z_{14} - z_1z_{16}, \\
z_6z_{14} - z_1z_{15}, & z_4z_{14} - z_0z_{17}, & z_3z_{14} - z_7z_{16}, & z_2z_{14} - z_6z_{15}, \\
z_0z_{14} - z_1z_{17}, & z_{12}z_{13} - z_5z_{17}, & z_{11}z_{13} - z_5z_{16}, & z_8z_{13} - z_{12}z_{17}, \\
z_7z_{13} - z_{11}z_{16}, & z_6z_{13} - z_{11}z_{15}, & z_2z_{13} - z_{12}z_{15}, & z_1z_{13} - z_{11}z_{14}, \\
z_0z_{13} - z_{11}z_{17}, & z_{11}z_{12} - z_{10}z_{13}, & z_{10}z_{12} - z_5z_{15}, & z_9z_{12} - z_{13}z_{17}, \\
z_7z_{12} - z_{10}z_{16}, & z_6z_{12} - z_{10}z_{15}, & z_3z_{12} - z_{13}z_{16}, & z_1z_{12} - z_{10}z_{14}, \\
z_0z_{12} - z_{10}z_{17}, & z_{10}z_{11} - z_5z_{14}, & z_9z_{11} - z_{13}z_{16}, & z_8z_{11} - z_{10}z_{17}, \\
z_6z_{11} - z_{10}z_{14}, & z_4z_{11} - z_{13}z_{17}, & z_2z_{11} - z_{10}z_{15}, & z_0z_{11} - z_{10}z_{16}, \\
z_9z_{10} - z_{11}z_{17}, & z_8z_{10} - z_{12}z_{15}, & z_7z_{10} - z_{11}z_{14}, & z_4z_{10} - z_{12}z_{17}, \\
z_3z_{10} - z_{11}z_{16}, & z_0z_{10} - z_{11}z_{15}, & z_8z_9 - z_{17}^2, & z_7z_9 - z_{16}^2, \\
z_6z_9 - z_{14}z_{17}, & z_5z_9 - z_{13}^2, & z_2z_9 - z_{15}z_{17}, & z_1z_9 - z_{14}z_{16}, \\
z_0z_9 - z_{16}z_{17}, & z_7z_8 - z_{14}z_{17}, & z_6z_8 - z_{15}^2, & z_5z_8 - z_{12}^2, \\
z_3z_8 - z_{16}z_{17}, & z_1z_8 - z_{14}z_{15}, & z_0z_8 - z_{15}z_{17}, & z_6z_7 - z_{14}^2,
\end{array}
$$

$$z_5z_7 - z_{11}^2, \qquad z_4z_7 - z_{16}z_{17}, \qquad z_2z_7 - z_{14}z_{15}, \qquad z_0z_7 - z_{14}z_{16},$$
$$z_5z_6 - z_{10}^2, \qquad z_4z_6 - z_{15}z_{17}, \qquad z_3z_6 - z_{14}z_{16}, \qquad z_0z_6 - z_{14}z_{15},$$
$$z_0z_5 - z_{10}z_{13}, \qquad z_3z_4 - z_9^2, \qquad z_2z_4 - z_8^2, \qquad z_1z_4 - z_{14}z_{17},$$
$$z_0z_4 - z_{17}^2, \qquad z_2z_3 - z_{14}z_{17}, \qquad z_1z_3 - z_7^2, \qquad z_0z_3 - z_{16}^2,$$
$$z_1z_2 - z_6^2, \qquad z_0z_2 - z_{15}^2, \qquad z_0z_1 - z_{14}^2, \qquad z_0^2 - z_{14}z_{17}.$$

Furthermore, W_P^{17} has the five singular points

$$P_1 := \pi(v_{0,0}), \quad P_2 := \pi(v_{0,2}), \quad P_3 := \pi(v_{2,0}), \quad P_4 := \pi(v_{2,2}), \quad P_5 := \pi(v),$$

that is $P_i = \{z_k = 0 \mid k \neq i\}$ for $1 \leq i \leq 5$.

PROPOSITION 8.1. *If $i = 1, 2, 3, 4$, the tangent cone $TC_{P_i} W_P^{17}$ to W_P^{17} at the point P_i is a cone over a Veronese surface.*

PROOF. Each point P_i, $i = 1, 2, 3, 4$, can be viewed as the origin of the open affine set $U_i := \{z_i \neq 0\}$. The ideal of the tangent cone $TC_{P_i}(W_P^{17} \cap U_i)$ is generated by the minimal degree homogeneous parts of all the polynomials in the ideal of $W_P^{17} \cap U_i$. By using Macaulay2, we obtain the following tangent cones.

$TC_{P_1}(W_P^{17} \cap U_1)$ has ideal generated by $z_{17}, z_{16}, z_{15}, z_{13}, z_{12}, z_9, z_8, z_4, z_3, z_2, z_0$, $z_{10}z_{11} - z_5z_{14}, z_6z_{11} - z_{10}z_{14}, z_7z_{10} - z_{11}z_{14}, z_6z_7 - z_{14}^2, z_5z_7 - z_{11}^2, z_5z_6 - z_{10}^2$. Hence $TC_{P_1} W_P^{17}$ is a cone with vertex P_1 over a Veronese surface in the $\mathbb{P}^5$ defined by $\{z_i = 0 \mid i = 0, 1, 2, 3, 4, 8, 9, 12, 13, 15, 16, 17\}$.

$TC_{P_2}(W_P^{17} \cap U_2)$ has ideal generated by $z_{17}, z_{16}, z_{14}, z_{13}, z_{11}, z_9, z_7, z_4, z_3, z_1, z_0$, $z_{10}z_{12} - z_5z_{15}, z_6z_{12} - z_{10}z_{15}, z_8z_{10} - z_{12}z_{15}, z_6z_8 - z_{15}^2, z_5z_8 - z_{12}^2, z_5z_6 - z_{10}^2$. Hence $TC_{P_2} W_P^{17}$ is a cone with vertex P_2 over a Veronese surface in the $\mathbb{P}^5$ defined by $\{z_i = 0 \mid i = 0, 1, 2, 3, 4, 7, 9, 11, 13, 14, 16, 17\}$.

$TC_{P_3}(W_P^{17} \cap U_3)$ has ideal generated by $z_{17}, z_{15}, z_{14}, z_{12}, z_{10}, z_8, z_6, z_4, z_2, z_1, z_0$, $z_{11}z_{13} - z_5z_{16}, z_7z_{13} - z_{11}z_{16}, z_9z_{11} - z_{13}z_{16}, z_7z_9 - z_{16}^2, z_5z_9 - z_{13}^2, z_5z_7 - z_{11}^2$. Hence $TC_{P_3} W_P^{17}$ is a cone with vertex P_3 over a Veronese surface in the $\mathbb{P}^5$ defined by $\{z_i = 0 \mid i = 0, 1, 2, 3, 4, 6, 8, 10, 12, 14, 15, 17\}$.

$TC_{P_4}(W_P^{17} \cap U_4)$ has ideal generated by $z_{16}, z_{15}, z_{14}, z_{11}, z_{10}, z_7, z_6, z_3, z_2, z_1, z_0$, $z_{12}z_{13} - z_5z_{17}, z_8z_{13} - z_{12}z_{17}, z_9z_{12} - z_{13}z_{17}, z_8z_9 - z_{17}^2, z_5z_9 - z_{13}^2, z_5z_8 - z_{12}^2$. Hence $TC_{P_4} W_P^{17}$ is a cone with vertex P_4 over a Veronese surface in the $\mathbb{P}^5$ defined by $\{z_i = 0 \mid i = 0, 1, 2, 3, 4, 6, 7, 10, 11, 14, 15, 16\}$. ∎

THEOREM 8.2. *The tangent cone $TC_{P_5} W_P^{17}$ to W_P^{17} at the point P_5 is a cone over a reducible sextic surface $M_6 \subset \mathbb{P}^7 \subset \mathbb{P}^{17}$, which is given by the union of four planes $\pi_1, \pi_2, \pi_1', \pi_2'$ and a quadric surface $Q \subset \mathbb{P}^3 \subset \mathbb{P}^7$. In particular, each one of the planes $\pi_1, \pi_2, \pi_1', \pi_2'$ intersects the quadric Q respectively along a line l_1, l_2, l_1', l_2',*
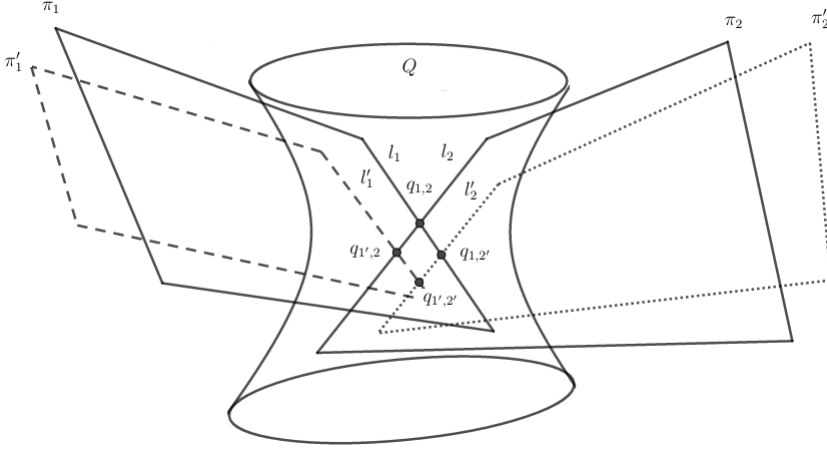
FIGURE 3. The reducible sextic surface $M_6 \subset \mathbb{P}^7$ given by the union of four planes π_1, π_2, π_1', π_2' and a quadric surface $Q \subset \mathbb{P}^3 \subset \mathbb{P}^7$, which intersect as in the statement of Theorems 8.2 and 10.3.

where l_i is disjoint from l_i', for $i = 1, 2$. In the other cases, the intersections of two of these lines identify four points, i.e., $q_{1,2} := l_1 \cap l_2$, $q_{1,2'} := l_1 \cap l_2'$, $q_{1',2} := l_1' \cap l_2$, and $q_{1',2'} := l_1' \cap l_2'$.

PROOF. The point P_5 can be viewed as the origin of the open affine set given by $U_5 := \{z_5 \neq 0\}$. The ideal of the tangent cone $TC_{P_5}(W_P^{17} \cap U_5)$ is generated by the minimal degree homogeneous parts of all the polynomials in the ideal of $W_P^{17} \cap U_5$. By using Macaulay2, we find that $TC_{P_5}(W_P^{17} \cap U_5)$ has ideal generated by the polynomials

$$z_{17},\ z_{16},\ z_{15},\ z_{14},\ z_9,\ z_8,\ z_7,\ z_6,\ z_0,\quad z_{11}z_{12} - z_{10}z_{13},$$
$$z_2z_{13},\ z_1z_{13},\ z_3z_{12},\ z_1z_{12},\ z_4z_{11},\ z_2z_{11},\ z_4z_{10},\ z_3z_{10},$$
$$z_3z_4,\ z_2z_4,\ z_1z_4,\ z_2z_3,\ z_1z_3,\ z_1z_2.$$

Hence $TC_{P_5}W_P^{17}$ is a cone with vertex P_5 over a surface M_6 contained in the $\mathbb{P}^7$ defined by $\{z_i = 0 \mid i = 0, 5, 6, 7, 8, 9, 14, 15, 16, 17\}$. This surface M_6 is the union of four planes π_1, π_2, π_1', π_2' and a quadric surface Q, where

$$\pi_1 := \{z_i = 0 \mid i = 0, 1, 3, 4, 5, 6, 7, 8, 9, 11, 13, 14, 15, 16, 17\},$$
$$\pi_2 := \{z_i = 0 \mid i = 0, 2, 3, 4, 5, 6, 7, 8, 9, 12, 13, 14, 15, 16, 17\},$$
$$\pi_1' := \{z_i = 0 \mid i = 0, 1, 2, 4, 5, 6, 7, 8, 9, 10, 12, 14, 15, 16, 17\},$$
$$\pi_2' := \{z_i = 0 \mid i = 0, 1, 2, 3, 5, 6, 7, 8, 9, 10, 11, 14, 15, 16, 17\},$$
$$Q := \{z_i = 0 \mid i = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 14, 15, 16, 17\} \cap \{z_{11}z_{12} - z_{10}z_{13} = 0\}.$$

We give an idea of M_6 in Figure 3. ■

By Proposition 8.1 and Theorem 8.2, we have that W_P^{17} has five non-similar points. For completeness, let us find their configuration. Let $l_{i,j} := \{z_k = 0 \mid k \neq i, j\}$ be the line joining the singular points P_i and P_j with $1 \leq i < j \leq 5$. By looking at the ideal of W_P^{17}, we deduce that the lines $l_{1,5}$, $l_{2,5}$, $l_{3,5}$, $l_{4,5}$ are contained in W_P^{17}, while the lines $l_{1,2}$, $l_{1,3}$, $l_{1,4}$, $l_{2,3}$, $l_{2,4}$, $l_{3,4}$ are not. Hence the five singular points P_1, P_2, P_3, P_4, P_5 of W_P^{17} are associated as in Table 3 of Appendix B.

REMARK 8.3. Let $bl : \mathrm{Bl}_{P_i=1,\dots 5}\,\mathbb{P}^{17} \to \mathbb{P}^{17}$ be the blow-up of $\mathbb{P}^{17}$ at the five singular points of W_P^{17} and let $\widetilde{W}$ be the strict transform of W_P^{17}. Then $\widetilde{W}$ intersects the exceptional divisor $bl^{-1}(P_5)$ along a surface which is isomorphic to M_6 and which has four singular points locally given by the intersection of three planes of $\mathbb{P}^4$, such that two of them intersect the third along two lines and intersect each other at a point which is the intersection of these two lines. Consequently, $\widetilde{W}$ is not a desingularization of W_P^{17}, since there are four singular points infinitely near to P_5. Therefore, it is not enough to blow-up W_P^{17} at P_1, P_2, P_3, P_4, P_5 to solve the singularities of W_P^{17}, as instead implicitly assumed by Fano and explicitly by Conte–Murre (see [7, Section 3.10]).

9. THE P-EF 3-FOLD OF GENUS 13

Let us study the Enriques–Fano threefold mentioned in [20, Remark 3]. We refer to Section A.8 of Appendix A for the computational techniques we will use. Let us take the smooth sextic del Pezzo surface $S_6 \subset \mathbb{P}^6_{[x_0:x_1:x_2:x_3:x_4:x_5:x_6]}$ defined in Section 7. Let us consider $\mathbb{P}^6$ as the hyperplane $\{y_0 = 0\} \subset \mathbb{P}^7_{[x_0:x_1:x_2:x_3:x_4:x_5:x_6:y]}$ and let V be the cone over S_6 with vertex $v := [0:0:0:0:0:0:0:1]$. With a similar proof as the one of [20, Lemma 3.1], one can see that V is a singular Fano threefold. Let t be the involution of $\mathbb{P}^7$ defined by $[x_0 : \cdots : x_6 : y] \mapsto [x_2 : x_4 : x_0 : x_5 : x_1 : x_3 : x_6 : -y]$. The locus of t-fixed points in $\mathbb{P}^7$ consists of two projective subspaces:

$$F_1 = \{x_0 + x_2 = x_1 + x_4 = x_3 + x_5 = x_6 = 0\} \cong \mathbb{P}^3,$$
$$F_2 = \{x_0 - x_2 = x_1 - x_4 = x_3 - x_5 = y = 0\} \cong \mathbb{P}^3.$$

In particular, we have that $F_1 \cap V = \{v\}$ and $F_2 \cap V = \{v_1, v_2, v_3, v_4\}$, where

$$v_1 := [1:1:1:1:1:1:1:0], \qquad v_2 := [1:-1:1:-1:-1:-1:1:0],$$
$$v_3 := [-1:1:-1:-1:1:-1:1:0], \quad v_4 := [-1:-1:-1:1:-1:1:1:0].$$

Thus, t induces an involution $\tau := t|_V$ of V with five fixed points. The quotient of V via the involution τ is an Enriques–Fano threefold of genus 13, which we will denote by W_P^{13}: one can deduce this by using a similar proof as the one of [20, Proposition 3.2]. In this case, the quotient map $\pi : V \to V/\tau = W_P^{13}$ is defined by the restriction on V

of the linear system $\mathcal{Q}$ of the quadric hypersurfaces of $\mathbb{P}^9$ of type

$$q_1(x_0 + x_2, x_1 + x_4, x_3 + x_5, x_6) + q_2(x_0 - x_2, x_1 - x_4, x_3 - x_5, y) = 0,$$

where q_1 and q_2 are quadratic homogeneous forms. The linear system $\mathcal{Q}$ defines a morphism $\varphi : \mathbb{P}^7 \to \mathbb{P}^{19}$ such that $[x_0 : \cdots : x_6 : y] \mapsto [Z_0 : \cdots : Z_{19}]$, where

$$
\begin{aligned}
&Z_0 = x_6^2, &&Z_1 = x_0^2 + x_2^2, &&Z_2 = x_1^2 + x_4^2, \\
&Z_3 = x_3^2 + x_5^2, &&Z_4 = (x_0 + x_2)x_6, &&Z_5 = (x_1 + x_4)x_6, \\
&Z_6 = (x_3 + x_5)x_6, &&Z_7 = x_0 x_1 + x_2 x_4, &&Z_8 = x_2 x_3 + x_0 x_5, \\
&Z_9 = x_1 x_3 + x_4 x_5, &&Z_{10} = (x_0 - x_2)y, &&Z_{11} = (x_1 - x_4)y, \\
&Z_{12} = (x_3 - x_5)y, &&Z_{13} = y^2, &&Z_{14} = 2x_0 x_2, \\
&Z_{15} = 2x_1 x_4, &&Z_{16} = 2x_3 x_5, &&Z_{17} = x_4 x_3 + x_1 x_5, \\
&Z_{18} = x_0 x_3 + x_2 x_5, &&Z_{19} = x_1 x_2 + x_0 x_4.
\end{aligned}
$$

By looking at the ideal of $V \subset \mathbb{P}^7$, we observe that the threefold $\varphi(V)$ is contained in a 13-dimensional projective subspace of $\mathbb{P}^{19}$ given by

$$H_{13} := \{Z_{14} = 2Z_0, \; Z_{15} = 2Z_0, \; Z_{16} = 2Z_0, \; Z_{17} = Z_4, \; Z_{18} = Z_5, \; Z_{19} = Z_6\}.$$

Thus, we obtain the morphism $\pi : V \to W_P^{13} = \varphi(V) \subset H_{13} \cong \mathbb{P}^{13}$ defined by $[x_0 : \cdots : x_6 : y] \mapsto [z_0 : \cdots : z_{13}]$, where

$$
\begin{aligned}
&z_0 = x_6^2, &&z_1 = x_0^2 + x_2^2, &&z_2 = x_1^2 + x_4^2, \\
&z_3 = x_3^2 + x_5^2, &&z_4 = (x_0 + x_2)x_6, &&z_5 = (x_1 + x_4)x_6, \\
&z_6 = (x_3 + x_5)x_6, &&z_7 = x_0 x_1 + x_2 x_4, &&z_8 = x_2 x_3 + x_0 x_5, \\
&z_9 = x_1 x_3 + x_4 x_5, &&z_{10} = (x_0 - x_2)y, &&z_{11} = (x_1 - x_4)y, \\
&z_{12} = (x_3 - x_5)y, &&z_{13} = y^2.
\end{aligned}
$$

By using Macaulay2, we find that the threefold W_P^{13} has ideal generated by the 42 quadratic polynomials

$$
\begin{aligned}
&z_4 z_5 - 2z_0 z_6 - z_2 z_6 + z_5 z_9, &&z_5^2 - z_6^2 - z_6 z_7 + z_5 z_8, \\
&2z_0 z_5 + z_3 z_5 - z_4 z_6 - z_6 z_9, &&z_4^2 - z_6^2 - z_6 z_7 + z_4 z_9, \\
&z_4 z_5 - 2z_0 z_6 - z_1 z_6 + z_4 z_8, &&-2z_0 z_5 - z_1 z_5 + z_4 z_6 + z_4 z_7, \\
&2z_0 z_4 + z_3 z_4 - z_5 z_6 - z_6 z_8, &&2z_0 z_4 + z_2 z_4 - z_5 z_6 - z_5 z_7, \\
&4z_0^2 - z_4^2 - z_5^2 + z_6 z_7, &&z_5 z_{10} - z_4 z_{11} + 2z_0 z_{12}, \\
&-z_6 z_{10} + 2z_0 z_{11} - z_4 z_{12}, &&2z_0 z_{10} - z_6 z_{11} + z_5 z_{12}, \\
&2z_0 z_4 - 2z_5 z_6 + 2z_0 z_9, &&2z_0 z_5 - 2z_4 z_6 + 2z_0 z_8, \\
&-2z_4 z_5 + 2z_0 z_6 + 2z_0 z_7, &&2z_0 z_3 + z_4^2 + z_5^2 - 2z_6^2 - z_6 z_7,
\end{aligned}
$$

$$2z_0z_2 + z_4^2 - z_5^2 - z_6z_7, \quad 2z_0z_1 - z_4^2 + z_5^2 - z_6z_7,$$

$$z_{12}^2 + 2z_0z_{13} - z_3z_{13}, \quad z_{11}z_{12} + z_4z_{13} - z_9z_{13},$$

$$z_{10}z_{12} - z_5z_{13} + z_8z_{13}, \quad z_4z_{10} - z_5z_{11} + z_7z_{12},$$

$$z_{11}^2 + 2z_0z_{13} - z_2z_{13}, \quad z_{10}z_{11} + z_6z_{13} - z_7z_{13},$$

$$- z_5z_{10} + z_9z_{11} - z_2z_{12}, \quad -z_4z_{10} + z_8z_{11} - z_6z_{12},$$

$$- z_6z_{10} + z_3z_{11} - z_9z_{12}, \quad z_{10}^2 + 2z_0z_{13} - z_1z_{13},$$

$$z_9z_{10} - z_5z_{11} + z_6z_{12}, \quad z_8z_{10} - z_4z_{11} + z_1z_{12},$$

$$z_7z_{10} - z_1z_{11} + z_4z_{12}, \quad z_3z_{10} - z_6z_{11} + z_8z_{12},$$

$$z_2z_{10} - z_7z_{11} + z_5z_{12}, \quad -z_4z_5 + 2z_0z_6 - z_3z_6 + z_8z_9,$$

$$2z_0z_5 - z_2z_5 - z_4z_6 + z_7z_9, \quad 2z_0z_4 - z_5z_7 - z_6z_8 + z_1z_9,$$

$$2z_0z_4 - z_1z_4 - z_5z_6 + z_7z_8, \quad -z_1z_5 + z_4z_6 + z_2z_8 - z_6z_9,$$

$$2z_4z_5 - 2z_0z_6 - z_1z_6 - z_2z_6 + z_3z_7, \quad z_2z_3 + z_5^2 - z_6z_7 - z_9^2,$$

$$z_1z_3 + z_4^2 - z_6z_7 - z_8^2, \quad z_1z_2 + z_4^2 + z_5^2 - z_6^2 - z_6z_7 - z_7^2.$$

Furthermore, W_P^{13} has the five singular points

$$P_1 := \pi(v_1) = [1 : 2 : 2 : 2 : 2 : 2 : 2 : 2 : 2 : 2 : 0 : 0 : 0 : 0],$$

$$P_2 := \pi(v_2) = [1 : 2 : 2 : 2 : 2 : -2 : -2 : -2 : -2 : 2 : 0 : 0 : 0 : 0],$$

$$P_3 := \pi(v_3) = [1 : 2 : 2 : 2 : -2 : 2 : -2 : -2 : 2 : -2 : 0 : 0 : 0 : 0],$$

$$P_4 := \pi(v_4) = [1 : 2 : 2 : 2 : -2 : -2 : 2 : 2 : -2 : -2 : 0 : 0 : 0 : 0],$$

$$P_5 := \pi(v) = [0 : 0 : 0 : 0 : 0 : 0 : 0 : 0 : 0 : 0 : 0 : 0 : 0 : 1].$$

PROPOSITION 9.1. *If $i = 1, 2, 3, 4$, the tangent cone $TC_{P_i} W_P^{13}$ to W_P^{13} at the point P_i is a cone over a Veronese surface.*

PROOF. Let us consider the following change of coordinates of $\mathbb{P}^{13}$:

$$z_0 = w_0, \quad z_i = w_i + 2w_0, \quad z_j = w_j, \quad i = 1, \dots, 9, \ j = 10, \dots, 13.$$

With respect to the new system of coordinates $[w_0 : \cdots : w_{13}]$ of $\mathbb{P}^{13}$, the point P_1 has coordinates $[1 : 0 : \cdots : 0]$ and can be viewed as the origin of the open affine set $U_0 := \{w_0 \neq 0\}$. The ideal of the tangent cone $TC_{P_1}(W_P^{13} \cap U_0)$ is generated by the minimal degree homogeneous parts of all the polynomials in the ideal of $W_P^{13} \cap U_0$. Via Macaulay, $TC_{P_1}(W_P^{13} \cap U_0)$ is found to have ideal generated by

$$- 9w_1 + 8w_7 + 8w_8 - 4w_9, \quad -9w_2 + 8w_7 - 4w_8 + 8w_9,$$

$$- 9w_3 - 4w_7 + 8w_8 + 8w_9, \quad -9w_4 + 2w_7 + 2w_8 - w_9,$$

$$- 9w_5 + 2w_7 - w_8 + 2w_9, \quad -9w_6 - w_7 + 2w_8 + 2w_9,$$

$w_{10} - w_{11} + w_{12}, \quad 9w_{11}w_{12} + 2w_7w_{13} + 2w_8w_{13} - 10w_9w_{13},$

$2w_7w_{11} - 10w_8w_{11} + 2w_9w_{11} - 10w_7w_{12} + 2w_8w_{12} + 2w_9w_{12},$

$6w_7w_{11} - 6w_8w_{11} - 18w_9w_{11} + 6w_7w_{12} - 6w_8w_{12} + 18w_9w_{12},$

$9w_{12}^2 + 4w_7w_{13} - 8w_8w_{13} - 8w_9w_{13}, \quad 9w_{11}^2 - 8w_7w_{13} + 4w_8w_{13} - 8w_9w_{13},$

$w_7^2 - 2w_7w_8 + w_8^2 - 2w_7w_9 - 2w_8w_9 + w_9^2.$

Hence $TC_{P_1} W_P^{13}$ is a cone with vertex at P_1 over a Veronese surface in

$\{w_0 = 0, \; -9w_1 + 8w_7 + 8w_8 - 4w_9 = 0, \; -9w_2 + 8w_7 - 4w_8 + 8w_9 = 0,$

$\quad -9w_3 - 4w_7 + 8w_8 + 8w_9 = 0, \; -9w_4 + 2w_7 + 2w_8 - w_9 = 0,$

$\quad -9w_5 + 2w_7 - w_8 + 2w_9 = 0, \; -9w_6 - w_7 + 2w_8 + 2w_9 = 0,$

$\quad w_{10} - w_{11} + w_{12} = 0\} \cong \mathbb{P}^5.$

Similar analysis holds for the points P_2, P_3, and P_4. ∎

THEOREM 9.2. *The tangent cone $TC_{P_5} W_P^{13}$ to W_P^{13} at the point P_5 is a cone over a reducible quintic surface M_5, which is given by the union of five planes π_0, π_1, π_2, π_3, π_4, such that the four planes π_1, π_2, π_3, π_4 intersect the plane π_0 along the four edges of a quadrilateral.*

PROOF. The point P_5 can be viewed as the origin of the open affine set given by $U_{13} := \{z_{13} \neq 0\}$. The ideal of the tangent cone $TC_{P_5}(W_P^{13} \cap U_{13})$ is generated by the minimal degree homogeneous parts of all the polynomials in the ideal of $W_P^{13} \cap U_{13}$. By using Macaulay2, we find that $TC_{P_5}(W_P^{13} \cap U_{13})$ has ideal generated by the polynomials

$z_6 - z_7, \quad z_5 - z_8, \quad z_4 - z_9, \quad z_2 - z_3, \quad z_1 - z_3, \quad 2z_0 - z_3,$

$z_9z_{10} - z_8z_{11} + z_7z_{12}, \quad z_8z_{10} - z_9z_{11} + z_3z_{12}, \quad z_7z_{10} - z_3z_{11} + z_9z_{12},$

$z_3z_{10} - z_7z_{11} + z_8z_{12}, \quad z_8^2 - z_9^2, \quad z_7^2 - z_9^2, \quad z_3^2 - z_9^2, \quad z_7z_8 - z_3z_9,$

$z_3z_8 - z_7z_9, \quad z_3z_7 - z_8z_9.$

Hence $TC_{P_5} W_P^{13}$ is a cone with vertex at P_5 over a surface M_5 contained in the $\mathbb{P}^6$ defined by $\{z_{13} = 0, z_6 = z_7, z_5 = z_8, z_4 = z_9, z_2 = z_3, z_1 = z_3, 2z_0 = z_3\}$. This surface M_5, an idea of which is given in Figure 4, is the union of the five planes

$\pi_0 := \{z_i = 0 \mid i \neq 10, 11\},$

$\pi_1 := \{2z_0 = z_1 = z_2 = z_3 = z_4 = z_5 = z_6 = z_7 = z_8 = z_9,$

$\qquad z_{10} = z_{11} - z_{12}, \; z_{13} = 0\},$

$\pi_2 := \{2z_0 = z_1 = z_2 = z_3 = z_4 = -z_5 = -z_6 = -z_7 = -z_8 = z_9,$

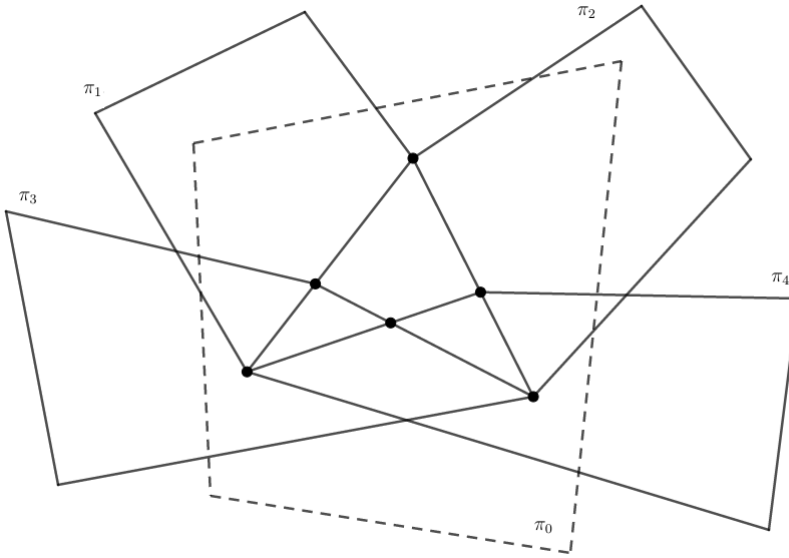$\qquad z_{10} = z_{12} - z_{11}, \; z_{13} = 0\},$

FIGURE 4. The reducible quintic surface $M_5 \subset \mathbb{P}^6$ given by the union of five planes $\pi_0, \pi_1, \pi_2,$ π_3, π_4, which intersect as in the statement of Theorem 9.2.

$$\pi_3 := \{2z_0 = z_1 = z_2 = z_3 = -z_4 = z_5 = -z_6 = -z_7 = z_8 = -z_9,$$
$$z_{10} = -z_{11} - z_{12}, \ z_{13} = 0\},$$
$$\pi_4 := \{2z_0 = z_1 = z_2 = z_3 = -z_4 = -z_5 = z_6 = z_7 = -z_8 = -z_9,$$
$$z_{10} = z_{11} + z_{12}, \ z_{13} = 0\}. \qquad \blacksquare$$

By Proposition 9.1 and Theorem 9.2, we have that W_P^{13} has five non-similar points. For completeness, let us find their configuration. Let $l_{i,j}$ be the line joining the singular points P_i and P_j for $1 \leq i < j \leq 5$. Thanks to Macaulay2, we find that the lines $l_{1,5}$, $l_{2,5}, l_{3,5}, l_{4,5}$ are contained in W_P^{13}, while the lines $l_{1,2}, l_{1,3}, l_{1,4}, l_{2,3}, l_{2,4}, l_{3,4}$ are not. Hence the five singular points P_1, P_2, P_3, P_4, P_5 of W_P^{13} are associated as in Table 3 of Appendix B.

REMARK 9.3. Let $bl : \mathrm{Bl}_{P_i=1,\dots 5} \mathbb{P}^{13} \to \mathbb{P}^{13}$ be the blow-up of $\mathbb{P}^{13}$ at the five singular points of W_P^{13} and let $\widetilde{W}$ be the strict transform of W_P^{13}. Then $\widetilde{W}$ intersects the exceptional divisor $bl^{-1}(P_5)$ along a surface isomorphic to M_5, which has six singular points locally given by the intersection of three planes of $\mathbb{P}^4$, such that two of them intersect the third along two lines and intersect each other at a point which is intersection of these two lines. Therefore, $\widetilde{W}$ is not a desingularization of W_P^{13}, since there are six

singular points infinitely near to P_5. Therefore, it is not enough to blow-up W_P^{13} at P_1, P_2, P_3, P_4, P_5 to solve the singularities of W_P^{13}, as instead implicitly assumed by Fano and explicitly by Conte–Murre (see [7, Section 3.10]).

10. The KLM-EF 3-fold of genus 9

Let us study the Enriques–Fano threefold described in [16, Section 13]. We refer to Section A.9 of Appendix A for the computational techniques we will use. Let $\mathcal{S}$ be the linear system of the sextic surfaces of $\mathbb{P}^3_{[t_0:\cdots:t_3]}$ having double points along the six edges $l_{ij} := \{t_i = t_j = 0\}$ of a fixed tetrahedron $T := \{t_0 t_1 t_2 t_3 = 0\}$, for $0 \leq i < j \leq 3$. Let us denote the vertices of T by $v_i := \{t_k = 0 \mid k \neq i\}$, for $0 \leq i \leq 3$. Then $\mathcal{S}$ defines a rational map $\nu_{\mathcal{S}} : \mathbb{P}^3 \dashrightarrow \mathbb{P}^{13}_{[w_0,\ldots,w_{13}]}$ whose image is the Enriques–Fano threefold $W_F^{13} = W_{\mathrm{BS}}^{13}$ studied in Section 5 (see Theorem 1.3).

Let us *fix* a general element $\Sigma \in \mathcal{S}$ and let us take its image $S := \nu_{\mathcal{S}}(\Sigma)$. Then there exists a hyperplane $H_{12} \cong \mathbb{P}^{12} \subset \mathbb{P}^{13}$ such that $S = W_F^{13} \cap H_{12}$. By the generality of Σ, we may assume that H_{12} does not pass through the singular points P_1, P_2, P_3, P_4, P_1', P_2', P_3', P_4', and so that $H_{12} = \{\sum_{i=0}^{13} a_i w_i = 0\}$, where a_0, a_1, a_3, a_4, a_9, a_{10}, a_{12}, a_{13} are not equal to zero. In particular, we may suppose that $a_0 = 1$. Let us blow-up $\mathbb{P}^3$ at the vertices of T: we obtain a smooth threefold Y' and a birational morphism $bl' : Y' \to \mathbb{P}^3$ with exceptional divisors $E_i := (bl')^{-1}(v_i)$, for $0 \leq i \leq 3$. If H denotes the pullback on Y' of the hyperplane class on $\mathbb{P}^3$, the strict transform of Σ on Y' is linearly equivalent to $6H - 3\sum_{i=0}^{3} E_i$. Let us blow-up Y' along the strict transforms $\tilde{l}_{ij}$ of the edges of T, for $0 \leq i < j \leq 3$: we obtain a smooth threefold Y'' and a birational morphism $bl'' : Y'' \to Y'$ with exceptional divisors $F_{ij} := (bl'')^{-1}(\tilde{l}_{ij})$, for $0 \leq i < j \leq 3$. Let Σ'' be the strict transform on Y'' of Σ. Then

$$\Sigma'' \sim 6H - 3\sum_{i=0}^{3} \tilde{E}_i - 2 \sum_{0 \leq i < j \leq 3} F_{ij},$$

where $\tilde{E}_i$ denotes the strict transform of E_i, for $0 \leq i \leq 3$, and H denotes the pullback $bl''^* H$, by abuse of notation. Let $\nu'' : Y'' \dashrightarrow W_F^{13} \subset \mathbb{P}^{13}$ be the birational map defined by the linear system $|\mathcal{O}_{Y''}(\Sigma'')|$ and let us take $E_3 := \nu''(F_{23} \cap \Sigma'')$. The hyperplane sections of $W_F^{13} \subset \mathbb{P}^{13}$ containing $\nu''(F_{23})$ correspond to the divisors on Y'' linearly equivalent to

$$6H - 3\sum_{i=0}^{3} \tilde{E}_i - 3F_{23} - \sum_{\substack{0 \leq i < j \leq 3 \\ (i,j) \neq (2,3)}} 2F_{ij}.$$

Since

$$\langle \nu''(F_{23}) \rangle = \{w_5 = w_6 = w_7 = w_8 = w_9 = w_{10} = w_{11} = w_{12} = w_{13} = 0\} \cong \mathbb{P}^4,$$

then we have that $\langle E_3 \rangle = H_{12} \cap \langle \nu''(F_{23}) \rangle \cong \mathbb{P}^3$, and so that $E_3 = S \cap \langle E_3 \rangle$, which is defined by the equations

$$w_0 + a_1 w_1 + a_2 w_2 + a_3 w_3 + a_4 w_4 = 0,$$
$$w_5 = w_6 = w_7 = w_8 = w_9 = w_{10} = w_{11} = w_{12} = w_{13} = 0,$$
$$w_1 w_3 + a_1 w_1 w_4 + a_2 w_2 w_4 + a_3 w_3 w_4 + a_4 w_4^2 = 0,$$
$$w_2^2 + a_1 w_1 w_4 + a_2 w_2 w_4 + a_3 w_3 w_4 + a_4 w_4^2 = 0.$$

Thus, E_3 is a quartic elliptic curve, since it is the complete intersection of two quadric surfaces of $\langle E_3 \rangle \cong \mathbb{P}^3$. If $\rho_{\langle E_3 \rangle} : \mathbb{P}^{13} \dashrightarrow \mathbb{P}^9$ denotes the projection of $\mathbb{P}^{13}$ from $\langle E_3 \rangle \cong \mathbb{P}^3$, then $W^9_{KLM} := \rho_{\langle E_3 \rangle}(W^{13}_F) \subset \mathbb{P}^9$ is the KLM-EF 3-fold of genus 9.

REMARK 10.1. For the construction of W^9_{KLM}, we have *fixed* a general sextic $\Sigma \in S$. The hyperplane sections of $W^9_{KLM} \subset \mathbb{P}^9$ correspond to the hyperplane sections of $W^{13}_F \subset \mathbb{P}^{13}$ containing E_3, which are images via ν_S of the sextic surfaces R of S which are tangent to Σ along the two branches of Σ intersecting at l_{23}. Then W^9_{KLM} is the image of $\mathbb{P}^3$ via the rational map defined by the sublinear system $\mathcal{R} \subset S$ of these sextic surfaces R.

By using Macaulay2, we find that the projection map is given by

$$\left[w_0 : w_1 : w_2 : w_3 : w_4 : w_5 : w_6 : w_7 : w_8 : w_9 : w_{10} : w_{11} : w_{12} : w_{13} \right]$$

$$\downarrow \rho_{\langle E_3 \rangle}$$

$$\left[w_0 + a_1 w_1 + a_2 w_2 + a_3 w_3 + a_4 w_4 : w_5 : w_6 : w_7 : w_8 : w_9 : w_{10} : w_{11} : w_{12} : w_{13} \right]$$

and that the ideal of $W^9_{KLM} \subset \mathbb{P}^9_{[z_0:z_1:z_2:z_3:z_4:z_5:z_6:z_7:z_8:z_9]}$ is generated by the following 16 polynomials of degree 2 or 3:

$z_6 z_8 - z_5 z_9, \quad z_3 z_8 - z_2 z_9, \quad z_7^2 - z_5 z_9, \quad z_4 z_7 - z_2 z_9,$

$z_3 z_7 - z_1 z_9, \quad z_2 z_7 - z_1 z_8, \quad z_4 z_6 - z_1 z_9, \quad z_2 z_6 - z_1 z_7,$

$z_4 z_5 - z_1 z_8, \quad z_3 z_5 - z_1 z_7, \quad z_2 z_3 - z_1 z_4,$

$z_1 z_2 + a_1 z_1 z_3 + a_2 z_1 z_4 + a_3 z_2 z_4 + a_4 z_3 z_4 - z_0 z_7,$

$z_2^2 z_9 + a_1 z_1 z_4 z_9 + a_2 z_2 z_4 z_9 + a_3 z_4^2 z_8 + a_4 z_4^2 z_9 - z_0 z_8 z_9,$

$z_1^2 z_9 + a_1 z_3^2 z_6 + a_2 z_1 z_3 z_9 + a_3 z_1 z_4 z_9 + a_4 z_3^2 z_9 - z_0 z_6 z_9,$

$z_2^2 z_5 + a_3 z_2^2 z_8 - a_2 a_3 z_2 z_4 z_8 - z_0 z_5 z_8 + a_2 z_0 z_7 z_8 - a_1^2 z_1^2 z_9$
$\quad + (a_4 - a_2^2 - a_1 a_3) z_2^2 z_9 + + 2 a_1^2 a_2 z_1 z_3 z_9 + a_1 (2 a_2^2 - a_4) z_1 z_4 z_9$
$\quad + (2 a_1 a_2 a_3 - a_2 a_4) z_2 z_4 z_9 + 2 a_1 a_2 a_4 z_3 z_4 z_9 + + a_1 z_0 z_5 z_9 - a_1 a_2 z_0 z_7 z_9,$

$z_1^2 z_5 + a_1 z_1^2 z_6 - z_0 z_5 z_6 + a_2 z_1^2 z_7 + (a_4 - a_1 a_2) z_1^2 z_9 - a_3^2 z_2^2 z_9 + a_1 a_2 a_3 z_1 z_3 z_9$
$\quad - a_3 (a_4 - a_2^2) z_1 z_4 z_9 + a_2 a_3^2 z_2 z_4 z_9 + a_2 a_3 a_4 z_3 z_4 z_9 + a_3 z_0 z_5 z_9 - a_2 a_3 z_0 z_7 z_9.$

Let us take the images of the eight quadruple points of W_F^{13}, by denoting them, by abuse of notation, in the following way:

$$P_1 := \rho_{\langle E_3 \rangle}(P_1') = \{z_k = 0 \mid k \neq 5\}, \quad P_2 := \rho_{\langle E_3 \rangle}(P_2') = \{z_k = 0 \mid k \neq 9\}$$

$$P_3 := \rho_{\langle E_3 \rangle}(P_3) = \{z_k = 0 \mid k \neq 6\}, \quad P_4 := \rho_{\langle E_3 \rangle}(P_4) = \{z_k = 0 \mid k \neq 8\},$$

$$P_5 := \rho_{\langle E_3 \rangle}(P_1) = \rho_{\langle E_3 \rangle}(P_2) = \rho_{\langle E_3 \rangle}(P_3') = \rho_{\langle E_3 \rangle}(P_4') = \{z_k = 0 \mid k \neq 0\}.$$

PROPOSITION 10.2. *If $i = 1, 2, 3, 4$, the tangent cone $TC_{P_i}W_{KLM}^9$ to W_{KLM}^9 at the point P_i is a cone over a Veronese surface.*

PROOF. Each point P_i, for $i = 1, 2, 3, 4$, can be viewed as the origin of the open affine set $U_{j(i)} := \{z_{j(i)} \neq 0\}$, where $j(1) = 5, j(2) = 9, j(3) = 6, j(4) = 8$. The ideal of the tangent cone $TC_{P_i}(W_{KLM}^9 \cap U_{j(i)})$ is generated by the minimal degree homogeneous parts of all the polynomials in the ideal of $W_{KLM}^9 \cap U_{j(i)}$. Thanks to Macaulay2, we obtain the following tangent cones.

$TC_{P_1}(W_{KLM}^9 \cap U_5)$ has ideal generated by $z_9, z_4, z_3, z_7^2 - z_6z_8, z_2z_7 - z_1z_8,$ $z_2z_6 - z_1z_7, z_2^2 - z_0z_8, z_1z_2 - z_0z_7, z_1^2 - z_0z_6$. Hence $TC_{P_1}W_{KLM}^9$ is a cone with vertex P_1 over a Veronese surface in $\{z_i = 0 \mid i = 3, 4, 5, 9\} \cong \mathbb{P}^5$.

$TC_{P_2}(W_{KLM}^9 \cap U_9)$ has ideal generated by $z_5, z_2, z_1, z_7^2 - z_6z_8, z_4z_7 - z_3z_8,$ $z_4z_6 - z_3z_7, a_4z_4^2 - z_0z_8, a_4z_3z_4 - z_0z_7, a_4z_3^2 - z_0z_6$. Hence $TC_{P_2}W_{KLM}^9$ is a cone with vertex P_2 over a Veronese surface in $\{z_i = 0 \mid i = 1, 2, 5, 9\} \cong \mathbb{P}^5$.

$TC_{P_3}(W_{KLM}^9 \cap U_6)$ has ideal generated by $z_8, z_4, z_2, z_7^2 - z_5z_9, z_3z_7 - z_1z_9,$ $z_3z_5 - z_1z_7, a_1z_3^2 - z_0z_9, a_1z_1z_3 - z_0z_7, a_1z_1^2 - z_0z_5$. Hence $TC_{P_3}W_{KLM}^9$ is a cone with vertex P_3 over a Veronese surface in $\{z_i = 0 \mid i = 2, 4, 6, 8\} \cong \mathbb{P}^5$.

$TC_{P_4}(W_{KLM}^9 \cap U_8)$ has ideal generated by $z_6, z_3, z_1, z_7^2 - z_5z_9, z_4z_7 - z_2z_9,$ $z_4z_5 - z_2z_7, a_3z_4^2 - z_0z_9, a_3z_2z_4 - z_0z_7, a_3z_2^2 - z_0z_5$. Hence $TC_{P_4}W_{KLM}^9$ is a cone with vertex P_4 over a Veronese surface in $\{z_i = 0 \mid i = 1, 3, 6, 8\} \cong \mathbb{P}^5$. ∎

THEOREM 10.3. *The tangent cone $TC_{P_5}W_{KLM}^9$ to W_{KLM}^9 at the point P_5 is a cone over a reducible sextic surface $M_6 \subset \mathbb{P}^7 \subset \mathbb{P}^9$, which is given by the union of four planes $\pi_1, \pi_2, \pi_1', \pi_2'$ and a quadric surface $Q \subset \mathbb{P}^3 \subset \mathbb{P}^7$. In particular, each one of the planes $\pi_1, \pi_2, \pi_1', \pi_2'$ intersects the quadric surface Q respectively along a line l_1, l_2, l_1', l_2', where l_i is disjoint from l_i', for $i = 1, 2$. In the other cases, the intersections of two of these lines identify four points, which are $q_{1,2} := l_1 \cap l_2, q_{1,2'} := l_1 \cap l_2',$ $q_{1',2} := l_1' \cap l_2$, and $q_{1',2'} := l_1' \cap l_2'$.*

PROOF. The point P_5 can be viewed as the origin of the open affine set given by $U_0 := \{z_0 \neq 0\}$. The ideal of the tangent cone $TC_{P_5}(W_{KLM}^9 \cap U_0)$ is generated by the minimal degree homogeneous parts of all the polynomials in the ideal of $W_{KLM}^9 \cap U_0$.

By using Macaulay2, one finds that $TC_{P_5}(W^9_{KLM} \cap U_0)$ has ideal generated by the polynomials

$$z_7, \ z_8z_9, \ z_6z_9, \ z_5z_9, \ z_2z_9, \ z_1z_9, \ z_6z_8, \ z_5z_8, \ z_3z_8,$$

$$z_1z_8, \ z_5z_6, \ z_4z_6, \ z_2z_6, \ z_4z_5, \ z_3z_5, \ z_2z_3 - z_1z_4.$$

Hence $TC_{P_5}W^9_{KLM}$ is a cone with vertex P_5 over a surface M_6 contained in the $\mathbb{P}^7$ given by $\{z_i = 0 \mid i = 0, 7\}$. The surface M_6 is the union of four planes $\pi_1, \pi_2, \pi'_1,$ π'_2 and a quadric surface Q, where

$$\pi_1 := \{z_i = 0 \mid i = 0, 1, 2, 5, 6, 7, 8\}, \quad \pi_2 := \{z_i = 0 \mid i = 0, 1, 3, 5, 6, 7, 9\},$$
$$\pi'_1 := \{z_i = 0 \mid i = 0, 3, 4, 6, 7, 8, 9\}, \quad \pi'_2 := \{z_i = 0 \mid i = 0, 2, 4, 5, 7, 8, 9\},$$
$$Q := \{z_i = 0 \mid i = 0, 5, 6, 7, 8, 9\} \cap \{z_2z_3 - z_1z_4 = 0\}.$$

We obtain the same situation as the one described in Theorem 8.2, and so a sextic surface M_6 as in Figure 3. ∎

By Proposition 10.2 and Theorem 10.3, we have that P_1, P_2, P_3, P_4, P_5 are non-similar singular points of W^9_{KLM}. For completeness, let us find their configuration. Let $l_{i,j}$ be the line joining the singular points P_i and P_j for $1 \leq i < j \leq 5$. Then we have $l_{1,2} = \{z_k = 0 \mid k \neq 5, 9\}, l_{1,3} = \{z_k = 0 \mid k \neq 5, 6\}, l_{1,4} = \{z_k = 0 \mid k \neq 5, 8\},$ $l_{1,5} = \{z_k = 0 \mid k \neq 0, 5\}, l_{2,3} = \{z_k = 0 \mid k \neq 6, 9\}, l_{2,4} = \{z_k = 0 \mid k \neq 8, 9\},$ $l_{2,5} = \{z_k = 0 \mid k \neq 0, 9\}, l_{3,4} = \{z_k = 0 \mid k \neq 6, 8\}, l_{3,5} = \{z_k = 0 \mid k \neq 0, 6\},$ $l_{4,5} = \{z_k = 0 \mid k \neq 0, 8\}$. By looking at the ideal of $W^9_{KLM} \subset \mathbb{P}^9$, we deduce that the lines $l_{1,3}, l_{1,4}, l_{1,5}, l_{2,3}, l_{2,4}, l_{2,5}, l_{3,5}, l_{4,5}$ are contained in W^9_{KLM}, while $l_{1,4}$ and $l_{2,3}$ are not. So the five singular points P_1, P_2, P_3, P_4, P_5 of W^9_{KLM} are associated as in Table 3 of Appendix B.

11. PROJECTIVE NORMALITY

Some authors define an *Enriques–Fano threefold* just as a threefold satisfying the following assumption (see for example [16, Definition 1.3]).

ASSUMPTION (∗). *Let $W \subset \mathbb{P}^N$ be a non-degenerate threefold whose general hyperplane section S is an Enriques surface and such that W is not a cone over S.*

If the pair $(W, \mathcal{L} := |\mathcal{O}_W(S)|)$ satisfies Assumption (∗), it is enough to take its *normalization* $\nu : W^\nu \to W$ to obtain an Enriques–Fano threefold in the general sense, that is $(W^\nu, \nu^*\mathcal{L})$. Indeed, an element of $\nu^*\mathcal{L}$ is ample, since it is the pullback of a very ample divisor of $\mathcal{L}$ via the finite birational morphism $\nu : W^\nu \to W$ (see [17, Theorem 1.2.13]). Moreover, if $(W^\nu, \nu^*\mathcal{L})$ were a (polarized) generalized cone,

W^ν would contain a 3-dimensional family of curves of degree 1 with respect to the given polarization such that they all pass through a point; thus, $W \subset \mathbb{P}^N$ would be the union of lines through a point, contradicting Assumption (∗).

Furthermore, we observe that if a pair $(X, \mathcal{L})$ is an Enriques–Fano threefold such that the elements of $\mathcal{L}$ are very ample divisors on X, then $\mathcal{L}$ defines an embedding $\phi_{\mathcal{L}} : X \hookrightarrow \mathbb{P}^p$ and $\phi_{\mathcal{L}}(X) \subset \mathbb{P}^p$ is a threefold satisfying Assumption (∗).

An example of "Enriques–Fano threefold" in the sense of Assumption (∗) is the KLM-EF 3-fold $W_{KLM}^9 \subset \mathbb{P}^9$: instead of proving the normality of this threefold, Knutsen–Lopez–Muñoz study properties of its normalization (see [16, Proposition 13.1]). We will see below that the KLM-EF 3-fold actually is (projectively) normal.

Also the rational F-EF 3-folds $W_F^{p=6,7,9,13} \subset \mathbb{P}^p$ are "Enriques–Fano threefold" in the sense of Assumption (∗): indeed, their normality is unproved, even if Fano assumed normality at the beginning of his work. The normality of the non-rational F-EF 3-fold W_F^4 is unproved too; however, it does not exactly satisfy Assumption (∗), since its hyperplane sections are not Enriques surfaces, but their minimal desingularizations are (see [9, p. 275]). We will see below that the rational F-EF 3-folds of genus 7, 9, and 13 actually are (projectively) normal.

Instead, the BS-EF 3-folds and the P-EF 3-folds are normal by construction, since they are quotient of normal threefolds under the action of a finite group defined by a certain involution with a finite number of fixed points (see [10, Proposition 2.15]). In particular, the BS-EF 3-folds with very ample hyperplane sections satisfy Assumption (∗) in the projective space in which they are embedded, while the other eight BS-EF 3-folds are Enriques–Fano threefolds satisfying exactly the abstract definition. Furthermore, as we saw in Sections 8 and 9 the P-EF 3-folds $W_P^{p=13,17}$ can be embedded in $\mathbb{P}^p$ and they also satisfy Assumption (∗).

DEFINITION 11.1. Let $\mathcal{R}$ be a 3-dimensional linear system of quadric surfaces of $\mathbb{P}^3$. Let us suppose that $\mathcal{R}$ is sufficiently general, i.e., $\mathcal{R}$ is base point free and, if l is a double line for $Q \in \mathcal{R}$, then Q is the unique quadric in $\mathcal{R}$ containing l. A *Reye congruence* is a surface obtained as the set

$$\{l \in \mathbb{G}(1,3) \mid l \text{ is contained in a pencil contained in } \mathcal{R}\},$$

where $\mathbb{G}(1,3)$ denotes the Grassmannian variety of lines in $\mathbb{P}^3$.

THEOREM 11.2. *Let $W \subset \mathbb{P}^N$ be a threefold satisfying Assumption (∗). If $S \subset \mathbb{P}^{N-1}$ is linearly normal and if either $N \geq 7$ or $N = 6$ and S is not a Reye congruence, then $h^1(\mathcal{O}_W) = 0$ and $W \subset \mathbb{P}^N$ is projectively normal.*

PROOF. Since the case where $N = 6$ and S is a Reye congruence is excluded, we have that $S \subset \mathbb{P}^{N-1}$ is projectively normal (see [13, Theorem 1.1]). Thus, by using the

arguments of [8, Lemmas 1.5, 1.6, and 1.7] (which are inspired by the ones of [11, pp. 10–11]), we obtain that $h^1(\mathcal{O}_W) = 0$ and that $W \subset \mathbb{P}^N$ is projectively normal. ∎

PROPOSITION 11.3. *Let $W \subset \mathbb{P}^N$ be a threefold satisfying Assumption (∗). If $W \subset \mathbb{P}^N$ is linearly normal and $h^1(\mathcal{O}_W) = 0$, then $S \subset \mathbb{P}^{N-1}$ is linearly normal.*

PROOF. We have to show that $h^0(\mathcal{O}_S(1)) = h^0(\mathcal{O}_{\mathbb{P}^{N-1}}(1)) = N$. This follows by the exact sequence
$$0 \to \mathcal{O}_W \to \mathcal{O}_W(1) \to \mathcal{O}_S(1) \to 0,$$
since $h^0(\mathcal{O}_W) = 1$, $h^1(\mathcal{O}_W) = 0$ and $h^0(\mathcal{O}_W(1)) = h^0(\mathcal{O}_{\mathbb{P}^N}(1)) = N + 1$ by hypothesis. ∎

COROLLARY 11.4. *Let $W \subset \mathbb{P}^N$ be a threefold satisfying Assumption (∗). If $W \subset \mathbb{P}^N$ is linearly normal and $h^1(\mathcal{O}_W) = 0$, then $W \subset \mathbb{P}^N$ is projectively normal (except when $N = 6$ and S is a Reye congruence).*

PROOF. See Theorem 11.2 and Proposition 11.3. ∎

PROPOSITION 11.5. *Let $W \subset \mathbb{P}^p$ be a threefold satisfying Assumption (∗) such that p is the genus of a curve section of W. Then $W \subset \mathbb{P}^p$ and $S \subset \mathbb{P}^{p-1}$ are linearly normal.*

PROOF. By Riemann–Roch on S we obtain $h^0(\mathcal{O}_S(1)) = p$. From $W \subset \mathbb{P}^p$, we have that $h^0(\mathcal{O}_W(1)) \geq p + 1$. On the other hand, from the exact sequence
$$0 \to \mathcal{O}_W \to \mathcal{O}_W(1) \to \mathcal{O}_S(1) \to 0$$
one gets $h^0(\mathcal{O}_W(1)) \leq p + 1$ and hence equality holds. ∎

COROLLARY 11.6. *The following Enriques–Fano threefolds are projectively normal:*
$$W_{KLM}^9 \subset \mathbb{P}^9, \quad W_F^{p=7,9,13} \subset \mathbb{P}^p, \quad W_{BS}^{p=7,8,9,10,13} \xrightarrow{\phi_{\mathcal{L}}} \mathbb{P}^p, \quad W_P^{p=13,17} \subset \mathbb{P}^p.$$

PROOF. See Theorem 11.2 and Proposition 11.5. ∎

We cannot say the same for the F-EF 3-fold $W_F^6 \subset \mathbb{P}^6$, since its hyperplane sections are Reye congruences (see [6, Proposition 3] and [12, Section 3]). As for the BS-EF 3-fold $W_{BS}^6 \xrightarrow{\phi_{\mathcal{L}}} \mathbb{P}^6$, Macaulay2 enables us to find that its hyperplane section $S \subset \mathbb{P}^5$ is not contained in quadric hypersurfaces of $\mathbb{P}^5$ (see Code A.1 of Appendix A): this is equivalent to saying that $S \subset \mathbb{P}^5$ is projectively normal (use Riemann–Roch and see [13, Theorem 1.1]), and so we have that $W_{BS}^6 \subset \mathbb{P}^6$ is projectively normal too (see Theorem 11.2). Thus, we obtain Theorem 1.6.

A. Macaulay2 codes

In the following, we will collect the input codes used in Macaulay2 for the computational analysis of this paper. We will essentially use the package *Cremona* of Staglianò (see [23]) [1].

A.1. *Computational analysis of the BS-EF 3-fold of genus 6*

```
Macaulay2, version 1.11
with packages: ConwayPolynomials, Elimination, IntegralClosure, InverseSystems,
LLLBases, PrimaryDecomposition, ReesAlgebra, TangentCone
i1 : needsPackage "Cremona";
i2 : PP3xPP3 = ZZ/10000019[x_0,x_1,x_2,x_3]**ZZ/10000019[y_0,y_1,y_2,y_3];
i3 : X = ideal{ x_0*y_0-7*x_1*y_1+4*x_2*y_2+2*x_3*y_3,
x_0*y_0-6*x_1*y_1+2*x_2*y_2+3*x_3*y_3, x_0*y_0-x_1*y_1-7*x_2*y_2+7*x_3*y_3};
i4 : PP9 = ZZ/10000019[Z_0..Z_9];
i5 : phi = rationalMap map(PP3xPP3,PP9,matrix{{x_0*y_0,x_1*y_1,x_2*y_2,x_3*y_3,
x_0*y_1+x_1*y_0,x_0*y_2+x_2*y_0,x_0*y_3+x_3*y_0,x_1*y_2+x_2*y_1,x_1*y_3+x_3*y_1,
x_2*y_3+x_3*y_2}});
i6 : (dim(image phi) -1, degree(image phi)) == (6,10)
i7 : image phi ==
ideal{-2*Z_1*Z_5*Z_6+Z_4*Z_6*Z_7+Z_4*Z_5*Z_8-2*Z_0*Z_7*Z_8+4*Z_0*Z_1*Z_9-Z_4^2*Z_9,
-2*Z_2*Z_4*Z_6+Z_5*Z_6*Z_7+4*Z_0*Z_2*Z_8-Z_5^2*Z_8+Z_4*Z_5*Z_9-2*Z_0*Z_7*Z_9,
-4*Z_1*Z_2*Z_6+Z_6*Z_7^2+2*Z_2*Z_4*Z_8-Z_5*Z_7*Z_8+2*Z_1*Z_5*Z_9-Z_4*Z_7*Z_9,
-2*Z_3*Z_4*Z_5+4*Z_0*Z_3*Z_7-Z_6^2*Z_7+Z_5*Z_6*Z_8+Z_4*Z_6*Z_9-2*Z_0*Z_8*Z_9,
-4*Z_1*Z_3*Z_5+2*Z_3*Z_4*Z_7-Z_6*Z_7*Z_8+Z_5*Z_8^2+2*Z_1*Z_6*Z_9-Z_4*Z_8*Z_9,
-4*Z_2*Z_3*Z_4+2*Z_3*Z_5*Z_7+2*Z_2*Z_6*Z_8-Z_6*Z_7*Z_9-Z_5*Z_8*Z_9+Z_4*Z_9^2,
-4*Z_1*Z_2*Z_3+Z_3*Z_7^2+2*Z_2*Z_8^2-Z_7*Z_8*Z_9+Z_1*Z_9^2,
-4*Z_0*Z_2*Z_3+Z_3*Z_5^2+2*Z_2*Z_6^2-Z_5*Z_6*Z_9+Z_0*Z_9^2,
-4*Z_0*Z_1*Z_3+Z_3*Z_4^2+Z_1*Z_6^2-Z_4*Z_6*Z_8+Z_0*Z_8^2,
-4*Z_0*Z_1*Z_2+Z_2*Z_4^2+Z_1*Z_5^2-Z_4*Z_5*Z_7+Z_0*Z_7^2}
i8 : phiX = ideal{Z_2-Z_3,Z_1-Z_3,Z_0-Z_3,
-2*Z_1*Z_5*Z_6+Z_4*Z_6*Z_7+Z_4*Z_5*Z_8-2*Z_0*Z_7*Z_8+4*Z_0*Z_1*Z_9-Z_4^2*Z_9,
-2*Z_2*Z_4*Z_6+Z_5*Z_6*Z_7+4*Z_0*Z_2*Z_8-Z_5^2*Z_8+Z_4*Z_5*Z_9-2*Z_0*Z_7*Z_9,
-4*Z_1*Z_2*Z_6+Z_6*Z_7^2+2*Z_2*Z_4*Z_8-Z_5*Z_7*Z_8+2*Z_1*Z_5*Z_9-Z_4*Z_7*Z_9,
-2*Z_3*Z_4*Z_5+4*Z_0*Z_3*Z_7-Z_6^2*Z_7+Z_5*Z_6*Z_8+Z_4*Z_6*Z_9-2*Z_0*Z_8*Z_9,
-4*Z_1*Z_3*Z_5+2*Z_3*Z_4*Z_7-Z_6*Z_7*Z_8+Z_5*Z_8^2+2*Z_1*Z_6*Z_9-Z_4*Z_8*Z_9,
-4*Z_2*Z_3*Z_4+2*Z_3*Z_5*Z_7+2*Z_2*Z_6*Z_8-Z_6*Z_7*Z_9-Z_5*Z_8*Z_9+Z_4*Z_9^2,
-4*Z_1*Z_2*Z_3+Z_3*Z_7^2+2*Z_2*Z_8^2-Z_7*Z_8*Z_9+Z_1*Z_9^2,
-4*Z_0*Z_2*Z_3+Z_3*Z_5^2+2*Z_2*Z_6^2-Z_5*Z_6*Z_9+Z_0*Z_9^2,
-4*Z_0*Z_1*Z_3+Z_3*Z_4^2+Z_1*Z_6^2-Z_4*Z_6*Z_8+Z_0*Z_8^2,
-4*Z_0*Z_1*Z_2+Z_2*Z_4^2+Z_1*Z_5^2-Z_4*Z_5*Z_7+Z_0*Z_7^2};
i9 : (dim oo -1, degree oo, oo == phi(X) ) == (3, 10, true)
i10 : H6 = ideal{Z_2-Z_3,Z_1-Z_3,Z_0-Z_3};
i11 : PP6 = ZZ/10000019[w_0..w_6];
i12 : inclusion = rationalMap map(PP6,PP9,matrix{{w_0,w_0,w_0,w_0,w_1,w_2,w_3,
w_4,w_5,w_6}});
i13 : image oo == H6
i14 : pigreca = phi*(rationalMap map(PP9,PP6, sub(matrix inverseMap(inclusion||H6), PP9) ))
```

[1] For more information, visit the website http://www2.macaulay2.com/Macaulay2/doc/ Macaulay2-1.21/share/doc/Macaulay2/Cremona/html/index.html.

```
i15 : pigreca(X) == inclusion^*(phiX)
i16 : WB6 = ideal{
-2*w_0*w_2*w_3+w_1*w_3*w_4+w_1*w_2*w_5-2*w_0*w_4*w_5+4*w_0^2*w_6-w_1^2*w_6,
-2*w_0*w_1*w_3+w_2*w_3*w_4+4*w_0^2*w_5-w_2^2*w_5+w_1*w_2*w_6-2*w_0*w_4*w_6,
-4*w_0^2*w_3+w_3*w_4^2+2*w_0*w_1*w_5-w_2*w_4*w_5+2*w_0*w_2*w_6-w_1*w_4*w_6,
-2*w_0*w_1*w_2+4*w_0^2*w_4-w_3^2*w_4+w_2*w_3*w_5+w_1*w_3*w_6-2*w_0*w_5*w_6,
-4*w_0^2*w_2+2*w_0*w_1*w_4-w_3*w_4*w_5+w_2*w_5^2+2*w_0*w_3*w_6-w_1*w_5*w_6,
-4*w_0^2*w_1+2*w_0*w_2*w_4+2*w_0*w_3*w_5-w_3*w_4*w_6-w_2*w_5*w_6+w_1*w_6^2,
-4*w_0^3+w_0*w_4^2+w_0*w_5^2-w_4*w_5*w_6+w_0*w_6^2,
-4*w_0^3+w_0*w_2^2+w_0*w_3^2-w_2*w_3*w_6+w_0*w_6^2,
-4*w_0^3+w_0*w_1^2+w_0*w_3^2-w_1*w_3*w_5+w_0*w_5^2,
-4*w_0^3+w_0*w_1^2+w_0*w_2^2-w_1*w_2*w_4+w_0*w_4^2};
i17 : WB6 == pigreca(X)
i18 : (dim ooo -1, degree ooo) == (3, 10)
i19 : P1 = ideal{w_1-2*w_0,w_2-2*w_0,w_3-2*w_0,w_4-2*w_0,w_5-2*w_0,w_6-2*w_0};
i20 : P2 = ideal{w_1+2*w_0,w_2+2*w_0,w_3+2*w_0,w_4-2*w_0,w_5-2*w_0,w_6-2*w_0};
i21 : P3 = ideal{w_1+2*w_0,w_2-2*w_0,w_3-2*w_0,w_4+2*w_0,w_5+2*w_0,w_6-2*w_0};
i22 : P4 = ideal{w_1-2*w_0,w_2+2*w_0,w_3+2*w_0,w_4+2*w_0,w_5+2*w_0,w_6-2*w_0};
i23 : P5 = ideal{w_1-2*w_0,w_2+2*w_0,w_3-2*w_0,w_4+2*w_0,w_5-2*w_0,w_6+2*w_0};
i24 : P6 = ideal{w_1+2*w_0,w_2-2*w_0,w_3+2*w_0,w_4+2*w_0,w_5-2*w_0,w_6+2*w_0};
i25 : P7 = ideal{w_1+2*w_0,w_2+2*w_0,w_3-2*w_0,w_4-2*w_0,w_5+2*w_0,w_6+2*w_0};
i26 : P8 = ideal{w_1-2*w_0,w_2-2*w_0,w_3+2*w_0,w_4-2*w_0,w_5+2*w_0,w_6+2*w_0};
i27 : -- let us see if the lines lij joining the points Pi and Pj
-- are contained in the threefold WB6
l12 = ideal{(toMap(saturate(P1*P2),1,1)).matrix};
i28 : (l12 + WB6 == l12) == true
i29 : l13 = ideal{(toMap(saturate(P1*P3),1,1)).matrix};
i30 : (l13 + WB6 == l13) == true
i31 : l14 = ideal{(toMap(saturate(P1*P4),1,1)).matrix};
i32 : (l14 + WB6 == l14) == true
i33 : l15 = ideal{(toMap(saturate(P1*P5),1,1)).matrix};
i34 : (l15 + WB6 == l15) == true
i35 : l16 = ideal{(toMap(saturate(P1*P6),1,1)).matrix};
i36 : (l16 + WB6 == l16) == true
i37 : l17 = ideal{(toMap(saturate(P1*P7),1,1)).matrix};
i38 : (l17 + WB6 == l17) == true
i39 : l18 = ideal{(toMap(saturate(P1*P8),1,1)).matrix};
i40 : (l18 + WB6 == l18) == true
i41 : -- etc...
-- let us now change the coordinates of PP6
-- in order to have P1 = [1:0...0]
PP6'=ZZ/10000019[z_0..z_6];
i42 : W' = sub(WB6, {(gens PP6)_0 =>(gens PP6')_0,
(gens PP6)_1 =>(gens PP6')_1+2*(gens PP6')_0,
(gens PP6)_2 =>(gens PP6')_2+2*(gens PP6')_0,
(gens PP6)_3 =>(gens PP6')_3+2*(gens PP6')_0,
(gens PP6)_4 =>(gens PP6')_4+2*(gens PP6')_0,
(gens PP6)_5 =>(gens PP6')_5+2*(gens PP6')_0,
(gens PP6)_6 =>(gens PP6')_6+2*(gens PP6')_0})
i43 : W'U0 = sub(oo, {(gens PP6')_0 => 1})
i44 : ConeP1 = sub(tangentCone oo, {(gens PP6')_0 => (gens PP6)_0,
(gens PP6')_1 =>(gens PP6)_1-2*(gens PP6)_0,
```

```
(gens PP6')_2 =>(gens PP6)_2-2*(gens PP6)_0,
(gens PP6')_3 =>(gens PP6)_3-2*(gens PP6)_0,
(gens PP6')_4 =>(gens PP6)_4-2*(gens PP6)_0,
(gens PP6')_5 =>(gens PP6)_5-2*(gens PP6)_0,
(gens PP6')_6 =>(gens PP6)_6-2*(gens PP6)_0})
i45 : degree oo == 4
i46 : -- similarly for P2,P3,P4,P5,P6,P7,P8
-- we observe now that WB6 is not contained in a quadric hypersurface of PP6
rationalMap toMap(WB6,2,1)
i47 : -- let us also see that a general hyperplane section S is not
-- contained in a quadric hypersurface of PP5, where
-- S = ideal{random(1,PP6)}+WB6
-- for example:
S = ideal{w_0-w_1+72*w_2-13*w_3+4*w_4+8*w_5+35*w_6}+WB6
i48: PP5 = ZZ/10000019[t_0..t_5]
i49 : inc = rationalMap map(PP5,PP6,matrix{{t_0-72*t_1+13*t_2-4*t_3-8*t_4-35*t_5,
t_0,t_1,t_2,t_3,t_4,t_5}})
i50 : image oo == ideal{S_0}
i51 : inc^*S
i52 : (dim oo -1, degree oo) == (2, 10)
i53 : toMap(ooo,2,1)
```

A.2. *Computational analysis of the BS-EF 3-fold of genus 7*

```
Macaulay2, version 1.11
with packages: ConwayPolynomials, Elimination, IntegralClosure, InverseSystems,
LLLBases, PrimaryDecomposition, ReesAlgebra, TangentCone
i1 : needsPackage "Cremona";
i2 : PP1xPP1xPP1xPP1 =
ZZ/10000019[x_0,x_1]**ZZ/10000019[y_0,y_1]**ZZ/10000019[z_0,z_1]**ZZ/10000019[t_0,t_1];
i3 : a0001=1;
i4 : a0010=1;
i5 : a0100=1;
i6 : a1000=1;
i7 : a1110=1;
i8 : a1101=1;
i9 : a1011=1;
i10 : a0111=1;
i11 : X = ideal{a0001*x_0*y_0*z_0*t_1+a0010*x_0*y_0*z_1*t_0+a0100*x_0*y_1*z_0*t_0+
a1000*x_1*y_0*z_0*t_0+a1110*x_1*y_1*z_1*t_0+a1101*x_1*y_1*z_0*t_1+
a1011*x_1*y_0*z_1*t_1+a0111*x_0*y_1*z_1*t_1};
i12 : phi = rationalMap map(PP1xPP1xPP1xPP1, ZZ/10000019[w_0..w_7],
matrix(PP1xPP1xPP1xPP1,{{x_1*y_1*z_1*t_1, x_1*y_0*z_0*t_1, x_0*y_0*z_1*t_1,
x_1*y_0*z_1*t_0, x_0*y_0*z_0*t_0, x_0*y_1*z_1*t_0,
x_1*y_1*z_0*t_0, x_0*y_1*z_0*t_1}}));
i13 : WB7 = phi(X);
i14 : (dim oo -1, degree oo) == (3,12)
i15 : PP7 = ring WB7;
i16 : P1 = ideal{w_1, w_2, w_3, w_4, w_5, w_6, w_7};
i17 : P2 = ideal{w_0, w_2, w_3, w_4, w_5, w_6, w_7};
i18 : P3 = ideal{w_0, w_1, w_3, w_4, w_5, w_6, w_7};
i19 : P4 = ideal{w_0, w_1, w_2, w_4, w_5, w_6, w_7};
```

```
i20 : P1' = ideal{w_0, w_1, w_2, w_3, w_5, w_6, w_7};
i21 : P2' = ideal{w_0, w_1, w_2, w_3, w_4, w_6, w_7};
i22 : P3' = ideal{w_0, w_1, w_2, w_3, w_4, w_5, w_7};
i23 : P4' = ideal{w_0, w_1, w_2, w_3, w_4, w_5, w_6};
i24 : -- let us see if the lines lij joining the points Pi and Pj
-- are contained in the threefold WB7
l12 = ideal{(toMap(saturate(P1*P2),1,1)).matrix};
i25 : (l12 + WB7 == l12) == true
i26 : l13 = ideal{(toMap(saturate(P1*P3),1,1)).matrix};
i27 : (l13 + WB7 == l13) == true
i28 : l14 = ideal{(toMap(saturate(P1*P4),1,1)).matrix};
i29 : (l14 + WB7 == l14) == true
i30 : l11' = ideal{(toMap(saturate(P1*P1'),1,1)).matrix};
i31 : (l11' + WB7 == l11') == false
i32 : l12' = ideal{(toMap(saturate(P1*P2'),1,1)).matrix};
i33 : (l12' + WB7 == l12') == true
i34 : l13' = ideal{(toMap(saturate(P1*P3'),1,1)).matrix};
i35 : (l13' + WB7 == l13') == true
i36 : l14' = ideal{(toMap(saturate(P1*P4'),1,1)).matrix};
i37 : (l14' + WB7 == l14') == true
i38 : -- etc...
sub(WB7, {(gens PP7)_0=>1});
i39 : ConeP1 = tangentCone oo
i40 : degree oo == 4
i41 : sub(WB7, {(gens PP7)_1=>1});
i42 : ConeP2 = tangentCone oo
i43 : degree oo == 4
i44 : -- etc.. similarly for P3,P4,P5,P1',P2',P3',P4'
```

A.3. *Computational analysis of the BS-EF 3-fold of genus 9*

```
Macaulay2, version 1.11
with packages: ConwayPolynomials, Elimination, IntegralClosure, InverseSystems,
LLLBases, PrimaryDecomposition, ReesAlgebra, TangentCone
i1 : needsPackage "Cremona";
i2 : PP5 = ZZ/10000019[x_0, x_1, x_2, y_3, y_4, y_5];
i3 : s1 = x_0^2-3*x_1^2+2*x_2^2;
i4 : s2 = 3*x_0^2-8*x_1^2+5*x_2^2;
i5 : r1 = 3*y_3^2-8*y_4^2+5*y_5^2;
i6 : r2 = y_3^2-3*y_4^2+2*y_5^2;
i7 : X = ideal{s1+r1, s2+r2};
i8 : (dim oo -1, degree oo) == (3,4)
i9 : PP11 = ZZ/10000019[Z_0..Z_11];
i10 : phi = rationalMap map(PP5, PP11, matrix(PP5,{{x_0^2, x_1^2, x_2^2, x_0*x_1,
x_0*x_2, x_1*x_2, y_3^2, y_4^2, y_5^2, y_3*y_4, y_3*y_5, y_4*y_5}}));
i11 : phi(X)
i12 : (dim oo -1, degree oo) == (3,16)
i13 : H9 = ideal{ooo_0, ooo_1}
i14 : phi(X) + H9 == phi(X)
i15 : PP9 = ZZ/10000019[w_0..w_9];
i16 : inclusion = rationalMap map(PP9,PP11, matrix(PP9,{{w_0+21*w_4-55*w_5+34*w_6,
w_0+8*w_4-21*w_5+13*w_6, w_0, w_1, w_2, w_3, w_4, w_5, w_6, w_7, w_8, w_9 }}));
i17 : image oo == H9
```

```
i18 : WB9 = inclusion^* (phi(X));
i19 : (dim oo -1, degree oo) == (3, 16)
i20 : rationalMap map(PP11,PP9, sub(matrix inverseMap(inclusion||H9), PP11))
i21 : pigreca = phi* oo
i23 : fixedPlanex = associatedPrimes (X+ideal{x_0,x_1,x_2});
i24 : fixedPlaney = associatedPrimes (X+ideal{y_3,y_4,y_5});
i25 : P1 = inclusion^* phi(fixedPlaney#0);
i26 : P4 = inclusion^* phi(fixedPlaney#1);
i27 : P2 = inclusion^* phi(fixedPlaney#2);
i28 : P3 = inclusion^* phi(fixedPlaney#3);
i29 : P1' = inclusion^* phi(fixedPlanex#0);
i30 : P4' = inclusion^* phi(fixedPlanex#1);
i31 : P2' = inclusion^* phi(fixedPlanex#2);
i32 : P3' = inclusion^* phi(fixedPlanex#3);
i33 : -- let us see if the lines lij joining the points Pi and Pj
-- are contained in the threefold WB9
l12 = ideal{(toMap(saturate(P1*P2),1,1)).matrix};
i34 : (l12 + WB9 == l12) == false
i35 : l13 = ideal{(toMap(saturate(P1*P3),1,1)).matrix};
i36 : (l13 + WB9 == l13) == false
i37 : l14 = ideal{(toMap(saturate(P1*P4),1,1)).matrix};
i38 : (l14 + WB9 == l14) == false
i39 : l11' = ideal{(toMap(saturate(P1*P1'),1,1)).matrix};
i40 : (l11' + WB9 == l11') == true
i41 : l12' = ideal{(toMap(saturate(P1*P2'),1,1)).matrix};
i42 : (l12' + WB9 == l12') == true
i43 : l13' = ideal{(toMap(saturate(P1*P3'),1,1)).matrix};
i44 : (l13' + WB9 == l13') == true
i45 : l14' = ideal{(toMap(saturate(P1*P4'),1,1)).matrix};
i46 : (l14' + WB9 == l14') == true
i47 : -- etc..
proj1 = rationalMap toMap(P2,1,1);
i48 : proj2 = rationalMap toMap(proj1(P3),1,1);
i49 : proj3 = rationalMap toMap(proj2(proj1(P4)),1,1);
i50 : proj4 = rationalMap toMap(proj3(proj2(proj1(P2'))),1,1);
i51 : proj5 = rationalMap toMap(proj4(proj3(proj2(proj1(P3')))),1,1);
i52 : proj6 = rationalMap toMap(proj5(proj4(proj3(proj2(proj1(P4'))))),1,1);
i53 : proj = proj1*proj2*proj3*proj4*proj5*proj6;
i54 : proj(WB9)
i55 : PP3 = ring oo;
i56 : isBirational( proj|WB9 )
i57 : septics = rationalMap map( PP3, PP9, matrix(inverseMap( proj|WB9 )));
i58 : time image oo == WB9
i59 : comp = associatedPrimes(ideal septics)
i60 : l3' = comp#0;
i61 : l2' = comp#1;
i62 : r21 = comp#2;
i63 : r11 = comp#3;
i64 : r31 = comp#4;
i65 : l1' = comp#5;
i66 : r23 = comp#6;
i67 : r13 = comp#7;
```

```
i68 : r33 = comp#8;
i69 : r22 = comp#9;
i70 : r12 = comp#10;
i71 : r32 = comp#11;
i72 : l1 = comp#12;
i73 : l2 = comp#13;
i74 : l3 = comp#14;
i75 : -- trihedron T' :
f1' = ideal{(gens PP3)_3};
i76 : f2' = ideal{(gens PP3)_1+(gens PP3)_3};
i77 : f3' = ideal{(gens PP3)_2+(gens PP3)_3};
i78 : f1'+f2' == l3'
i79 : f1'+f3' == l2'
i80 : f2'+f3' == l1'
i81 : v'= saturate(f1'+f2'+f3')
i82 : -- trihedron T :
f1 = ideal{(gens PP3)_0-55*(gens PP3)_1+34*(gens PP3)_2};
i83 : f2 = ideal{(gens PP3)_0 - 21*(gens PP3)_1 +13*(gens PP3)_2};
i84 : f3 = ideal{(gens PP3)_0};
i85 : f1+f2 == l3
i86 : f1+f3 == l2
i87 : f2+f3 == l1
i88 : v = saturate(l1+l2+l3)
i89 : r11 == f1+f1'
i90 : r12 == f1+f2'
i91 : r13 == f1+f3'
i92 : r21 == f2+f1'
i93 : r22 == f2+f2'
i94 : r23 == f2+f3'
i95 : r31 == f3+f1'
i96 : r32 == f3+f2'
i97 : r33 == f3+f3'
i98 : -- general septic surface of the linear system :
K = septics^* ideal{random(1,PP9)};
i99 : (dim oo -1, degree oo) == (2,7)
i100 : -- K has double point along l1,l2,l3,l1',l2',l3' :
(minors(1,jacobian(K))+l1 == l1) == true
i101 : (minors(1,jacobian(K))+l2 == l2) == true
i102 : (minors(1,jacobian(K))+l3 == l3) == true
i103 : (minors(1,jacobian(K))+l1' == l1') == true
i104 : (minors(1,jacobian(K))+l2' == l2') == true
i105 : (minors(1,jacobian(K))+l3' == l3') == true
i106 : -- K has triple point at v and v' :
(minors(1,jacobian(jacobian(K)))+minors(1,jacobian(K))+v == v) == true
i107 : (minors(1,jacobian(jacobian(K)))+minors(1,jacobian(K))+v' == v') == true
i108 : -- remark
septics(f1) == P2
i109 : septics(f1') == P2'
i110 : septics(f2) == P3
i111 : septics(f2') == P3'
i112 : septics(f3) == P4
i113 : septics(f3') == P4'
```

A.4. *Computational analysis of the BS-EF 3-fold of genus 13*

```
Macaulay2, version 1.11
with packages: ConwayPolynomials, Elimination, IntegralClosure, InverseSystems,
LLLBases, PrimaryDecomposition, ReesAlgebra, TangentCone
i1 : needsPackage "Cremona";
i2 : PP1x = ZZ/10000019[x_0,x_1];
i3 : PP1y = ZZ/10000019[y_0,y_1];
i4 : PP1z = ZZ/10000019[z_0,z_1];
i5 : X = PP1x ** PP1y ** PP1z;
i6 : use X;
i7 : pigreca = rationalMap map(X, ZZ/10000019[w_0..w_13], matrix{{x_0^2*y_0^2*z_0^2,
x_0^2*y_0^2*z_1^2, x_0^2*y_0*y_1*z_0*z_1, x_0^2*y_1^2*z_0^2, x_0^2*y_1^2*z_1^2,
x_0*x_1*y_0^2*z_0*z_1, x_0*x_1*y_0*y_1*z_0^2, x_0*x_1*y_0*y_1*z_1^2,
x_0*x_1*y_1^2*z_0*z_1, x_1^2*y_0^2*z_0^2, x_1^2*y_0^2*z_1^2, x_1^2*y_0*y_1*z_0*z_1,
x_1^2*y_1^2*z_0^2, x_1^2*y_1^2*z_1^2}});
i8 : WB13 = image pigreca;
i9 : (dim oo -1, degree oo) == (3, 24)
i10 : PP13 = ring WB13;
i11 : P1 = pigreca(ideal{x_1,y_0,z_0});
i12 : P2 = pigreca(ideal{x_1,y_1,z_1});
i13 : P3 = pigreca(ideal{x_0,y_1,z_0});
i14 : P4 = pigreca(ideal{x_0,y_0,z_1});
i15 : P1' = pigreca(ideal{x_0,y_1,z_1});
i16 : P2' = pigreca(ideal{x_0,y_0,z_0});
i17 : P3' = pigreca(ideal{x_1,y_0,z_1});
i18 : P4' = pigreca(ideal{x_1,y_1,z_0});
i19 : -- let us see if the lines lij joining the points Pi and Pj
-- are contained in the threefold WB13
l12 = ideal{(toMap(saturate(P1*P2),1,1)).matrix};
i20 : (l12 + WB13 == l12) == false
i21 : l13 = ideal{(toMap(saturate(P1*P3),1,1)).matrix};
i22 : (l13 + WB13 == l13) == false
i23 : l14 = ideal{(toMap(saturate(P1*P4),1,1)).matrix};
i24 : (l14 + WB13 == l14) == false
i25 : l11' = ideal{(toMap(saturate(P1*P1'),1,1)).matrix};
i26 : (l11' + WB13 == l11') == false
i27 : l12' = ideal{(toMap(saturate(P1*P2'),1,1)).matrix};
i28 : (l12' + WB13 == l12') == true
i29 : l13' = ideal{(toMap(saturate(P1*P3'),1,1)).matrix};
i30 : (l13' + WB13 == l13') == true
i31 : l14' = ideal{(toMap(saturate(P1*P4'),1,1)).matrix};
i32 : (l14' + WB13 == l14') == true
i33 : -- etc..
PP3 = ZZ/10000019[t_0..t_3];
i34 : sexties = rationalMap map(PP3,PP13, matrix{{t_0*t_1^3*t_2*t_3, t_0^2*t_1^2*t_2^2,
t_0^2*t_1^2*t_2*t_3, t_0^2*t_1^2*t_3^2, t_0^3*t_1*t_2*t_3, t_0*t_1^2*t_2^2*t_3,
t_0*t_1^2*t_2*t_3^2, t_0^2*t_1*t_2^2*t_3, t_0^2*t_1*t_2*t_3^2, t_1^2*t_2^2*t_3^2,
t_0*t_1*t_2^3*t_3, t_0*t_1*t_2^2*t_3^2, t_0*t_1*t_2*t_3^3, t_0^2*t_2^2*t_3^2}});
i35 : WF13 = image oo
i36 : (WF13 == WB13) == true
```

A.5. *Computational analysis of the BS-EF 3-fold of genus 8*

```
Macaulay2, version 1.11
with packages: ConwayPolynomials, Elimination, IntegralClosure, InverseSystems,
LLLBases, PrimaryDecomposition, ReesAlgebra, TangentCone
i1 : needsPackage "Cremona";
i2 : PP4 = ZZ/10000019[x_0..x_4];
i3 : Q = ideal{x_0^2-x_1^2 -x_2^2+x_3^2};
i4 : R = ideal{2*x_0^2-x_1^2-3*x_2^2+2*x_3^2};
i5 : fixedconic1 = ideal{x_2,x_3,x_4^2-R_0};
i6 : fixedconic2 = ideal{x_0,x_1,x_4^2+R_0};
i7 : four = associatedPrimes (fixedconic1+Q)
i8 : p1 = four#0;
i9 : p2 = four#1;
i10 : p1' = four#2;
i11 : p2' = four#3;
i12 : four' = associatedPrimes (fixedconic2+Q)
i13 : p3 = four'#0;
i14 : p4 = four'#1;
i15 : p3' = four'#2;
i16 : p4' = four'#3;
i17 : PP9 = ZZ/10000019[z_0..z_9];
i18 : phi = rationalMap map(PP4,PP9,matrix{{x_4^2*x_0+x_0*R_0,x_4^2*x_1+x_1*R_0,
x_4^2*x_2-x_2*R_0,x_4^2*x_3-x_3*R_0,x_4*x_0^2,x_4*x_1^2,
x_4*x_2^2,x_4*x_3^2,x_4*x_0*x_1,x_4*x_2*x_3}});
i19 : phiY = phi(Q);
i20 : H8 = ideal{phiY_0}
i21 : PP8 = ZZ/10000019[w_0,w_1,w_2,w_3,w_4,w_5,w_6,w_7,w_8];
i22 : inclusion = rationalMap map(PP8,PP9,
matrix(PP8,{{w_0,w_1,w_2,w_3,w_4+w_5-w_6,w_4,w_5,w_6,w_7,w_8}}));
i23 : H8 == image inclusion
i24 : WB8 = inclusion^* phiY;
i25 : (dim oo -1, degree oo) == (3,14)
i26 : P1 = inclusion^* phi(p1)
i27 : P2 = inclusion^* phi(p2)
i28 : P3 = inclusion^* phi(p3)
i29 : P4 = inclusion^* phi(p4)
i30 : P1' = inclusion^* phi(p1')
i31 : P2' = inclusion^* phi(p2')
i32 : P3' = inclusion^* phi(p3')
i33 : P4' = inclusion^* phi(p4')
i34 : -- let us see if the lines lij joining the points Pi and Pj
-- are contained in the threefold WB8
l12 = ideal{(toMap(saturate(P1*P2),1,1)).matrix};
i35 : (l12 + WB8 == l12) == true
i36 : l13 = ideal{(toMap(saturate(P1*P3),1,1)).matrix};
i37 : (l13 + WB8 == l13) == true
i38 : l14 = ideal{(toMap(saturate(P1*P4),1,1)).matrix};
i39 : (l14 + WB8 == l14) == true
i40 : l11' = ideal{(toMap(saturate(P1*P1'),1,1)).matrix};
i41 : (l11' + WB8 == l11') == false
i42 : l12' = ideal{(toMap(saturate(P1*P2'),1,1)).matrix};
```

```
i43 : (l12' + WB8 == l12') == false
i44 : l13' = ideal{(toMap(saturate(P1*P3'),1,1)).matrix};
i45 : (l13' + WB8 == l13') == true
i46 : l14' = ideal{(toMap(saturate(P1*P4'),1,1)).matrix};
i47 : (l14' + WB8 == l14') == true
i48 : -- etc...
proj1 = rationalMap toMap(P1,1,1);
i49 : proj1' = rationalMap toMap(proj1(P1'),1,1);
i50 : proj2 = rationalMap toMap(proj1'(proj1(P2)),1,1);
i51 : proj3 = rationalMap toMap(proj2(proj1'(proj1(P3))),1,1);
i52 : proj3' = rationalMap toMap(proj3(proj2(proj1'(proj1(P3')))),1,1);
i53 : proj = proj1*proj1'*proj2*proj3*proj3'
i54 : isBirational(proj | WB8)
i55 : PP3 = target proj;
i56 : septies = rationalMap map( PP3, PP8, matrix(inverseMap(proj|WB8)) )
i57 : image oo == WB8
i58 : baseL = associatedPrimes ideal septies
i59 : e0= baseL#0;
i60 : l1= baseL#1;
i61 : l2= baseL#2;
i62 : s1= baseL#3;
i63 : s2= baseL#4;
i64 : l2'= baseL#5;
i65 : l1'= baseL#6;
i66 : l0= baseL#7;
i67 : r1= baseL#8;
i68 : t1= baseL#9;
i69 : r2= baseL#10;
i70 : t2= baseL#11;
i71 : C= baseL#12;
i72 : v = saturate(l1+l2+l0);
i73 : q1 = saturate(l1+r1+s1+e0+l2')
i74 : q2 = saturate(l2+r2+s2+e0+l1')
i75 : ar = saturate(r1+r2+l0)
i76 : as = saturate(s1+s2+l0)
i77 : at = saturate(t1+t2+l0)
i78 : a1 = saturate(l1+t1)
i79 : a2 = saturate(l2+t2)
i80 : b1 = saturate(r1+t1+C)
i81 : b2 = saturate(r2+t2+C)
i82 : c1 = saturate(s1+t1)
i83 : c2 = saturate(s2+t2)
i84 : q1' = saturate(l1'+t1)
i85 : q2' = saturate(l2'+t2)
i86 : -- general septic surface of the linear system :
N = septies^* ideal{random(1,PP8)};
i87 : (dim oo -1, degree oo) == (2, 7)
i88 : -- N is double along l0,l1,l2,l1',l2',C
(minors(1,jacobian(N))+ l1 == l1) == true
i89 : (minors(1,jacobian(N))+ l2 == l2) == true
i90 : (minors(1,jacobian(N))+ l2' == l2') == true
i91 : (minors(1,jacobian(N))+ l1' == l1') == true
```

```
i92 : (minors(1,jacobian(N))+ l0 == l0) == true
i93 : (minors(1,jacobian(N))+ C == C) == true
i94 : -- N is triple at v
(minors(1,jacobian(jacobian(N)))+minors(1,jacobian(N))+ v == v) == true
i95 : -- N is quadruple at q1 and q2
(minors(1,jacobian(jacobian(jacobian(N))))+minors(1,jacobian(jacobian(N)))+
minors(1,jacobian(N))+ q1 == q1) == true
i96 : (minors(1,jacobian(jacobian(jacobian(N))))+minors(1,jacobian(jacobian(N)))+
minors(1,jacobian(N))+ q2 == q2) == true
```

A.6. Computational analysis of the BS-EF 3-fold of genus 10

```
Macaulay2, version 1.11
with packages: ConwayPolynomials, Elimination, IntegralClosure, InverseSystems,
LLLBases, PrimaryDecomposition, ReesAlgebra, TangentCone
i1 : needsPackage "Cremona";
i2 : PP2=ZZ/10000019[u_0,u_1,u_2];
i3 : PP6 = ZZ/10000019[x_0,x_1,x_2,x_3,x_4,x_5,x_6];
i4 : cubics3points = rationalMap map(PP2, PP6, matrix{{u_1^2*u_2,
u_1*u_2^2, u_0^2*u_2,u_0*u_2^2, u_0^2*u_1,u_0*u_1^2, u_0*u_1*u_2}});
i5 : S6 = image cubics3points
i6 : PP1 = ZZ/10000019[y_0,y_1];
i7 : PP1xPP6= PP1 ** PP6;
i8 : pr2 = rationalMap(PP1xPP6,PP6, matrix{{x_0,x_1,x_2,x_3,x_4,x_5,x_6}});
i9 : PP10 = ZZ/10000019[w_0..w_10];
i10 : phi = rationalMap map(PP1xPP6,PP10, matrix{{y_0^2*x_6,y_0^2*x_0+y_0^2*x_2,
y_0^2*x_1+y_0^2*x_4,y_0^2*x_3+y_0^2*x_5,y_1^2*x_6,y_1^2*x_0+y_1^2*x_2,
y_1^2*x_1+y_1^2*x_4,y_1^2*x_3+y_1^2*x_5,y_0*y_1*x_0-y_0*y_1*x_2,
y_1*y_0*x_1-y_1*y_0*x_4,y_1*y_0*x_3-y_1*y_0*x_5}});
i11 : PP1xS6 = pr2^* S6;
i12 : WB10 = phi(PP1xS6);
i13 : (dim WB10 -1, degree WB10) == (3,18)
i14 : ideal{WB10_0,WB10_1,2*WB10_2,WB10_3,WB10_4,2*WB10_5,WB10_6,WB10_7,WB10_8,
2*WB10_9,WB10_10,WB10_11,WB10_12,2*WB10_13,WB10_14,2*WB10_15,2*WB10_16,
4*WB10_17,4*WB10_18,4*WB10_19}
i15 : oo == WB10
i16 : P1 = ideal{w_0,w_1,w_2,w_3,w_5-2*w_4,w_6-2*w_4,w_7-2*w_4,w_8,w_9,w_10};
i17 : P2 = ideal{w_0,w_1,w_2,w_3,w_5-2*w_4,w_6+2*w_4,w_7+2*w_4,w_8,w_9,w_10};
i18 : P3 = ideal{w_0,w_1,w_2,w_3,w_5+2*w_4,w_6-2*w_4,w_7+2*w_4,w_8,w_9,w_10};
i19 : P4 = ideal{w_0,w_1,w_2,w_3,w_5+2*w_4,w_6+2*w_4,w_7-2*w_4,w_8,w_9,w_10};
i20 : P1' = ideal{w_1-2*w_0,w_2-2*w_0,w_3-2*w_0,w_4,w_5,w_6,w_7,w_8,w_9,w_10};
i21 : P2' = ideal{w_1-2*w_0,w_2-2*w_0,w_3+2*w_0,w_4,w_5,w_6,w_7,w_8,w_9,w_10};
i22 : P3' = ideal{w_1+2*w_0,w_2-2*w_0,w_3+2*w_0,w_4,w_5,w_6,w_7,w_8,w_9,w_10};
i23 : P4' = ideal{w_1+2*w_0,w_2+2*w_0,w_3-2*w_0,w_4,w_5,w_6,w_7,w_8,w_9,w_10};
i24 : -- let us see if the lines lij joining the points Pi and Pj
-- are contained in the threefold WB10
l12 = ideal{(toMap(saturate(P1*P2),1,1)).matrix};
i25 : (l12 + WB10 == l12) == true
i26 : l13 = ideal{(toMap(saturate(P1*P3),1,1)).matrix};
i27 : (l13 + WB10 == l13) == true
i28 : l14 = ideal{(toMap(saturate(P1*P4),1,1)).matrix};
i29 : (l14 + WB10 == l14) == true
```

```
i30 : l11' = ideal{(toMap(saturate(P1*P1'),1,1)).matrix};
i31 : (l11' + WB10 == l11') == true
i32 : l12' = ideal{(toMap(saturate(P1*P2'),1,1)).matrix};
i33 : (l12' + WB10 == l12') == false
i34 : l13' = ideal{(toMap(saturate(P1*P3'),1,1)).matrix};
i35 : (l13' + WB10 == l13') == false
i36 : l14' = ideal{(toMap(saturate(P1*P4'),1,1)).matrix};
i37 : (l14' + WB10 == l14') == false
i38 : -- etc...
proj1 = rationalMap toMap(P1,1,1);
i39 : proj2 = rationalMap toMap(proj1(P2),1,1);
i40 : proj3 = rationalMap toMap(proj2(proj1(P3)),1,1);
i41 : proj4 = rationalMap toMap(proj3(proj2(proj1(P4))),1,1);
i42 : proj1' = rationalMap toMap(proj4(proj3(proj2(proj1(P1')))),1,1);
i43 : proj2' = rationalMap toMap(proj1'(proj4(proj3(proj2(proj1(P2'))))),1,1);
i44 : proj3' = rationalMap toMap(proj2'(proj1'(proj4(proj3(proj2(proj1(P3')))))),1,1);
i45 : proj = proj1*proj2*proj3*proj4*proj1'*proj2'*proj3'
i46 : isBirational(proj | WB10)
i47 : PP3 = target proj;
i48 : sexties = rationalMap map( PP3, PP10, matrix(inverseMap(proj|WB10)) )
i49 : image oo == WB10
i50 : baseL = associatedPrimes ideal sexties
i51 : l23 = baseL#0
i52 : r1 = baseL#1
i53 : l12 = baseL#2
i54 : r3 = baseL#3
i55 : l13 = baseL#4
i56 : r2 = baseL#5
i57 : l02 = baseL#6
i58 : l03 = baseL#7
i59 : l01 = baseL#8
i60 : v1 = baseL#9
i61 : v2 = baseL#10
i62 : v3 = baseL#11
i63 : f0 =ideal{(gens PP3)_0};
i64 : f1 =ideal{(gens PP3)_1+(gens PP3)_2+(gens PP3)_3};
i65 : f2=ideal{(gens PP3)_1-(gens PP3)_2+(gens PP3)_3};
i66 : f3 =ideal{(gens PP3)_1+(gens PP3)_2-(gens PP3)_3};
i67 : plane = ideal{(gens PP3)_1-(gens PP3)_2-(gens PP3)_3};
i68 : l12 == f1+f2
i69 : l13 == f1+f3
i70 : l23 == f2+f3
i71 : l01 == f0+f1
i72 : l02 == f0+f2
i73 : l03 == f0+f3
i74 : r1 == plane+f1
i75 : r2 == plane+f2
i76 : r3 == plane+f3
i77 : v0 = f1+f2+f3+plane
i78 : v1 == f0+f2+f3
i79 : v2 == f0+f1+f3
i80 : v3 == f0+f1+f2
```

```
i81 : q1 = saturate(l01+r1)
i82 : q2 = saturate(l02+r2)
i83 : q3 = saturate(l03+r3)
i84 : -- general element of the linear system defining sexties :
M = sexties^* ideal{random(1,PP10)};
i85 : (dim oo -1, degree oo)
i86 : -- M has double points along r1,r2,r3 :
(minors(1,jacobian(M))+r1 == r1) == true
i87 : (minors(1,jacobian(M))+r2 == r2) == true
i88 : (minors(1,jacobian(M))+r3 == r3) == true
i89 : -- M has triple points at v1,v2,v3 :
(minors(1,jacobian(jacobian(M)))+minors(1,jacobian(M))+ v1 == v1) == true
i90 : (minors(1,jacobian(jacobian(M)))+minors(1,jacobian(M))+ v2 == v2) == true
i91 : (minors(1,jacobian(jacobian(M)))+minors(1,jacobian(M))+ v3 == v3) == true
i92 : -- v0 is a quadruple point of M :
(minors(1,jacobian(jacobian(jacobian(M))))+minors(1,jacobian(jacobian(M)))+
minors(1,jacobian(M))+ v0 == v0) == true
```

A.7. Computational analysis of the P-EF 3-fold of genus 17

```
Macaulay2, version 1.11
with packages: ConwayPolynomials, Elimination, IntegralClosure, InverseSystems,
LLLBases, PrimaryDecomposition, ReesAlgebra, TangentCone
i1 : needsPackage "Points";
i2 : needsPackage "Cremona";
i3 : PP1 = ZZ/10000019[u_0,u_1];
i4 : PP1'= ZZ/10000019[v_0,v_1];
i5 : P1P1 = PP1 ** PP1';
i6 : PP9 = ZZ/10000019[y_{0,0},y_{0,1},y_{0,2},y_{1,0},y_{1,1},
y_{1,2},y_{2,0},y_{2,1},y_{2,2},x];
i7 : antiCanonicalEmbeddingP = rationalMap map(P1P1,PP9, matrix{{u_1^2*v_1^2,
u_1^2*v_0*v_1, u_1^2*v_0^2,u_1*u_0*v_1^2,u_1*u_0*v_0*v_1,u_1*u_0*v_0^2,
u_0^2*v_1^2,u_0^2*v_0*v_1,u_0^2*v_0^2,0}});
i8 : P = image oo
i9 : (dim P -1, degree P) == (2,8)
i10 : numgens P
i11 : V = ideal{P_1,P_2,P_3,P_4,P_5,P_6,P_7,P_8,P_9,P_10,P_11,P_12,
P_13,P_14,P_15,P_16,P_17,P_18,P_19,P_20}
i12 : (dim V -1, degree V) == (3, 8)
i13 : PP29 = ZZ/10000019[Z_0..Z_29];
i14 : phi = rationalMap map(PP9, PP29, matrix(PP9, {{y_{1,1}^2, y_{0,0}^2, y_{0,2}^2,
y_{2,0}^2, y_{2,2}^2, x^2, y_{0,1}^2, y_{1,0}^2, y_{1,2}^2, y_{2,1}^2, y_{0,1}*x,
y_{1,0}*x, y_{1,2}*x, y_{2,1}*x, y_{0,0}*y_{1,1}, y_{0,2}*y_{1,1}, y_{2,0}*y_{1,1},
y_{2,2}*y_{1,1}, y_{0,1}*y_{1,0}, y_{0,1}*y_{1,2}, y_{1,0}*y_{2,1}, y_{1,2}*y_{2,1},
y_{0,0}*y_{0,2}, y_{0,0}*y_{2,0}, y_{0,2}*y_{2,2}, y_{2,0}*y_{2,2}, y_{0,1}*y_{2,1},
y_{0,0}*y_{2,2}, y_{0,2}*y_{2,0}, y_{1,0}*y_{1,2} }}));
i15 : phi(V)
i16 : H17 = ideal{Z_18-Z_14, Z_19-Z_15, Z_20-Z_16, Z_21-Z_17, Z_22-Z_6, Z_23-Z_7,
Z_24-Z_8, Z_25-Z_9, Z_26-Z_0, Z_27-Z_0, Z_28-Z_0, Z_29-Z_0};
i17 : phi(V) + H17 == phi(V)
i18 : PP17=ZZ/10000019[z_0..z_17];
i19 : inclusion = rationalMap map(PP17, PP29, matrix(PP17, {{z_0,z_1,z_2,z_3,z_4,z_5,z_6,
```

```
z_7,z_8,z_9,z_10,z_11,z_12,z_13,z_14,z_15,z_16,z_17, z_14,z_15,z_16,z_17,z_6,z_7,z_8,
z_9,z_0,z_0,z_0,z_0 }}));
i20 : image oo == H17
i21 : WP17 = inclusion^* (phi(V))
i22 : (dim oo -1, degree oo) == (3, 32)
i23 : pigreca = rationalMap map(PP9,PP17, matrix(PP9, {{y_{1,1}^2, y_{0,0}^2, y_{0,2}^2,
y_{2,0}^2, y_{2,2}^2, x^2, y_{0,1}^2, y_{1,0}^2, y_{1,2}^2, y_{2,1}^2, y_{0,1}*x,
y_{1,0}*x, y_{1,2}*x, y_{2,1}*x, y_{0,0}*y_{1,1}, y_{0,2}*y_{1,1},
y_{2,0}*y_{1,1}, y_{2,2}*y_{1,1} }}));
i24 : pigreca(V) == WP17
i25 : sub(WP17, {(gens PP17)_1=>1});
i26 : ConeP1 = tangentCone oo
i27 : degree oo == 4
i28 : sub(WP17, {(gens PP17)_2=>1});
i29 : ConeP2 = tangentCone oo
i30 : degree oo == 4
i31 : sub(WP17, {(gens PP17)_3=>1});
i32 : ConeP3 = tangentCone oo
i33 : degree oo == 4
i34 : sub(WP17, {(gens PP17)_4=>1});
i35 : ConeP4 = tangentCone oo
i36 : degree oo == 4
i37 : sub(WP17, {(gens PP17)_5=>1});
i38 : ConeP5 = tangentCone oo
i39 : degree oo == 6
i40 : M6 = ConeP5+ideal{(gens PP17)_5}
i41 : time irredCompM6 = associatedPrimes M6;
i42 : plane1 = irredCompM6#0
i43 : plane2 = irredCompM6#1
i44 : plane1' = irredCompM6#2
i45 : plane2' = irredCompM6#3
i46 : Q = irredCompM6#4
i47 : line1 = Q+plane1;
i48 : line1' = Q+plane1';
i49 : line2 = Q+plane2;
i50 : line2' = Q+plane2';
i51 : (dim(line1+line1')-1) == -1
i52 : (dim(line2+line2')-1) == -1
i53 : q12 = saturate(line1+line2)
i54 : q12' = saturate(line1+line2')
i55 : q1'2 = saturate(line1'+line2)
i56 : q1'2' = saturate(line1'+line2')
```

A.8. Computational analysis of the P-EF 3-fold of genus 13

```
Macaulay2, version 1.11
with packages: ConwayPolynomials, Elimination, IntegralClosure, InverseSystems,
LLLBases, PrimaryDecomposition, ReesAlgebra, TangentCone
i1 : needsPackage "Cremona";
i2 : PP2=ZZ/10000019[u_0,u_1,u_2];
i3 : a1 = ideal{u_1,u_2};
i4 : a2 = ideal{u_0,u_2};
```

```
i5 : a3 = ideal{u_0,u_1};
i6 : cubics3points = rationalMap toMap(saturate(a1*a2*a3),3,1);
i7 : DelPezzo6ic = image cubics3points
i8 : (dim DelPezzo6ic -1, degree DelPezzo6ic)
i9 : PP6 = ring DelPezzo6ic;
i10 : PP7 = ZZ/10000019[x_0,x_1,x_2,x_3,x_4,x_5,x_6,y];
i11 : inclusion = rationalMap map(PP6,PP7, matrix{{(gens PP6)_0,(gens PP6)_1,
(gens PP6)_2,(gens PP6)_3,(gens PP6)_4,(gens PP6)_5,(gens PP6)_6,0}});
i12 : S6 = inclusion(DelPezzo6ic)
i13 : numgens S6 == 10
i14 : V = ideal{S6_1,S6_2,S6_3,S6_4,S6_5,S6_6,S6_7,S6_8,S6_9};
i15 : (dim V -1, degree V) == (3, 6)
i16 : F1 = ideal{(gens PP7)_0+(gens PP7)_2, (gens PP7)_1+(gens PP7)_4,
(gens PP7)_3+(gens PP7)_5, (gens PP7)_6};
i17 : v = (associatedPrimes (F1+V))#0
i18 : oo == ideal{x_0,x_1,x_2,x_3,x_4,x_5,x_6}
i19 : F2 = ideal{(gens PP7)_0-(gens PP7)_2, (gens PP7)_1-(gens PP7)_4,
(gens PP7)_3-(gens PP7)_5, y};
i20 : F2intV = associatedPrimes saturate(F2+V);
i21 : v1 = F2intV#0
i22 : v2 = F2intV#3
i23 : v3 = F2intV#2
i24 : v4 = F2intV#1
i25 : PP13 = ZZ/10000019[z_0..z_13];
i26 : pigreco = rationalMap map(PP7,PP13, matrix{{x_6^2, x_0^2+x_2^2, x_1^2+x_4^2,
x_3^2+x_5^2, (x_0+x_2)*x_6, (x_1+x_4)*x_6, (x_3+x_5)*x_6, x_0*x_1+x_2*x_4, x_2*x_3+x_0*x_5,
x_1*x_3+x_4*x_5, (x_0-x_2)*y, (x_1-x_4)*y, (x_3-x_5)*y, y^2}});
i27 : PP19 = ZZ/10000019[Z_0..Z_19];
i28 : phi = rationalMap map(PP7,PP19,matrix{{x_6^2, x_0^2+x_2^2, x_1^2+x_4^2,
x_3^2+x_5^2, (x_0+x_2)*x_6, (x_1+x_4)*x_6, (x_3+x_5)*x_6, x_0*x_1+x_2*x_4,
x_2*x_3+x_0*x_5, x_1*x_3+x_4*x_5, (x_0-x_2)*y, (x_1-x_4)*y, (x_3-x_5)*y, y^2,
2*x_0*x_2, 2*x_1*x_4, 2*x_3*x_5, x_4*x_3+x_1*x_5, x_0*x_3+x_2*x_5, x_1*x_2+x_0*x_4}});
i29 : phi(V)
i30 : phiV = sub(phi(V), {Z_14 => 2*Z_0,Z_15 => 2*Z_0,Z_16 => 2*Z_0, Z_19 => Z_6,
Z_18 => Z_5, Z_17 => Z_4})
i31 : PP13' = ZZ/10000019[Z_0..Z_13];
i32 : ideal(submatrix(gens (sub(ooo, PP13')), {6..47}))
i33 : WP13 = sub(oo, { (gens PP13')_0=>(gens PP13)_0, (gens PP13')_1=>(gens PP13)_1,
(gens PP13')_2=>(gens PP13)_2, (gens PP13')_3=>(gens PP13)_3,
(gens PP13')_4=>(gens PP13)_4, (gens PP13')_5=>(gens PP13)_5,
(gens PP13')_6=>(gens PP13)_6, (gens PP13')_7=>(gens PP13)_7,
(gens PP13')_8=>(gens PP13)_8, (gens PP13')_9=>(gens PP13)_9,
(gens PP13')_10=>(gens PP13)_10, (gens PP13')_11=>(gens PP13)_11,
(gens PP13')_12=>(gens PP13)_12, (gens PP13')_13=>(gens PP13)_13 })
i34 : (dim oo -1, degree oo) == (3, 24)
i35 : WP13 == pigreco(V)
i36 : P1 = ideal{z_1 -2*z_0,z_2 -2*z_0,z_3 -2*z_0,z_4 -2*z_0,z_5 -2*z_0,
z_6 -2*z_0,z_7 -2*z_0,z_8 -2*z_0,z_9 -2*z_0,z_10,z_11,z_12,z_13};
i37 : P2 = ideal{z_1 -2*z_0,z_2 -2*z_0,z_3 -2*z_0,z_4 -2*z_0,z_5 +2*z_0,
z_6 +2*z_0,z_7 +2*z_0,z_8 +2*z_0,z_9 -2*z_0,z_10,z_11,z_12,z_13};
i38 : P3 = ideal{z_1 -2*z_0,z_2 -2*z_0,z_3 -2*z_0,z_4 +2*z_0,z_5 -2*z_0,
z_6 +2*z_0,z_7 +2*z_0,z_8 -2*z_0,z_9 +2*z_0,z_10,z_11,z_12,z_13};
```

```
i39 : P4 = ideal{z_1 -2*z_0,z_2 -2*z_0,z_3 -2*z_0,z_4 +2*z_0,z_5 +2*z_0,
z_6 -2*z_0,z_7 -2*z_0,z_8 +2*z_0,z_9 +2*z_0,z_10,z_11,z_12,z_13};
i40 : P5 = pigreco(v);
i41 : -- let us see if the lines lij joining the points Pi and Pj
-- are contained in the threefold WP13
l12 = ideal{(toMap(saturate(P1*P2),1,1)).matrix};
i42 : (l12 + WP13 == l12 ) == false
i43 : l13 = ideal{(toMap(saturate(P1*P3),1,1)).matrix};
i44 : (l13 + WP13 == l13) == false
i45 : l14 = ideal{(toMap(saturate(P1*P4),1,1)).matrix};
i46 : (l14 + WP13 == l14 ) == false
i47 : l15 = ideal{(toMap(saturate(P1*P5),1,1)).matrix};
i48 : (l15 + WP13 == l15) == true
i49 : l23 = ideal{(toMap(saturate(P2*P3),1,1)).matrix};
i50 : (l23 + WP13 == l23) == false
i51 : l24 = ideal{(toMap(saturate(P2*P4),1,1)).matrix};
i52 : (l24 + WP13 == l24) == false
i53 : l25 = ideal{(toMap(saturate(P2*P5),1,1)).matrix};
i54 : (l25 + WP13 == l25) == true
i55 : l34 = ideal{(toMap(saturate(P3*P4),1,1)).matrix};
i56 : (l34 + WP13 == l34) == false
i57 : l35 = ideal{(toMap(saturate(P3*P5),1,1)).matrix};
i58 : (l35 + WP13 == l35) == true
i59 : l45 = ideal{(toMap(saturate(P4*P5),1,1)).matrix};
i60 : (l45 + WP13 == l45) == true
i61 : W' = sub(WP13, {(gens PP13)_0=>(gens PP13')_0,
(gens PP13)_1=>(gens PP13')_1+2*(gens PP13')_0,
(gens PP13)_2=>(gens PP13')_2+2*(gens PP13')_0,
(gens PP13)_3=>(gens PP13')_3+2*(gens PP13')_0,
(gens PP13)_4=>(gens PP13')_4+2*(gens PP13')_0,
(gens PP13)_5=>(gens PP13')_5+2*(gens PP13')_0,
(gens PP13)_6=>(gens PP13')_6+2*(gens PP13')_0,
(gens PP13)_7=>(gens PP13')_7+2*(gens PP13')_0,
(gens PP13)_8=>(gens PP13')_8+2*(gens PP13')_0,
(gens PP13)_9=>(gens PP13')_9+2*(gens PP13')_0,
(gens PP13)_10=>(gens PP13')_10, (gens PP13)_11=>(gens PP13')_11,
(gens PP13)_12=>(gens PP13')_12, (gens PP13)_13=>(gens PP13')_13});
i62 : W'U0 = sub(oo, {(gens PP13')_0 => 1});
i63 : tangentCone W'U0 == ideal{-9*Z_1+8*Z_7+8*Z_8-4*Z_9, -9*Z_2+8*Z_7-4*Z_8+8*Z_9,
-9*Z_3-4*Z_7+8*Z_8+8*Z_9, -9*Z_4+2*Z_7+2*Z_8-Z_9, -9*Z_5+2*Z_7-Z_8+2*Z_9,
-9*Z_6-Z_7+2*Z_8+2*Z_9, Z_10-Z_11+Z_12, 9*Z_12^2-(-4*Z_7+8*Z_8+8*Z_9)*Z_13,
9*Z_11^2-(8*Z_7-4*Z_8+8*Z_9)*Z_13, 9*Z_11*Z_12+(2*Z_7+2*Z_8-10*Z_9)*Z_13,
(2*Z_7-10*Z_8+2*Z_9)*Z_11+(-10*Z_7+2*Z_8+2*Z_9)*Z_12,
(6*Z_7-6*Z_8-18*Z_9)*Z_11+(6*Z_7-6*Z_8+18*Z_9)*Z_12,
Z_7^2-2*Z_7*Z_8+Z_8^2-2*Z_7*Z_9-2*Z_8*Z_9+Z_9^2}
i64 : degree (tangentCone W'U0) == 4
i65 : sub(WP13, {(gens PP13)_13=>1});
i66 : ConeP5 = tangentCone oo
i67 : degree oo == 5
i68 : TCOW'U13 = ideal{ z_6-z_7, z_5-z_8, z_4-z_9, z_2-z_3, z_1-z_3, 2*z_0-z_3,
z_9*z_10-z_8*z_11+z_7*z_12, z_8*z_10-z_9*z_11+z_3*z_12,
z_7*z_10-z_3*z_11+z_9*z_12, z_3*z_10-z_7*z_11+z_8*z_12,
```

```
z_8^2-z_9^2, z_7*z_8-z_3*z_9, z_3*z_8-z_7*z_9,
z_7^2-z_9^2, z_3*z_7-z_8*z_9, z_3^2-z_9^2 }
i69 : (ConeP5 == oo ) == true
i70 : M5 = ConeP5+ideal{(gens PP13)_13}
i71 : (dim oo -1, degree oo) == (2, 5)
i72 : irredCompM5 = associatedPrimes M5;
i73 : plane0=irredCompM5#0
i74 : plane1=irredCompM5#1
i75 : plane2=irredCompM5#2
i76 : plane3=irredCompM5#3
i77 : plane4=irredCompM5#4
i78 : (dim(plane0+plane1)-1, degree (plane0+plane1)) == (1,1)
i79 : (dim(plane0+plane2)-1, degree (plane0+plane2)) == (1,1)
i80 : (dim(plane0+plane3)-1, degree (plane0+plane3)) == (1,1)
i81 : (dim(plane0+plane4)-1, degree (plane0+plane4)) == (1,1)
i82 : (dim(plane1+plane2)-1, degree (plane1+plane2)) == (0,1)
i83 : (dim(plane1+plane3)-1, degree (plane1+plane3)) == (0,1)
i84 : (dim(plane1+plane4)-1, degree (plane1+plane4)) == (0,1)
i85 : (dim(plane2+plane3)-1, degree (plane2+plane3)) == (0,1)
i86 : (dim(plane2+plane4)-1, degree (plane2+plane4)) == (0,1)
i87 : (dim(plane3+plane4)-1, degree (plane3+plane4)) == (0,1)
```

A.9. Computational analysis of the KLM-EF 3-fold of genus 9

```
Macaulay2, version 1.11
with packages: ConwayPolynomials, Elimination, IntegralClosure, InverseSystems,
LLLBases, PrimaryDecomposition, ReesAlgebra, TangentCone
i1 : needsPackage "Cremona";
i2 : PP3 = ZZ/10000019[t_0..t_3];
i3 : PP13 = ZZ/10000019[w_0..w_13];
i4 : sexties = rationalMap map(PP3,PP13, matrix{{t_0*t_1^3*t_2*t_3, t_0^2*t_1^2*t_2^2,
t_0^2*t_1^2*t_2*t_3, t_0^2*t_1^2*t_3^2, t_0^3*t_1*t_2*t_3, t_0*t_1^2*t_2^2*t_3,
t_0*t_1^2*t_2*t_3^2, t_0^2*t_1*t_2*t_3^2, t_1^2*t_2^2*t_3^2,
t_0*t_1*t_2^3*t_3, t_0*t_1*t_2^2*t_3^2, t_0*t_1*t_2*t_3^3, t_0^2*t_2^2*t_3^2}});
i5 : WF13 = image sexties
i6 : (dim WF13 -1, degree WF13) == (3, 24)
i7 : P1 = ideal{w_0,w_1,w_2,w_3,w_5,w_6,w_7,w_8,w_9,w_10,w_11,w_12,w_13};
i8 : tangentCone sub(WF13, {(gens PP13)_4=>1})
i9 : degree oo == 4
i10 : P2 = ideal{w_1,w_2,w_3,w_4,w_5,w_6,w_7,w_8,w_9,w_10,w_11,w_12,w_13};
i11 : tangentCone sub(WF13, {(gens PP13)_0=>1})
i12 : degree oo == 4
i13 : P3 = ideal{w_0,w_1,w_2,w_3,w_4,w_5,w_6,w_7,w_8,w_9,w_11,w_12,w_13};
i14 : tangentCone sub(WF13, {(gens PP13)_10=>1})
i15 : degree oo == 4
i16 : P4 = ideal{w_0,w_1,w_2,w_3,w_4,w_5,w_6,w_7,w_8,w_9,w_10,w_11,w_13};
i17 : tangentCone sub(WF13, {(gens PP13)_12=>1})
i18 : degree oo == 4
i19 : P1' = ideal{w_0,w_1,w_2,w_3,w_5,w_4,w_6,w_7,w_8,w_10,w_11,w_12,w_13};
i20 : tangentCone sub(WF13, {(gens PP13)_9=>1})
i21 : degree oo == 4
i22 : P2' = ideal{w_0,w_1,w_2,w_3,w_4,w_5,w_6,w_7,w_8,w_9,w_10,w_11,w_12};
```

```
i23 : tangentCone sub(WF13, {(gens PP13)_13=>1})
i24 : degree oo == 4
i25 : P3' = ideal{w_0,w_1,w_2,w_4,w_5,w_6,w_7,w_8,w_9,w_10,w_11,w_12,w_13};
i26 : tangentCone sub(WF13, {(gens PP13)_3=>1})
i27 : degree oo == 4
i28 : P4' = ideal{w_0,w_2,w_3,w_4,w_5,w_6,w_7,w_8,w_9,w_10,w_11,w_12,w_13};
i29 : tangentCone sub(WF13, {(gens PP13)_1=>1})
i30 : degree oo == 4
i31 : J = jacobian((map sexties).matrix);
i32 : JJ = jacobian(J);
i33 : JJl23 = sub(JJ,{(gens PP3)_2=> 0, (gens PP3)_3 =>0})
i34 : SPANnuF23 = ideal{w_5,w_6,w_7,w_8,w_9,w_10,w_11,w_12,w_13};
i35 : -- H12 = ideal{random(1,PP13)};
-- for example
H12 = ideal{w_0+11*w_1+2*w_2+3*w_3+5*w_4+4*w_5+6*w_6-7*w_7-8*w_8-9*w_9+
10*w_10-11*w_11+12*w_12+13*w_13};
i36 : S = H12+WF13;
i37 : E3 = saturate(S+SPANnuF23)
i38 : (dim oo -1, degree oo, genus oo) == (1, 4, 1)
i39 : SPANE3 = ideal{E3_0,E3_1,E3_2,E3_3,E3_4,E3_5,E3_6,E3_7,E3_8,E3_9};
i40 : PP9 = ZZ/10000019[z_0..z_9];
i41 : projE3 = rationalMap map(PP13,PP9, matrix{{SPANE3_9,SPANE3_8,SPANE3_7,SPANE3_6,
SPANE3_5,SPANE3_4,SPANE3_3,SPANE3_2,SPANE3_1,SPANE3_0}})
i42 : KLM = projE3(WF13)
i43 : (dim oo -1, degree oo) == (3, 16)
i44 : isBirational((projE3|WF13)||KLM) == true
i45 : sub(KLM, {(gens PP9)_5=>1});
i46 : Conep1 = tangentCone oo
i47 : degree oo
i48 : sub(KLM, {(gens PP9)_9=>1});
i49 : Conep2 = tangentCone oo
i50 : degree oo
i51 : sub(KLM, {(gens PP9)_6=>1});
i52 : Conep3 = tangentCone oo
i53 : degree oo
i54 : sub(KLM, {(gens PP9)_8=>1});
i55 : Conep4 = tangentCone oo
i56 : degree oo
i57 : sub(KLM, {(gens PP9)_0=>1});
i58 : Conep5 = tangentCone oo
i59 : degree oo
i60 : M6 = Conep5+ideal{(gens PP9)_0}
i61 : irredCompM6 = associatedPrimes M6;
i62 : plane1 = irredCompM6#0
i63 : plane2 = irredCompM6#1
i64 : plane2' = irredCompM6#2
i65 : plane1' = irredCompM6#3
i66 : Q = irredCompM6#4
i67 : line1 = Q+plane1;
i68 : line1' = Q+plane1';
i69 : line2 = Q+plane2;
i70 : line2' = Q+plane2';
```

```
i71 : dim(line1+line1')-1 == -1
i72 : dim(line2+line2')-1 == -1
i73 : q12 = saturate(line1+line2)
i74 : q12' = saturate(line1+line2')
i75 : q1'2 = saturate(line1'+line2)
i76 : q1'2' = saturate(line1'+line2')
```

B. Configurations of the singularities of some Enriques–Fano threefolds of genus 6, 7, 8, 9, 10, 13, 17

In this appendix, we graphically represent the configurations of the singular points of the F-EF 3-folds $W_F^{p=6,7,9,13}$, the BS-EF 3-folds $W_{BS}^{p=6,7,8,9,10,13}$, the P-EF 3-folds $W_P^{p=13,17}$, and the KLM-EF 3-fold W_{KLM}^9.
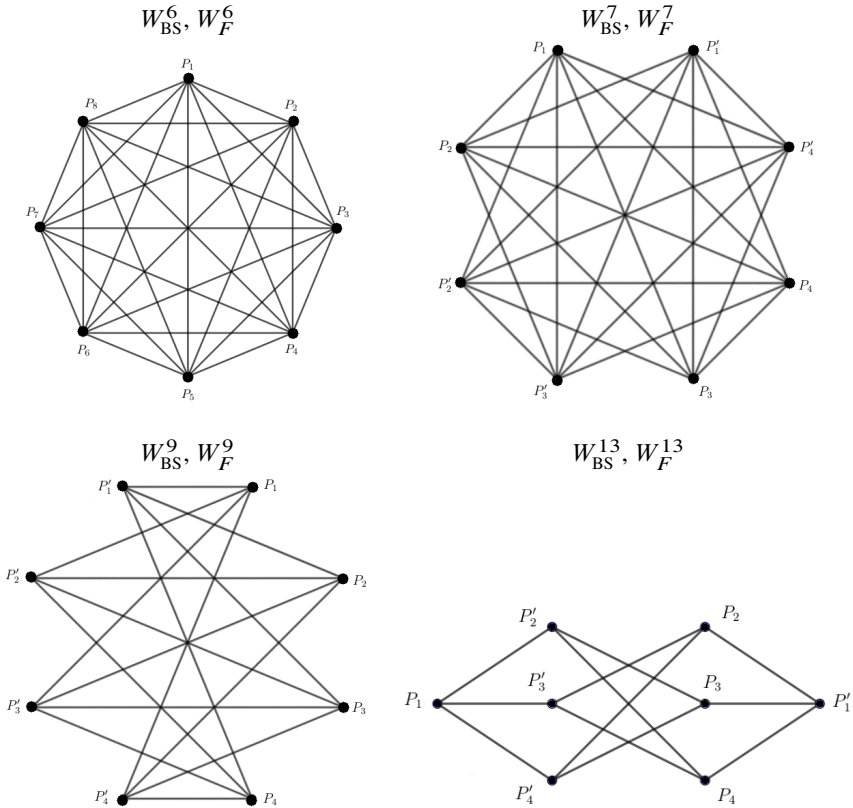
Table 1. Configurations of the eight quadruple points of the Enriques–Fano threefolds $W_F^p \subset \mathbb{P}^p$ and $W_{BS}^p \xrightarrow{\phi_{\mathcal{L}}} \mathbb{P}^p$, for $p = 6, 7, 9, 13$.
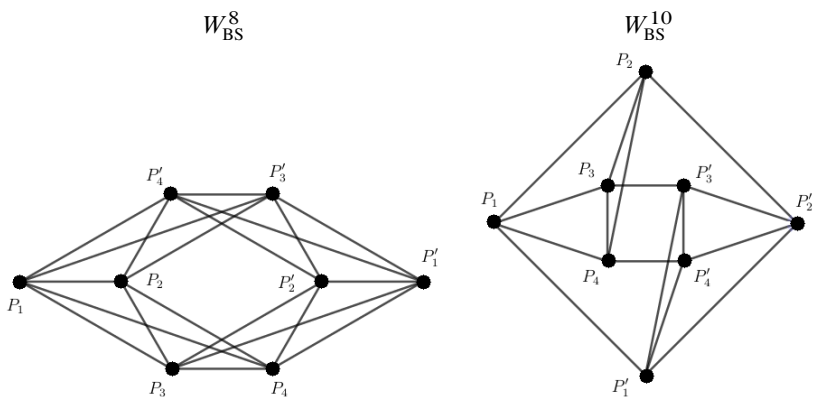
TABLE 2. Configurations of the eight quadruple points of the Enriques–Fano threefolds $W_{\mathrm{BS}}^{p} \overset{\phi_{\mathcal{L}}}{\hookrightarrow}$ $\mathbb{P}^{p}$, for $p = 8, 10$.
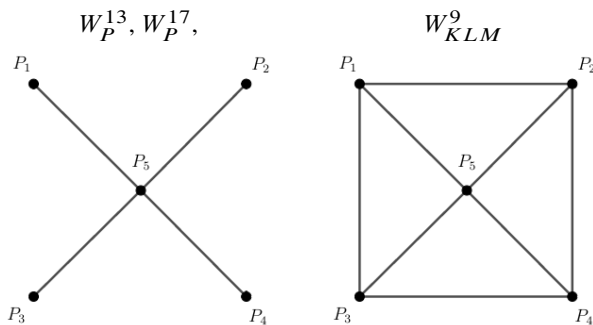

TABLE 3. Configurations of the five singular points of the Enriques–Fano threefolds W_{P}^{13}, W_{P}^{17}, and W_{KLM}^{9}.

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