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Complex Variable(s) Functions. – *Domains of holomorphy in Stein spaces*, by VIOREL VÎJÎITU, communicated on 10 November 2022.

ABSTRACT. – We prove an interpolation theorem for domains in Stein spaces that are locally Stein outside rare analytic sets and improve several existing results in this area. This is then applied to the Levi problem in Stein spaces.

KEYWORDS. – Domain of holomorphy, pseudoconcave set, envelope of holomorphy, Stein space.

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1. INTRODUCTION

Let X be a complex space (always reduced and with countable topology). By X_{reg} we denote the set of manifold points of X, $X_{\text{sg}} = X \setminus X_{\text{reg}}$ the analytic set of singular points, X^* the normalization of X, and $\pi : X^* \to X$ the corresponding finite, holomorphic surjective map.

Also we put $X'_{\text{sg}} = \pi(X^*_{\text{sg}})$; it is an analytic subset of X_{sg}, and $X'_{\text{sg}} \subset X$ has codimension at least two. If $T \subset X$ is a set, then denote $\pi^{-1}(T)$ by T^*.

An open set $\Omega \subset X$ is said to be *locally Stein at a subset* Γ of X if any point of Γ has an open neighborhood U in X such that $U \cap \Omega$ is Stein. If we may choose $\Gamma = \partial\Omega$, then we term Ω *locally Stein* (in X).

A subset Σ of X is said to be

- *pseudoconcave* if Σ is closed and its complement is locally Stein;
- *locally complete pluripolar* if every point of Σ has a connected, open neighborhood U on which there exists a plurisubharmonic (psh) function φ, which is not identically $-\infty$ on any irreducible component of U, and $\Sigma \cap U = \{\varphi = -\infty\}$. (For instance, nowhere dense, locally closed analytic subsets of X are locally complete pluripolar.)

Henceforth, unless explicitly stated, we consider only complex spaces of pure dimension.

Now we can state a few of our results that we apply later to remove the compacity assumptions in [3] (cf. Theorems 4 and 5).

The following result extends an interpolation theorem [16] that is recovered for X normal.

THEOREM 1. *Let E be a holomorphic vector bundle over a Stein space X. Let Ω be an open set of X with Ω^* locally Stein at $\partial\Omega^* \setminus X^*_{\mathrm{sg}}$. Then, for any discrete subset Λ of $\Omega \setminus X_{\mathrm{sg}}$ whose closure in X is disjoint with X'_{sg}, every section of E over Λ extends to a holomorphic section of E over Ω.*

Then we generalize a theorem due to Scheja [12, Satz 3] from normal spaces to arbitrary complex spaces. This will be applied in Section 4.

THEOREM 2. *For every hypersurface Σ of a Stein space X, there exists a holomorphic function on $X \setminus \Sigma$ that does not extend holomorphically to X.*

The following statement can be seen as a complement to [13]. Its proof is an immediate consequence of Proposition 3.

PROPOSITION 1. *Let Y be a complex manifold and $\Sigma \subset Y$ a closed, locally complete pluripolar set. Then, for any open set Ω of Y that is locally Stein at $\partial\Omega \setminus \Sigma$, the set $\breve{\Omega}$ of interior points of the union $\Sigma \cup \Omega$ is locally Stein.*

Let X be a complex space, not necessarily of pure dimension. An open set Ω of X is *not* a *domain of holomorphy* if there exist an irreducible component X' of X and two nonempty open subsets U' and V' of X' such that the following holds.

> *The analytic set U' is irreducible, not contained in Ω, $V' \subseteq \Omega \cap U'$, and for any holomorphic function $f \in \mathcal{O}_X(\Omega)$ there exists a holomorphic function $g \in \mathcal{O}_{X'}(U')$ such that $g|_{V'} = f|_{V'}$.*

EXAMPLE 1. In $\mathbb{C}^2$ consider the Stein curves $\mathrm{A} = \mathbb{C} \times \{0\}$ and $\mathrm{B} = \{0\} \times \mathbb{C}$, which intersect only at the origin of $\mathbb{C}^2$. Obviously, $X = \mathrm{A} \cup \mathrm{B}$ is a reduced Stein space of dimension one and X is not irreducible.

Now, let $\Omega = (\mathrm{A} \setminus \{(1,0)\}) \cup (\mathrm{B} \setminus \{(0,1)\})$, which is a Stein open subset X, and put $\mathrm{V} = \mathrm{A} \setminus \{(1,0),(0,0)\}$ and $\mathrm{U} = (\mathrm{A} \setminus \{(1,0)\}) \cup \mathrm{B}$.

It follows that U and V are domains of X, $V \subset \Omega \cap U$, and $U \neq \Omega$.

Here we show that, in spite of the fact that Ω is a Stein open subset of the Stein space X, for any holomorphic function $f \in \mathcal{O}(\Omega)$ there exists a holomorphic function $g \in \mathcal{O}(\mathrm{U})$ such that $f = g$ on V.

Indeed, f induces two holomorphic functions, $f_1 \in \mathcal{O}(\mathrm{A} \setminus \{(1,0)\})$ and $f_2 \in \mathcal{O}(\mathrm{B} \setminus \{(0,1)\})$, such that $f_1(0,0) = f_2(0,0)$. Then the function $g : \mathrm{U} \to \mathbb{C}$, which equals f_1 on $\mathrm{A} \setminus \{(1,0)\}$ and $g = f_1(0,0)$ on B, is holomorphic on U and coincides with f on V.

Therefore, the ordinary definition of domains of holomorphy in complex manifolds does no carry over *ad litteram* to complex spaces.

EXAMPLE 2. Let X be the Segre cone of $\mathbb{C}^4$ (see [6]), namely $X = \{z_1 z_2 = z_3 z_4\}$. Note that X is a connected, normal Stein space of dimension two.

Let H be the hypersurface $\{z_2 = z_3 = 0\}$ of X. Then $X \setminus \mathrm{H}$ is domain of holomorphy in X, but $X \setminus \mathrm{H}$ is not Stein.

Indeed, first observe that $X \setminus \mathrm{H} \simeq \mathbb{C} \times (\mathbb{C}^2 \setminus \{(0,0)\})$, where the biholomorphism is induced by the holomorphic map

$$\mathbb{C} \times \mathbb{C}^2 \ni (s, u, v) \mapsto (su, v, u, sv) = (z_1, z_2, z_3, z_4) \in X.$$

Second, in order to check that $X \setminus \mathrm{H}$ is a domain of holomorphy in X, consider the holomorphic function $f : X \setminus \mathrm{H} \to \mathbb{C}$ defined by the formula

$$f(z) = \begin{cases} (z_1 - z_4)/(z_2 - z_3) & \text{if } z_2 \neq z_3, \\ -z_1/z_3 & \text{if } z_2 = z_3 \neq 0. \end{cases}$$

We claim that f is singular at any point of H. For this, let $\xi \in \mathrm{H}$ and notice that $\xi = (s, 0, 0, t)$ for some $s, t \in \mathbb{C}$. Then, to settle the claim, we produce a sequence $(\xi_n)_n$ of points in $X \setminus \mathrm{H}$ that converges to ξ and such that $f(\xi_n) = -n$ for all $n \in \mathbb{N}$. This is done as follows.

If $s \neq 0$ or $t \neq 0$, then let $\xi_n = (s, t/n, s/n, t)$ for all $n \in \mathbb{N}$.

If $s = t = 0$, then let $\xi_n = (1/n^2, 1/n^2, 1/n^3, 1/n)$ for all $n \in \mathbb{N}$.

EXAMPLE 3. Let X be a two-dimensional Stein space with isolated singularities, and let X_{nn} denote the analytic set of non-normal points of X. Then $X \setminus X_{\mathrm{nn}}$ is a domain of holomorphy in X.

Indeed, to prove this, observe that $X_{\mathrm{nn}} \subseteq X_{\mathrm{sg}}$. Then, thanks to Markoe [11], the space X is not normal at a point $a \in X_{\mathrm{sg}}$ if and only if $\operatorname{prof}(\mathcal{O}_{X,a}) = 1$.

Now by Bănică and Stănăşilă [2, p. 365], for every point $a_j \in X_{\mathrm{nn}}$ there exists a holomorphic function $f_j \in \mathcal{O}(X \setminus \{a_j\})$ that cannot be extended holomorphically across a_j.

Therefore, for $\varepsilon_j > 0$ sufficiently small, the series $\sum \varepsilon_j f_j$ converges uniformly on compact subsets of $X \setminus X_{\mathrm{nn}}$ to a holomorphic function f on $X \setminus X_{\mathrm{nn}}$ that is singular at every point a_j.

A space that fulfills the above condition is the Stein surface due to Harvey [8]; namely X is the image of the proper holomorphic mapping

$$h : \mathbb{C}^2 \to \mathbb{C}^4, \quad (z, w) \mapsto (z^2, z^3, w, zw).$$

In this case, $X_{\text{nn}} = X_{\text{sg}} = \{0\}$, X is irreducible, locally irreducible, the map h is a homeomorphism onto X so that h is the normalization of X, and $X \setminus \{0\}$ is a domain of holomorphy in X.

A direct proof of the last assertion above follows readily because the holomorphic function $f : X \setminus \{0\} \to \mathbb{C}$ given by

$$f(z) = \begin{cases} z_2/z_1 & \text{if } z_1 \neq 0, \\ z_4/z_3 & \text{if } z_3 \neq 0 \end{cases}$$

does not extend holomorphically to the whole space X. Otherwise, if $\tilde{f} \in \mathcal{O}(X)$ is a holomorphic extension of f, then the holomorphic function

$$g : X \to \mathbb{C}^2, \quad z = (z_1, z_2, z_3, z_4) \mapsto \big(\tilde{f}(z), z_3\big)$$

would be an inverse for h, which cannot hold since X is not smooth. In particular, this shows also that X is not normal at the origin.

The behavior of the notion of domain of holomorphy with respect to normalization is given below (cf. Proposition 5).

PROPOSITION 2. *Let Ω be an open subset of a Stein space X. If Ω^* is a domain of holomorphy in X^*, then Ω is a domain of holomorphy in X.*

2. SOME USEFUL RESULTS

In this section, we collect some facts that will be applied in Sections 3 and 4. First, we quote a particular case of a theorem proved in [15].

PROPOSITION 3. *Let Y be a complex manifold and $\Sigma \subset Y$ a closed, locally complete pluripolar set. Then, for every pseudoconcave set $A \subset Y \setminus \Sigma$ its closure $\bar{A}$ in Y is pseudoconcave in Y.*

COROLLARY 1. *Let Y be a complex manifold and $\Sigma \subset Y$ a closed, locally complete pluripolar set. If Ω is an open subset of Y such that Ω is locally Stein at $\partial\Omega \setminus \Sigma$ and $\partial\Omega \setminus \Sigma$ is dense in $\partial\Omega$, then Ω is locally Stein.*

PROOF. Let $\breve{\Omega}$ be the interior of the union $\Omega \cup \Sigma$. Obviously, $\Omega \subset \breve{\Omega}$. Then the proof concludes by Proposition 3 and the topological property

$$\breve{\Omega} \setminus \Omega = \partial\Omega \setminus \overline{\partial\Omega \setminus \Sigma}.$$

Indeed, let us check first the inclusion "$\subseteq$". Take $y_0 \in \breve{\Omega} \setminus \Omega$. Since Σ is nowhere dense in Y, it follows that $y_0 \in \overline{\Omega}$, hence $y_0 \in \partial\Omega$. Let V be a connected open neighborhood of y_0 in Y with $V \subset \Omega \cup \Sigma$. Thus $V \setminus \Sigma \subset \Omega$ so that there is a closed subset B of V such that $B \subset \Sigma$ and $\Omega \cap V = V \setminus B$. Then $V \cap \partial\Omega = B$, hence $V \cap (\partial\Omega \setminus \Sigma) = \emptyset$, whence $y_0 \notin \overline{\partial\Omega \setminus \Sigma}$.

For the reverse inclusion, let $y_0 \in \partial\Omega \setminus \overline{\partial\Omega \setminus \Sigma}$, and then let V be a connected open neighborhood of y_0 in Y with $V \cap (\partial\Omega \setminus \Sigma) = \emptyset$. Hence $(V \setminus \Sigma) \cap \partial\Omega = \emptyset$. But $V \setminus \Sigma$ is connected and as Σ is nowhere dense in Y, it follows that $V \setminus \Sigma \subset \Omega$, whence $V \subset \Omega \cup \Sigma$, that is $V \subset \breve{\Omega}$, *a fortiori* $y_0 \in \breve{\Omega}$. ∎

From [16, 17], we quote the following interpolation theorem.

THEOREM 3. *Let E be a holomorphic vector bundle over a normal Stein space X. Let $\Omega \subset X$ be an open set that is locally Stein at $\partial\Omega \setminus X_{\mathrm{sg}}$.*

Then, for any discrete set $\Lambda \subset \Omega$ whose closure in X is disjoint with X_{sg}, every section of E over Λ extends to a holomorphic section of E over D.

We apply Corollary 1 and Theorem 3 to prove the following result.

PROPOSITION 4. *Let X be a normal Stein space. Let $\Omega \subset X$ be an open set. Then the following assertions hold true.*

(1) *If Ω is a domain of holomorphy in X, then Ω is locally Stein at the set $\partial\Omega \setminus X_{\mathrm{sg}}$, and for every analytic set $\Sigma \subset X$ of codimension at least two, the set $\partial\Omega \setminus \Sigma$ is dense in $\partial\Omega$.*

(2) *If there exists a closed, locally complete pluripolar set Σ of X with $\partial\Omega \setminus \Sigma$ dense in $\partial\Omega$ and such that Ω is locally Stein at $\partial\Omega \setminus \Sigma$, then Ω is a domain of holomorphy.*

PROOF. Ad (1). Recall that in any Stein manifold, a domain of holomorphy means precisely a connected, non empty Stein open set.

Then, granting the Riemann extension theorem, the above fact implies readily that Ω is locally Stein at every point of $\partial\Omega \setminus X_{\mathrm{sg}}$.

Now let Σ be an analytic subset of X of codimension at least two. In order to derive a contradiction, assume that $\partial\Omega \setminus \Sigma$ is not dense in $\partial\Omega$. Therefore, there is a domain W of X with $W \cap \partial\Omega \neq \emptyset$ and $(W \setminus \Sigma) \cap \partial\Omega = \emptyset$. Since $W \setminus \Sigma$ is connected, either $(W \setminus \Sigma) \cap \overline{\Omega} = \emptyset$ so that $W \cap \overline{\Omega} \subset \Sigma$, which is not possible since Σ is nowhere dense in X, or $W \setminus \Sigma \subset \Omega$ so that by Riemann's extension theorem, Ω is not a domain of holomorphy. Both ends contradict the hypothesis, hence $\partial\Omega \setminus \Sigma$ is dense in $\partial\Omega$.

Ad (2). In order to apply Theorem 3, we need to check that Ω is locally Stein at $\partial\Omega \setminus X_{\mathrm{sg}}$ and that $\partial\Omega \setminus X_{\mathrm{sg}}$ is dense in $\partial\Omega$.

The first part follows by Corollary 1. The second part is done by *reductio ad absurdum*. So assume that $\partial\Omega \setminus X_{\mathrm{sg}}$ is not dense in $\partial\Omega$. Then there is x_0 in $\partial\Omega$ such that, after shrinking X around x_0, if necessary, one may take $\Omega = X \setminus B$ for a closed subset B of X_{sg}. Now the hypothesis says that $B \setminus \Sigma$ is dense in B and $X \setminus B$ is locally Stein at $B \setminus \Sigma$, whence $B \setminus \Sigma = \emptyset$ by Riemann's extension theorem since X is normal and $B \subset X_{\text{sg}}$. Therefore, $B \subset \Sigma$ from which we infer that $B = \emptyset$, which contradicts the existence of x_0.

Now, with Theorem 3 at hand, the statement concludes again by contradiction and routine arguments. (If V is a domain of X with $V \cap \partial\Omega \neq \emptyset$ and $\{U_\alpha\}$ is the family of connected components of $V \cap \Omega$, then $\{U_\alpha\}$ is the family of connected components of $(V \setminus X_{\text{sg}}) \cap \Omega$, and by (‡) in the subsequent proof of Proposition 1 in Section 3, for any α there is a boundary point of Ω in $V \setminus X_{\text{sg}}$ that is an accumulation point of $U_\alpha \setminus X_{\text{sg}}$.) ∎

Corollary 2. *Let X be a normal Stein space. Then an open set $\Omega \subset X$ is a domain of holomorphy if and only if Ω is locally Stein at $\partial\Omega \setminus X_{\text{sg}}$ and the set $\partial\Omega \setminus X_{\text{sg}}$ is dense in $\partial\Omega$.*

The following result extends a similar one due to Hirschowitz [9] that is recovered for normal spaces.

Corollary 3. *The complement of every hypersurface of a Stein space is a domain of holomorphy.*

Proof. This follows by Proposition 2 and Corollary 2. ∎

Proposition 5. *Let $\pi : X \to Y$ be a finite, surjective, holomorphic map of normal Stein spaces X and Y. Then an open subset Ω of Y is a domain of holomorphy in Y if and only if $\pi^{-1}(\Omega)$ is a domain of holomorphy in X.*

Proof. For the "only if" assertion observe first that, for every open set D of Y, the closure of $\pi^{-1}(D)$ equals the pull-back through π of the closure of D. Hence we get that $\partial\pi^{-1}(\Omega) = \pi^{-1}(\partial\Omega)$. Therefore, with $\Sigma = \pi^{-1}(Y_{\text{sg}})$, which is an analytic subset of X of codimension at least two, we deduce the density assertion from (2) of Proposition 3. On the other hand, since for every Stein open subset of Y its pull-back through π is Stein, we conclude by statement (2) of Proposition 4.

Now, consider the "if" part. We prove this by contradiction, so assume that Ω is not a domain of holomorphy in Y, hence there is a point $y_0 \in \partial\Omega$ and open sets U and V as in Definition 1, V is connected and contains y_0. Let $x_0 \in X$ with $\pi(x_0) = y_0$ and let V^* be a connected open neighborhood of x_0 such that $\pi(V^*) \subset V$. Let U^* be any nonempty connected component of $V^* \cap \pi^{-1}(\Omega)$. There is a boundary point x^* of $\pi^{-1}(\Omega)$ in V^* that is an accumulation point of U^*. (See the topological fact (‡) in the subsequent proof of Proposition 2.)

Further, by Theorem 3 there exists a holomorphic function $h \in \mathcal{O}(\pi^{-1}(\Omega))$ whose restriction to U^* is unbounded.

Now, by routine arguments, there is a monic holomorphic polynomial $P(t, y) = t^d + a_1(y)t^{d-1} + \cdots + a_d(y)$ of some degree $d \in \mathbb{N}$ (given by the number of sheets of π above a neighborhood of y_0), where the coefficients are holomorphic functions on Ω, such that $P(h(x), \pi(x)) = 0$ for all $x \in \pi^{-1}(\Omega)$.

Let $\tilde{a}_1, \ldots, \tilde{a}_d$ be the holomorphic "extensions" to V of the holomorphic functions $a_1, \ldots, a_d \in \mathcal{O}(\Omega)$, respectively.

Hence setting $\tilde{P}(t, y) = t^d + \tilde{a}_1(y)t^{d-1} + \cdots + \tilde{a}_d(y)$, by the identity theorem of holomorphic functions one has $\tilde{P}(h(x), \pi(x)) = 0$ for all $x \in U^*$, which is not possible. Thus the proof of proposition. ∎

In the final part of this section, we recall a few more facts from [7].

Let $\widetilde{\mathcal{O}}_X$ denote the coherent sheaf of germs of weakly holomorphic functions in X. Recall that $\widetilde{\mathcal{O}}_X = \pi_\star(\mathcal{O}_{X^\star})$ and $\widetilde{\mathcal{O}}_X$ is a subsheaf of the sheaf of meromorphic functions $\mathcal{M}_X$ on X.

Let $\mathcal{I} \subset \mathcal{O}_X$ be the coherent ideal sheaf of universal denominators. At stalk level, for every $x \in X$, $\mathcal{I}_x = \{h_x \in \mathcal{O}_x : h_x\widetilde{\mathcal{O}}_x \subseteq \mathcal{O}_x\}$.

The vanishing set $V(\mathcal{I})$ of $\mathcal{I}$ coincides with the analytic set X_{nn} of not normal points of X so that $V(\mathcal{I}) \subset X_{\mathrm{sg}}$; hence $V(\mathcal{I})$ is nowhere dense.

A holomorphic function $h \in \mathcal{O}(X)$ is said to be *active* if, for every point $x \in X$, the germ h_x is not a zero divisor of $\mathcal{O}_{X,x}$. Also, if a germ g_x of a holomorphic function defined in a neighborhood of a point $x \in X$ is not a zero divisor, then this property is maintained in a neighborhood of x.

It can be shown that $h \in \mathcal{O}(X)$ is active, precisely when h does not vanish identically on any irreducible component of X.

Proposition 6. *Let X be a Stein space of finite dimension. Then there are finitely many holomorphic functions in $\mathcal{I}(X)$ whose common zero set is $V(\mathcal{I})$. Besides, there is an active holomorphic function in $\mathcal{I}(X)$.*

Proof. The first part is a standard application of induction over the dimension of $X \setminus V(\mathcal{I})$ and Cartan's Theorem B. (It is perhaps important to note that, here we really need Theorem B for coherent ideals with nilpotents.)

Now consider finitely many holomorphic functions $h_1, \ldots, h_m$ in $\mathcal{I}(X)$ whose common zero set is $V(\mathcal{I})$. (As a matter of fact, if n is the complex dimension of X, then we may take $m = n + 1$.) Since $V(\mathcal{I})$ does not contain any irreducible component of X, there is a discrete set $\{x_j\}$ of X such that for any irreducible component X' of X there is some $x_j \in X' \setminus V(\mathcal{I})$. For any j consider the set $B_j \subset \mathbb{C}^m$ of all $(c_1, \ldots, c_m)$ such that $c_1h_1 + \cdots + c_mh_m$ does not vanish at x_j. This set is open and dense in $\mathbb{C}^m$.

Therefore, by Baire's theorem, there are $c_1, \dots, c_m \in \mathbb{C}$ such that the holomorphic function $h := c_1 h_1 + \cdots + c_m h_m$ does no vanish at any point of $\{x_j\}$; hence h is active, whence the proof of proposition. ■

3. The proofs

Proof of Theorem 1. Of course, Cartan's Theorem B settles the case when Λ is finite.

Now, assume that Λ is infinite and write $\Lambda = \{x_k : k \in \mathbb{N}\}$. For any index k, select a point $x_k^* \in D^* \setminus X_{\text{sg}}^*$ such that $\pi(x_k^*) = x_k$. This is possible since $\pi(X_{\text{sg}}^*) \subset X_{\text{sg}}$ and $\Lambda \subset D \setminus X_{\text{sg}}$.

Since π is proper, the set $\Lambda^* = \{x_k^* : k \in \mathbb{N}\}$ is discrete in $D^* \setminus X_{\text{sg}}^*$, and because $\bar{\Lambda} \cap X_{\text{sg}} = \emptyset$, the closure of Λ^* in X^* is disjoint with X_{sg}^*.

Let $E^\star$ be the pull-back of E through π. For each index $k \in \mathbb{N}$, consider any vector $\mathbf{v}_k \in E_{x_k}$ and let $\mathbf{v}_k^*$ be in the fiber of E^* over x_k^* that maps onto $\mathbf{v}_k$.

By Proposition 6, there are finitely many holomorphic functions $h_1, \dots, h_m$ in $\mathcal{I}(X)$ whose common zero set is $V(\mathcal{I})$. Therefore, for every open subset U of X and weakly holomorphic function $g \in \widetilde{\mathcal{O}}(U)$, the products $h_j g$ are holomorphic functions in $\mathcal{O}(U)$.

Since $X_{\text{reg}} \cap V(\mathcal{I}) = \emptyset$, for each k there is an index j such that $h_j(x_k) \neq 0$. Hence we may select complex numbers γ_{jk} such that, for each k one has

$$\sum_{j=1}^{m} \gamma_{jk} h_j(x_k) = 1.$$

By Theorem 3, there are holomorphic sections σ_j of E^* over D^* satisfying $\sigma_j(x_k^*) = \gamma_{kj} \mathbf{v}_k^*$ for all k.

Therefore, $\sigma = h_1\sigma_1 + \cdots + h_m\sigma_m$ becomes a holomorphic section of E over D, and this σ is what we want, whence the proof of theorem. ■

Proof of Proposition 2. Note that $X'_{\text{sg}} = \pi(X_{\text{sg}}^*)$ is an analytic subset of X of codimension at least two (because X has pure dimension). Hence $\pi^{-1}(X'_{\text{sg}})$ has codimension at least two in X^*, so that the set $\partial\Omega^* \setminus \pi^{-1}(X'_{\text{sg}})$ is dense in $\partial\Omega^*$ according to Proposition 4.

From this we infer that $\partial\Omega \setminus X'_{\text{sg}}$ is dense in $\partial\Omega$. This is a consequence of the following straightforward assertion.

(†) Let $\varphi : S \to T$ be a proper, surjective map between Hausdorff topological spaces S and T. Then, for every open set $D \subset T$ one has $\varphi(\partial\varphi^{-1}(D)) = \partial D$.

Therefore, given a point $a \in \partial\Omega \setminus X'_{\text{sg}}$, by Theorem 1 it follows that, for every sequence $(x_\nu)_\nu$ of points in $\Omega \setminus X_{\text{sg}}$ converging to a and for every sequence $(c_\nu)_\nu$ of complex numbers, there is a holomorphic function $h \in \mathcal{O}(\Omega)$ such that $h(x_\nu) = c_\nu$ for all ν.

To proceed with the proof, in order to reach a contradiction, assume that Ω is not a domain of holomorphy, and let U and V be as in definition.

For the sake of simplicity, let X be irreducible so that X^* is connected. (Otherwise, we have to work with a connected component of X^* that maps onto the irreducible component of X that contains U.)

We will show that there are a holomorphic function $f \in \mathcal{O}(\Omega)$ and a sequence $(x_\nu)_\nu$ of points in $\Omega \cap U$ converging to a boundary point of Ω that belongs to U for which f is unbounded on $(x_\nu)_\nu$.

Indeed, since U is irreducible, its preimage $U^* = \pi^{-1}(U)$ is connected, and the irreducible components $\{G_j\}_j$ of $\Omega \cap U$ are in one-to-one correspondence with the connected components $\{G_j^*\}_j$ of $\Omega^* \cap U^*$.

Recall the following fact (cf. [5, the lemma on p. 50]).

(‡) Let S be a topological space, locally arcwise-connected and locally compact. Let D and U be open subsets of S such that $U \cap \partial D \neq \emptyset$. If U is connected, then, for any connected component $W \neq \emptyset$ of $D \cap U$, there is a boundary point s^* of D in U that is an accumulation point of W.

Indeed, select two points $s_0 \in W$, $s_1 \in U \cap \partial D$, and let $\gamma : [0, 1] \to U$ be a continuous path such that $\gamma(0) = s_0$ and $\gamma(1) = s_1$. Let t^* be the supremum of all $t \in [0, 1]$ such that $\gamma([0, t]) \subset W$. Clearly, $t^* \in (0, 1]$ and $s^* = \gamma(t^*)$ is as desired.

In the complex manifold X^*_{reg}, we have the open set $\Omega^* \setminus X^*_{\text{sg}}$, the domain $U^* \setminus X^*_{\text{sg}}$, and the connected components $G_k^* \setminus X^*_{\text{sg}}$ of $(U^* \setminus X^*_{\text{sg}}) \cap (\Omega^* \setminus X^*_{\text{sg}})$.

By (‡), for every k there is a boundary point b_k of $\Omega^* \setminus X^*_{\text{sg}}$ (with respect to X^*_{reg}) that is an accumulation point of $G_k^* \cap (\Omega^* \setminus X^*_{\text{sg}})$. We choose an index k such that $G_k \cap V \neq \emptyset$ and let b be the corresponding b_k. Then, because $\Omega^* \setminus \pi^{-1}(X_{\text{sg}})$ is dense in $\Omega^* \setminus X^*_{\text{sg}}$, there is a sequence $(x^*_\nu)_\nu$ of points in $G_k^* \setminus \pi^{-1}(X_{\text{sg}})$ converging to b.

Set $a = \pi(b)$, and $x_\nu = \pi(x^*_\nu)$ for all $\nu \in \mathbb{N}$.

Then the existence of the holomorphic function $f \in \mathcal{O}(\Omega)$ that is unbounded on $(x_\nu)_\nu$ follows now by Theorem 1. In particular, there cannot be a holomorphic function $g \in \mathcal{O}(U)$ such that $f|_V = g|_V$ as this would imply that f and g agree on G_k, so that f would be bounded on $(x_\nu)_\nu$. The proof of the proposition is complete. ■

PROOF OF THEOREM 2. Let n be the complex dimension of X. We keep the notations as in the proof of Theorem 1, and borrow freely some ideas from Scheja's paper [12]. Recall that $\pi : X^* \to X$ denotes the normalization map of X, and $A^* = \pi^{-1}(A)$.

By using the identity theorem for holomorphic functions on irreducible complex spaces, it is easily seen that we may assume A irreducible.

Now the proof continues by case analysis.

Case 1. Suppose that A is not contained in $V(\mathcal{I})$. Since A is irreducible, $A \cap V(\mathcal{I})$ is nowhere dense in A.

Let h be an active holomorphic function on X that vanishes on A. Hence $h^* = h \circ \pi$ is a holomorphic function on X^* that vanishes on A^* and h^* does not vanish identically on any connected component of X^*.

Let B^* be the union of the (at most countable) irreducible components B_j^* of $\{h^* = 0\}$ that are not contained in A^*, and let $\nu_j \in \mathbb{N}$ be the vanishing order of h^* on B_j^*. Hence the analytic set $B^* \cap A^*$ has dimension $\leq n-2$ so that $A_0 = A \setminus \pi(B^* \cap A^*)$ is a Zariski dense open subset of A. Besides, $\Gamma = A_0 \setminus V(\mathcal{I})$ is dense in A.

We show that, for every point $a \in \Gamma$, there is a holomorphic function $f \in \mathcal{O}(X \setminus A)$ that is unbounded in any neighborhood of a.

To check this, let $a^* \in A^* \setminus B^*$ such that $\pi(a^*) = a$. By [12, Satz 4], there is a holomorphic function $g^* \in \mathcal{O}(X^*)$ that vanishes of order at least ν_j on each B_j^* and $g^*(a^*) \neq 0$. Therefore, the meromorphic function g^*/h^* on X^* is holomorphic on $X^* \setminus A^*$ and has a pole at a^*.

Let θ be the weakly holomorphic function on $X \setminus A$ induced by g^*/h^*.

Since $a \notin V(\mathcal{I})$, by Proposition 6, there exists a holomorphic function $h_1 \in \mathcal{I}(X)$ such that $h_1(a) \neq 0$. It follows that $f = h_1\theta$ is holomorphic on $X \setminus A$ and satisfies the following property.

> For every irreducible component Y of X passing through a and for every open neighborhood V of a in Y, f is unbounded on $V \setminus A$.

In particular, f does not extend holomorphically across a.

Case 2. Let $A \subset V(\mathcal{I})$. We follow the recipe from Case 1. Here we select the active holomorphic function h to be an element of $\mathcal{I}(X)$.

We show that for any point $a \in A_0$ there is a holomorphic function on $X \setminus A$ that is unbounded about a.

Indeed, as in Case 1, we produce the weakly holomorphic function θ exactly as there. It is, then, readily seen that $h\theta^2$, which is holomorphic on $X \setminus A$, satisfies the same properties as the function f, whence the proof of theorem. ∎

4. On envelopes of holomorphy

An *envelope of holomorphy* of a complex space X is a pair $(X^\natural, \chi)$ of a Stein space $X^\natural$ and a holomorphic map $\chi : X \to X^\natural$ such that the canonically induced map

$$\mathcal{O}(X^\natural) \to \mathcal{O}(X), \quad f \mapsto f \circ \chi,$$

is bijective. If χ is understood from the context, we simply say that $X^\natural$ is the envelope of holomorphy of X.

The pair $(X^\natural, \chi)$, if it exists, is unique up to a natural isomorphism.

Observe that if $\mathcal{O}(X)$ separates points and gives local coordinates (this is the case when X is an open subset of a Stein space) and its envelope $X^\sharp$ exists, then there is an open immersion $\iota : X \to X^\sharp$ so that one may view X as an open subset of $X^\sharp$.

Another way of saying that X has an envelope of holomorphy is that $\mathcal{O}(X)$ is a Stein algebra [4].

Proposition 7. *Let X be a complex space such that $\mathcal{O}(X)$ separates points and gives local coordinates. If X has an envelope of holomorphy $(X^\natural, \chi)$ and $\theta : Y \to X^\natural$ is the normalization map, then Y is the envelope of holomorphy of $\theta^{-1}(\iota(X))$.*

Proof. Note that if X is a relatively compact open subset of a Stein space Z, Proposition 7 reduces to [3, Lemma 3.2]. The relative compacity has been used there in the following way. For a point $z \in Z$, consider the minimal number n_z of generators of $\widetilde{\mathcal{O}}_{Z,z}$ as an $\mathcal{O}_{Z,z}$-module.

The function $Z \ni z \mapsto n_z \in \mathbb{N}$ is upper semi-continuous, and since X is relatively compact in Z one has $\sup_{z\in X} n_z < \infty$. By routine arguments, it follows that every weakly holomorphic function on X is quotient of two holomorphic functions on X.

This idea does not work when X is not relatively compact, but we are saved by Proposition 6.

Now, for the commodity of the reader, we sketch a proof of Proposition 7 that goes along the lines in [3]. All we need to show is the following claim.

> Every weakly holomorphic function $f \in \widetilde{\mathcal{O}}(X)$ has a unique weakly holomorphic extension $F \in \widetilde{\mathcal{O}}(X^\sharp)$.

Indeed, let $f \in \widetilde{\mathcal{O}}(X)$. Let $h \in \mathcal{I}(X^\sharp)$ be an active holomorphic function. Thus fh is holomorphic on X. Let $g \in \mathcal{O}(X^\sharp)$ be its (unique) extension. Then $F = g/h \in \mathcal{M}(X^\sharp)$. We check that F is weakly holomorphic on $X^\sharp$.

For this to be true we prove that the pole set B of the meromorphic function $F \circ \theta$ on $X^\sharp$ is the empty set. Otherwise, B would be a hypersurface, and, then, since $f \circ \theta$ is holomorphic on $X^\sharp$, the intersection $\theta(B) \cap X$ is the empty set. Now by Theorem 2, there is a holomorphic function $f_\sharp \in \mathcal{O}(X^\sharp \setminus B)$ that cannot be extended holomorphically to $X^\sharp$, hence as $X \subset X^\sharp \setminus B$, $f_\sharp$ restricted to X cannot be extended holomorphically to $X^\sharp$, contradicting the fact that $X^\sharp$ is the envelope of holomorphy of X.

In conclusion, B is the empty set, whence the claim. This finishes the proof of the proposition. ∎

Remark. We may apply Proposition 7 for X an open subset of a Stein space Z such that X is locally Stein at every point of $\partial X \setminus Z_{\mathrm{sg}}$. This happens if either X is an increasing union of Stein open subsets, or X is a domain of holomorphy, or X satisfies the subsequent principal hypersection condition.

Therefore, we improve [3, Theorem 3.2] as follows.

Theorem 4. *A locally Stein domain X of a Stein space Z is Stein if and only if X has an envelope of holomorphy.*

Proof. Note that, under the above hypothesis, for every point $a \in \partial X \setminus Z_{\mathrm{sg}}$ and for every sequence $(x_k)_k$ of points in X that converges to a, granting Theorem 1, there is a holomorphic function $f \in \mathcal{O}(X)$ that is unbounded on the sequence $(x_k)_k$. Then, with Proposition 7 at hand, the proof in [3] works *ad litteram*.

In the same vein, in order to state an improved version of [3, Theorem 3.14] we give the following *ad hoc* definition.

Let Z be a Stein space of pure dimension n. An open set $X \subset Z$ satisfies the *principal hypersection* condition, (PH) in short, if, for every *active* holomorphic function h on Z, the trace of its zero set on X is Stein.

Obviously, this condition is interesting only in dimensions at least three. Notice that (PH)-condition is weaker than the hypersection condition in [3]. ■

Theorem 5. *Let Z be a Stein space of pure dimension at least three. Then a domain X of Z that satisfies the* PH*-condition is Stein if and only if X has an envelope of holomorphy.*

Proof. We proceed as in the proof of Theorem 4, and for this we need X be locally Stein at every point of $\partial X \setminus Z_{\mathrm{sg}}$. But this follows by routine arguments from the following fact (cf. [14, Theorem 3]). ■

Proposition 8. *A domain $D \subset \mathbb{C}^n$ $(n \geq 3)$ is Stein if, for any point $a \in \partial D$, there is a dense subset $\Lambda(a)$ of the unit sphere in $\mathbb{C}^n$ such that $D \cap H_\lambda$ is Stein for all $\lambda \in \Lambda(a)$, where $H_\lambda = \{z \in \mathbb{C}^n : \langle z - a, \lambda \rangle = 0\}$.*

5. A few open problems

In the circle of ideas discussed in this article, we would like to state some open problems.

(1) *Is is true that, in any normal Stein surface, every domain of holomorphy is Stein?*

(2) *Let X be any complex space such that $\mathcal{O}(X)$ separates points. Does it follow that every meromorphic function on X is quotient of two holomorphic functions on X?*

This is true for X Stein (with singularities) [1], or when X is an open subset of a Stein manifold [10].

(3) *Let Z be any irreducible Stein space. Let X be an open subset of Z such that, for any principal hypersurface Y of Z, the intersection $X \cap Y$ is Stein and Runge in Y. Does it follow that Ω is Stein?*

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