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Functional Analysis, Integral Transforms. – *Integral operators in Hölder spaces on upper Ahlfors regular sets*, by Massimo Lanza de Cristoforis, communicated on 10 November 2022.

Abstract. – Volume and layer potentials are integrals on a subset Y of the Euclidean space $\mathbb{R}^n$ that depend on a variable in a subset X of $\mathbb{R}^n$. Here we present a unified approach to some results by assuming that X and Y are subsets of a metric space M and that Y is equipped with a measure ν that satisfies upper Ahlfors growth conditions that include non-doubling measures. We prove continuity statements in the frame of (generalized) Hölder spaces upon variation both of the density functions on Y and of the off-diagonal potential kernel and $T1$ theorems that generalize corresponding results of J. García-Cuerva and A. E. Gatto in case $X = Y$ for kernels that include the standard ones.

Keywords. – Non-doubling measures, metric spaces, Hölder spaces, singular and weakly singular integrals, potential theory in metric spaces.

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1. Introduction

Volume and layer potentials are integrals on a subset Y of the Euclidean space $\mathbb{R}^n$ that depend on a variable in a subset X of $\mathbb{R}^n$. Typically, X and Y are either measurable subsets of $\mathbb{R}^n$ with the n-dimensional Lebesgue measure, or manifolds imbedded in $\mathbb{R}^n$, or boundaries of open subsets of $\mathbb{R}^n$ with the surface measure and X may well be different from Y. Here we present a unified approach to some results by assuming that X and Y are subsets of a metric space (M, d) and that Y is equipped with a measure ν that satisfies upper Ahlfors growth conditions that include non-doubling measures introduced below. Let (M, d) be a metric space and let X, Y be subsets of M.

$$\text{Let } \mathcal{N} \text{ be a } \sigma\text{-algebra of parts of } Y,\ \mathcal{B}_Y \subseteq \mathcal{N}.$$
$$\text{Let } \nu \text{ be a measure on } \mathcal{N}. \tag{1.1}$$
$$\text{Let } \nu(B(x,r) \cap Y) < +\infty \quad \forall (x,r) \in X \times]0,+\infty[.$$

Here $\mathcal{B}_Y$ denotes the σ-algebra of the Borel subsets of Y and

$$B(\xi, r) \equiv \{\eta \in M : d(\xi,\eta) < r\}, \quad B(\xi, r] \equiv \{\eta \in M : d(\xi,\eta) \le r\}, \tag{1.2}$$

for all $(\xi, r) \in M \times]0, +\infty[$. We plan to consider continuous off-diagonal kernels K from $(X \times Y) \setminus D_{X \times Y}$ to $\mathbb{C}$, where $D_{X \times Y}$ denotes the diagonal set $\{(x, y) \in X \times Y : x = y\}$ and formulate reasonable assumptions so that the integral operators defined by

$$\int_{Y \setminus \{x\}} K(x, y)\varphi(y)\, d\nu(y) \quad \forall x \in X \tag{1.3}$$

are bounded from a space $C_b^{0,\beta}(Y)$ for some $\beta \in]0, 1]$ of bounded Hölder continuous functions on Y to a space of (generalized) Hölder continuous functions on X (see the appendix for the Hölder spaces). In particular, we plan to extend the work of García-Cuerva and Gatto [10, 11], Gatto [13] who have considered standard kernels in case $X = Y = M$ and proved $T1$ theorems.

We assume that $\upsilon_Y \in]0, +\infty[$ and we consider two types of assumptions on ν. The first assumption is that Y is upper υ_Y-Ahlfors regular with respect to X, i.e., that

(1.4) there exist $r_{X,Y,\upsilon_Y} \in]0, +\infty]$, $c_{X,Y,\upsilon_Y} \in]0, +\infty[$ such that
$\nu(B(x, r) \cap Y) \leq c_{X,Y,\upsilon_Y} r^{\upsilon_Y}$
for all $x \in X$ and $r \in]0, r_{X,Y,\upsilon_Y}[$.

In case $X = Y$, we just say that Y is upper υ_Y-Ahlfors regular and this is the assumption that has been considered by García-Cuerva and Gatto [10, 11], Gatto [12, 13] in case $X = Y$. See also Edmunds, Kokilashvili, and Meskhi [7, Chap. 6] in the frame of Lebsgue spaces.

Then we consider a stronger version of the upper Ahlfors regularity. Namely, we assume that Y is strongly upper υ_Y-Ahlfors regular with respect to X, i.e., that

(1.5) there exist $r_{X,Y,\upsilon_Y} \in]0, +\infty]$, $c_{X,Y,\upsilon_Y} \in]0, +\infty[$ such that
$\nu\big((B(x, r_2) \setminus B(x, r_1)) \cap Y\big) \leq c_{X,Y,\upsilon_Y} (r_2^{\upsilon_Y} - r_1^{\upsilon_Y})$
for all $x \in X$ and $r_1, r_2 \in [0, r_{X,Y,\upsilon_Y}[$ with $r_1 < r_2$,

where we understand that $B(x, 0) \equiv \emptyset$ (in case $X = Y$, we just say that Y is strongly upper υ_Y-Ahlfors regular). So, for example, if Y is the boundary of an open Lipschitz bounded subset of $M = \mathbb{R}^n$ and ν is the usual $(n-1)$-dimensional measure, then Y is upper $(n-1)$-Ahlfors regular with respect to $\mathbb{R}^n$ and if Y is the boundary of an open bounded subset of $M = \mathbb{R}^n$ of class C^1, then Y is strongly upper $(n-1)$-Ahlfors regular with respect to Y. The condition (1.5) of strong upper υ_Y-Ahlfors regularity reveals to be useful in the analysis of limiting exponents.

Here we note that both the conditions above of upper Ahlfors regularity include cases in which ν does not satisfy a doubling condition. We note that Dyn'kin [5] (see also Dyn'kin [6]) has considered the strong upper Ahlfors regularity condition (1.5) in case $X = Y$ is a curve in $M = \mathbb{R}^2$ and for the specific choice $r_2 = 5r$, $r_1 = 1r$ for

$r \in]0, +\infty[$. The author is not aware of other references on condition (1.5). We plan to consider "potential type" kernels as in the following definition (see also [4]).

DEFINITION 1.6. Let $X, Y \subseteq M$. Let $s_1, s_2, s_3 \in \mathbb{R}$. We denote by $\mathcal{K}_{s_1,s_2,s_3}(X \times Y)$ the set of continuous functions K from $(X \times Y) \setminus D_{X\times Y}$ to $\mathbb{C}$ such that

$$\|K\|_{\mathcal{K}_{s_1,s_2,s_3}(X\times Y)} \equiv \sup\left\{d(x,y)^{s_1}\left|K(x,y)\right| : (x,y) \in X \times Y,\ x \neq y\right\} + \sup\left\{\frac{d(x',y)^{s_2}}{d(x',x'')^{s_3}}\left|K(x',y) - K(x'',y)\right| : x', x'' \in X,\ x' \neq x'',\ y \in Y \setminus B\big(x', 2d(x',x'')\big)\right\} < +\infty.$$

For $s_2 = s_1 + s_3$, one has the so-called class of standard kernels that is the case in which García-Cuerva and Gatto [10, 11] and Gatto [13] have proved $T1$ theorems for the integral operators with kernel K in case of weakly singular, singular, and hyper-singular integral operators with $X = Y$.

Here we extend some of those results also to case $s_2 \neq s_1 + s_3$, and for certain exponents we assume the above strong upper υ_Y-Ahlfors regularity condition to deal with certain limiting cases.

More precisely, we prove Proposition 5.2 on the dependence of the integral in (1.3) in a generalized Hölder space upon variation both of the kernel K in the class of kernels of Definition 1.6 and of the function φ in $L^\infty_\nu(Y)$. Here we mention that in the critical case $s_2 = \upsilon_Y$, we have to resort to the condition (1.5) of strong upper υ_Y-Ahlfors regularity and that the target space of the integral operator is a generalized Hölder space.

We prove the (generalized) Hölder inequality of Proposition 5.11 that implies the validity of the $T1$ Theorem 5.17, that in turn implies the continuity of the integral in (1.3) upon variation both of the kernel K in a subclass of the class of kernels of Definition 1.6 and of the function φ in the Hölder space $C^{0,\beta}_b(Y)$. Here we mention that in the critical case $s_2 = \upsilon_Y + \beta$ we have to resort to the condition (1.5) of strong upper υ_Y-Ahlfors regularity and that the target space of the integral operator is a generalized Hölder space.

In Proposition 6.3, we prove the continuity for integral operators with kernels of the form $Z(x,y)(g(y) - g(x))$, where Z is singular or hypersingular and g is a β-Hölder continuous function on $X \cup Y$. Such operators find application in the proof of the boundary behavior of the double layer potential and in particular of the tangential gradient of the double layer potential (cf., e.g., Colton and Kress [2, p. 56], Dondi and the author [4, §8], and Dalla Riva, Musolino, and the author [3, Thm. 4.35]).

Finally, we prove the $T1$ theorem of Proposition 7.5 and the corresponding continuity Theorem 7.12 for the integral operator of (1.3) upon perturbation of both the kernel K and the density φ in the singular case $s_1 = \upsilon_Y$. Here we mention that in the critical case $s_2 = \upsilon_Y + \beta$ we have resorted to the condition (1.5) of strong upper υ_Y-Ahlfors regularity.

In most of the literature, potentials and corresponding applications have been considered in case $M = \mathbb{R}^n$, and Y is a subset of $\mathbb{R}^n$ with $\upsilon_Y = (n-1)$ for layer potentials and with $\upsilon_Y = n$ for volume potentials. Far less seems to have been developed in case $\upsilon_Y < (n-1)$ (cf. Selvaggi and Sisto [22]) and the results above, as well as the above-mentioned results of García-Cuerva and Gatto offer a theoretical basis for case $\upsilon_Y < (n-1)$.

2. Preliminaries on weakly singular integral operators

An off-diagonal function in $X \times Y$ is a function from $(X \times Y) \setminus D_{X\times Y}$ to $\mathbb{C}$. We now wish to consider a specific class of off-diagonal kernels.

Definition 2.1. Let X and Y be subsets of M. Let $s \in \mathbb{R}$. We denote by $\mathcal{K}_{s,X\times Y}$ the set of continuous functions K from $(X \times Y) \setminus D_{X\times Y}$ to $\mathbb{C}$ such that

$$\|K\|_{\mathcal{K}_{s,X\times Y}} \equiv \sup_{(x,y)\in(X\times Y)\setminus D_{X\times Y}} \left|K(x,y)\right| d(x,y)^s < +\infty.$$

The elements of $\mathcal{K}_{s,X\times Y}$ are said to be kernels of potential type s in $X \times Y$.

We now introduce the space

$$B(X) \equiv \{f \in \mathbb{C}^X : f \text{ is bounded}\}, \quad \|f\|_{B(X)} \equiv \sup_X |f| \quad \forall f \in B(X)$$

of bounded functions in X with the sup-norm. By the Hölder inequality, one can prove the following (see also Prössdorf [21, p. 49]).

Theorem 2.2 (Of Hille–Tamarkin for potential operators). *Let X, Y be subsets of M. Let ν be as in (1.1). Let $s \in \mathbb{R}$. Let $d(x,\cdot)^{-s}$ belong to $L^1_\nu(Y \setminus \{x\})$ for all $x \in X$. Let*

$$\sup_{x\in X} \int_{Y\setminus\{x\}} d(x,y)^{-s}\, d\nu(y) < +\infty.$$

If $(K,\varphi) \in \mathcal{K}_{s,X\times Y} \times L^\infty_\nu(Y)$, then the function $A[K,\varphi]$ from X to $\mathbb{C}$ defined by

$$A[K,\varphi](x) \equiv \int_{Y\setminus\{x\}} K(x,y)\varphi(y)\, d\nu(y) \quad \forall x \in X$$

belongs to $B(X)$. *Moreover, the bilinear map from* $\mathcal{K}_{s,X\times Y}\times L^\infty_\nu(Y)$ *to* $B(X)$, *which takes* (K,φ) *to* $A[K,\varphi]$ *is continuous and*

$$\|A[K,\varphi]\|_{B(X)}\le\sup_{x\in X}\int_{Y\setminus\{x\}}d(x,y)^{-s}\,d\nu(y)\|K\|_{\mathcal{K}_{s,X\times Y}}\|\varphi\|_{L^\infty_\nu(Y)}$$

for all $(K,\varphi)\in\mathcal{K}_{s,X\times Y}\times L^\infty_\nu(Y)$.

3. Integrability of the function $d(x,\cdot)^{-s}$

In this section, we analyze the integrability of $d(x,y)^{-s}$, both in case Y is upper υ_Y-Ahlfors regular as in Gatto's work [13, p. 104] and in case Y is strongly upper υ_Y-Ahlfors regular. The proofs below are based on the use of the distribution function (while those of Gatto [13, p. 104] are based on a dyadic decomposition).

Lemma 3.1. *Let* $X, Y\subseteq M$. *Let* ν *be as in* (1.1). *Let* $s\in\,]0,+\infty[$. *Then*

$$\begin{aligned}\int_{(B(x,r_2)\setminus B(x,r_1))\cap(Y\setminus\{x\})}&d(x,y)^{-s}\,d\nu(y)\\&=s\int_0^{r_2^{-1}}t^{s-1}\nu\big((Y\setminus\{x\})\cap\big(B(x,r_2)\setminus B(x,r_1)\big)\big)\,dt\\&\quad+s\int_{r_2^{-1}}^{r_1^{-1}}t^{s-1}\nu\big((Y\setminus\{x\})\cap\big(B(x,t^{-1})\setminus B(x,r_1)\big)\big)\,dt\end{aligned}$$

and

$$\begin{aligned}\int_{(B(x,r_2)\setminus B(x,r_1))\cap(Y\setminus\{x\})}&d(x,y)^{s}\,d\nu(y)\\&=s\int_0^{r_1}t^{s-1}\nu\big((Y\setminus\{x\})\cap\big(B(x,r_2)\setminus B(x,r_1)\big)\big)\,dt\\&\quad+s\int_{r_1}^{r_2}t^{s-1}\nu\big((Y\setminus\{x\})\cap\big(B(x,r_2)\setminus B(x,t]\big)\big)\,dt\end{aligned}$$

for all $x\in X$ *and* $r_1,r_2\in[0,+\infty]$ *with* $r_1<r_2$, *where we understand that* $r_1^{-1}\equiv+\infty$ *and* $B(x,r_1)\equiv\emptyset$ *if* $r_1=0$ *and that* $r_2^{-1}\equiv 0$ *and* $B(x,r_2)\equiv M$ *if* $r_2=+\infty$ *(see* (1.2) *for the definition of* $B(x,t]$*).*

Proof. We first consider the first equality of the statement. Let $x\in X$, $r_1,r_2\in[0,+\infty]$ with $r_1<r_2$. Since the function $d(x,\cdot)^{-1}$ is continuous in $Y\setminus\{x\}$, a known result of real analysis implies that

$$\int_{(B(x,r_2)\setminus B(x,r_1))\cap(Y\setminus\{x\})}d(x,y)^{-s}\,d\nu(y)=s\int_0^{+\infty}t^{s-1}m_{d(x,\cdot)^{-1}}(t)\,dt,$$

where

$$m_{d(x,\cdot)^{-1}}(t) \equiv \nu\big(\{y \in \big(B(x,r_2) \setminus B(x,r_1)\big) \cap \big(Y \setminus \{x\}\big) : d(x,y)^{-1} > t\}\big)$$

for all $t \in [0,+\infty[$ is the distribution function associated to $d(x,\cdot)^{-1}$ (cf., e.g., Folland [9, Prop. 6.24]). Next we note that

$$\begin{aligned} m_{d(x,\cdot)^{-1}}(t) &= \nu\big(\{y \in \big(B(x,r_2) \setminus B(x,r_1)\big) \cap \big(Y \setminus \{x\}\big) : d(x,y)^{-1} > t\}\big) \\ &= \nu\big(\{y \in \big(B(x,r_2) \setminus B(x,r_1)\big) \cap \big(Y \setminus \{x\}\big) : d(x,y) < t^{-1}\}\big) \\ &= \nu\big(\big(\big(Y \setminus \{x\}\big) \cap B(x,t^{-1})\big) \cap \big(B(x,r_2) \setminus B(x,r_1)\big)\big). \end{aligned}$$

We also note that if $t^{-1} \geq r_2$, i.e., $t \leq r_2^{-1}$, then

$$B(x,t^{-1}) \cap \big(B(x,r_2) \setminus B(x,r_1)\big) = B(x,r_2) \setminus B(x,r_1),$$

and that in case $r_1 > 0$ if $0 < t^{-1} \leq r_1$, i.e., $t \geq r_1^{-1}$, then

$$B(x,t^{-1}) \cap \big(B(x,r_2) \setminus B(x,r_1)\big) = \emptyset,$$

and that if $r_1 < t^{-1} < r_2$, i.e., $r_2^{-1} < t < r_1^{-1}$ with the usual understanding if $r_1 = 0$ or if $r_2 = +\infty$, then

$$B(x,t^{-1}) \cap \big(B(x,r_2) \setminus B(x,r_1)\big) = B(x,t^{-1}) \setminus B(x,r_1).$$

Then we have

$$\begin{aligned} & s\int_0^{+\infty} t^{s-1} m_{d(x,\cdot)^{-1}}(t)\,dt \\ &= s\int_0^{+\infty} t^{s-1}\nu\big(\big(\big(Y \setminus \{x\}\big) \cap B(x,t^{-1})\big) \cap \big(B(x,r_2) \setminus B(x,r_1)\big)\big)\,dt \\ &= s\int_0^{r_2^{-1}} t^{s-1}\nu\big(\big(Y \setminus \{x\}\big) \cap \big(B(x,r_2) \setminus B(x,r_1)\big)\big)\,dt \\ &\quad + s\int_{r_2^{-1}}^{r_1^{-1}} t^{s-1}\nu\big(\big(Y \setminus \{x\}\big) \cap \big(B(x,t^{-1}) \setminus B(x,r_1)\big)\big)\,dt + s\int_{r_1^{-1}}^{+\infty} t^{s-1}\nu(\emptyset)\,dt, \end{aligned}$$

where the first addendum in the right-hand side is absent if $r_2 = +\infty$ and the last addendum in the right-hand side is absent if $r_1 = 0$ and is equal to zero in case $r_1 > 0$.

We now consider the second equality of the statement. Let $x \in X$, $r_1, r_2 \in [0,+\infty]$ with $r_1 < r_2$. Since the function $d(x,\cdot)$ is continuous in $Y \setminus \{x\}$, we have

$$\int_{(B(x,r_2)\setminus B(x,r_1))\cap(Y\setminus\{x\})} d(x,y)^s\,d\nu(y) = s\int_0^{+\infty} t^{s-1} m_{d(x,\cdot)}(t)\,dt,$$

where

$$m_{d(x,\cdot)}(t) \equiv \nu\big(\{y \in \big(B(x,r_2) \setminus B(x,r_1)\big) \cap \big(Y \setminus \{x\}\big) : d(x,y) > t\}\big)$$

for all $t \in [0,+\infty[$ is the distribution function associated to $d(x,\cdot)$. Next we note that if $t \in]0,r_1[$, then

$$\begin{aligned} m_{d(x,\cdot)}(t) &= \nu\big(\{y \in \big(B(x,r_2) \setminus B(x,r_1)\big) \cap \big(Y \setminus \{x\}\big) : d(x,y) > t\}\big) \\ &= \nu\big(\big(B(x,r_2) \setminus B(x,r_1)\big) \cap \big(Y \setminus \{x\}\big)\big). \end{aligned}$$

We also note that if $t \in [r_1,r_2]$, then

$$\begin{aligned} m_{d(x,\cdot)}(t) &= \nu\big(\{y \in \big(B(x,r_2) \setminus B(x,r_1)\big) \cap \big(Y \setminus \{x\}\big) : d(x,y) > t\}\big) \\ &= \nu\big(\big(B(x,r_2) \setminus B(x,t]\big) \cap \big(Y \setminus \{x\}\big)\big) \end{aligned}$$

and that if $t \in]r_2,+\infty[$, we have

$$m_{d(x,\cdot)}(t) = \nu\big(\{y \in \big(B(x,r_2) \setminus B(x,r_1)\big) \cap \big(Y \setminus \{x\}\big) : d(x,y) > t\}\big) = 0.$$

Hence, the formula of the statement holds true. ∎

We are now ready to prove the following for upper Ahlfors regular sets.

LEMMA 3.2. *Let $X, Y \subseteq M$. Let $\upsilon_Y \in]0,+\infty[$. Let ν be as in (1.1). Let Y be upper υ_Y-Ahlfors regular with respect to X. Then the following statements hold.*

(i) $\nu(\{x\}) = 0$ *for all $x \in X \cap Y$.*

(ii)

$$\int_{B(x,r)\cap Y} d(x,y)^{-s}\,d\nu(y) \leq \frac{c_{X,Y,\upsilon_Y}\,\upsilon_Y}{\upsilon_Y - s} r^{\upsilon_Y - s} \quad \forall s \in]0,\upsilon_Y[$$

and

$$\int_{B(x,r)\cap Y} d(x,y)^{-s}\,d\nu(y) \leq c_{X,Y,\upsilon_Y} r^{\upsilon_Y - s} \quad \forall s \in]-\infty,0]$$

for all $x \in X$ and $r \in]0, r_{X,Y,\upsilon_Y}[$.

PROOF. (i) follows by the inequality

$$\nu\big(\{x\}\big) \leq \nu\big(B(x,r) \cap Y\big) \leq c_{X,Y,\upsilon_Y} r^{\upsilon_Y} \quad \forall r \in]0, r_{X,Y,\upsilon_Y}[,$$

for all $x \in X \cap Y$, which holds by the upper υ_Y-Ahlfors regularity of Y with respect to X. Indeed, it suffices to take the limit as r tends to 0^+.

We now turn to prove statement (ii). If $s = 0$, then (ii) is an immediate consequence of the upper υ_Y-Ahlfors regularity of Y with respect to X. If $s \in]0, \upsilon_Y[$, Lemma 3.1 implies that

$$\begin{aligned}
&\int_{B(x,r)\cap(Y\setminus\{x\})} d(x,y)^{-s}\,d\nu(y) \\
&= s\int_0^{r^{-1}} t^{s-1}\nu\big((Y\setminus\{x\})\cap B(x,r)\big)\,dt \\
&\quad + s\int_{r^{-1}}^{+\infty} t^{s-1}\nu\big((Y\setminus\{x\})\cap B(x,t^{-1})\big)\,dt \\
&\le s\int_0^{r^{-1}} t^{s-1}\,dt\, c_{X,Y,\upsilon_Y} r^{\upsilon_Y} + s\int_{r^{-1}}^{+\infty} t^{s-1} c_{X,Y,\upsilon_Y}(t^{-1})^{\upsilon_Y}\,dt \\
&= c_{X,Y,\upsilon_Y}\left\{r^{\upsilon_Y} r^{-s} + s\int_{r^{-1}}^{+\infty} t^{s-1-\upsilon_Y}\,dt\right\} \\
&\le c_{X,Y,\upsilon_Y}\left\{r^{\upsilon_Y - s} - \frac{s}{s-\upsilon_Y} r^{-(s-\upsilon_Y)}\right\} \\
&= c_{X,Y,\upsilon_Y} r^{\upsilon_Y - s}\left(1 - \frac{s}{s-\upsilon_Y}\right) = c_{X,Y,\upsilon_Y}\frac{\upsilon_Y}{\upsilon_Y - s} r^{\upsilon_Y - s},
\end{aligned}$$

and thus statement (ii) holds true. If $s \in]-\infty, 0[$, Lemma 3.1 implies that

$$\begin{aligned}
&\int_{B(x,r)\cap(Y\setminus\{x\})} d(x,y)^{-s}\,d\nu(y) \\
&\quad = (-s)\int_0^{r} t^{(-s)-1}\nu\big((B(x,r)\setminus B(x,t])\cap(Y\setminus\{x\})\big)\,dt \\
&\quad \le (-s)\int_0^{r} t^{(-s)-1}\nu\big(B(x,r)\cap(Y\setminus\{x\})\big)\,dt \\
&\quad \le (-s)\int_0^{r} t^{(-s)-1} c_{X,Y,\upsilon_Y} r^{\upsilon_Y}\,dt = c_{X,Y,\upsilon_Y} r^{\upsilon_Y - s}
\end{aligned}$$

and thus statement (ii) holds true. ■

In the case of strongly upper Ahlfors regular sets, we can also prove the following.

LEMMA 3.3. *Let $X, Y \subseteq M$. Let $\upsilon_Y \in]0, +\infty[$. Let ν be as in (1.1). Let Y be strongly upper υ_Y-Ahlfors regular with respect to X. Then the following statements hold.*

(i) $\nu(\{y \in Y : d(x,y) = r\}) = 0$ *for all $x \in X$ and $r \in]0, r_{X,Y,\upsilon_Y}[$.*

(ii) *If $s \in \mathbb{R}\setminus\{\upsilon_Y\}$, then*

$$\int_{(B(x,r_2)\setminus B(x,r_1))\cap Y} d(x,y)^{-s}\,d\nu(y) \le \frac{c_{X,Y,\upsilon_Y}\upsilon_Y}{\upsilon_Y - s}(r_2^{\upsilon_Y - s} - r_1^{\upsilon_Y - s})$$

for all $x \in X$ *and* $r_1, r_2 \in]0, r_{X,Y,\upsilon_Y}[$ *with* $r_1 < r_2$.

(iii)
$$\int_{(B(x,r_2)\setminus B(x,r_1))\cap Y} d(x,y)^{-\upsilon_Y}\, d\nu(y) \le c_{X,Y,\upsilon_Y}\, \upsilon_Y \log \frac{r_2}{r_1}$$
for all $x \in X$ *and* $r_1, r_2 \in]0, r_{X,Y,\upsilon_Y}[$ *with* $r_1 < r_2$.

PROOF. Statement (i) follows by the inequality
$$\nu\big(\{y \in Y : d(x,y) = r\}\big) \le \nu\big((B(x,r_2) \setminus B(x,r_1)) \cap Y\big) \le c_{X,Y,\upsilon_Y}(r_2^{\upsilon_Y} - r_1^{\upsilon_Y})$$
for all $x \in X$ and $r_1, r_2 \in [0, r_{X,Y,\upsilon_Y}[$ with $r_1 < r < r_2$, which holds by the strong upper υ_Y-Ahlfors regularity of Y with respect to X. Indeed, it suffices to take the limit as r_1 tends to r^- and r_2 tends to r^+.

Next we turn to prove statements (ii) and (iii). If $s \in]0, +\infty[$, Lemma 3.1 implies that
$$\begin{aligned}
&\int_{(B(x,r_2)\setminus B(x,r_1))\cap(Y\setminus\{x\})} d(x,y)^{-s}\, d\nu(y)\\
&\quad\le s\int_0^{r_2^{-1}} t^{s-1}\nu\big(Y \cap \big(B(x,r_2) \setminus B(x,r_1)\big)\big)\, dt\\
&\qquad + s\int_{r_2^{-1}}^{r_1^{-1}} t^{s-1}\nu\big(Y \cap \big(B(x,t^{-1}) \setminus B(x,r_1)\big)\big)\, dt\\
&\quad\le s\int_0^{r_2^{-1}} t^{s-1}\, dt\, c_{X,Y,\upsilon_Y}(r_2^{\upsilon_Y} - r_1^{\upsilon_Y}) + s\int_{r_2^{-1}}^{r_1^{-1}} t^{s-1} c_{X,Y,\upsilon_Y}\big((t^{-1})^{\upsilon_Y} - r_1^{\upsilon_Y}\big)\, dt\\
&\quad= c_{X,Y,\upsilon_Y}\left\{(r_2^{\upsilon_Y} - r_1^{\upsilon_Y})r_2^{-s} + s\int_{r_2^{-1}}^{r_1^{-1}} t^{s-1-\upsilon_Y}\, dt - r_1^{\upsilon_Y} s\int_{r_2^{-1}}^{r_1^{-1}} t^{s-1}\, dt\right\}.
\end{aligned}$$
We now consider separately case $s \in]0, +\infty[\setminus \{\upsilon_Y\}$ of statement (ii) and case $s = \upsilon_Y$ of statement (iii). Let $s \in]0, +\infty[\setminus \{\upsilon_Y\}$. Then we have
$$\begin{aligned}
&\int_{(B(x,r_2)\setminus B(x,r_1))\cap(Y\setminus\{x\})} d(x,y)^{-s}\, d\nu(y)\\
&\quad\le c_{X,Y,\upsilon_Y}\Big\{r_2^{\upsilon_Y - s} - r_1^{\upsilon_Y} r_2^{-s} + \frac{s}{s-\upsilon_Y}(r_1^{-(s-\upsilon_Y)} - r_2^{-(s-\upsilon_Y)})\\
&\qquad\qquad - r_1^{\upsilon_Y}\big((r_1^{-1})^s - (r_2^{-1})^s\big)\Big\}\\
&\quad= c_{X,Y,\upsilon_Y}\left\{r_2^{\upsilon_Y - s}\left(1 - \frac{s}{s-\upsilon_Y}\right) + r_1^{\upsilon_Y - s}\left(\frac{s}{s-\upsilon_Y} - 1\right)\right\}\\
&\quad= c_{X,Y,\upsilon_Y}\frac{\upsilon_Y}{\upsilon_Y - s}\left\{r_2^{\upsilon_Y - s} - r_1^{\upsilon_Y - s}\right\},
\end{aligned}$$

and thus statement (ii) holds true for $s \in]0, +\infty[\setminus \{\upsilon_Y\}$. Now let $s = \upsilon_Y$. Then we have

$$
\begin{aligned}
&\int_{(B(x,r_2)\setminus B(x,r_1))\cap(Y\setminus\{x\})} d(x,y)^{-\upsilon_Y}\, d\nu(y) \\
&\leq c_{X,Y,\upsilon_Y}\left\{(r_2^{\upsilon_Y} - r_1^{\upsilon_Y})r_2^{-\upsilon_Y} + \upsilon_Y \int_{r_2^{-1}}^{r_1^{-1}} t^{\upsilon_Y-1-\upsilon_Y}\, dt - r_1^{\upsilon_Y}\upsilon_Y \int_{r_2^{-1}}^{r_1^{-1}} t^{\upsilon_Y-1}\, dt\right\} \\
&= c_{X,Y,\upsilon_Y}\left\{1 - r_1^{\upsilon_Y} r_2^{-\upsilon_Y} + \upsilon_Y \log\frac{r_1^{-1}}{r_2^{-1}} - r_1^{\upsilon_Y}\left((r_1^{-1})^{\upsilon_Y} - (r_2^{-1})^{\upsilon_Y}\right)\right\} \\
&= c_{X,Y,\upsilon_Y}\, \upsilon_Y \log\frac{r_2}{r_1}
\end{aligned}
$$

and thus statement (iii) holds true. Finally, we consider case $s \in]-\infty, 0]$ of (ii). If $s = 0$, statement (ii) is an immediate consequence of the strong upper υ_Y-Ahlfors regularity of Y with respect to X. Now let $s < 0$. Lemma 3.1 implies that

$$
\begin{aligned}
&\int_{(B(x,r_2)\setminus B(x,r_1))\cap(Y\setminus\{x\})} d(x,y)^{-s}\, d\nu(y) \\
&\quad \leq (-s)\int_0^{r_1} t^{(-s)-1}\nu\big(Y \cap \big(B(x,r_2)\setminus B(x,r_1)\big)\big)\, dt \\
&\qquad + (-s)\int_{r_1}^{r_2} t^{(-s)-1}\nu\big(Y \cap \big(B(x,r_2)\setminus B(x,t)\big)\big)\, dt \\
&\quad \leq r_1^{(-s)} c_{X,Y,\upsilon_Y}(r_2^{\upsilon_Y} - r_1^{\upsilon_Y}) + (-s)\int_{r_1}^{r_2} t^{(-s)-1} c_{X,Y,\upsilon_Y}(r_2^{\upsilon_Y} - t^{\upsilon_Y})\, dt \\
&\quad = c_{X,Y,\upsilon_Y}\left\{(r_2^{\upsilon_Y} - r_1^{\upsilon_Y})r_1^{-s} + (-s)\int_{r_1}^{r_2} t^{(-s)-1}\, dt\, r_2^{\upsilon_Y}\right. \\
&\qquad\qquad \left. - (-s)\int_{r_1}^{r_2} t^{(-s)-1+\upsilon_Y}\, dt\right\} \\
&\quad = c_{X,Y,\upsilon_Y}\left\{r_2^{\upsilon_Y} r_1^{-s} - r_1^{\upsilon_Y-s} + r_2^{\upsilon_Y}(r_2^{(-s)} - r_1^{(-s)})\right. \\
&\qquad\qquad \left. + \frac{s}{-s+\upsilon_Y}(r_2^{-s+\upsilon_Y} - r_1^{-s+\upsilon_Y})\right\} \\
&\quad = c_{X,Y,\upsilon_Y}\left\{r_2^{\upsilon_Y-s} - r_1^{\upsilon_Y-s} + \frac{s}{\upsilon_Y - s} r_2^{\upsilon_Y-s} - \frac{s}{\upsilon_Y - s} r_1^{\upsilon_Y-s}\right\} \\
&\quad = c_{X,Y,\upsilon_Y}\left\{r_2^{\upsilon_Y-s}\left(1 + \frac{s}{\upsilon_Y - s}\right) - r_1^{\upsilon_Y-s}\left(1 + \frac{s}{\upsilon_Y - s}\right)\right\} \\
&\quad = c_{X,Y,\upsilon_Y}\frac{\upsilon_Y}{\upsilon_Y - s}\{r_2^{\upsilon_Y-s} - r_1^{\upsilon_Y-s}\}.
\end{aligned}
$$

■

Then we can prove the following basic inequalities for the integral on an upper Ahlfors regular set Y and on the intersection of Y with balls with center at a point x of X of the powers of $d(x,y)^{-1}$ with exponent $s \in]-\infty, \upsilon_Y[$.

LEMMA 3.4. *Let $X, Y \subseteq M$. Let $\upsilon_Y \in]0,+\infty[$. Let ν be as in (1.1). Let Y be upper υ_Y-Ahlfors regular with respect to X. Then the following statements hold.*

(i) *Let $\nu(Y) < +\infty$. If $s \in]0, \upsilon_Y[$, then*

$$c'_{s,X,Y} \equiv \sup_{x\in X} \int_Y \frac{d\nu(y)}{d(x,y)^s} \leq \nu(Y)a^{-s} + c_{X,Y,\upsilon_Y} \frac{\upsilon_Y}{\upsilon_Y - s} a^{\upsilon_Y - s} \tag{3.5}$$

for all $a \in]0, r_{X,Y,\upsilon_Y}[$. If $s = 0$, then

$$c'_{0,X,Y} \equiv \sup_{x\in X} \int_Y \frac{d\nu(y)}{d(x,y)^0} = \nu(Y).$$

(ii) *Let $\nu(Y) < +\infty$ whenever $r_{X,Y,\upsilon_Y} < +\infty$. If $s \in]-\infty, \upsilon_Y[$, then*

$$c''_{s,X,Y} \equiv \sup_{(x,t)\in X\times]0,+\infty[} t^{s-\upsilon_Y} \int_{B(x,t)\cap Y} \frac{d\nu(y)}{d(x,y)^s} < +\infty.$$

PROOF. (i) If $x \in X \cap Y$, then $\nu(\{x\}) = 0$ and thus a commonly accepted abuse of notation allows us to write

$$\int_Y \frac{d\nu(y)}{d(x,y)^s} = \int_{Y\setminus\{x\}} \frac{d\nu(y)}{d(x,y)^s}.$$

If instead $x \in X \setminus Y$, then $Y = Y \setminus \{x\}$ and we have

$$\int_Y \frac{d\nu(y)}{d(x,y)^s} = \int_{Y\setminus\{x\}} \frac{d\nu(y)}{d(x,y)^s}.$$

If $s > 0$, Lemma 3.2 (ii) implies that

$$\begin{aligned}\int_{Y\setminus\{x\}} \frac{d\nu(y)}{d(x,y)^s} &\leq \int_{Y\setminus B(x,a)} \frac{d\nu(y)}{d(x,y)^s} + \int_{Y\cap(B(x,a)\setminus\{x\})} \frac{d\nu(y)}{d(x,y)^s} \\ &\leq \nu(Y)a^{-s} + c_{X,Y,\upsilon_Y} \frac{\upsilon_Y}{\upsilon_Y - s} a^{\upsilon_Y - s} \quad \forall a \in]0, r_{X,Y,\upsilon_Y}[.\end{aligned}$$

If $s = 0$, then statement (i) is trivial and thus the proof of (i) is complete.

(ii) By the same remark at the beginning of the proof of (i) and by Lemma 3.2 (ii), we have

$$\begin{aligned}\int_{B(x,t)\cap Y} \frac{d\nu(y)}{d(x,y)^s} &= \int_{Y\cap(B(x,t)\setminus\{x\})} \frac{d\nu(y)}{d(x,y)^s} \\ &\leq c_{X,Y,\upsilon_Y} \max\left\{1, \frac{\upsilon_Y}{\upsilon_Y - s}\right\} t^{\upsilon_Y - s} \quad \forall t \in]0, r_{X,Y,\upsilon_Y}[.\end{aligned}$$

If instead $r_{X,Y,\upsilon_Y} < +\infty$ and $t \in [r_{X,Y,\upsilon_Y}, +\infty[$, then case $t \in]0, r_{X,Y,\upsilon_Y}[$ implies that

$$\begin{aligned}
&t^{s-\upsilon_Y} \int_{B(x,t)\cap Y} \frac{d\nu(y)}{d(x,y)^s} \\
&= t^{s-\upsilon_Y} \int_{Y\cap(B(x,t)\setminus\{x\})} \frac{d\nu(y)}{d(x,y)^s} \\
&\le t^{s-\upsilon_Y} \limsup_{\rho\to r^-_{X,Y,\upsilon_Y}} \left\{ \int_{Y\cap(B(x,t)\setminus B(x,\rho))} \frac{d\nu(y)}{d(x,y)^s} + \int_{Y\cap(B(x,\rho)\setminus\{x\})} \frac{d\nu(y)}{d(x,y)^s} \right\} \\
&\le t^{s-\upsilon_Y} \limsup_{\rho\to r^-_{X,Y,\upsilon_Y}} \max_{\eta\in[\rho,t]} \eta^{-s}\nu(Y) + r^{s-\upsilon_Y}_{X,Y,\upsilon_Y} \limsup_{\rho\to r^-_{X,Y,\upsilon_Y}} \int_{Y\cap(B(x,\rho)\setminus\{x\})} \frac{d\nu(y)}{d(x,y)^s} \\
&\le t^{s-\upsilon_Y} \max_{\eta\in[r_{X,Y,\upsilon_Y},t]} \eta^{-s}\nu(Y) + r^{s-\upsilon_Y}_{X,Y,\upsilon_Y} \limsup_{\rho\to r^-_{X,Y,\upsilon_Y}} c_{X,Y,\upsilon_Y} \max\left\{1, \frac{\upsilon_Y}{\upsilon_Y - s}\right\} \rho^{\upsilon_Y - s} \\
&\le r^{-\upsilon_Y}_{X,Y,\upsilon_Y} \nu(Y) + c_{X,Y,\upsilon_Y} \max\left\{1, \frac{\upsilon_Y}{\upsilon_Y - s}\right\},
\end{aligned}$$

and thus the proof of (ii) is complete. ■

We now estimate the integral of the powers of $d(x,y)^{-1}$ with exponent $s \in [\upsilon_Y, +\infty[$ on the complement in Y of balls with center at a point x of X.

LEMMA 3.6. *Let X, $Y \subseteq M$. Let $\upsilon_Y \in]0, +\infty[$. Let ν be as in (1.1), $\nu(Y) < +\infty$. Then the following statements hold.*

(i) *Let Y be upper υ_Y-Ahlfors regular with respect to X. If $s \in]\upsilon_Y, +\infty[$, then*

$$c'''_{s,X,Y} \equiv \sup_{(x,t)\in X\times]0,+\infty[} t^{s-\upsilon_Y} \int_{Y\setminus B(x,t)} \frac{d\nu(y)}{d(x,y)^s} < +\infty.$$

(ii) *Let Y be strongly upper υ_Y-Ahlfors regular with respect to X. Then*

$$c^{iv}_{X,Y} \equiv \sup_{(x,t)\in X\times]0,1/e[} |\log t|^{-1} \int_{Y\setminus B(x,t)} \frac{d\nu(y)}{d(x,y)^{\upsilon_Y}} < +\infty.$$

PROOF. (i) Let $x \in X$. We first consider case $r_{X,Y,\upsilon_Y} < +\infty$. If $t \in]0, r_{X,Y,\upsilon_Y}[$, Lemma 3.1 with $r_1 = t$, $r_2 = +\infty$, and the rule of change of variables in the integrals imply that

$$\begin{aligned}
&t^{s-\upsilon_Y} \int_{Y\setminus B(x,t)} \frac{d\nu(y)}{d(x,y)^s} \\
&\quad = t^{s-\upsilon_Y} s \int_t^{+\infty} (1/u)^{s-1} \nu\big((Y\setminus\{x\}) \cap \big(B(x,u)\setminus B(x,t)\big)\big) u^{-2}\, du
\end{aligned}$$

$$
\begin{aligned}
&= t^{s-\upsilon_Y} s \int_t^{r_{X,Y,\upsilon_Y}} u^{-s-1} \nu\big((Y \setminus \{x\}) \cap (B(x,u) \setminus B(x,t))\big)\, du \\
&\quad + t^{s-\upsilon_Y} s \int_{r_{X,Y,\upsilon_Y}}^{+\infty} u^{-s-1} \nu\big((Y \setminus \{x\}) \cap (B(x,u) \setminus B(x,t))\big)\, du \\
&\le t^{s-\upsilon_Y} s \int_t^{r_{X,Y,\upsilon_Y}} c_{X,Y,\upsilon_Y} u^{-s-1+\upsilon_Y}\, du + t^{s-\upsilon_Y} s \Big[\frac{u^{-s}}{-s}\Big]_{u=r_{X,Y,\upsilon_Y}}^{+\infty} \nu(Y) \\
&= t^{s-\upsilon_Y} \frac{s c_{X,Y,\upsilon_Y}}{\upsilon_Y - s} [u^{\upsilon_Y - s}]_{u=t}^{u=r_{X,Y,\upsilon_Y}} + t^{s-\upsilon_Y} r_{X,Y,\upsilon_Y}^{-s} \nu(Y) \\
&\le \frac{s c_{X,Y,\upsilon_Y}}{s - \upsilon_Y} + r_{X,Y,\upsilon_Y}^{-\upsilon_Y} \nu(Y).
\end{aligned}
$$

If instead $t \in [r_{X,Y,\upsilon_Y}, +\infty[$, then we have

$$
t^{s-\upsilon_Y} \int_{Y \setminus B(x,t)} \frac{d\nu(y)}{d(x,y)^s} \le t^{s-\upsilon_Y} t^{-s} \nu(Y) \le r_{X,Y,\upsilon_Y}^{-\upsilon_Y} \nu(Y),
$$

and thus the proof of statement (i) in case $r_{X,Y,\upsilon_Y} < +\infty$ is complete. If $r_{X,Y,\upsilon_Y} = +\infty$, we proceed as above without the integral from r_{X,Y,υ_Y} to $+\infty$.

(ii) Let $t_* \equiv \frac{1}{2} \min\{1/e, r_{X,Y,\upsilon_Y}\}$. By Lemma 3.3 (iii), we have

$$
\begin{aligned}
\int_{Y \setminus B(x,t)} \frac{d\nu(y)}{d(x,y)^{\upsilon_Y}} &= \int_{Y \setminus B(x,t_*)} \frac{d\nu(y)}{d(x,y)^{\upsilon_Y}} + \int_{(B(x,t_*) \setminus B(x,t)) \cap Y} \frac{d\nu(y)}{d(x,y)^{\upsilon_Y}} \\
&\le t_*^{-\upsilon_Y} \nu(Y) + c_{X,Y,\upsilon_Y} \upsilon_Y \log \frac{t_*}{t} \\
&\le t_*^{-\upsilon_Y} \nu(Y) + c_{X,Y,\upsilon_Y} \upsilon_Y (\log t_* - \log t) \\
&\le |\log t| \bigg(\frac{t_*^{-\upsilon_Y} \nu(Y) + c_{X,Y,\upsilon_Y} \upsilon_Y |\log t_*|}{|\log t|} + c_{X,Y,\upsilon_Y} \upsilon_Y \bigg) \\
&\le |\log t| \big(t_*^{-\upsilon_Y} \nu(Y) + (1 + |\log t_*|) c_{X,Y,\upsilon_Y} \upsilon_Y \big)
\end{aligned}
$$

for all $t \in]0, t_*[$. If $t \in [t_*, 1/e[$, then we have

$$
|\log t|^{-1} \int_{Y \setminus B(x,t)} \frac{d\nu(y)}{d(x,y)^{\upsilon_Y}} \le |\log t|^{-1} t^{-\upsilon_Y} \nu(Y) \le \big|\log(1/e)\big|^{-1} t_*^{-\upsilon_Y} \nu(Y)
$$

and thus the proof of (ii) is complete. ■

4. Weakly singular potential operators in spaces of essentially bounded functions on upper Ahlfors regular subsets of M

We now prove an “action statement” by exploiting the Hille–Tamarkin theorem in case Y is upper υ_Y-Ahlfors regular with respect to X.

PROPOSITION 4.1. *Let $X, Y \subseteq M$. Let $\upsilon_Y \in]0,+\infty[$, $s \in [0, \upsilon_Y[$. Let ν be as in (1.1), $\nu(Y) < +\infty$. Let Y be upper υ_Y-Ahlfors regular with respect to X. Then the following statements hold.*

(i) *If $(K, \varphi) \in \mathcal{K}_{s,X\times Y} \times L^\infty_\nu(Y)$, then the function $K(x,\cdot)\varphi(\cdot)$ is integrable in Y for all $x \in X$ and the function $A[K,\varphi]$ defined by*

$$A[K,\varphi](x) \equiv \int_Y K(x,y)\varphi(y)\,d\nu(y) \quad \forall x \in X \tag{4.2}$$

is bounded.

(ii) *The bilinear map from $\mathcal{K}_{s,X\times Y} \times L^\infty_\nu(Y)$ to $B(X)$, which takes (K,φ) to $A[K,\varphi]$, is continuous and the following inequality holds:*

$$\sup_X \left|A[K,\varphi]\right| \le c'_{s,X,Y}\|K\|_{\mathcal{K}_{s,X\times Y}}\|\varphi\|_{L^\infty_\nu(Y)} \tag{4.3}$$

for all $(K,\varphi) \in \mathcal{K}_{s,X\times Y} \times L^\infty_\nu(Y)$ (see (3.5) for $c'_{s,X,Y}$).

PROOF. By Lemma 3.4 (i), we have

$$\sup_{x\in X}\int_Y d(x,y)^{-s}\,d\nu(y) = c'_{s,X,Y} < +\infty.$$

Then the Hille–Tamarkin theorem (Theorem 2.2) for potential operators implies the continuity of $A[\cdot,\cdot]$ from $\mathcal{K}_{s,X\times Y} \times L^\infty_\nu(Y)$ to $B(X)$ and the validity of inequality (4.3). Hence, statements (i) and (ii) hold true. ■

5. CONDITIONS OF ACTION INTO GENERALIZED HÖLDER SPACES FOR WEAKLY SINGULAR POTENTIAL OPERATORS ACTING ON ESSENTIALLY BOUNDED FUNCTIONS IN UPPER AHLFORS REGULAR SUBSETS OF M

Next we consider off-diagonal kernels K as in Definition 1.6. One can easily verify that $(\mathcal{K}_{s_1,s_2,s_3}(X\times Y), \|\cdot\|_{\mathcal{K}_{s_1,s_2,s_3}(X\times Y)})$ is a Banach space. By Definition 1.6, if $s_1, s_2, s_3 \in \mathbb{R}$, we have $\mathcal{K}_{s_1,s_2,s_3}(X\times Y) \subseteq \mathcal{K}_{s_1,X\times Y}$ and

$$\|K\|_{\mathcal{K}_{s_1,X\times Y}} \le \|K\|_{\mathcal{K}_{s_1,s_2,s_3}(X\times Y)} \quad \forall K \in \mathcal{K}_{s_1,s_2,s_3}(X\times Y).$$

Next we introduce a function that we need for a generalized Hölder norm. For each $\theta \in]0,1]$, we define the function $\omega_\theta(\cdot)$ from $[0,+\infty[$ to itself by setting

$$\omega_\theta(r) \equiv \begin{cases} 0 & r = 0, \\ r^\theta|\ln r| & r \in]0, r_\theta], \\ r_\theta^\theta|\ln r_\theta| & r \in]r_\theta, +\infty[, \end{cases}$$

where

$$r_\theta \equiv e^{-1/\theta} \quad \forall \theta \in]0,1].$$

Obviously, $\omega_\theta(\cdot)$ is concave and satisfies condition (A.1) of the appendix. We also note that if $\mathbb{D} \subseteq M$, then the continuous embedding

$$C_b^{0,\theta}(\mathbb{D}) \subseteq C_b^{0,\omega_\theta(\cdot)}(\mathbb{D}) \subseteq C_b^{0,\theta'}(\mathbb{D})$$

holds for all $\theta' \in]0,\theta[$. We refer to the appendix for the notation of (generalized) Hölder spaces. In particular, the subscript b denotes that we are considering the intersection of a (generalized) Hölder space with the space $B(\mathbb{D})$ of the bounded functions in $\mathbb{D}$. Next we introduce the following elementary lemma, which we exploit later and which can be proved by the triangular inequality.

LEMMA 5.1. *If $x', x'' \in M$, $x' \neq x''$, $y \in M \setminus B(x', 2d(x', x''))$, then*

$$\frac{1}{2} d(x', y) \leq d(x'', y) \leq 2d(x', y).$$

We now consider the properties of an integral operator with a kernel in the class $\mathcal{K}_{s_1,s_2,s_3}(X \times Y)$ and acting on essentially bounded functions on Y.

PROPOSITION 5.2. *Let $X, Y \subseteq M$. Let $\upsilon_Y \in]0,+\infty[$. Let ν be as in (1.1), $\nu(Y) < +\infty$. Let Y be upper υ_Y-Ahlfors regular with respect to X. Let*

$$s_1 \in [\upsilon_Y - 1, \upsilon_Y[, \quad s_1 \geq 0, \quad s_2 \in [0,+\infty[, \quad s_3 \in]0,1].$$

If $s_2 = \upsilon_Y$, we further require that Y be strongly upper υ_Y-Ahlfors regular with respect to X.

If $s_2 > \upsilon_Y$, we further require that $s_2 < \upsilon_Y + s_3$.

Let ϖ be the map from $[0,+\infty[$ to itself defined by $\varpi(0) \equiv 0$ and

$$\varpi(r) \equiv \begin{cases} r^{\min\{\upsilon_Y - s_1, s_3\}} & \text{if } s_2 < \upsilon_Y, \\ \max\left\{r^{\upsilon_Y - s_1}, \omega_{s_3}(r)\right\} & \text{if } s_2 = \upsilon_Y, \\ r^{\min\{\upsilon_Y - s_1, s_3 + \upsilon_Y - s_2\}} & \text{if } s_2 > \upsilon_Y, \end{cases} \qquad \forall r \in]0,+\infty[. \tag{5.3}$$

Then the bilinear map from

$$\mathcal{K}_{s_1,s_2,s_3}(X \times Y) \times L^\infty_\nu(Y) \quad \text{to} \quad C_b^{0,\varpi(\cdot)}(X),$$

which takes (K,φ) to $A[K,\varphi]$, is continuous.

PROOF. We first note that the inequality (4.3), the elementary inequality

$$\|K\|_{\mathcal{K}_{s_1},X\times Y} \leq \|K\|_{\mathcal{K}_{s_1 s_2,s_3}(X\times Y)},$$

the membership of s_1 in $[0, \upsilon_Y[$, and Lemma 3.4 imply that $c'_{s_1,X,Y}$ is finite and that

$$\sup_{x\in X}\left|\int_Y K(x,y)\varphi(y)\,d\nu(y)\right| \leq \|K\|_{\mathcal{K}_{s_1,X\times Y}}\|\varphi\|_{L^\infty_\nu(Y)}c'_{s_1,X,Y} \leq \|K\|_{\mathcal{K}_{s_1 s_2,s_3}(X\times Y)}\|\varphi\|_{L^\infty_\nu(Y)}c'_{s_1,X,Y} \tag{5.4}$$

for all $(K,\varphi) \in \mathcal{K}_{s_1,s_2,s_3}(X\times Y)\times L^\infty_\nu(Y)$. Next we turn to estimate the (generalized) Hölder constant of $A[K,\varphi]$. We first note that $\upsilon_Y - s_1 \in]0,1]$ and that if $s_2 > \upsilon_Y$, then we also have

$$0 < \upsilon_Y - s_2 + s_3 < s_3 \leq 1.$$

Now let $x', x'' \in X$, $x' \neq x''$. By the above inequality (5.4), the function $A[K,\varphi]$ is bounded. Thus there is no loss of generality in assuming that

$$0 < 3d(x',x'') \leq r_{s_3} \leq e^{-1/s_3} \leq 1/e$$

(cf. Remark A.2 of the appendix). We plan to split the integral on Y that appears in the definition of our integral operator $A[K,\varphi]$ into two parts. Namely, the part of Y in the ball $B(x', 2d(x',x''))$ and the part of Y outside of the same ball. Thus we write

$$\begin{aligned}
&\left|A[K,\varphi](x') - A[K,\varphi](x'')\right| \\
&\leq \left|\int_{B(x',2d(x',x''))\cap Y} K(x',y)\varphi(y)\,d\nu(y)\right. \\
&\qquad \left. - \int_{B(x',2d(x',x''))\cap Y} K(x'',y)\varphi(y)\,d\nu(y)\right| \\
&\qquad + \left|\int_{Y\setminus B(x',2d(x',x''))} \left[K(x',y) - K(x'',y)\right]\varphi(y)\,d\nu(y)\right| \\
&\leq \|\varphi\|_{L^\infty_\nu(Y)}\left\{\int_{B(x',2d(x',x''))\cap Y} \left|K(x',y)\right|d\nu(y)\right. \\
&\qquad + \int_{B(x',2d(x',x''))\cap Y} \left|K(x'',y)\right|d\nu(y) \\
&\qquad \left. + \int_{Y\setminus B(x',2(x',x''))} \left|K(x',y) - K(x'',y)\right|d\nu(y)\right\}.
\end{aligned}$$

Since $s_1 \in [0, \upsilon_Y[$, we would like to apply Lemma 3.4 (ii) in order to estimate the first two integrals in the right-hand side. However, we note that in the second one we have $|K(x'',y)|$, while the center of the ball of integration is in x' and not in x'' that is the first argument in $|K(x'',y)|$. Thus we observe that

$$B\big(x', 2d(x',x'')\big) \subseteq B\big(x'', 3d(x',x'')\big)$$

and that the triangular inequality implies that

$$
\begin{aligned}
(5.5)\qquad & \left|A[K,\varphi](x') - A[K,\varphi](x'')\right| \\
&\leq \|\varphi\|_{L^\infty_\nu(Y)} \left\{ \int_{B(x',2d(x',x''))\cap Y} \left|K(x',y)\right| d\nu(y) \right. \\
&\qquad + \int_{B(x'',3d(x',x''))\cap Y} \left|K(x'',y)\right| d\nu(y) \\
&\qquad \left. + \int_{Y\setminus B(x',2d(x',x''))} \left|K(x',y) - K(x'',y)\right| d\nu(y) \right\}.
\end{aligned}
$$

Since $s_1 \in [0, \upsilon_Y[$, Lemma 3.4 (ii) implies that

$$
\begin{aligned}
(5.6)\qquad & \int_{B(x',2d(x',x''))\cap Y} \left|K(x',y)\right| d\nu(y) + \int_{B(x'',3d(x',x''))\cap Y} \left|K(x'',y)\right| d\nu(y) \\
&\leq \|K\|_{\mathcal{K}_{s_1,s_2,s_3}(X\times Y)} \left\{ \int_{B(x',2d(x',x''))\cap Y} \frac{d\nu(y)}{d(x',y)^{s_1}} \right. \\
&\qquad \left. + \int_{B(x'',3d(x',x''))\cap Y} \frac{d\nu(y)}{d(x'',y)^{s_1}} \right\} \\
&\leq \|K\|_{\mathcal{K}_{s_1,s_2,s_3}(X\times Y)} 2c''_{s_1,X,Y} 3^{\upsilon_Y - s_1} d(x',x'')^{\upsilon_Y - s_1}.
\end{aligned}
$$

Hence, we can estimate the integrals in inequality (5.5) on the part of Y in the ball $B(x', 2d(x', x''))$ in terms of the power $d(x', x'')^{\upsilon_Y - s_1}$. We now try to estimate the integral on the part of Y that is outside of the same ball. To do so, we observe that

$$
\begin{aligned}
(5.7)\qquad & \int_{Y\setminus B(x',2d(x',x''))} \left|K(x',y) - K(x'',y)\right| d\nu(y) \\
&\leq \|K\|_{\mathcal{K}_{s_1,s_2,s_3}(X\times Y)} \int_{Y\setminus B(x',2d(x',x''))} \frac{d(x',x'')^{s_3}}{d(x',y)^{s_2}}\, d\nu(y)
\end{aligned}
$$

for all $s_2 \in [0, +\infty[$ and $s_3 \in]0, 1]$. If $s_2 \in [0, \upsilon_Y[$, Lemma 3.4 (i) implies that

$$
(5.8)\qquad \int_{Y\setminus B(x',2d(x',x''))} \frac{d(x',x'')^{s_3}}{d(x',y)^{s_2}}\, d\nu(y) \leq c'_{s_2,X,Y} d(x',x'')^{s_3}.
$$

Then the above inequalities (5.5), (5.6), (5.7), and (5.8) imply that we can estimate $|A[K,\varphi](x') - A[K,\varphi](x'')|$ in terms of the powers $d(x', x'')^{\upsilon_Y - s_1}$ and $d(x', x'')^{s_3}$ for $3d(x', x'') \leq r_{s_3} < 1$. Since we can estimate $\sup_Y |A[K,\varphi]|$ by means of inequality (5.4), Remark A.2 of the appendix implies the validity of the statement.

If $s_2 = \upsilon_Y$, Lemma 3.6 (ii) implies that

$$
\begin{aligned}
(5.9)\qquad \int_{Y\setminus B(x',2d(x',x''))} \frac{d(x',x'')^{s_3}}{d(x',y)^{s_2}}\, d\nu(y) &\leq c^{iv}_{X,Y} d(x',x'')^{s_3} \left|\ln\left(2d(x',x'')\right)\right| \\
&\leq c^{iv}_{X,Y} d(x',x'')^{s_3} \left(1 + \left|\ln d(x',x'')\right|\right).
\end{aligned}
$$

Then the above inequalities (5.5), (5.6), (5.7), and (5.9) imply that we can estimate $|A[K,\varphi](x') - A[K,\varphi](x'')|$ in terms of

$$d(x',x'')^{\upsilon_Y - s_1}, \quad d(x',x'')^{s_3}, \quad d(x',x'')^{s_3}\left|\ln\left(d(x',x'')\right)\right|$$

for $3d(x',x'') \leq r_{s_3} < 1$. Since we can estimate $\sup_Y |A[K,\varphi]|$ by means of inequality (5.4), Remark A.2 of the appendix implies the validity of the statement.

If $s_2 \in]\upsilon_Y, +\infty[$, Lemma 3.6 (i) implies that

$$\text{(5.10)} \quad \int_{Y\setminus B(x',2d(x',x''))} \frac{d(x',x'')^{s_3}}{d(x',y)^{s_2}}\, d\nu(y) \leq c'''_{s_2,X,Y} 2^{\upsilon_Y - s_2} d(x',x'')^{\upsilon_Y - s_2 + s_3}.$$

Then the above inequalities (5.5), (5.6), (5.7), and (5.10) imply that we can estimate $|A[K,\varphi](x') - A[K,\varphi](x'')|$ in terms of $d(x',x'')^{\upsilon_Y - s_1}$ and $d(x',y)^{\upsilon_Y - s_2 + s_3}$ for $3d(x',x'') \leq r_{s_3} < 1$. Since we can estimate $\sup_Y |A[K,\varphi]|$ by means of inequality (5.4), Remark A.2 of the appendix implies the validity of the statement. ■

In case the density or moment φ is Hölder continuous in Y, then we can prove the following.

PROPOSITION 5.11. *Let $X, Y \subseteq M$. Let*

$$\upsilon_Y \in]0,+\infty[, \quad s_1 \in [0,\upsilon_Y[, \quad \beta \in]0,1], \quad s_2 \in [\beta,+\infty[, \quad s_3 \in]0,1].$$

Let ν be as in (1.1), $\nu(Y) < +\infty$. Let Y be upper υ_Y-Ahlfors regular with respect to X.

If $s_2 - \beta = \upsilon_Y$, we further require that Y be strongly upper υ_Y-Ahlfors regular with respect to X.

If $s_2 - \beta > \upsilon_Y$, we further require that $s_3 + \upsilon_Y - (s_2 - \beta) > 0$.

Then there exists $c > 0$ such that the function $A[K,\varphi]$ defined by (4.2) satisfies the following inequality:

$$\text{(5.12)} \quad \left|A[K,\varphi](x') - A[K,\varphi](x'')\right| \leq c\|K\|_{\mathcal{K}_{s_1,s_2,s_3}(X\times Y)}\|\varphi\|_{C_b^{0,\beta}(Y)}\,\omega\big(d(x',x'')\big) + \left|A[K,1](x') - A[K,1](x'')\right| \sup_Y |\varphi| \quad \forall x',x'' \in X,\ d(x',x'') \leq \frac{1}{3} r_{s_3},$$

for all $(K,\varphi) \in \mathcal{K}_{s_1,s_2,s_3}(X\times Y) \times C_b^{0,\beta}(Y)$, where $\omega(0) \equiv 0$ and

$$\text{(5.13)} \quad \omega(r) \equiv \begin{cases} r^{\min\{\upsilon_Y - s_1 + \beta, s_3\}} & \text{if } s_2 - \beta < \upsilon_Y, \\ \max\left\{r^{\upsilon_Y - s_1 + \beta}, \omega_{s_3}(r)\right\} & \text{if } s_2 - \beta = \upsilon_Y, \\ r^{\min\{\upsilon_Y - s_1 + \beta, s_3 + \upsilon_Y - (s_2 - \beta)\}} & \text{if } s_2 - \beta > \upsilon_Y, \end{cases} \quad \forall r \in]0,+\infty[.$$

PROOF. Let $(K,\varphi) \in \mathcal{K}_{s_1,s_2,s_3}(X \times Y) \times C_b^{0,\beta}(Y)$. Since $s_1 \in [0, \upsilon_Y[$, the Hille–Tamarkin proposition (Proposition 4.1) implies that $K(x,\cdot)\varphi(\cdot)$ is integrable in Y for all x in X and that the function $A[K,\varphi]$ is bounded in X. Since the statement is trivial for $d(x',x'') = 0$, it suffices to assume that $d(x',x'') > 0$. Since $\varphi \in C_b^{0,\beta}(Y)$, the McShane extension theorem (Theorem A.3 of the appendix) implies that there exists $\widetilde{\varphi} \in C_b^{0,\beta}(M)$ such that

$$\sup_M |\widetilde{\varphi}| = \sup_Y |\varphi|, \quad |\widetilde{\varphi} : M|_\beta = |\varphi : Y|_\beta, \quad \|\widetilde{\varphi}\|_{C_b^{0,\beta}(M)} = \|\varphi\|_{C_b^{0,\beta}(Y)}. \tag{5.14}$$

By the triangular inequality, we have

$$\begin{aligned} &\big|A[K,\varphi](x') - A[K,\varphi](x'')\big| \\ &\quad\leq \left| \int_Y \big[K(x',y) - K(x'',y)\big]\big(\widetilde{\varphi}(y) - \widetilde{\varphi}(x')\big)\, d\nu(y) \right| \\ &\qquad + \big|\widetilde{\varphi}(x')\big| \left| \int_Y \big[K(x',y) - K(x'',y)\big]\, d\nu(y) \right|. \end{aligned} \tag{5.15}$$

Since $\sup_M |\widetilde{\varphi}| = \sup_Y |\varphi|$, second addendum in the right-hand side is less or equal to

$$\big|A[K,1](x') - A[K,1](x'')\big| \sup_Y |\varphi|.$$

We now estimate the first addendum. The idea is to split it into two parts. Namely, the part of Y in the ball $B(x', 2d(x',x''))$ and the part of Y outside of the same ball. Thus we write

$$\begin{aligned} &\left| \int_Y \big[K(x',y) - K(x'',y)\big]\big(\widetilde{\varphi}(y) - \widetilde{\varphi}(x')\big)\, d\nu(y) \right| \\ &\quad\leq \int_{B(x',2d(x',x''))\cap Y} \big|K(x',y)\big|\, d(y,x')^\beta\, d\nu(y) |\widetilde{\varphi} : M|_\beta \\ &\qquad + \int_{B(x',2d(x',x''))\cap Y} \big|K(x'',y)\big|\, d(y,x')^\beta\, d\nu(y) |\widetilde{\varphi} : M|_\beta \\ &\qquad + \int_{Y\setminus B(x',2d(x',x''))} \big|K(x',y) - K(x'',y)\big|\, d(y,x')^\beta\, d\nu(y) |\widetilde{\varphi} : M|_\beta. \end{aligned}$$

Since $s_1 \in [0, \upsilon_Y[$, we would like to apply Lemma 3.4 (ii) in order to estimate the first two integrals in the right-hand side. However, we note that in the second one we have $|K(x'',y)|$, while the center of the ball of integration is in x' and not in x'' that is the first argument in $|K(x'',y)|$. It is enough to observe that

$$B\big(x', 2d(x',x'')\big) \subseteq B\big(x'', 3d(x',x'')\big).$$

We also observe that the factor $d(y,x')^\beta$ of $|K(x'',y)|$ in the second integral in the right-hand side contains $d(y,x')$ and instead we would like to have $d(y,x'')$, because the first argument of $|K(x'',y)|$ is x'' and not x'. It is enough to remember the Yensen inequality

$$d(y,x')^\beta \leq d(y,x'')^\beta + d(x',x'')^\beta,$$

and we deduce that

$$\begin{aligned}
&\left| \int_Y \left[K(x',y) - K(x'',y)\right]\left(\tilde{\varphi}(y) - \tilde{\varphi}(x')\right) d\nu(y) \right| \\
&\quad \leq \int_{B(x',2d(x',x''))\cap Y} \left|K(x',y)\right| d(y,x')^\beta \, d\nu(y) |\tilde{\varphi} : M|_\beta \\
&\qquad + \int_{B(x'',3d(x',x''))\cap Y} \left|K(x'',y)\right| d(y,x')^\beta \, d\nu(y) |\tilde{\varphi} : M|_\beta \\
&\qquad + \int_{Y\setminus B(x',2d(x',x''))} \left|K(x',y) - K(x'',y)\right| d(y,x')^\beta \, d\nu(y) |\tilde{\varphi} : M|_\beta \\
&\quad \leq \|K\|_{\mathcal{K}_{s_1,s_2,s_3}(X\times Y)} |\tilde{\varphi} : M|_\beta \\
&\qquad \times \left\{ \int_{B(x',2d(x',x''))\cap Y} \frac{d\nu(y)}{d(y,x')^{s_1-\beta}} \right. \\
&\qquad\quad + \int_{B(x'',3d(x',x''))\cap Y} \frac{d(x',x'')^\beta \, d\nu(y)}{d(y,x'')^{s_1}} \\
&\qquad\quad + \int_{B(x'',3d(x',x''))\cap Y} \frac{d\nu(y)}{d(y,x'')^{s_1-\beta}} \\
&\qquad\quad \left. + \int_{Y\setminus B(x',2d(x',x''))} \frac{d(x',x'')^{s_3} d(x',y)^\beta \, d\nu(y)}{d(x',y)^{s_2}} \right\}.
\end{aligned}$$

Then by Lemma 3.4 (ii), we have

$$\begin{aligned}
&\left| \int_Y \left[K(x',y) - K(x'',y)\right]\left(\tilde{\varphi}(y) - \tilde{\varphi}(x')\right) d\nu(y) \right| \\
&\quad \leq \|K\|_{\mathcal{K}_{s_1,s_2,s_3}(X\times Y)} |\tilde{\varphi} : M|_\beta \\
&\qquad \times \left\{ c''_{s_1-\beta,X,Y} 2^{\upsilon_Y - s_1 + \beta} d(x',x'')^{\upsilon_Y - s_1 + \beta} \right. \\
&\qquad\quad + d(x',x'')^\beta c''_{s_1,X,Y} 3^{\upsilon_Y - s_1} d(x',x'')^{\upsilon_Y - s_1} \\
&\qquad\quad + c''_{s_1-\beta,X,Y} 3^{\upsilon_Y - s_1 + \beta} d(x',x'')^{\upsilon_Y - s_1 + \beta} \\
&\qquad\quad \left. + d(x',x'')^{s_3} \int_{Y\setminus B(x',2d(x',x''))} \frac{d\nu(y)}{d(x',y)^{s_2-\beta}} \right\}.
\end{aligned} \tag{5.16}$$

At this point, we distinguish three cases. If $s_2 - \beta \in [0, \upsilon_Y[$, then Lemma 3.4 (i) implies that

$$\int_{Y\setminus B(x',2d(x',x''))} \frac{d\nu(y)}{d(x',y)^{s_2-\beta}} \leq \int_Y \frac{d\nu(y)}{d(x',y)^{s_2-\beta}} \leq c'_{s_2-\beta,X,Y}.$$

Since $0 < d(x',x'') \leq r_{s_3} \leq e^{-1/s_3} \leq 1/e < 1$, we have

$$d(x',x'')^{\upsilon_Y - s_1+\beta} \leq d(x',x'')^{\min\{\upsilon_Y-s_1+\beta,s_3\}},$$
$$d(x',x'')^{s_3} \leq d(x',x'')^{\min\{\upsilon_Y-s_1+\beta,s_3\}},$$

and thus inequality (5.16) implies that we can estimate the first term in the right-hand side of (5.15) in terms of the power $d(x',x'')^{\min\{\upsilon_Y-s_1+\beta,s_3\}}$. Hence, equalities (5.14), inequalities (5.15) and (5.16) imply that there exists $c > 0$ such that inequality (5.12) holds with

$$\omega(r) = r^{\min\{\upsilon_Y-s_1+\beta,s_3\}} \quad \forall r \in]0,+\infty[$$

and the proof of this case is complete.

If $s_2 - \beta = \upsilon_Y$, then Lemma 3.6 (ii) and inequality $0 < 2d(x',x'') \leq 1/e$ imply that

$$\int_{Y\setminus B(x',2d(x',x''))} \frac{d\nu(y)}{d(x',y)^{s_2-\beta}} \leq c^{iv}_{X,Y} \left|\ln\left(2d(x',x'')\right)\right|$$
$$\leq c^{iv}_{X,Y}\left(1+\left|\ln d(x',x'')\right|\right),$$

and thus inequality (5.16) implies that we can estimate the first term in the right-hand side of (5.15) in terms of the powers $d(x',x'')^{\upsilon_Y-s_1+\beta}$, $d(x',x'')^{s_3}$ and of $d(x',x'')^{s_3}|\ln d(x',x'')|$. Hence, equalities (5.14), inequalities (5.15) and (5.16) imply that there exists $c > 0$ such that inequality (5.12) holds with

$$\omega(r) = \max\left\{r^{\upsilon_Y-s_1+\beta}, r^{s_3}, \omega_{s_3}(r)\right\} \quad \forall r \in]0,+\infty[.$$

Since $\max\{r^{\upsilon_Y-s_1+\beta}, r^{s_3}, \omega_{s_3}(r)\} = \max\{r^{\upsilon_Y-s_1+\beta}, \omega_{s_3}(r)\}$ for $r \in]0, r_{s_3}[$, the proof of this case is complete.

If $s_2 - \beta > \upsilon_Y$, then Lemma 3.6 (i) implies that

$$\int_{Y\setminus B(x',2d(x',x''))} \frac{d\nu(y)}{d(x',y)^{s_2-\beta}} \leq c'''_{s_2-\beta,X,Y} 2^{\upsilon_Y-(s_2-\beta)} d(x',x'')^{\upsilon_Y-(s_2-\beta)}.$$

Since $0 < d(x',x'') \leq r_{s_3} \equiv e^{-1/s_3} \leq 1/e < 1$, and $s_3 + \upsilon_Y - (s_2-\beta) > 0$, we have

$$d(x',x'')^{\upsilon_Y-s_1+\beta} \leq d(x',x'')^{\min\{\upsilon_Y-s_1+\beta,s_3+\upsilon_Y-(s_2-\beta)\}},$$
$$d(x',x'')^{s_3+\upsilon_Y-(s_2-\beta)} \leq d(x',x'')^{\min\{\upsilon_Y-s_1+\beta,s_3+\upsilon_Y-(s_2-\beta)\}},$$

and thus inequality (5.16) implies that we can estimate the first term in the right-hand side of (5.15) in terms of $d(x', x'')^{\min\{\upsilon_Y - s_1 + \beta, s_3 + \upsilon_Y - (s_2 - \beta)\}}$. Hence, equalities (5.14), inequalities (5.15) and (5.16) imply that there exists $c > 0$ such that inequality (5.12) holds with

$$\omega(r) = r^{\min\{\upsilon_Y - s_1 + \beta, s_3 + \upsilon_Y - (s_2 - \beta)\}} \quad \forall r \in]0, +\infty[,$$

and thus the proof of this last case is complete. ■

Then we have the following immediate consequence of Proposition 5.11 (see also Proposition 4.1 and Remark A.2), that can be considered a "$T1$ theorem" for weakly singular integral operators acting in Hölder spaces of the sort of a corresponding result of Gatto [13, Thm. 1] who considered case $X = Y$, Y upper υ_Y-Ahlfors regular in the case $s_2 = s_1 + s_3$, $r_{X,Y,\upsilon_Y} = +\infty$ that we also consider as a specific case in the statement below. Thus the following theorem can be considered an extension of Gatto's theorem [13, Thm. 1].

Theorem 5.17. *Let $X, Y \subseteq M$. Let*

$$\upsilon_Y \in]0, +\infty[, \quad s_1 \in [0, \upsilon_Y[, \quad \beta \in]0, 1], \quad s_2 \in [\beta, +\infty[, \quad s_3 \in]0, 1].$$

Let ν be as in (1.1), $\nu(Y) < +\infty$. Let Y be upper υ_Y-Ahlfors regular with respect to X.

If $s_2 - \beta = \upsilon_Y$, we further require that Y be strongly upper υ_Y-Ahlfors regular with respect to X.

If $s_2 - \beta > \upsilon_Y$, we further require that

$$s_3 + \upsilon_Y - (s_2 - \beta) > 0.$$

Let ω be as in (5.13). Let $K \in \mathcal{K}_{s_1,s_2,s_3}(X \times Y)$. Then the following statements are equivalent.

(i) *The linear operator $A[K, \cdot]$ from $C_b^{0,\beta}(Y)$ to $C_b^{0,\omega(\cdot)}(X)$ that takes φ to $A[K, \varphi]$ is continuous.*

(ii) *The function $A[K, 1]$ from X to $\mathbb{C}$ that takes x to $A[K, 1](x)$ belongs to $C_b^{0,\omega(\cdot)}(X)$.*

Under the assumptions of Theorem 5.17, one could consider the vector space $\mathcal{K}^{\omega(\cdot)}_{s_1,s_2,s_3}(X \times Y)$ of those $K \in \mathcal{K}_{s_1,s_2,s_3}(X \times Y)$ such that $A[K, 1]$ belongs to $C^{0,\omega(\cdot)}(X)$, introduce the norm

$$\|K\|_{\mathcal{K}^{\omega(\cdot)}_{s_1,s_2,s_3}(X \times Y)} \equiv \|K\|_{\mathcal{K}_{s_1,s_2,s_3}(X \times Y)} + |A[K, 1] : X|_{\omega(\cdot)}$$

for all $K \in \mathcal{K}^{\omega(\cdot)}_{s_1,s_2,s_3}(X \times Y)$, and conclude that $A[\cdot, \cdot]$ is bilinear and continuous from $\mathcal{K}^{\omega(\cdot)}_{s_1,s_2,s_3}(X \times Y) \times C_b^{0,\beta}(Y)$ to $C_b^{0,\omega(\cdot)}(X)$ (cf. Proposition 5.11).

6. Analysis of an integral operator with a specific kernel

Let X, Y be subsets of M. Let ν be as in (1.1). We plan to analyze the integral operator

$$Q[Z,g,1](x) \equiv \int_Y Z(x,y)\big(g(x)-g(y)\big)\,d\nu(y) \quad \forall x \in X, \tag{6.1}$$

where Z belongs to a class $\mathcal{K}_{s_1,s_2,s_3}(X\times Y)$ as in Definition 1.6 and g is a $\mathbb{C}$-valued function in Y. We exploit the operator in (6.1) in the next section and we note that operators as in (6.1) appear in the applications (cf., e.g., Colton and Kress [2, p. 56] and Dondi and the author [4, §8]). In order to estimate the Hölder quotient of $Q[Z,g,1]$, we need to introduce a further norm for kernels.

Definition 6.2. Let $X, Y \subseteq M$. Let ν be as in (1.1). Let $s_1, s_2, s_3 \in \mathbb{R}$. We set

$$\begin{aligned}\mathcal{K}^\sharp_{s_1,s_2,s_3}(X\times Y) \equiv \Big\{ &K \in \mathcal{K}_{s_1,s_2,s_3}(X\times Y) : \\ &K(x,\cdot) \text{ is } \nu-\text{integrable in } Y\setminus B(x,r) \text{ for all } (x,r)\in X\times\,]0,+\infty[, \\ &\sup_{x\in X}\sup_{r\in]0,+\infty[}\left|\int_{Y\setminus B(x,r)} K(x,y)\,d\nu(y)\right| < +\infty\Big\}\end{aligned}$$

and

$$\begin{aligned}\|K\|_{\mathcal{K}^\sharp_{s_1,s_2,s_3}(X\times Y)} &\equiv \|K\|_{\mathcal{K}_{s_1,s_2,s_3}(X\times Y)} \\ &+ \sup_{x\in X}\sup_{r\in]0,+\infty[}\left|\int_{Y\setminus B(x,r)} K(x,y)\,d\nu(y)\right| \quad \forall K\in\mathcal{K}^\sharp_{s_1,s_2,s_3}(X\times Y).\end{aligned}$$

Clearly, $(\mathcal{K}^\sharp_{s_1,s_2,s_3}(X\times Y), \|\cdot\|_{\mathcal{K}^\sharp_{s_1,s_2,s_3}(X\times Y)})$ is a normed space. By definition, $\mathcal{K}^\sharp_{s_1,s_2,s_3}(X\times Y)$ is continuously embedded into the space $\mathcal{K}_{s_1,s_2,s_3}(X\times Y)$. We are now ready to prove the following statement on the Hölder continuity of $Q[Z,g,1]$ that extends some work of Gatto [13, Proof of Thm. 3, Thm. 4]. Here we note that $C^{0,\beta}(X\cup Y)$ is endowed with the seminorm $|\cdot : X\cup Y|_\beta$.

Proposition 6.3. *Let X, $Y \subseteq M$. Let*

$$\upsilon_Y \in\,]0,+\infty[, \quad \beta\in\,]0,1], \quad s_1\in[\beta,\upsilon_Y+\beta[, \quad s_2\in[\beta,+\infty[, \quad s_3\in\,]0,1].$$

Let ν be as in (1.1), $\nu(Y)<+\infty$. Then the following statements hold.

(i) *If $s_1<\upsilon_Y$, then the following statements hold.*

(a) *If $s_2-\beta>\upsilon_Y$, $s_2<\upsilon_Y+\beta+s_3$, and Y is upper υ_Y-Ahlfors regular with respect to X, then the bilinear map from*

$$\mathcal{K}_{s_1,s_2,s_3}(X\times Y)\times C^{0,\beta}(X\cup Y) \quad \text{to} \quad C_b^{0,\min\{\beta,\upsilon_Y+s_3+\beta-s_2\}}(X),$$

which takes (Z,g) to $Q[Z,g,1]$, is continuous.

(aa) *If* $s_2 - \beta = \upsilon_Y$ *and* Y *is strongly upper* υ_Y*-Ahlfors regular with respect to* X*, then the bilinear map from*

$$\mathcal{K}_{s_1,s_2,s_3}(X \times Y) \times C^{0,\beta}(X \cup Y) \quad to \quad C_b^{0,\max\{r^\beta,\omega_{s_3}(r)\}}(X),$$

which takes (Z,g) *to* $Q[Z,g,1]$*, is continuous.*

(aaa) *If* $s_2 - \beta < \upsilon_Y$ *and* Y *is upper* υ_Y*-Ahlfors regular with respect to* X*, then the bilinear map from*

$$\mathcal{K}_{s_1,s_2,s_3}(X \times Y) \times C^{0,\beta}(X \cup Y) \quad to \quad C_b^{0,\min\{\beta,s_3\}}(X),$$

which takes (Z,g) *to* $Q[Z,g,1]$*, is continuous.*

(ii) *If* $s_1 = \upsilon_Y$*, then the following statements hold.*

(b) *If* $s_2 - \beta > \upsilon_Y$*,* $s_2 < \upsilon_Y + \beta + s_3$*, and* Y *is upper* υ_Y*-Ahlfors regular with respect to* X*, then the bilinear map from*

$$\mathcal{K}^\sharp_{s_1,s_2,s_3}(X \times Y) \times C^{0,\beta}(X \cup Y) \quad to \quad C_b^{0,\min\{\beta,\upsilon_Y+s_3+\beta-s_2\}}(X),$$

which takes (Z,g) *to* $Q[Z,g,1]$*, is continuous.*

(bb) *If* $s_2 - \beta = \upsilon_Y$ *and* Y *is strongly upper* υ_Y*-Ahlfors regular with respect to* X*, then the bilinear map from*

$$\mathcal{K}^\sharp_{s_1,s_2,s_3}(X \times Y) \times C^{0,\beta}(X \cup Y) \quad to \quad C_b^{0,\max\{r^\beta,\omega_{s_3}(r)\}}(X),$$

which takes (Z,g) *to* $Q[Z,g,1]$*, is continuous.*

(bbb) *If* $s_2 - \beta < \upsilon_Y$ *and* Y *is upper* υ_Y*-Ahlfors regular with respect to* X*, then the bilinear map from*

$$\mathcal{K}^\sharp_{s_1,s_2,s_3}(X \times Y) \times C^{0,\beta}(X \cup Y) \quad to \quad C_b^{0,\min\{\beta,s_3\}}(X),$$

which takes (Z,g) *to* $Q[Z,g,1]$*, is continuous.*

(iii) *If* $s_1 > \upsilon_Y$*, then the following statements hold.*

(c) *If* $s_2 - \beta > \upsilon_Y$*,* $s_2 < \upsilon_Y + \beta + s_3$*, and* Y *is upper* υ_Y*-Ahlfors regular with respect to* X*, then the bilinear map from*

$$\mathcal{K}_{s_1,s_2,s_3}(X \times Y) \times C^{0,\beta}(X \cup Y) \quad to \quad C_b^{0,\min\{\upsilon_Y+\beta-s_1,\upsilon_Y+s_3+\beta-s_2\}}(X),$$

which takes (Z,g) *to* $Q[Z,g,1]$*, is continuous.*

(cc) *If* $s_2 - \beta = \upsilon_Y$ *and* Y *is strongly upper* υ_Y*-Ahlfors regular with respect to* X*, then the bilinear map from*

$$\mathcal{K}_{s_1,s_2,s_3}(X \times Y) \times C^{0,\beta}(X \cup Y) \quad \text{to} \quad C_b^{0,\max\{r^{\upsilon_Y+\beta-s_1},\omega_{s_3}(r)\}}(X),$$

which takes (Z, g) *to* $Q[Z, g, 1]$*, is continuous.*

(ccc) *If* $s_2 - \beta < \upsilon_Y$ *and* Y *is upper* υ_Y*-Ahlfors regular with respect to* X*, then the bilinear map from*

$$\mathcal{K}_{s_1,s_2,s_3}(X \times Y) \times C^{0,\beta}(X \cup Y) \quad \text{to} \quad C_b^{0,\min\{\upsilon_Y-(s_1-\beta),s_3\}}(X),$$

which takes (Z, g) *to* $Q[Z, g, 1]$*, is continuous.*

PROOF. By the elementary inequality

$$\begin{aligned}&\big|Z(x, y)\big(g(x) - g(y)\big)\big| \\ &\leq \frac{|g : X \cup Y|_\beta}{d(x, y)^{s_1-\beta}} \|Z\|_{\mathcal{K}_{s_1,s_2,s_3}(X\times Y)} \quad \forall (x, y) \in (X \times Y) \setminus D_{X\times Y}\end{aligned}$$

for all $(Z, g) \in \mathcal{K}_{s_1,s_2,s_3}(X \times Y) \times C^{0,\beta}(X \cup Y)$, the map from

$$\mathcal{K}_{s_1,s_2,s_3}(X \times Y) \times C^{0,\beta}(X \cup Y) \quad \text{to} \quad \mathcal{K}_{s_1-\beta,X\times Y}$$

that takes (Z, g) to the kernel $Z(x, y)(g(x) - g(y))$ is bilinear and continuous. Since $s_1 - \beta \in [0, \upsilon_Y[$, Proposition 4.1 (ii) implies that the map $Q[\cdot, \cdot, 1]$ is bilinear and continuous from $\mathcal{K}_{s_1,s_2,s_3}(X \times Y) \times C^{0,\beta}(X \cup Y)$ to $B(X)$ under the assumptions of all the statements (i)–(iii).

We now turn to estimate the Hölder quotient of $Q[Z, g, 1]$, under the assumptions of all the statements (i)–(iii). Let $x', x'' \in X$. By Remark A.2 of the appendix, it suffices to consider case $0 < 3d(x', x'') \leq r_{s_3} \leq e^{-1/s_3} (\leq 1/e < 1)$. Then Lemma 3.4 and the inclusion $B(x', 2d(x', x'')) \subseteq B(x'', 3d(x', x''))$ imply that

(6.4)

$$\begin{aligned}&\big|Q[Z, g, 1](x') - Q[Z, g, 1](x'')\big| \\ &= \left|\int_Y Z(x', y)\big(g(y) - g(x')\big)\, d\nu(y) - \int_Y Z(x'', y)\big(g(y) - g(x'')\big)\, d\nu(y)\right| \\ &\leq \int_{Y\cap B(x',2d(x',x''))} \big|Z(x', y)\big| \big|g(y) - g(x')\big|\, d\nu(y) \\ &\quad + \int_{Y\cap B(x',2d(x',x''))} \big|Z(x'', y)\big| \big|g(y) - g(x'')\big|\, d\nu(y) \\ &\quad + \left|\int_{Y\setminus B(x',2d(x',x''))} Z(x', y)\big(g(y) - g(x')\big) - Z(x'', y)\big(g(y) - g(x'')\big)\, d\nu(y)\right|\end{aligned}$$

$$
\begin{aligned}
&\leq \|Z\|_{\mathcal{K}_{s_1,X\times Y}} |g:X\cup Y|_\beta \left\{ \int_{Y\cap B(x',2d(x',x''))} \frac{d\nu(y)}{d(x',y)^{s_1-\beta}} \right. \\
&\qquad\qquad \left. + \int_{Y\cap B(x'',3d(x',x''))} \frac{d\nu(y)}{d(x'',y)^{s_1-\beta}} \right\} \\
&\quad + \left| \int_{Y\setminus B(x',2d(x',x''))} Z(x',y)\big[(g(y)-g(x')) - (g(y)-g(x''))\big]\,d\nu(y) \right| \\
&\quad + \left| \int_{Y\setminus B(x',2d(x',x''))} \big[Z(x',y)-Z(x'',y)\big](g(y)-g(x''))\,d\nu(y) \right| \\
&\leq \|Z\|_{\mathcal{K}_{s_1,X\times Y}} |g:X\cup Y|_\beta c''_{s_1-\beta,X,Y} \\
&\quad \times \left\{ (2d(x',x''))^{\upsilon_Y-(s_1-\beta)} + (3d(x',x''))^{\upsilon_Y-(s_1-\beta)} \right\} \\
&\quad + \left| \int_{Y\setminus B(x',2d(x',x''))} Z(x',y)\,d\nu(y) \right| |g:X\cup Y|_\beta d(x',x'')^\beta \\
&\quad + \|Z\|_{\mathcal{K}_{s_1,s_2,s_3}(X\times Y)} |g:X\cup Y|_\beta \int_{Y\setminus B(x',2d(x',x''))} \frac{d(x',x'')^{s_3}}{d(x',y)^{s_2}} d(y,x'')^\beta\,d\nu(y).
\end{aligned}
$$

We now turn to estimate the second addendum in the right-hand side of (6.4). If $s_1 < \upsilon_Y$, then Lemma 3.4 (i) implies that

$$
\begin{aligned}
\left| \int_{Y\setminus B(x',2d(x',x''))} Z(x',y)\,d\nu(y) \right| &|g:X\cup Y|_\beta d(x',x'')^\beta \\
&\leq \|Z\|_{\mathcal{K}_{s_1,X\times Y}} c'_{s_1,X,Y} |g:X\cup Y|_\beta d(x',x'')^\beta.
\end{aligned}
\tag{6.5}
$$

If $s_1 = \upsilon_Y$, then the definition of the norm in $\mathcal{K}^\sharp_{s_1,s_2,s_3}(X\times Y)$ implies that

$$
\begin{aligned}
\left| \int_{Y\setminus B(x',2d(x',x''))} Z(x',y)\,d\nu(y) \right| &|g:X\cup Y|_\beta d(x',x'')^\beta \\
&\leq \|Z\|_{\mathcal{K}^\sharp_{s_1,s_2,s_3}(X\times Y)} |g:X\cup Y|_\beta d(x',x'')^\beta.
\end{aligned}
\tag{6.6}
$$

If $s_1 > \upsilon_Y$, then Lemma 3.6 (i) implies that

$$
\begin{aligned}
\left| \int_{Y\setminus B(x',2d(x',x''))} Z(x',y)\,d\nu(y) \right| &|g:X\cup Y|_\beta d(x',x'')^\beta \\
&\leq \|Z\|_{\mathcal{K}_{s_1,X\times Y}} c'''_{s_1,X,Y} |g:X\cup Y|_\beta d(x',x'')^{\upsilon_Y-s_1} d(x',x'')^\beta.
\end{aligned}
\tag{6.7}
$$

We now turn to estimate the last integral in the right-hand side of (6.4) by exploiting Lemma 3.6. To do so, however, we need to replace the factor $d(y,x'')^\beta$ by a constant multiple of $d(y,x')^\beta$. Thus we note that the elementary Lemma 5.1 implies that

$$
d(y,x'') \leq 2d(x',y) \quad \forall y\in Y\setminus B\big(x',2d(x',x'')\big)
$$

and we conclude that

$$
\begin{aligned}
(6.8)\qquad & \int_{Y\setminus B(x',2d(x',x''))} \frac{d(x',x'')^{s_3}}{d(x',y)^{s_2}} d(y,x'')^{\beta}\, d\nu(y) \\
& \leq 2^{\beta} \int_{Y\setminus B(x',2d(x',x''))} \frac{d\nu(y)}{d(x',y)^{s_2-\beta}} d(x',x'')^{s_3}.
\end{aligned}
$$

We now distinguish three cases. If $s_2-\beta > \upsilon_Y$, then Lemma 3.6 (i) implies that

$$
\begin{aligned}
(6.9)\qquad & \int_{Y\setminus B(x',2d(x',x''))} \frac{d\nu(y)}{d(x',y)^{s_2-\beta}} d(x',x'')^{s_3} \\
& \leq c'''_{s_2-\beta,X,Y}\, d(x',x'')^{\upsilon_Y-(s_2-\beta)} d(x',x'')^{s_3} \\
& = c'''_{s_2-\beta,X,Y}\, d(x',x'')^{\upsilon_Y+s_3+\beta-s_2}.
\end{aligned}
$$

If $s_2-\beta = \upsilon_Y$, then Lemma 3.6 (ii) implies that

$$
\begin{aligned}
(6.10)\qquad & \int_{Y\setminus B(x',2d(x',x''))} \frac{d\nu(y)}{d(x',y)^{s_2-\beta}} d(x',x'')^{s_3} \\
& \leq c^{iv}_{X,Y} \left|\log\left(2d(x',x'')\right)\right| d(x',x'')^{s_3} \\
& \leq c^{iv}_{X,Y} \left|\log d(x',x'')\right| d(x',x'')^{s_3} \left(1+\frac{\log 2}{\left|\log d(x',x'')\right|}\right) \\
& \leq 2c^{iv}_{X,Y} \left|\log d(x',x'')\right| d(x',x'')^{s_3}.
\end{aligned}
$$

If $s_2-\beta < \upsilon_Y$, then Lemma 3.4 (i) implies that

$$
(6.11)\qquad \int_{Y\setminus B(x',2d(x',x''))} \frac{d\nu(y)}{d(x',y)^{s_2-\beta}} d(x',x'')^{s_3} \leq c'_{s_2-\beta,X,Y}\, d(x',x'')^{s_3}.
$$

We are now ready to estimate $|Q[Z,g,1](x')-Q[Z,g,1](x'')|$. We first consider statement (i), where $s_1 < \upsilon_Y$.

If $s_2-\beta > \upsilon_Y$, inequalities (6.4), (6.5), (6.8), and (6.9) imply that we can estimate $|Q[Z,g,1](x')-Q[Z,g,1](x'')|$ in terms of the powers

$$
d(x',x'')^{\upsilon_Y-(s_1-\beta)}, \quad d(x',x'')^{\beta}, \quad d(x',x'')^{\upsilon_Y+s_3+\beta-s_2}.
$$

Then we observe that condition $0 < d(x',x'') \leq r_{s_3} \leq e^{-1/s_3} \leq 1/e < 1$, as well as the inequalities $\upsilon_Y-(s_1-\beta) > 0$, $\upsilon_Y+s_3+\beta-s_2 > 0$, imply that

$$
\begin{aligned}
d(x',x'')^{\upsilon_Y-(s_1-\beta)} &\leq d(x',x'')^{\min\{\upsilon_Y-(s_1-\beta),\beta,\upsilon_Y+s_3+\beta-s_2\}}, \\
d(x',x'')^{\beta} &\leq d(x',x'')^{\min\{\upsilon_Y-(s_1-\beta),\beta,\upsilon_Y+s_3+\beta-s_2\}}, \\
d(x',x'')^{\upsilon_Y+s_3+\beta-s_2} &\leq d(x',x'')^{\min\{\upsilon_Y-(s_1-\beta),\beta,\upsilon_Y+s_3+\beta-s_2\}}.
\end{aligned}
$$

Since $\beta < \upsilon_Y-(s_1-\beta)$, we conclude that statement (a) holds true.

If $s_2 - \beta = \upsilon_Y$, inequalities (6.4), (6.5), (6.8), and (6.10) imply that we can estimate $|Q[Z, g, 1](x') - Q[Z, g, 1](x'')|$ in terms of

$$d(x', x'')^{\upsilon_Y - (s_1 - \beta)}, \quad d(x', x'')^{\beta}, \quad \omega_{s_3}\big(d(x', x'')\big).$$

Then we observe that condition

$$0 < d(x', x'') \le r_{s_3} \le e^{-1/s_3} \le 1/e < 1$$

implies that

$$\begin{aligned} d(x', x'')^{\upsilon_Y - (s_1 - \beta)} &\le \max\left\{d(x', x'')^{\upsilon_Y - (s_1 - \beta)}, d(x', x'')^{\beta}, \omega_{s_3}\big(d(x', x'')\big)\right\}, \\ d(x', x'')^{\beta} &\le \max\left\{d(x', x'')^{\upsilon_Y - (s_1 - \beta)}, d(x', x'')^{\beta}, \omega_{s_3}\big(d(x', x'')\big)\right\}, \\ \big|\log d(x', x'')\big| d(x', x'')^{s_3} &\le \max\left\{d(x', x'')^{\upsilon_Y - (s_1 - \beta)}, d(x', x'')^{\beta}, \omega_{s_3}\big(d(x', x'')\big)\right\}. \end{aligned}$$

Since $\beta < \upsilon_Y - (s_1 - \beta)$, we conclude that statement (aa) holds true.

If $s_2 - \beta < \upsilon_Y$, inequalities (6.4), (6.5), (6.8), and (6.11) imply that we can estimate $|Q[Z, g, 1](x') - Q[Z, g, 1](x'')|$ in terms of the powers

$$d(x', x'')^{\upsilon_Y - (s_1 - \beta)}, \quad d(x', x'')^{\beta}, \quad d(x', x'')^{s_3}.$$

Then we observe that condition

$$0 < d(x', x'') \le r_{s_3} \le e^{-1/s_3} \le 1/e < 1$$

implies that

$$\begin{aligned} d(x', x'')^{\upsilon_Y - (s_1 - \beta)} &\le d(x', x'')^{\min\{\upsilon_Y - (s_1 - \beta), \beta, s_3\}}, \\ d(x', x'')^{\beta} &\le d(x', x'')^{\min\{\upsilon_Y - (s_1 - \beta), \beta, s_3\}}, \\ d(x', x'')^{s_3} &\le d(x', x'')^{\min\{\upsilon_Y - (s_1 - \beta), \beta, s_3\}}. \end{aligned}$$

Since $\beta < \upsilon_Y - (s_1 - \beta)$, we conclude that statement (aaa) holds true.

The proof of statements (ii) and (iii) can be completed by arguments that are similar to those of statement (i). Thus we only sketch the proofs.

So we now prove (ii), where $s_1 = \upsilon_Y$. If $s_2 - \beta > \upsilon_Y$, inequalities (6.4), (6.6), (6.8), and (6.9) imply that we can estimate $|Q[Z, g, 1](x') - Q[Z, g, 1](x'')|$ in terms of the powers

$$d(x', x'')^{\upsilon_Y - (s_1 - \beta)}, \quad d(x', x'')^{\beta}, \quad d(x', x'')^{\upsilon_Y + s_3 + \beta - s_2},$$

where $\upsilon_Y - (s_1 - \beta) = \beta$. Then inequalities $\upsilon_Y - (s_1 - \beta) = \beta > 0$ and $\upsilon_Y + s_3 + \beta - s_2 > 0$ imply that statement (b) holds true.

If $s_2 - \beta = \upsilon_Y$, inequalities (6.4), (6.6), (6.8), and (6.10) imply that we can estimate $|Q[Z,g,1](x') - Q[Z,g,1](x'')|$ in terms of

$$d(x',x'')^{\upsilon_Y-(s_1-\beta)}, \quad d(x',x'')^{\beta}, \quad \omega_{s_3}\big(d(x',x'')\big),$$

where $\upsilon_Y - (s_1 - \beta) = \beta$. Hence, statement (bb) holds true.

If $s_2 - \beta < \upsilon_Y$, inequalities (6.4), (6.6), (6.8), and (6.11) imply that we can estimate $|Q[Z,g,1](x') - Q[Z,g,1](x'')|$ in terms of the powers

$$d(x',x'')^{\upsilon_Y-(s_1-\beta)}, \quad d(x',x'')^{\beta}, \quad d(x',x'')^{s_3},$$

where $\upsilon_Y - (s_1 - \beta) = \beta$. Then inequality $\upsilon_Y - (s_1 - \beta) = \beta > 0$ implies that statement (bbb) holds true.

Finally, we consider statement (iii), where $s_1 > \upsilon_Y$.

If $s_2 - \beta > \upsilon_Y$, inequalities (6.4), (6.7), (6.8), and (6.9) imply that we can estimate $|Q[Z,g,1](x') - Q[Z,g,1](x'')|$ in terms of the powers

$$d(x',x'')^{\upsilon_Y-(s_1-\beta)}, \quad d(x',x'')^{\upsilon_Y+s_3+\beta-s_2}.$$

Then inequalities $\upsilon_Y - (s_1 - \beta) > 0$ and $\upsilon_Y + s_3 + \beta - s_2 > 0$ imply that statement (c) holds true.

If $s_2 - \beta = \upsilon_Y$, inequalities (6.4), (6.7), (6.8), and (6.10) imply that we can estimate $|Q[Z,g,1](x') - Q[Z,g,1](x'')|$ in terms of

$$d(x',x'')^{\upsilon_Y-(s_1-\beta)}, \quad \omega_{s_3}\big(d(x',x'')\big).$$

Hence, statement (cc) holds true.

If $s_2 - \beta < \upsilon_Y$, inequalities (6.4), (6.7), (6.8), and (6.11) imply that we can estimate $|Q[Z,g,1](x') - Q[Z,g,1](x'')|$ in terms of the powers

$$d(x',x'')^{\upsilon_Y-(s_1-\beta)}, \quad d(x',x'')^{s_3}.$$

Hence, statement (ccc) holds true. ∎

It is interesting to note that although the integrand in (6.1) that defines $Q[Z,g,1](x)$ displays a weak singularity at $y = x$ when (Z,g) belongs to $\mathcal{K}_{\upsilon_Y,X\times Y} \times C^{0,\beta}(X \cup Y)$, the estimates of the Hölder quotient of $Q[Z,g,1]$ of Proposition 6.3 (ii) require that $Z \in \mathcal{K}^{\sharp}_{\upsilon_Y,s_2,s_3}(X \times Y)$, i.e., we can estimate

$$\sup_{x\in X}\ \sup_{r\in]0,+\infty[} \left| \int_{Y\setminus B(x,r)} Z(x,y)\,d\nu(y) \right|$$

and that the singularity of $Z(x,y)$ at $y = x$ is not weak. Due to the importance of such estimate, one can understand the importance of the following classical definition.

DEFINITION 6.12. Let $X, Y \subseteq M$. Let ν be as in (1.1). Let $s_1 \in \mathbb{R}$. If $K \in \mathcal{K}_{s_1,X\times Y}$ and if $K(x,\cdot)$ is ν-integrable in $Y \setminus B(x,r)$ for all $(x,r) \in X \times]0,+\infty[$, then we set

$$A^\sharp[K,1](x) \equiv \sup_{r\in]0,+\infty[} \left| \int_{Y\setminus B(x,r)} K(x,y)\,d\nu(y) \right| \quad \forall x \in X.$$

The function $A^\sharp[K,1]$ is said to be the maximal function associated to the kernel K.

So the estimates of the Hölder constant of $Q[Z,g,1]$ of Proposition 6.3 (ii) requires that we can estimate the maximal function $A^\sharp[Z,1]$ associated to the kernel Z.

EXAMPLE 6.13. Let $n \geq 2$ be a natural number and let Ω be a bounded open subset of class C^1 of $\mathbb{R}^n$. Let Z_{dl} be the kernel of the double layer potential on $\partial\Omega$ corresponding to the fundamental solution of the Laplace operator. Let ν be the ordinary surface measure on $\partial\Omega$. Then $\partial\Omega$ is strongly upper $(n-1)$-Ahlfors regular (with respect to $\partial\Omega$) and one can verify the following.

(i) If $s_3, \beta \in]0,1[$, $\beta + s_3 > 1$ and Ω is of class C^{1,s_3}, then the tangential gradient $\mathrm{grad}_x Z_{dl}(x,y)$ belongs to $(\mathcal{K}_{n-s_3,n,s_3}(\partial\Omega\times\partial\Omega))^n$ (cf. [16, §4]) and Proposition 6.3 (iii)(c) implies that $Q[\mathrm{grad}_x Z_{dl},\cdot,1]$ is linear and continuous from $C^{0,\beta}(\partial\Omega)$ to $C^{0,s_3+\beta-1}(\partial\Omega,\mathbb{R}^n)$ (see Miranda [20, Statement 15.VI], where the author mentions a result of Giraud [14]. For case $n=2$, see Fichera and De Vito [8, Statement LXXXIII]).

(ii) If $s_3 \in]0,1[$, $\beta = 1$ and Ω is of class C^{1,s_3}, then the tangential gradient $\mathrm{grad}_x Z_{dl}(x,y)$ belongs to $(\mathcal{K}_{n-s_3,n,s_3}(\partial\Omega\times\partial\Omega))^n$ (cf. [16, §4]) and Proposition 6.3 (iii)(cc) implies that $Q[\mathrm{grad}_x Z_{dl},\cdot,1]$ is linear and continuous from $C^{0,\beta}(\partial\Omega)$ to $C^{0,\omega_{s_3}(\cdot)}(\partial\Omega,\mathbb{R}^n)$.

Actually, Proposition 6.3 can be applied to analyze the properties of the double layer potential corresponding to more general second order elliptic differential operators with constant coefficients, but we have no room to show it here (see [16]).

EXAMPLE 6.14. Let $n \geq 2$ be a natural number. Let

$$X = \mathbb{B}_n(0,1) \equiv \{x \in \mathbb{R}^n : |x| < 1\}, \quad Y = \mathbb{R}^n.$$

Let $\delta, \gamma \in]0,+\infty[$. Then Y is strongly upper n-Ahlfors regular with respect to X and the (nonstandard) kernel

$$L_{\delta,\gamma}(x,y) \equiv \frac{\left|\sin(|x-y|^{-\delta})\right|^{\frac{1}{\delta+1}}}{|x-y|^\gamma} \quad \forall (x,y) \in X \times Y \setminus D_{X\times Y}$$

belongs to the class $\mathcal{K}_{\gamma,\gamma+1,\frac{1}{\delta+1}}(X \times Y)$. If $\beta \in]\frac{\delta}{\delta+1}, 1]$, $\gamma \in]n, n+\beta-\frac{\delta}{\delta+1}[$, then Proposition 6.3 (iii)(c) implies that $Q[L_{\delta,\gamma}, \cdot, 1]$ is linear and continuous from $C_b^{0,\beta}(Y)$ to $C_b^{0,\min\{n+\beta-\gamma, n+\beta-\gamma-\frac{\delta}{\delta+1}\}}(X) = C_b^{0,n+\beta-\gamma-\frac{\delta}{\delta+1}}(X)$.

7. Singular integral operators on subsets of M in spaces of Hölder continuous functions

Let X, Y be subsets of M. Let ν be as in (1.1). Then under reasonable assumptions on a $\mathbb{C}$-valued function K in $(X \times Y) \setminus D_{X\times Y}$ and on a $\mathbb{C}$-valued function φ in Y, the integral

$$\int_Y K(x, y)\varphi(y)\, d\nu(y) \tag{7.1}$$

may exist in the sense of the principal value, i.e., the limit

$$\text{p.v.} \int_Y K(x, y)\varphi(y)\, d\nu(y) \equiv \lim_{\varepsilon\to 0} \int_{Y\setminus B(x,\varepsilon)} K(x, y)\varphi(y)\, d\nu(y)$$

may exist and may define a linear operator from a function space of functions defined on Y to a function space of functions defined on X.

We plan to analyze the case in which Y is (strongly) upper υ_Y-Ahlfors regular with respect to X, K is a kernel of potential type υ_Y, and φ is Hölder continuous and bounded.

Then under additional reasonable assumptions that ensure that the above integral in (7.1) exists in the sense of the principal value also for the constant function $\varphi = 1$, the classical idea is to observe that if $\tilde{\varphi}$ is an extension to M of φ, then

$$\begin{aligned} \text{p.v.} \int_Y & K(x, y)\varphi(y)\, d\nu(y) \\ &= \text{p.v.} \int_Y K(x, y)\big(\tilde{\varphi}(y) - \tilde{\varphi}(x)\big)\, d\nu(y) + \tilde{\varphi}(x)\text{p.v.} \int_Y K(x, y)\, d\nu(y) \end{aligned} \tag{7.2}$$

for all $x \in X$ and to consider separately the first and the second integral that appear in the right-hand side of (7.2). In order to estimate the Hölder norm of the first integral in the right-hand side of (7.2) in terms of a norm of K and of the Hölder norm of $\tilde{\varphi}$, we plan to exploit Proposition 6.3. Then in order to estimate the Hölder norm of the second integral in the right-hand side of (7.2), we plan to introduce another norm for K.

We now turn to consider the first integral in the right-hand side of (7.2) and we introduce the following consequence of Proposition 6.3 (ii) that implies the convergence of the first integral in the right-hand side of (7.2).

PROPOSITION 7.3. *Let X, $Y \subseteq M$. Let*

$$\upsilon_Y \in]0, +\infty[, \quad \beta \in]0, 1[, \quad \beta \leq \upsilon_Y, \quad s_2 \in [\beta, +\infty[, \quad s_3 \in]0, 1].$$

Let ν be as in (1.1), $\nu(Y) < +\infty$. Then the following statements hold.

(i) *If $s_2 > \upsilon_Y + \beta$, $s_2 \leq \upsilon_Y + s_3$, and Y is upper υ_Y-Ahlfors regular with respect to X, then the bilinear map from $\mathcal{K}^\sharp_{\upsilon_Y,s_2,s_3}(X \times Y) \times C^{0,\beta}(X \cup Y)$ to $C^{0,\beta}_b(X)$ which takes (K, ψ) to the function*

$$\int_Y K(x, y)\big(\psi(y) - \psi(x)\big)\, d\nu(y) \quad \forall x \in X \tag{7.4}$$

is continuous.

(ii) *If $s_2 = \upsilon_Y + \beta$, $\beta < s_3$, and Y is strongly upper υ_Y-Ahlfors regular with respect to X, then the bilinear map from $\mathcal{K}^\sharp_{\upsilon_Y,s_2,s_3}(X \times Y) \times C^{0,\beta}(X \cup Y)$ to $C^{0,\beta}_b(X)$ which takes (K, ψ) to the function in (7.4) is continuous.*

(iii) *If $s_2 < \upsilon_Y + \beta$, $\beta \leq s_3$, and Y is upper υ_Y-Ahlfors regular with respect to X, then the bilinear map from $\mathcal{K}^\sharp_{\upsilon_Y,s_2,s_3}(X \times Y) \times C^{0,\beta}(X \cup Y)$ to $C^{0,\beta}_b(X)$ which takes (K, ψ) to the function in (7.4) is continuous.*

PROOF. (i) Since

$$s_2 - \beta > \upsilon_Y, \quad s_2 \leq \upsilon_Y + s_3 < \upsilon_Y + s_3 + \beta,$$

then Proposition 6.3 (ii)(b) implies that the bilinear map from

$$\mathcal{K}^\sharp_{\upsilon_Y,s_2,s_3}(X \times Y) \times C^{0,\beta}(X \cup Y) \quad \text{to} \quad C_b^{0,\min\{\beta,\upsilon_Y+s_3+\beta-s_2\}}(X),$$

which takes (K, ψ) to the function in (7.4) is continuous. Since $s_2 \leq \upsilon_Y + s_3$, we have

$$\upsilon_Y + s_3 + \beta - s_2 \geq \beta$$

and accordingly $C_b^{0,\min\{\beta,\upsilon_Y+s_3+\beta-s_2\}}(X)$ equals the space $C^{0,\beta}_b(X)$. Hence, statement (i) holds true.

(ii) Since $s_2 - \beta = \upsilon_Y$, then Proposition 6.3 (ii)(bb) implies that the bilinear map from

$$\mathcal{K}^\sharp_{\upsilon_Y,s_2,s_3}(X \times Y) \times C^{0,\beta}(X \cup Y) \quad \text{to} \quad C_b^{0,\max\{r^\beta,\omega_{s_3}(r)\}}(X)$$

which takes (K, ψ) to the function in (7.4) is continuous. Since $\beta < s_3$,

$$C_b^{0,\max\{r^\beta,\omega_{s_3}(r)\}}(X)$$

equals $C^{0,\beta}_b(X)$. Hence, statement (ii) holds true.

(iii) Since $s_2 - \beta < \upsilon_Y$, Proposition 6.3 (ii)(bbb) implies that the bilinear map from

$$\mathcal{K}^\sharp_{\upsilon_Y,s_2,s_3}(X \times Y) \times C^{0,\beta}(X \cup Y) \quad \text{to} \quad C_b^{0,\min\{\beta,s_3\}}(X),$$

which takes (K, ψ) to the function in (7.4), is continuous. Since $\beta \leq s_3$, statement (iii) holds true. ■

Before we turn to consider the second integral in the right-hand side of (7.2), we try to understand for which kernels K as in the previous Proposition 7.3 the principal value in the left-hand side of equality (7.2) exists for all $x \in X$ and defines a linear and continuous operator from $C_b^{0,\beta}(Y)$ to $C_b^{0,\beta}(X)$. We do so by means of the following, that can be considered a "$T1$ theorem" for singular integral operators acting in Hölder spaces of the sort of corresponding results of Lemarié [17] and Meyer [19] for $X = Y = \mathbb{R}^n$ and of Gatto's theorem [13, Thm. 3] who considered case $X = Y$, Y upper υ_Y-Ahlfors regular in the case $s_2 = \upsilon_Y + s_3$, $r_{X,Y,\upsilon_Y} = +\infty$. Thus the following proposition can be considered an extension of Gatto's theorem [13, Thm. 3].

PROPOSITION 7.5. *Let $X, Y \subseteq M$. Let*

$$\upsilon_Y \in]0, +\infty[, \quad \beta \in]0, 1[, \quad \beta \leq \upsilon_Y, \quad s_2 \in [\beta, +\infty[, \quad s_3 \in]0, 1].$$

Let ν be as in (1.1), $\nu(Y) < +\infty$.

If $s_2 > \upsilon_Y + \beta$, we assume that $s_2 \leq \upsilon_Y + s_3$ and that Y is upper υ_Y-Ahlfors regular with respect to X.

If $s_2 = \upsilon_Y + \beta$, we assume that $\beta < s_3$ and that Y is strongly upper υ_Y-Ahlfors regular with respect to X.

If $s_2 < \upsilon_Y + \beta$, we assume that $\beta \leq s_3$ and that Y is upper υ_Y-Ahlfors regular with respect to X.

Let $K \in \mathcal{K}^\sharp_{\upsilon_Y, s_2, s_3}(X \times Y)$. Then the following statements are equivalent.

(i) *The principal value*

$$A[K, \varphi](x) \equiv \text{p.v.} \int_Y K(x, y)\varphi(y)\, d\nu(y)$$

exists in $\mathbb{C}$ for all $x \in X$ and $\varphi \in C_b^{0,\beta}(Y)$, the function $A[K, \varphi]$ from X to $\mathbb{C}$ that takes x to $A[K, \varphi](x)$ belongs to $C_b^{0,\beta}(X)$ for all $\varphi \in C_b^{0,\beta}(Y)$, and the linear operator $A[K, \cdot]$ from $C_b^{0,\beta}(Y)$ to $C_b^{0,\beta}(X)$ that takes φ to $A[K, \varphi]$ is continuous.

(ii) *The principal value*

$$A[K, 1](x) \equiv \text{p.v.} \int_Y K(x, y)\, d\nu(y)$$

exists in $\mathbb{C}$ for all $x \in X$ and the function $A[K, 1]$ from X to $\mathbb{C}$ that takes x to $A[K, 1](x)$ belongs to $C_b^{0,\beta}(X)$.

If statements (i) and (ii) hold, then the following equality holds:

$$\text{(7.6)} \qquad \text{p.v.}\int_Y K(x,y)\varphi(y)\,d\nu(y) = \int_Y K(x,y)\big(\tilde{\varphi}(y)-\tilde{\varphi}(x)\big)\,d\nu(y) + \tilde{\varphi}(x)\text{p.v.}\int_Y K(x,y)\,d\nu(y)$$

for all $x \in X$, $\varphi \in C_b^{0,\beta}(Y)$ and for all $\tilde{\varphi} \in C_b^{0,\beta}(M)$ such that $\tilde{\varphi}_{|Y} = \varphi$.

Proof. If $\varphi \in C_b^{0,\beta}(Y)$, then there exists at least an extension $\tilde{\varphi} \in C_b^{0,\beta}(M)$ of φ to M (see the McShane extension theorem (Theorem A.3)) and we have

$$\text{(7.7)} \qquad \int_{Y\setminus B(x,\varepsilon)} K(x,y)\varphi(y)\,d\nu(y) = \int_{Y\setminus B(x,\varepsilon)} K(x,y)\big(\tilde{\varphi}(y)-\tilde{\varphi}(x)\big)\,d\nu(y) + \tilde{\varphi}(x)\int_{Y\setminus B(x,\varepsilon)} K(x,y)\,d\nu(y)$$

for all $\varepsilon \in]0,+\infty[$ and $x \in X$. By our assumptions and by Proposition 7.3, the function $K(x,y)(\tilde{\varphi}(y)-\tilde{\varphi}(x))$ is ν-integrable in the variable $y \in Y \setminus \{x\}$ and accordingly

$$\text{(7.8)} \qquad \lim_{\varepsilon\to 0}\int_{Y\setminus B(x,\varepsilon)} K(x,y)\big(\tilde{\varphi}(y)-\tilde{\varphi}(x)\big)\,d\nu(y) = \int_Y K(x,y)\big(\tilde{\varphi}(y)-\tilde{\varphi}(x)\big)\,d\nu(y)$$

for each $x \in X$. Then by taking the limit in equality (7.7) as ε tends to 0, we deduce that the principal value p.v. $\int_Y K(x,y)\varphi(y)\,d\nu(y)$ exists in $\mathbb{C}$ for all $x \in X$ if and only if the principal value

$$\text{p.v.}\int_Y \tilde{\varphi}(x)K(x,y)\,d\nu(y)$$

exists in $\mathbb{C}$ for all $x \in X$ and that in case of existence we have

$$\text{(7.9)} \qquad \text{p.v.}\int_Y K(x,y)\varphi(y)\,d\nu(y) = \int_Y K(x,y)\big(\tilde{\varphi}(y)-\tilde{\varphi}(x)\big)\,d\nu(y) + \text{p.v.}\int_Y \tilde{\varphi}(x)K(x,y)\,d\nu(y).$$

If statement (i) holds true, then by taking $\varphi = 1$, we deduce the validity of (ii). Then the equality (7.9) and equality

$$\text{(7.10)} \qquad \text{p.v.}\int_Y \tilde{\varphi}(x)K(x,y)\,d\nu(y) = \lim_{\varepsilon\to 0}\int_Y \tilde{\varphi}(x)K(x,y)\,d\nu(y) = \tilde{\varphi}(x)\lim_{\varepsilon\to 0}\int_Y K(x,y)\,d\nu(y),$$

for all $x \in X$, $\varphi \in C^{0,\beta}_b(Y)$ and for all $\widetilde{\varphi} \in C^{0,\beta}_b(M)$ such that $\widetilde{\varphi}_{|Y} = \varphi$, imply the validity of equality (7.6) of the statement.

Conversely, if statement (ii) holds true, then equality (7.10) implies that the principal value

$$\text{p.v.} \int_Y \widetilde{\varphi}(x) K(x,y)\, d\nu(y)$$

exists in $\mathbb{C}$ for all $x \in X$, $\varphi \in C^{0,\beta}_b(Y)$ and for all $\widetilde{\varphi} \in C^{0,\beta}_b(M)$ such that $\widetilde{\varphi}_{|Y} = \varphi$. Then the argument above implies that the principal value p.v. $\int_Y K(x,y)\varphi(y)\, d\nu(y)$ exists in $\mathbb{C}$ and that equality (7.9) holds for all $x \in X$, $\varphi \in C^{0,\beta}_b(Y)$ and for all $\widetilde{\varphi} \in C^{0,\beta}_b(M)$ such that $\widetilde{\varphi}_{|Y} = \varphi$. Then equalities (7.9) and (7.10) imply the validity of equality (7.6) of the statement.

We now turn to show that the linear operator $A[K,\cdot]$ is continuous from $C^{0,\beta}_b(Y)$ to $C^{0,\beta}_b(X)$. It suffices to show that $A[K,\cdot]$ is bounded on the unit ball $B_{C^{0,\beta}_b(Y)}(0,1)$ of $C^{0,\beta}_b(Y)$. By the McShane extension theorem (Theorem A.3), the set

$$\{\widetilde{\varphi}_{|X\cup Y} : \varphi \in B_{C^{0,\beta}_b(Y)}(0,1)\}$$

is bounded in $C^{0,\beta}_b(X \cup Y)$.

Then Proposition 7.3 implies that the set of the first addenda of equality (7.6) of the statement as $\varphi \in B_{C^{0,\beta}_b(Y)}(0,1)$ is bounded in $C^{0,\beta}_b(X)$.

Then the continuity of the restriction operator from $C^{0,\beta}_b(X \cup Y)$ to $C^{0,\beta}_b(X)$, the membership of $A[K,1]$ in $C^{0,\beta}_b(X)$, and the continuity of the pointwise product in $C^{0,\beta}_b(X)$ imply that the set of the second addendums of equality (7.6) of the statement as $\varphi \in B_{C^{0,\beta}_b(Y)}(0,1)$ is bounded in $C^{0,\beta}_b(X)$.

Hence, equality (7.6) of the statement implies that the set of the $A[K,\varphi]$ such that $\varphi \in B_{C^{0,\beta}_b(Y)}(0,1)$ is bounded in $C^{0,\beta}_b(X)$ and thus proof of the statement is complete. ■

Proposition 7.5 suggests to introduce the following class of potential-type kernels to estimate the Hölder norm of second integral in the right-hand side of (7.2).

DEFINITION 7.11. Let $X, Y \subseteq M$. Let ν be as in (1.1). Let $s_1, s_2, s_3 \in \mathbb{R}$, $\theta \in]0,1]$. We set

$$\mathcal{K}^{\sharp 0,\theta}_{s_1,s_2,s_3}(X \times Y) \equiv \Big\{ K \in \mathcal{K}^{\sharp}_{s_1,s_2,s_3}(X \times Y) : \\ \text{p.v.} \int_Y K(x,y)\, d\nu(y) \in \mathbb{C} \text{ for all } x \in X, \\ \Big| \text{p.v.} \int_Y K(\cdot,y)\, d\nu(y) : X \Big|_\theta < +\infty \Big\}$$

and

$$\|K\|_{\mathcal{K}^{\sharp 0,\theta}_{s_1,s_2,s_3}(X\times Y)} \equiv \|K\|_{\mathcal{K}^{\sharp}_{s_1,s_2,s_3}(X\times Y)} + \left|\text{p.v.}\int_Y K(\cdot,y)\,d\nu(y) : X\right|_\theta$$

for all $K \in \mathcal{K}^{\sharp 0,\theta}_{s_1,s_2,s_3}(X\times Y)$.

By definition of $\mathcal{K}^{\sharp 0,\theta}_{s_1,s_2,s_3}(X\times Y)$, we have

$$\left\|\text{p.v.}\int_Y K(\cdot,y)\,d\nu(y)\right\|_{C^{0,\theta}_b(X)} \leq \|K\|_{\mathcal{K}^{\sharp 0,\theta}_{s_1,s_2,s_3}(X\times Y)},$$

for all $K \in \mathcal{K}^{\sharp 0,\theta}_{s_1,s_2,s_3}(X\times Y)$. Then by combining Propositions 7.3 and 7.5, we deduce the validity of the following theorem.

THEOREM 7.12. *Let $X, Y \subseteq M$. Let*

$$\upsilon_Y \in]0,+\infty[, \quad \beta \in]0,1[, \quad \beta \leq \upsilon_Y, \quad s_2 \in [\beta,+\infty[, \quad s_3 \in]0,1].$$

Let ν be as in (1.1), $\nu(Y) < +\infty$.

If $s_2 > \upsilon_Y + \beta$, we assume that $s_2 \leq \upsilon_Y + s_3$ and that Y is upper υ_Y-Ahlfors regular with respect to X.

If $s_2 = \upsilon_Y + \beta$, we assume that $\beta < s_3$ and that Y is strongly upper υ_Y-Ahlfors regular with respect to X.

If $s_2 < \upsilon_Y + \beta$, we assume that $\beta \leq s_3$ and that Y is upper υ_Y-Ahlfors regular with respect to X.

Then the bilinear map A from

$$\mathcal{K}^{\sharp 0,\beta}_{\upsilon_Y,s_2,s_3}(X\times Y) \times C^{0,\beta}_b(Y) \quad \text{to} \quad C^{0,\beta}_b(X)$$

that takes (K,φ) to the function

$$A[K,\varphi](x) = \text{p.v.}\int_Y K(x,y)\varphi(y)\,d\nu(y) \quad \forall x \in X$$

is continuous.

PROOF. By the definition of $\mathcal{K}^{\sharp 0,\beta}_{\upsilon_Y,s_2,s_3}(X\times Y)$, Proposition 7.5 implies that the principal value that defines $A[K,\varphi](x)$ exists in $\mathbb{C}$ for all $x \in X$, and that the function $A[K,\varphi]$ from X to $\mathbb{C}$ that takes x to $A[K,\varphi](x)$ belongs to $C^{0,\beta}_b(X)$ for all $K \in \mathcal{K}^{\sharp 0,\beta}_{\upsilon_Y,s_2,s_3}(X\times Y)$ and $\varphi \in C^{0,\beta}_b(Y)$.

Since $A[\cdot,\cdot]$ is bilinear, it suffices to show that $A[\cdot,\cdot]$ is bounded on the product

$$B \equiv B_{\mathcal{K}^{\sharp 0,\beta}_{\upsilon_Y,s_2,s_3}(X\times Y)}(0,1) \times B_{C^{0,\beta}_b(Y)}(0,1)$$

of the unit balls in $\mathcal{K}^{\sharp 0,\beta}_{\upsilon_Y,s_2,s_3}(X\times Y)$ and $C^{0,\beta}_b(Y)$, respectively. By the McShane extension theorem (Theorem A.3), the set

$$B_{\mathcal{K}^{\sharp 0,\beta}_{\upsilon_Y,s_2,s_3}(X\times Y)}(0,1)\times\left\{\widetilde{\varphi}_{|X\cup Y}:\varphi\in B_{C^{0,\beta}_b(Y)}(0,1)\right\}$$

is bounded in $\mathcal{K}^{\sharp 0,\beta}_{\upsilon_Y,s_2,s_3}(X\times Y)\times C^{0,\beta}_b(X\cup Y)$. Then Proposition 7.3 and the continuous imbedding of $\mathcal{K}^{\sharp 0,\beta}_{\upsilon_Y,s_2,s_3}(X\times Y)$ into $\mathcal{K}^{\sharp}_{\upsilon_Y,s_2,s_3}(X\times Y)$ imply that the set of the first terms in the right-hand side of equality (7.6) as $(K,\varphi)\in B$ is bounded in $C^{0,\beta}_b(X)$.

Then the continuity of the restriction operator from $C^{0,\beta}_b(X\cup Y)$ to $C^{0,\beta}_b(X)$, the definition of norm in $\mathcal{K}^{\sharp 0,\beta}_{\upsilon_Y,s_2,s_3}(X\times Y)$, and the continuity of the pointwise product in $C^{0,\beta}_b(X)$ imply that the set of the second addendums of equality (7.6) as $(K,\varphi)\in B$ is bounded in $C^{0,\beta}_b(X)$.

Hence, equality (7.6) implies that the set of the $A[K,\varphi]$ such that $(K,\varphi)\in B$ is bounded in $C^{0,\beta}_b(X)$ and thus proof of the statement is complete. ■

A. Generalized Hölder spaces

Let ω be a function from $[0,+\infty[$ to itself such that

$$\begin{aligned}&\omega(0)=0,\quad \omega(r)>0\quad \forall r\in]0,+\infty[,\\ &\omega \text{ is increasing},\quad \lim_{r\to 0^+}\omega(r)=0,\\ &\text{and}\quad \sup_{(a,t)\in[1,+\infty[\times]0,+\infty[}\frac{\omega(at)}{a\omega(t)}<+\infty.\end{aligned}\tag{A.1}$$

If f is a function from a subset $\mathbb{D}$ of M to $\mathbb{C}$, then we denote by $|f:\mathbb{D}|_{\omega(\cdot)}$ the $\omega(\cdot)$-Hölder constant of f, which is delivered by the formula

$$|f:\mathbb{D}|_{\omega(\cdot)}\equiv\sup\left\{\frac{|f(x)-f(y)|}{\omega\big(d(x,y)\big)}:x,y\in\mathbb{D},\ x\neq y\right\}.$$

If $|f:\mathbb{D}|_{\omega(\cdot)}<\infty$, we say that f is $\omega(\cdot)$-Hölder continuous. Sometimes, we simply write $|f|_{\omega(\cdot)}$ instead of $|f:\mathbb{D}|_{\omega(\cdot)}$. The subset of $C^0(\mathbb{D})$ whose functions are $\omega(\cdot)$-Hölder continuous is denoted by $C^{0,\omega(\cdot)}(\mathbb{D})$ and $|f:\mathbb{D}|_{\omega(\cdot)}$ is a semi-norm on $C^{0,\omega(\cdot)}(\mathbb{D})$. Then we consider the space

$$C^{0,\omega(\cdot)}_b(\mathbb{D})\equiv C^{0,\omega(\cdot)}(\mathbb{D})\cap B(\mathbb{D})$$

with the norm

$$\|f\|_{C^{0,\omega(\cdot)}_b(\mathbb{D})}\equiv\sup_{x\in\mathbb{D}}|f(x)|+|f|_{\omega(\cdot)}\quad\forall f\in C^{0,\omega(\cdot)}_b(\mathbb{D}).$$

Remark A.2. Let ω be as in (A.1). Let $\mathbb{D}$ be a subset of M. Let f be a bounded function from $\mathbb{D}$ to $\mathbb{C}$, $a \in]0, +\infty[$. Then,

$$\sup_{x,y\in\mathbb{D},\, d(x,y)\geq a} \frac{\big|f(x) - f(y)\big|}{\omega\big(d(x,y)\big)} \leq \frac{2}{\omega(a)} \sup_{\mathbb{D}} |f|.$$

In the case in which $\omega(\cdot)$ is the function r^α for some fixed $\alpha \in]0,1]$, a so-called Hölder exponent, we simply write $|\cdot : \mathbb{D}|_\alpha$ instead of $|\cdot : \mathbb{D}|_{r^\alpha}$, $C^{0,\alpha}(\mathbb{D})$ instead of $C^{0,r^\alpha}(\mathbb{D})$, $C_b^{0,\alpha}(\mathbb{D})$ instead of $C_b^{0,r^\alpha}(\mathbb{D})$, and we say that f is α-Hölder continuous provided that $|f : \mathbb{D}|_\alpha < \infty$.

We also mention the following immediate consequence of the extension theorem of McShane [18] (see Björk [1, Prop. 1], Kufner, John, and Fučík [15, Thm. 1.8.3]).

Theorem A.3. *Let (M,d) be a metric space, $Y \subset M$. Let $\alpha \in]0,1]$. If $\varphi \in C_b^{0,\alpha}(Y)$, then there exists $\widetilde{\varphi} \in C_b^{0,\alpha}(M)$ such that*

$$\sup_M |\widetilde{\varphi}| = \sup_Y |\varphi|, \quad |\widetilde{\varphi} : M|_\alpha = |\varphi : Y|_\alpha, \quad \|\widetilde{\varphi}\|_{C_b^{0,\alpha}(M)} = \|\varphi\|_{C_b^{0,\alpha}(Y)}.$$

Proof. If $\varphi \in C_b^{0,\alpha}(Y)$, then φ is uniformly continuous and admits a unique extension $\varphi^\sharp$ to the closure $\overline{Y}$. Then one can readily show that

$$\sup_{\overline{Y}} |\varphi^\sharp| = \sup_Y |\varphi|, \quad |\varphi^\sharp : \overline{Y}|_\alpha = |\varphi : Y|_\alpha.$$

Since $\overline{Y}$ is closed, the above-mentioned extension theorem of McShane implies that there exists $\widetilde{\varphi} \in C_b^{0,\alpha}(M)$ such that $\widetilde{\varphi}_{|\overline{Y}} = \varphi^\sharp$ and

$$\sup_M |\widetilde{\varphi}| = \sup_{\overline{Y}} |\varphi^\sharp|, \quad |\widetilde{\varphi} : M|_\alpha = |\varphi^\sharp : \overline{Y}|_\alpha.$$

Accordingly, $\widetilde{\varphi}_{|Y} = \varphi$ and the equalities of the statement follow. ∎

One could exploit the extension theorem of McShane to define an isometric extension operator from $C_b^{0,\alpha}(Y)$ to $C_b^{0,\alpha}(M)$. However, such extension operator is not necessarily linear.

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