Rend. Lincei Mat. Appl. 34 (2023), 159–173
DOI 10.4171/RLM/1002

Functional analysis. – *The conditional measures for the determinantal point process with the Bergman kernel*, by ALEXANDER I. BUFETOV, communicated on 10 November 2022.

ABSTRACT. – This note gives an explicit description of conditional measures for the determinantal point process with the Bergman kernel.

KEYWORDS. – Gaussian analytic function, determinantal measure, Bergman space, Palm measure, Gibbs property.

2020 MATHEMATICS SUBJECT CLASSIFICATION. – Primary 37A50; Secondary 37A60, 60G57, 46N30.

1. INTRODUCTION

1.1. Formulation of the main result

The aim of this note is to give an explicit formula (see formula (8) in Theorem 1.2) for the conditional measures of the zero set of the Gaussian analytic function under the condition that the configuration be fixed in the complement of a compact set. The conditional measure is an L-process in the sense of Borodin [1]; the kernel admits a simple explicit expression in terms of generalized Blaschke products corresponding to the fixed particles outside a compact set. Recall that, by the Peres–Virág theorem [21], the zero set of the Gaussian analytic function is a determinantal point process with the Bergman kernel; cf. (2). The main tool is the explicit representation obtained in [7] of the Radon–Nikodym derivative of the reduced Palm measure of our determinantal point process with respect to the process itself; the Radon–Nikodym derivative is found as a generalized multiplicative functional corresponding to the divergent Blaschke product over the particles of our configuration. The argument relies on the determinantal property of the process and the specific properties of the kernel, while not explicitly using the Gaussian property. The notation and conventions of this note follow [3, 7]. A detailed general introduction to Gaussian analytic functions and determinantal point processes may be found in Hough–Krishnapur–Peres–Virág [14].

We proceed to the precise formulations. Let $\mathbb{D}$ be the open unit disc. A configuration X on $\mathbb{D}$ is a subset of $\mathbb{D}$, possibly infinite, but without accumulation points in $\mathbb{D}$, or, equivalently, a purely atomic Radon measure on $\mathbb{D}$. The space $\mathrm{Conf}(\mathbb{D})$ is a complete separable metric space with respect to the vague topology on Radon measures (cf.

e.g. [3] and references therein). Let $a_n(\omega)$, $n \geq 0$, be independent standard complex Gaussian random variables, with expectation 0 and variance 1. The power series

$$\sum_{n=0}^{\infty} a_n(\omega) z^n \tag{1}$$

almost surely has radius of convergence 1; by the Peres–Virág theorem [21], the law $\mathbb{P}_K$ of the zero set of the series (1) is the determinantal measure on the space $\mathrm{Conf}(\mathbb{D})$ governed by the Bergman kernel

$$K(z, w) = \frac{1}{\pi(1 - z\bar{w})^2}, \quad z, w \in \mathbb{D}, \tag{2}$$

of orthogonal projection in the space $L_2(\mathbb{D})$ of square-integrable functions with respect to the usual Lebesgue measure onto the closed subspace of square-integrable holomorphic functions.

Consider a decomposition $\mathbb{D} = B \sqcup C$ of the unit disc into two disjoint Borel sets with B open and having compact closure in $\mathbb{D}$. The natural restriction map $\pi_C : X \mapsto X \cap C$ sends the measure $\mathbb{P}_K$ forward to its projection $\overline{\mathbb{P}_K^C}$; the $\overline{\mathbb{P}_K^C}$-almost surely defined conditional measures of $\mathbb{P}_K$ for a configuration Y on $\mathrm{Conf}(\mathbb{D})$ satisfying $Y = Y \cap C$ on the preimage $\pi_C^{-1}(Y)$ are denoted by $\mathbb{P}(\cdot|C; Y)$. [8, Lemma 1.11] states that if $\mathbb{P}$ is a determinantal point process induced by a positive Hermitian contraction, then so is its conditional measure (cf. Lyons [17] for the case of a discrete phase space); Lemma 1.11 also gives a limit procedure for finding the kernel governing the conditional measure. Our aim in this note is to give an explicit formula, see (8), for the conditional measure $\mathbb{P}(\cdot|C; Y)$. The starting point for the argument is [7, Theorem 1.4] that gives an explicit expression for the Radon–Nikodym derivative of the Palm measure $\mathbb{P}_K^q$ of $\mathbb{P}_K$ with respect to $\mathbb{P}_K$; the Radon–Nikodym derivative, cf. (11), is expressed in terms of a regularized multiplicative functional $\overline{\Psi}_q$, cf. (6), that we now write in a slightly different way. The next step is an expression of the conditional measure $\mathbb{P}(\cdot|C; Y)$ in terms of the multiplicative functional $\overline{\Psi}_q$.

We consider the unit disc as the Poincaré model for the Lobachevsky plane, and for $p \in \mathbb{D}$, $R > 0$ we let $D(p, R)$ stand for the Lobachevskian ball of radius R centered at p.

PROPOSITION 1.1. *For $\mathbb{P}_K$-almost every $X \in \mathrm{Conf}(\mathbb{D})$ and any $q \in \mathbb{D}$, the limit*

$$\widetilde{\Psi}_q(X) = \lim_{R \to \infty} \prod_{x \in X \cap D(q,R)} \left| \frac{x - q}{1 - \bar{x}q} \right|^2 \cdot \exp\left(\frac{\sqrt{-1}}{2\pi} \int_{D(q,R)} \left(1 - \left| \frac{z - q}{1 - \bar{z}q} \right|^2 \right) \frac{dz \wedge d\bar{z}}{(1 - |z|^2)^2} \right) \tag{3}$$

exists in $L_1(\mathrm{Conf}(\mathbb{D}), \mathbb{P}_K)$ as well as $\mathbb{P}_K$-almost surely along a subsequence.

We have

$$\int \widetilde{\Psi}_q(X)\, d\mathbb{P}_K(X) = \frac{e^{\gamma-1}}{2}, \tag{4}$$

where

$$\gamma = \lim_{n\to\infty}\left(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n\right) \tag{5}$$

is the Euler–Mascheroni constant.

Denote

$$\overline{\Psi}_q(X) = 2e^{1-\gamma}\widetilde{\Psi}_q(X). \tag{6}$$

In view of (4), we have

$$\overline{\Psi}_q(X) = 2e^{1-\gamma}\widetilde{\Psi}_q(X). \tag{7}$$

We are now ready to proceed to the formulation of the main result of this note, an explicit description of the conditional measures of the determinantal point process with the Bergman kernel.

THEOREM 1.2. *For $\overline{\mathbb{P}_K^C}$-almost every configuration Y on the disc, the conditional measure $\mathbb{P}_K(\cdot\,|C;Y)$ has the form*

$$\eta_{Y,0}\left(1 + \sum_{m=1}^{\infty} \det L_Y(q_j, q_k)_{j,k=1,\dots,m} \cdot \prod_{j=1}^{m}\left(\frac{\sqrt{-1}}{2\pi}\frac{dq_j \wedge d\bar{q}_j}{(1-|q_j|^2)^2}\right)\right), \tag{8}$$

where

$$L_Y(q_1, q_2) = \frac{\overline{\Psi}_{q_1}(Y)\overline{\Psi}_{q_2}(Y)}{1 - q_1\bar{q}_2} \tag{9}$$

and

$$\eta_{Y,0} = \frac{1}{\det(1 + L_Y)} = \mathbb{P}(\#_B = 0 \mid C; Y)$$

is the conditional probability that there are no particles in B.

Equivalently, the conditional measure is an L-ensemble in the sense of Borodin [1] (cf. also [2]), and the L-kernel is given by the formula (9).

REMARK. Theorem 1.2, with minimal modifications, directly applies also to determinantal point processes corresponding to Bergman spaces on the disc with respect to more general weights. Given a Bergman weight ω on the disc (cf. [7, Definition 3.2 and

Examples 6.1–6.2]), one considers the corresponding Bergman space, its reproducing kernel, and the corresponding determinantal process on the disc; and, under a technical condition, cf. [7, (3)], quite similarly to (3), one constructs the regularized Blaschke product, cf. [7, (5)]. The normalized Blaschke product plays the role of $\overline{\Psi}_q$ of (7). The formula (8) stays the same. Indeed, as we will see below, the proof of Theorem 1.2 only relies on the explicit form of the Radon–Nikodym derivative of our process with respect to its Palm measure obtained in [7, (5)] also for Bergman kernels associated to general Bergman weights. The equality (4) is of course specific to the classical Bergman kernel.

Remark. The approach of this note in reconstructing the conditional measures of a point process from the Radon–Nikodym derivatives of its Palm measures goes back to [3, 4], where, following Olshanski [19], it is proved that, for determinantal point processes governed by integrable kernels on $\mathbb{R}$, the conditional measure, subject to the condition that the configuration be fixed outside an interval, is an orthogonal polynomial ensemble. The generalization to point processes on $\mathbb{C}$ is achieved in [5–7]. Note, however, that the point processes considered in [3, 4] are rigid in the sense of Ghosh [10, 11], Ghosh and Peres [12]. The determinantal point process with the Bergman kernel is, however, insertion tolerant; cf. Holroyd and Soo [13]. The result of this note thus extends the program of [3] onto insertion tolerant processes.

Question. What are conditional measures for zero sets of more general Gaussian analytic functions? The argument in this note relies on the determinantal property and is not directly applicable. At the same time, convenient formulas for correlation functions of zero sets also exist beyond the determinantal context: for instance, in the recent paper [16], Katori and Shirai consider Gaussian Laurent series on the annulus and establish explicit formulas for the correlation functions of the corresponding random zero set as products of permanents and determinants. Is it possible to describe conditional measures in this case? Proposition 1.3 does not use the determinantal property, only the fact that the Radon–Nikodym derivative of our process with respect to its Palm measure is given by a multiplicative functional. The analysis of Katori and Shirai [16] uses the “elliptic extension of Cauchy’s evaluation of determinant due to Frobenius”; cf. [16, p. 1130] and references therein. Is it possible also to obtain an extension of the argument below for more general Gaussian analytic functions?

1.2. Palm measures of the point process $\mathbb{P}_K$

1.2.1. Correlation functions of point processes

Let E be a Polish space. A *configuration* on E is a countable or finite collection of points in E, called *particles*, considered without regard to order and subject to the additional requirement that every compact set contain only finitely many particles of a configuration. Let $\mathrm{Conf}(E)$ be the space of configurations on E. For a bounded Borel set $B \subset E$, let

$$\#_B : \mathrm{Conf}(E) \to \mathbb{N} \cup \{0\}$$

be the function that to a configuration assigns the number of its particles belonging to B. The random variables $\#_B$ over all bounded Borel sets $B \subset E$ determine the Borel sigma-algebra on $\mathrm{Conf}(E)$. A Borel probability measure $\mathbb{P}$ on $\mathrm{Conf}(E)$ is called *a point process* with phase space E. Recall that the point process $\mathbb{P}$ is said to admit correlation measures of order l if, for any continuous compactly supported function φ on E^l, the functional

$$\sum_{x_1,\dots,x_l \in X} \varphi(x_1, \dots, x_l)$$

is $\mathbb{P}$-integrable; the sum is taken over all ordered l-tuples of distinct particles in X. The l-th correlation measure ρ_l of the point process $\mathbb{P}$ is then defined by the formula

$$\mathbb{E}_{\mathbb{P}}\Big(\sum_{x_1,\dots,x_l \in X} \varphi(x_1, \dots, x_l)\Big) = \int_{E^l} \varphi(q_1, \dots, q_l) d\rho_l(q_1, \dots, q_l).$$

If all correlation measures of a point process are well defined and for any $m \in \mathbb{N}$ the m-th correlation measure is absolutely continuous with respect to m-th tensor power of the first one, then say that our point process admits correlation functions of all orders.

1.2.2. Campbell and Palm measures

Following Kallenberg [15] and Daley–Vere-Jones [9], we recall the definition of Campbell measures of point processes; the notation follows [3]. Let $\mathbb{P}$ be a point process on E admitting the first correlation measure $\rho_1^{\mathbb{P}}$. The *Campbell measure* $\mathcal{C}_{\mathbb{P}}$ of $\mathbb{P}$ is a sigma-finite measure on $E \times \mathrm{Conf}(E)$ such that for any Borel subsets $B \subset E$, $\mathcal{Z} \subset \mathrm{Conf}(E)$ we have

$$\mathcal{C}_{\mathbb{P}}(B \times \mathcal{Z}) = \int_{\mathcal{Z}} \#_B(X) d\mathbb{P}(X).$$

The Palm measure $\widehat{\mathbb{P}}^q$ is the canonical conditional measure, in the sense of Rohlin [22], of the Campbell measure $\mathcal{C}_{\mathbb{P}}$ with respect to the measurable partition of the space $E \times \mathrm{Conf}(E)$ into subsets $\{q\} \times \mathrm{Conf}(E)$, $q \in E$; cf. [3].

By definition, the Palm measure $\widehat{\mathbb{P}}^q$ is supported on the subset of configurations containing a particle at position q. Removing these particles, one defines the *reduced* Palm measure $\mathbb{P}^q$ as the push-forward of the Palm measure $\widehat{\mathbb{P}}^q$ under the erasing map $X \to X \setminus \{q\}$.

Iterating the definition, one arrives at iterated Campbell, Palm, and reduced Palm measures: the r-th *Campbell measure* $\mathcal{C}^r_{\mathbb{P}}$ of $\mathbb{P}$ is a sigma-finite measure on the product $E \times \cdots \times \mathbb{E} \times \mathrm{Conf}(E)$ of r copies of E and $\mathrm{Conf}(E)$ such that for any disjoint Borel subsets $B_1, \ldots, B_r \subset E$, $\mathcal{Z} \subset \mathrm{Conf}(E)$ we have

$$\mathcal{C}_{\mathbb{P}}(B_1 \times \cdots \times B_r \times \mathcal{Z}) = \int_{\mathcal{Z}} \#_{B_1}(X) \cdots \#_{B_r}(X) d\mathbb{P}(X).$$

Given distinct $q_1, \ldots, q_r \in E$, the Palm measure $\widehat{\mathbb{P}}^{q_1,\ldots,q_r}$ is the canonical conditional measure, in the sense of Rohlin [22], of the Campbell measure $\mathcal{C}^r_{\mathbb{P}}$ with respect to the measurable partition of the space $E \times \cdots \times E \times \mathrm{Conf}(E)$ into subsets $\{q_1, \ldots, q_r\} \times \mathrm{Conf}(E)$, $q \in E$; cf. [3]. By definition, the Palm measure $\widehat{\mathbb{P}}^{q_1,\ldots,q_r}$ is supported on the subset of configurations containing a particle at each position $q_1, \ldots, q_r$. Removing these particles, one defines the *reduced* Palm measure $\mathbb{P}^{q_1,\ldots,q_r}$ as the push-forward of the Palm measure $\widehat{\mathbb{P}}^q$ under the erasing map $X \to X \setminus \{q_1, \ldots, q_r\}$; see Kallenberg [15], whose formalism is also adopted in [3], for a more detailed exposition. As all conditional measures, reduced Palm measures $\mathbb{P}^q$ are *a priori* only defined for ρ_1-almost every q. In our context of determinantal point processes, for any distinct $q_1, \ldots, q_m \in E$, the Shirai–Takahashi theorem allows us to fix a convenient explicit Borel realization $\mathbb{P}^{q_1,\ldots,q_m}$ of reduced Palm measures.

1.2.3. Determinantal point processes

As before, let E be a Polish space, and let μ be a sigma-finite Borel measure on E. Recall that a Borel probability measure $\mathbb{P}$ on $\mathrm{Conf}(E)$ is called *determinantal* if there exists a locally trace class operator K acting in $L_2(E, \mu)$ such that for any bounded measurable function g, for which $g - 1$ is supported in a bounded set B, we have

$$\mathbb{E}_{\mathbb{P}} \prod_{x \in X} g(x) = \det\big(1 + (g-1)K\chi_B\big). \tag{10}$$

Here and elsewhere in similar formulas, 1 stands for the identity operator. The Fredholm determinant in (10) is well defined since K is locally of trace class. The equation (10) determines the measure $\mathbb{P}$ uniquely. We use the notation $\mathbb{P}_K$ for the determinantal measure induced by the operator K. By a theorem due to Macchì [18], Soshnikov [27], and Shirai–Takahashi [23] (cf. also [24, 25]), any Hermitian positive contraction that belongs to the local trace class defines a determinantal point process.

1.2.4. Generalized multiplicative functionals

Let g be a Borel function on E and let Ψ be a Borel function defined on a Borel subset $\mathcal{Z} \subset \mathrm{Conf}(E)$ and satisfying the following: if $X, Y \in \mathcal{Z}$ and there exist distinct particles

$p_1, \dots, p_r, q_1, \dots, q_s \in E$ such that

$$X \setminus \{p_1, \dots, p_r\} = Y \setminus \{q_1, \dots, q_s\},$$

then

$$\Psi(X) = \frac{g(p_1) \cdots g(p_r)}{g(q_1) \cdots g(q_s)} \Psi(Y).$$

In this case, we say that Ψ is a generalized multiplicative functional corresponding to the function g. The regularized multiplicative functional (3) is a particular case of a generalized multiplicative functional. If the point process $\mathbb{P}$ has trivial tail σ-algebra, then a generalized multiplicative functional corresponding to a function g, provided it exists, is $\mathbb{P}$-almost surely unique up to multiplication by a constant.

1.2.5. The characterization of Palm measures for $\mathbb{P}_K$

The starting point for the argument is [7, Theorem 1.4] that, in view of Proposition 1.1, can be formulated as follows: for any $q \in \mathbb{D}$, the reduced Palm measure $\mathbb{P}_K^q$ of our determinantal point process $\mathbb{P}_K$ with the Bergman kernel is given by the formula

$$\mathbb{P}_K^q = \overline{\Psi}_q \mathbb{P}_K. \tag{11}$$

In other words, the Radon–Nikodym derivative is given by a regularized multiplicative functional. The argument is completed by a general proposition describing the conditional measures for a point process whose Palm measures are expressed as a product of the original measure and a multiplicative functional.

1.3. Palm measures, multiplicative functionals and conditional measures

Let $\mathbb{P}$ be a point process with phase space E admitting correlation functions of all orders. We fix a Borel realization $\mathbb{P}^q$ of its reduced Palm measures and assume that there exists a symmetric positive Borel function $B(q_1, q_2)$, $q_1, q_2 \in E$, defined on $E \times E$ and such that for any $q \in E$ the Radon–Nikodym derivative $d\mathbb{P}^q / d\mathbb{P}$ is a generalized multiplicative functional corresponding to the function $B(q, \cdot)$.

PROPOSITION 1.3. *For any decomposition $E = B \sqcup C$ into two Borel sets with $\rho_1^{\mathbb{P}}(B) < +\infty$ and for $\overline{\mathbb{P}_C}$-almost every $Y \in \mathrm{Conf}(E; C)$, the conditional measure $\mathbb{P}_Y = \mathbb{P}(\cdot | Y, C)$ has the form*

$$\eta_{Y,0} \left(1 + \sum_{m=1}^{\infty} \prod_{1 \le i < j \le m} B(q_i, q_j) \cdot \prod_{i=1}^{m} \frac{d\mathbb{P}^{q_i}}{d\mathbb{P}}(Y) \, d\rho_1(q_i) \right),$$

where $\eta_{Y,0} = \mathbb{P}(\#_B(X) = 0 | Y; C)$.

Proposition 1.3, together with Proposition 1.1, directly implies Theorem 1.2 in view of the Cauchy identity

$$\det\left(\frac{1}{1-q_j\bar{q}_k}\right)_{j,k=1,\dots,n} = \frac{\prod_{1\le j<k\le n}|q_j-q_k|^2}{\prod_{1\le j<k\le n}|1-q_j\bar{q}_k|^2}.$$

It remains to prove Propositions 1.1 and 1.3.

2. Proof of Proposition 1.3

2.1. Conditional measures of point processes

Let E be a locally compact complete metric space, let $\mathrm{Conf}(E)$ be the space of configurations on E. Given a configuration $X \in \mathrm{Conf}(E)$ and a subset $C \subset E$, we let $X|_C$ stand for the restriction of X onto the subset C. We let $\overline{\mathbb{P}}^C$ be the push-forward measure under the natural projection $X \to X|_C$. Given a point process on E, that is, a Borel probability measure $\mathbb{P}$ on $\mathrm{Conf}(E)$, the measure $\mathbb{P}(\cdot|X;C)$ on $\mathrm{Conf}(E \setminus C)$ is defined as the conditional measure of $\mathbb{P}$ with respect to the condition that the restriction of our random configuration onto C coincides with $X|_C$. More formally, we consider the surjective restriction mapping $X \to X|_C$ from $\mathrm{Conf}(E)$ to $\mathrm{Conf}(C)$. Fibers of this mapping can be identified with $\mathrm{Conf}(E\setminus C)$ and conditional measures, in the sense of Rohlin [22], are precisely the measures $\mathbb{P}(\cdot|X;C)$. Let $\mathrm{Conf}(E;C)$ be the subset of those configurations on E all whose particles lie in C; in other words, the image of the natural projection $X \to X|_C$. By definition, we have

$$\mathbb{P} = \int_{\mathrm{Conf}(E;C)} \mathbb{P}(\cdot|Y;C)\,d\overline{\mathbb{P}}^C(Y).$$

The decomposition into conditional measures is by definition lifted onto the level of Campbell measures:

$$\mathfrak{C}_{\mathbb{P}} = \int_{\mathrm{Conf}(E;C)} \mathfrak{C}_{\mathbb{P}(\cdot|Y;C)}\,d\overline{\mathbb{P}}^C(Y). \tag{12}$$

2.2. Palm measures of different orders

As before, let $\mathbb{P}$ be a point process on a Polish space E. We assume that the point process $\mathbb{P}$ admits correlation functions of all orders and that the reduced Palm measures of $\mathbb{P}$ are almost surely absolutely continuous with respect to $\mathbb{P}$; it follows that reduced Palm measures of all orders are also almost surely absolutely continuous with respect to $\mathbb{P}$; almost surely is here understood with respect to the first correlation measure.

It is convenient to think that the space E is endowed with a sigma-finite Borel measure μ such that the first correlation measure of $\mathbb{P}$ is absolutely continuous with respect to μ; the m-th correlation measure of $\mathbb{P}$ then has the form

$$\rho_m(q_1, \ldots, q_m) d\mu(q_1) \cdots d\mu(q_m),$$

where ρ_m is the m-th correlation function.

For μ-almost any distinct points $p_1, \ldots, p_m, q_1, \ldots, q_r$ and $\mathbb{P}$-almost any configuration $X \in \mathrm{Conf}(E)$ not containing any of the points $p_1, \ldots, p_m, q_1, \ldots, q_r$ the following identity directly follows from the definition of the Palm measures:

$$\frac{\rho_{m+r}(p_1, \ldots, p_m, q_1, \ldots, q_r)}{\rho_r(q_1, \ldots, q_r)} \cdot \frac{d\mathbb{P}^{p_1,\ldots,p_m,q_1,\ldots,q_r}}{d\mathbb{P}^{q_1,\ldots,q_r}}(X) = \rho_m(p_1, \ldots, p_m) \frac{d\mathbb{P}^{p_1,\ldots,p_m}}{d\mathbb{P}}(X, q_1, \ldots, q_r). \tag{13}$$

2.3. *Palm measures and conditional measures*

As before, we consider a point process $\mathbb{P}$ on the phase space E; the point process $\mathbb{P}$ is assumed to admit correlation functions of all orders. Consider a decomposition $E = B \sqcup C$ of our phase space E as a disjoint union of two Borel sets. As before, we let $\overline{\mathbb{P}}^C$ be the push-forward measure under the natural projection $X \to X|_C$, and, for a configuration Y all whose particles lie in C, in this subsection, we write $\mathbb{P}_{[Y,C]} = \mathbb{P}(\cdot\,|Y; C)$. We take a natural m and let $\mathbb{P}$ be a point process whose reduced Palm measures of order m are $\rho_m^{\mathbb{P}}$-almost surely absolutely continuous with respect to $\mathbb{P}$.

From (12), it follows that, $\overline{\mathbb{P}}^C$-almost surely, the m-th correlation measure of the measure $\mathbb{P}_{[Y,C]}$ is absolutely continuous with respect to the m-th tensor power of the first correlation measure of $\mathbb{P}$ (note that for determinantal point processes governed by Hermitian contractions this requirement is automatically verified by [8, Lemma 1.11] on the preservation of the determinantal property under taking conditional measures; observe, however, that Lemma 1.11 is not used in this derivation of the explicit form of the conditional measures).

Let $\rho_{[Y,C],m}$ be the m-th correlation function of $\mathbb{P}_{[Y,C]}$. By definition, we have

$$\frac{\rho_{[Y,C],m}(q_1, \ldots, q_m)\, d\mathbb{P}_{[Y,C]}^{q_1,\ldots,q_m}}{d\mathbb{P}_{[Y,C]}}(Z) = \frac{\rho_m(q_1, \ldots, q_m) d\mathbb{P}^{q_1,\ldots,q_m}}{d\mathbb{P}}(Y \cup Z). \tag{14}$$

We now directly obtain the following.

COROLLARY 2.1. *Assume that the first correlation measure of the set B is finite. Then for $\overline{\mathbb{P}}^C$-almost any configuration Y, the conditional measure $\mathbb{P}_{[Y,C]}$ has the form*

$$\sum_{m=0}^{\infty} \eta_{[Y,C],m}(q_1, \ldots, q_m)\, d\mu(q_1) \cdots d\mu(q_m), \tag{15}$$

where $\eta_{[Y,C],0} = \mathbb{P}_{[Y,C]}(\varnothing)$ *is the conditional probability of the absence of particles in* B *and*

$$\eta_{[Y,C],m}(q_1,\dots,q_m) = \eta_{[Y,C],0}\cdot \frac{\rho_m(q_1,\dots,q_m)d\mathbb{P}^{q_1,\dots,q_m}}{d\mathbb{P}}(Y). \tag{16}$$

Proof. That the conditional measure $\mathbb{P}_{[Y,C]}$ is absolutely continuous with respect to the Poisson process of intensity μ follows from the fact that the measure $\mathbb{P}_{[Y,C]}$ is, $\overline{\mathbb{P}}_C$-almost surely, supported on the set of configurations with finitely many particles and that the m-th correlation measure of the measure $\mathbb{P}_{[Y,C]}$ is absolutely continuous with respect to the m-th tensor power of the first correlation measure of $\mathbb{P}$.

Now take a general measure of the form (15) for some Borel functions $\eta_{[Y,C],m}$, $m = 1,\dots$; the r-th Palm measure of our measure at the particles $p_1,\dots,p_r$ takes the form

$$M^{-1}(p_1,\dots,p_r)\sum_{m=0}^{\infty}\eta_{[Y,C],m+r}(p_1,\dots,p_r,q_1,\dots,q_m)\,d\mu(q_1)\cdots d\mu(q_m),$$

where $M(p_1,\dots,p_r)$ is a normalization constant.

We now write (14) with $Z=\varnothing$. By definition,

$$\rho_{[Y,C],m}(q_1,\dots,q_m)\cdot \mathbb{P}_{[Y,C]}^{q_1,\dots,q_m}(\varnothing) = \eta_{[Y,C],m}(q_1,\dots,q_m),$$

and the desired equality (16) follows. ∎

2.4. *Conclusion of the proof of Proposition 1.3*

Iterating (13), we arrive at the identity

$$\rho_m(p_1,\dots,p_m)\frac{d\mathbb{P}^{p_1,\dots,p_m}}{d\mathbb{P}}(X) =$$
$$\rho_1(p_1)\frac{d\mathbb{P}^{p_1}}{d\mathbb{P}}(X,p_2,\dots,p_m)\cdots\rho_1(p_{m-1})\frac{d\mathbb{P}^{p_{m-1}}}{d\mathbb{P}}(X,p_m)\cdot\rho_1(p_m)\frac{d\mathbb{P}^{p_m}}{d\mathbb{P}}(X).$$

Combining with (16), we obtain the expression

$$\eta_{Y,m}(q_1,\dots,q_m)$$
$$=\eta_{Y,0}\cdot\rho_1(q_1)\cdots\rho_m(q_m)\cdot\frac{d\mathbb{P}^{q_1}}{d\mathbb{P}}(Y)\cdot\frac{d\mathbb{P}^{q_2}}{d\mathbb{P}}(q_1,Y)\cdots\frac{d\mathbb{P}^{q_m}}{d\mathbb{P}}(q_1,\dots,q_{m-1},Y).$$

Substituting the expression of the Radon–Nikodym derivative as the multiplicative functional completes the proof. ∎

3. Proof of Proposition 1.1

Recall, cf. e.g. Simon [26], that the Hilbert–Carleman regularization $\det_2$ of the Fredholm determinant is introduced on finite rank operators by the formula

$$\det_2(1+A) = \exp(-\operatorname{tr} A)\det(1+A)$$

and then extended by continuity onto the space of Hilbert–Schmidt operators.

We first recall a well-known observation (cf. e.g. Osada–Shirai [20, p. 737]). Let

$$g\colon \mathbb{D} \to \mathbb{R}_+$$

be a nonnegative bounded Borel radial function, in other words, a function depending only on the absolute value of its argument. The eigenfunctions of the operator $\sqrt{g}K\sqrt{g}$ are precisely the functions $\sqrt{g}z^k$, $k \geq 0$, and the corresponding eigenvalue is

$$\frac{k+1}{\pi}\int_{\mathbb{D}} g(z)|z|^{2k}\,dz = (k+1)\int_0^1 \tilde{g}(\rho)\rho^k\,d\rho,$$

where $\tilde{g}(\rho) = g(\sqrt{\rho}e^{i\theta})$ for any θ.

It directly follows that the operator

$$K_1(z,w) = \sqrt{1-|z|^2}K(z,w)\sqrt{1-|w|^2}$$

is Hilbert–Schmidt, as its eigenvalues are

$$(k+1)\int_0^1 (1-\rho)\rho^k\,d\rho = \frac{1}{k+2}, \quad k = 0,1,\ldots.$$

By definition (3), we have

$$\mathbb{E}_{\mathbb{P}_K}\widetilde{\Psi} = \det_2(1+K_1). \tag{17}$$

Note here that

$$\frac{1}{\pi\bigl(1-|z|^2\bigr)} = \bigl(1-|z|^2\bigr)K(z,z).$$

For $r \in (0,1)$, set

$$\widetilde{\Psi}_r(X) = \prod_{x\in X, |x|<r} |x|^2 \cdot \exp\left(\frac{\sqrt{-1}}{2\pi}\int_{\{z:|z|<r\}} \frac{dz\wedge d\bar{z}}{(1-|z|^2)}\right).$$

By definition, we have

$$\mathbb{E}_{\mathbb{P}_K}\widetilde{\Psi}_r = \det_2(1+\chi_{\{z:|z|<r\}}K_1\chi_{\{z:|z|<r\}}).$$

Since

$$\chi_{\{z:|z|<r\}} K_1 \chi_{\{z:|z|<r\}} \to K_1$$

in the Hilbert–Schmidt norm as $r \to 1$, writing the Cauchy–Bunyakovsky–Schwarz inequality

$$\mathbb{E}_{\mathbb{P}_K} |\widetilde{\Psi} - \widetilde{\Psi}_r| \leq \sqrt{\mathbb{E}_{\mathbb{P}_K} |\widetilde{\Psi}_r|^2} \sqrt{\mathbb{E}_{\mathbb{P}_K} |\widetilde{\Psi} \widetilde{\Psi}_r^{-1} - 1|^2}$$

one directly checks the relation

$$\lim_{r\to 1} \mathbb{E}_{\mathbb{P}_K} \left| \widetilde{\Psi} - \widetilde{\Psi}_r \right| = 0,$$

which is to say that the limit

$$\widetilde{\Psi} = \lim_{r\to 1} \widetilde{\Psi}_r$$

exists in $L_1(\mathrm{Conf}(\mathbb{D}), \mathbb{P}_K)$ as well as almost surely along a subsequence.

We now compute the right-hand side of (17). In order to do so, we let $K^{(n)}$ be the orthogonal projection onto $\{1, z, \ldots, z^n\}$ in $L_2(\mathbb{D})$ and write

$$K_1^{(n)} = \sqrt{1 - |z|^2} K^{(n)} \sqrt{1 - |w|^2}.$$

We have $K_1^{(n)} \to K_1$ in the Hilbert–Schmidt norm as $n \to \infty$, whence

$$\det{}_2(1 + K_1) = \lim_{n\to\infty} \det(1 + K_1^{(n)}) \times \exp(-\operatorname{tr} K_1^{(n)}).$$

By definition, we have

$$\det(1 + K_1^{(n)}) = \prod_{k=0}^{n} \left(1 + \frac{1}{k+2}\right) = \frac{n+3}{2}.$$

and

$$\operatorname{tr}(K_1^{(n)}) = \frac{1}{2} + \cdots + \frac{1}{n+1}.$$

Summing up, cf. (5), we obtain

$$\det{}_2(1 + K_1) = \frac{e^{\gamma - 1}}{2}.$$

We, therefore, obtain an alternative representation of the Palm measure $\mathbb{P}_K^0$ with respect to the original measure:

$$\frac{d\mathbb{P}_K^0}{d\mathbb{P}}(X) = \frac{e^{\gamma-1}}{2} \lim_{r\to 1} \prod_{x\in X:|x|<r} |x|^2 \exp\left(\frac{\sqrt{-1}}{2\pi} \int_{\{z:|z|<r\}} \frac{dz \wedge d\bar{z}}{1 - |z|^2}\right).$$

Setting $D(z, R)$ to be the Lobachevskian ball centered at z and of Lobachevskian radius R, for any $q \in \mathbb{D}$ we rewrite

$$\frac{d\mathbb{P}_K^q}{d\mathbb{P}}(X) = \frac{e^{\gamma-1}}{2} \lim_{R\to\infty} \prod_{x\in D(q,R)\cap X} \left|\frac{x-q}{1-\bar{q}x}\right|^2$$
$$\times \exp\left(\frac{\sqrt{-1}}{2}\int_{D(q,R)} \left(1-\left|\frac{z-q}{1-\bar{q}z}\right|^2\right)\right) K(z,z)\, dz \wedge d\bar{z}.$$

Proposition 1.1 is proved. Theorem 1.2 is proved completely. ■

Acknowledgments. – I am deeply grateful to Dmitrii Khliustov, Alexey Klimenko, and Yanqi Qiu for useful discussions. I am deeply grateful to the referee for the very helpful suggestions. Part of this work was done during a visit to the Scuola Internazionale Superiore di Studi Avanzati in Trieste and to the Alma Mater Studiorum - University of Bologna. I am deeply grateful to both institutions for their warm hospitality.

Funding. – This research has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme, grant agreement No 647133 (ICHAOS), as well as from the ANR grant ANR-18-CE40-0035 REPKA. The author is supported by the grant 075-15-2021-602 of the Government of the Russian Federation.

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Received 5 January 2022,
and in revised form 14 July 2022

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