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Mathematical Analysis. – *Siegel disks on rational surfaces*, by Takato Uehara, communicated on 10 November 2022.

Abstract. – We show the existence of a rational surface automorphism of positive entropy with a given number of Siegel disks. Moreover, among automorphisms obtained from quadratic birational maps on the projective plane fixing irreducible cubic curves, we find out an automorphism of positive entropy with multiple Siegel disks.

Keywords. – Siegel disks, rational surfaces, biholomorphic automorphisms.

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1. Introduction

A Siegel disk for a holomorphic map on a complex manifold is a domain of the manifold preserved by the map such that the restriction to the domain is analytically conjugate to an irrational rotation (see Section 2). Siegel disks are interesting objects and have been constructed by many authors especially for automorphisms on rational manifolds with positive entropy. For example, McMullen [8] and Bedford–Kim [2, 3] constructed rational surfaces, namely, rational manifolds of dimension 2, admitting automorphisms of positive entropy with Siegel disks by considering a certain class of birational maps on the projective plane. Moreover, Oguiso–Perroni [9] constructed rational manifolds of dimension ≥ 4 admitting automorphisms of positive entropy with an arbitrarily high number of Siegel disks by using the product construction made of automorphisms on McMullen's rational surfaces and toric manifolds.

The automorphisms we considered in this paper not only have positive entropy but also preserve meromorphic volume forms. In this case, the interesting feature of each automorphism F is that it is obtained from birational map on $\mathbb{P}^2$ by blowing up finitely many points on the smooth locus of a cubic curve in $\mathbb{P}^2$ and that it falls into the category described by Bedford [1]. Moreover, every F-invariant Fatou component with finite volume turns out to be a *rotation domain*, and a Siegel disk corresponds to a rotation domain of rank 2 containing a fixed point of F (see [3]). In particular, the Fatou set of F is nonempty.

This paper presents two families of automorphisms of rational surfaces with Siegel disks. The first one preserves meromorphic volume forms whose pole divisors consist of three rational curves meeting at a single point. One of the main theorems is to show the existence of a rational surface automorphism of positive entropy with a given number of Siegel disks.

THEOREM 1.1. *For any $k \in \mathbb{Z}_{\geq 0}$, there exists a rational surface X and an automorphism $F : X \to X$ such that F has positive entropy $h_{\mathrm{top}}(F) > 0$ and F has exactly k fixed points at which Siegel disks are centered.*

The automorphism F mentioned in Theorem 1.1 is obtained from a birational map $f : \mathbb{P}^2 \to \mathbb{P}^2$ of degree $\max\{2, k-1\}$ by blowing up points on the smooth locus of a cubic curve C in $\mathbb{P}^2$. When $k \geq 3$, the curve C we considered is the union of three lines meeting at a single point.

Next, we consider the case where automorphisms are obtained from quadratic birational maps on $\mathbb{P}^2$ that fix a cubic curve C. Let $f : \mathbb{P}^2 \to \mathbb{P}^2$ be a birational map with its inverse $f^{-1} : \mathbb{P}^2 \to \mathbb{P}^2$ and its indeterminacy set $I(f)$, namely, the set of points on which f is not defined. We say that f *properly fixes* C if the indeterminacy sets $I(f^{\pm 1})$ of $f^{\pm 1}$ are both contained in the smooth locus C^* of C, and

$$f(C) := \overline{f\big(C \setminus I(f)\big)} = C.$$

It is known that a certain class of quadratic birational maps properly fixing C is lifted to automorphisms with positive entropy by blowing up finitely many points on C^* (see [2, 3, 5, 8, 11, 12]). Let $\mathcal{QF}(C)$ be the set of automorphisms $F : X \to X$ on rational surfaces X with positive entropy and with the property that there is a quadratic birational map $f : \mathbb{P}^2 \to \mathbb{P}^2$ that properly fixes C and a blowup $\pi : X \to \mathbb{P}^2$ of points on C^* such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{F} & X \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ \mathbb{P}^2 & \xrightarrow{f} & \mathbb{P}^2 \end{array}$$

commutes. Then, F preserves a meromorphic volume form whose pole divisor is the strict transform of C. In the case where C is non-reduced, Bedford–Kim [3] constructed $F \in \mathcal{QF}(C)$ with multiple Siegel disks, when C is a single line with multiplicity 3. On the other hand, McMullen [8] and Bedford–Kim [2] constructed $F \in \mathcal{QF}(C)$ with a single Siegel disk, when C is reduced but non-irreducible. In this article, we focus our attention on the case of irreducible cubic curves and obtain the following theorem.

THEOREM 1.2. *For a reduced irreducible cubic curve C on $\mathbb{P}^2$, if there is an automorphism $F \in \mathcal{QF}(C)$ having a Siegel disk, then C is a cuspidal cubic curve. Moreover, if C is a cuspidal cubic curve, then $F \in \mathcal{QF}(C)$ admits at most two fixed points at which Siegel disks are centered, and there is an automorphism $F \in \mathcal{QF}(C)$ having exactly two fixed points at which Siegel disks are centered.*

The existence of a Siegel disk for an automorphism F centered at x implies that the derivative $DF(x)$ of F at x has multiplicatively independent eigenvalues (μ, ν) with $|\mu| = |\nu| = 1$ (see Section 2). Conversely, results from transcendence theory guarantee that F has a Siegel disk centered at x under the assumption that the multiplicatively independent eigenvalues (μ, ν) with $|\mu| = |\nu| = 1$ are algebraic. Moreover, if algebraic eigenvalues (μ, ν) with $|\mu| = |\nu| = 1$ have Galois conjugates (μ_*, ν_*) satisfying $|\mu_* \nu_*| = 1$, but $|\mu_*/\nu_*| \neq 1$, then (μ, ν) are multiplicatively independent (see also [8]). Our task is thus to construct automorphisms whose derivatives have such a pair (μ, ν) of eigenvalues. Note that in our construction, the automorphisms are obtained from birational maps, and the birational maps considered here have explicit forms with parameters.

After preliminary studies in Section 2, Sections 3 and 4 are devoted to constructing automorphisms with Siegel disks in order to prove Theorems 1.2 and 1.1, respectively, and Sections 5 and 6 are devoted to proving two propositions needed in our discussion.

2. Preliminary

In this section, we briefly review some well-known facts about Siegel disks on complex surfaces, automorphisms on rational surfaces, and cubic curves on the projective plane used later. We refer to [5, 8, 11, 12], in which many of the results are proved.

First, we recall the definition of a Siegel disk on a complex surface (see [8]). For a unit disk $\Delta^2 := \{(x, y) \in \mathbb{C}^2 \mid |x| \leq 1,\ |y| \leq 1\}$, a linear automorphism $L : \Delta^2 \to \Delta^2$ given by $L(x, y) = (\mu x, \nu y)$ is called an *irrational rotation* if $|\mu| = |\nu| = 1$ and (μ, ν) are *multiplicatively independent*; that is, they satisfy $\mu^k \nu^l \neq 1$ for any $(k, l) \neq (0, 0) \in \mathbb{Z}^2$.

Definition 2.1. Let X be a complex surface and F an automorphism on X. A domain $U \subset X$ is called a *Siegel disk* for F centered at $p \in U$ if $F(U) = U$ and $F : (p, U) \to (p, U)$ is analytically conjugate to an irrational rotation $L : (0, \Delta^2) \to (0, \Delta^2)$.

It is obvious that the derivative $DF(p)$ of F at p is an irrational rotation when F has a Siegel disk centered at p. Conversely, results from the transcendence theory say that if $DF(p)$ is an irrational rotation with *algebraic eigenvalues*, then F has a Siegel disk centered at p (see [8]).

Next, we consider rational surfaces. Here, we assume that a rational surface X admits a birational morphism $\pi : X \to \mathbb{P}^2$ (see [5, 8, 11, 12]). Then, it is known that π is expressed as a composition

$$\pi : X = X_\rho \xrightarrow{\pi_\rho} X_{\rho-1} \xrightarrow{\pi_{\rho-1}} \cdots \xrightarrow{\pi_2} X_1 \xrightarrow{\pi_1} X_0 = X,$$

where $\pi_i : X_i \to X_{i-1}$ is the blowup of a point $p_i \in X_{i-1}$ with the exceptional curve $\mathcal{E}_i := \pi_i^{-1}(\{p_i\})$, which is isomorphic to $\mathbb{P}^1$. Since π_i induces an isomorphism $\pi_i|_{X_i \setminus \mathcal{E}_i} : X_i \setminus \mathcal{E}_i \to X_{i-1} \setminus \{p_i\}$, we will identify each point $x \in X_i \setminus \mathcal{E}_i$ with $\pi_i(x) \in X_{i-1} \setminus \{p_i\}$ in this article. Moreover, if p is a point on an exceptional curve, we sometimes say that p is an *infinitely near point* on $\mathbb{P}^2$, or a *point* on $\mathbb{P}^2$ for short. On the other hand, a point is said to be *proper* if it is *not* an infinitely near point. The total transform $E_i := \pi_\rho^* \circ \cdots \circ \pi_{i+1}^*(\mathcal{E}_i)$ is called the *exceptional divisor* over p_i. Then, π gives an expression of the cohomology group:

$$H^2(X;\mathbb{Z}) \cong \mathrm{Pic}(X) = \mathbb{Z}[H] \oplus \mathbb{Z}[E_1] \oplus \cdots \oplus \mathbb{Z}[E_\rho],$$

where H is the total transform $\pi^*(L)$ of a line L in $\mathbb{P}^2$. The intersection form on the cohomology group $H^2(X;\mathbb{Z})$ is given by

$$\begin{cases} \big([H],[H]\big) = 1 & \\ \big([E_i],[E_j]\big) = -\delta_{i,j} & (i,j = 1,\dots,\rho) \\ \big([H],[E_i]\big) = 0 & (i = 1,\dots,\rho). \end{cases}$$

Let $F : X \to X$ be an automorphism on X. Then, F induces the action $F^* : H^2(X;\mathbb{Z}) \to H^2(X;\mathbb{Z})$ on the cohomology group. By the theorems of Gromov and Yomdin, the *topological entropy of* F is given by $h_{\mathrm{top}}(F) = \log \lambda(F^*) \geq 0$, where $\lambda(F^*)$ is the spectral radius of F^*. Moreover, since F^* preserves the Kähler cone and the intersection form with signature $(1,\rho)$, it is seen that the characteristic polynomial of F is expressed as

$$\det(tI - F^*) = \begin{cases} R_F(t) & \big(\lambda(F^*) = 1\big) \\ R_F(t)S_F(t) & \big(\lambda(F^*) > 1\big), \end{cases}$$

where $R_F(t)$ is a product of cyclotomic polynomials and $S_F(t)$ is a Salem polynomial, namely, the minimal polynomial of a Salem number. Here, a *Salem number* is an algebraic unit $\delta > 1$ such that its conjugates include $\delta^{-1} < 1$ and the conjugates other than $\delta^{\pm 1}$ lie on the unit circle. Hence, if $\lambda(F^*) > 1$, then it is a root of $S_F(t) = 0$.

Now, we consider a cubic curve $C \subset \mathbb{P}^2$, that is, a reduced (possibly non-irreducible or singular) curve of degree three, with its smooth locus C^* (see [5, 8]). Denote by $\mathrm{Pic}^0(C) \subset \mathrm{Pic}(C)$ the subgroup consisting of divisor classes whose restrictions to each irreducible component of C have degree zero. Then, it is known that $\mathrm{Pic}^0(C) \cong \mathbb{C}/\Gamma$, where $\Gamma \subset \mathbb{C}$ is a lattice with rank given by either

(1) $\operatorname{rank}\Gamma = 2$ if C is smooth, or

(2) $\operatorname{rank}\Gamma = 1$ if C is a nodal cubic, or a conic with a transverse line, or three lines meeting in three points, or

(3) $\operatorname{rank}\Gamma = 0$ if C is a cuspidal cubic, or a conic with a tangent line, or three lines through a single point.

Let $V_1, \dots, V_r$ be the irreducible components of C. Note that $1 \le r \le 3$ as C is a cubic curve. Moreover, fix points $0_i \in V_i \cap C^*$ so that $\sum_{i=1}^{r} \deg V_i \cdot [0_i] = 0$; namely, the divisor $\sum_{i=1}^{r} \deg V_i \cdot 0_i$ is the restriction of a line $L \subset \mathbb{P}^2$ to C^*, where $\deg V_i \in \mathbb{Z}_{>0}$ is the degree of the component V_i in $\mathbb{P}^2$. For each $1 \le j \le r$, let $\kappa : V_j \cap C^* \to \mathrm{Pic}^0(C)$ be the map defined by $\kappa(p) = [p] - [0_j]$. Then, κ is a bijection, which gives the group structure on $V_j \cap C^*$ isomorphic to $\mathrm{Pic}^0(C) \cong \mathbb{C}/\Gamma$, with the property that three points $q_1, q_2, q_3 \in C^*$ satisfy $\sum_{i=1}^{3}[q_i] = 0$ if and only if $\sum_{i=1}^{3} \kappa(q_i) = 0$ and $\#\{i \mid q_i \in V_j\} = \deg V_j$ for any $1 \le j \le r$ (see [5]).

Let $f : \mathbb{P}^2 \to \mathbb{P}^2$ be a birational map on $\mathbb{P}^2$. In general, f admits the *indeterminacy set* $I(f)$, namely, the finite set on which f cannot be defined (see [11, 12]). Note that $I(f)$ is a cluster; that is, if $p \in I(f)$ is infinitely near to a point q, then $q \in I(f)$. All birational maps considered in this article are assumed to belong to the set $\mathcal{B}(C)$ of birational maps f *properly fixing* C; namely, $I(f^{\pm 1}) \subset C^*$ and $f(C) = C$. Here, if $I(f^{\pm 1})$ contain an infinitely near point p, then $p \in C^*$ means that p belongs to the strict transform $\pi^{-1}(C^*)$, where $\pi : X \to \mathbb{P}^2$ is a birational morphism such that p is proper on X. When $f \in \mathcal{B}(C)$, there is $\delta(f) \in \mathbb{C}^*$, called the *determinant* of f, such that $f^*\eta = \delta(f)\eta$, where η is a nowhere vanishing meromorphic 2-form on $\mathbb{P}^2$ having simple poles along C. The determinant $\delta(f)$ satisfies $\delta(f) = \mathrm{Det}\, Df(p)$ for any fixed point $p \in \mathbb{P}^2 \setminus C$ of f. Moreover, it should be noted that f preserves the smooth locus C^* under our assumption. Thus, f induces the actions $f_* : \mathrm{Pic}(C) \to \mathrm{Pic}(C)$ and $f_* : \mathrm{Pic}^0(C) \to \mathrm{Pic}^0(C)$. Through the Poincaré residue map, it turns out that the action f_* on $\mathrm{Pic}^0(C) \cong \mathbb{C}/\Gamma$ is given by $f_*(t) = \delta(f)t$ for $t \in \mathbb{C}/\Gamma$ (see [8]). Note that if $\mathrm{rank}\,\Gamma \ge 1$, then $\delta(f)$ must be a root of unity as $\delta(f)\Gamma = \Gamma$, while if $\mathrm{rank}\,\Gamma = 0$, then $\delta(f)$ may be an arbitrary nonzero complex number.

One of our interests is to construct automorphisms on rational surfaces. From birational maps on $\mathbb{P}^2$ satisfying a certain assumption, we obtain rational surface automorphisms.

PROPOSITION 2.2. *Assume that $C \subset \mathbb{P}^2$ is a reduced cubic curve.*

(1) *For a birational map $f : \mathbb{P}^2 \to \mathbb{P}^2$ in $\mathcal{B}(C)$, assume that any indeterminacy point $p \in I(f^{-1})$ satisfies $f^m(p) \in I(f)$ for some $m = m(p) \ge 0$. Then, there is a blowup $\pi : X \to \mathbb{P}^2$ of points on C^* such that π lifts $f : \mathbb{P}^2 \to \mathbb{P}^2$ to an automorphism $F : X \to X$.*

(2) *Assume that a birational map $f : \mathbb{P}^2 \to \mathbb{P}^2$ in $\mathcal{B}(C)$ is lifted to an automorphism $F : X \to X$ by a blowup $\pi : X \to \mathbb{P}^2$ of points on C^*. Then, any indeterminacy point $p \in I(f^{-1})$ satisfies $f^k(p) \notin I(f)$ with $0 \le k < m_p$ and $f^{m_p}(p) \in I(f)$ for some $m_p \ge 0$. Moreover, π admits an expression*

$$\pi = \pi_0 \circ \nu : X \to \mathbb{P}^2,$$

where $\pi_0 : X_0 \to \mathbb{P}^2$ *is the blowup of the points* $\{f^k(p) \mid p \in I(f^{-1}), 0 \le k \le m_p\}$ *on* C^**, and* $\nu : X \to X_0$ *is a birational morphism. Furthermore, the blowup* $\pi_0 : X_0 \to \mathbb{P}^2$ *lifts* f *to an automorphism* $F_0 : X_0 \to X_0$.

Proof. (1) (see [12]). Let $(p, q) \in I(f^{-1}) \times I(f)$ be a pair of *proper* points so that $f^n(p) = q$ with $n = \min\{m \in \mathbb{N} \mid f^m(p') = q' \text{ for } (p', q') \in I(f^{-1}) \times I(f)\}$. Under our assumption, such a pair (p, q) exists, and from the minimality of n, the orbit $\{f^i(p)\}_{i=0}^n$ consists of distinct proper points on the smooth locus C^*. Now, let $X_0 \to \mathbb{P}^2$ be the blowup of $\{f^i(p)\}_{i=0}^n$. This blowup lifts $f : \mathbb{P}^2 \to \mathbb{P}^2$ to a birational map $f_0 : X_0 \to X_0$, which satisfies

$$I(f_0^{-1}) = I(f^{-1}) \setminus \{p\}, \quad I(f_0) = I(f) \setminus \{q\}.$$

Note that $\#I(f^{-1}) = \#I(f)$. Hence, as long as $\#I(f_0^{-1}) = \#I(f_0) > 0$, one can repeat the argument by replacing $f : \mathbb{P}^2 \to \mathbb{P}^2$ with $f_0 : X_0 \to X_0$. In the end, a resulting map becomes an automorphism. See [12] for a more detailed discussion.

(2) (see [11]). We notice that if $p \in I(f^{-1})$ satisfies $f^k(p) \notin I(f)$ for $0 \le k \le m-1$, then $f^m(p)$ is a well-defined point in $I(f^{-m})$. As π lifts f^m to the automorphism F^m, the point $f^m(p)$ must be blown up by π. Since the number of points blown up by π is finite, there is $m_p \ge 0$ such that $f^k(p) \notin I(f)$ for $0 \le k \le m_p - 1$ and $f^{m_p}(p) \in I(f)$. Moreover, π blows up the points $\{f^k(p) \mid p \in I(f^{-1}), 0 \le k \le m_p\}$, and hence π admits the expression $\pi = \pi_0 \circ \nu : X \to \mathbb{P}^2$. The blowup π_0 lifts f to an automorphism F_0 from a similar argument in the proof of (1). See [11] for a more detailed discussion. ∎

Definition 2.3. For a birational map $f \in \mathcal{B}(C)$ satisfying the assumption in Proposition 2.2 (1), the blowup π_0 given in Proposition 2.2 (2) is called the *proper blowup* for f.

Remark 2.4. Let $f : \mathbb{P}^2 \to \mathbb{P}^2$ be a birational map lifted to an automorphism $F : X \to X$ by a blowup $\pi : X \to \mathbb{P}^2$. With the identification of a point $p \in X$ with $\pi(p) \in \mathbb{P}^2$ under the assumption that $\pi(p) \notin I(\pi^{-1})$, the dynamical behavior of F around p is the same as that of f around the corresponding point. In particular, F has a Siegel disk centered at p if and only if so does f.

The next lemma is used to calculate the cohomological actions of automorphisms.

Lemma 2.5. *Let* π *be the proper blowup for* f*, which lifts* f *to an automorphism* F*, and let* $p_1, \dots, p_\rho$ *be the points blown up by* π *and* E_l *the exceptional divisor over* p_l*. If a point* p_i *satisfies* $p_i \notin I(f^{-1})$*, then the action* F^* *of* F *sends* E_i *to* E_j *for some* $j \ne i$.

PROOF. Under the notations given in the proof of Proposition 2.2 (1), we may assume that $p_i = f^k(p) \notin I(f^{-1})$ for some $k \geq 1$, as the other cases can be treated in a similar manner. Note that $f^m(p) \notin I(f^{-1})$ for any $0 \leq m \leq k$ in this case. As is mentioned in the proof of Proposition 2.2 (1), the blowup $X_0 \to \mathbb{P}^2$ of $\{f^i(p)\}_{i=0}^n$ lifts f to $f_0 : X_0 \to X_0$, and then f_0 sends $\mathcal{E}^{k-1}$ to $\mathcal{E}^k$, where $\mathcal{E}^l$ is the exceptional curve over $f^l(p)$. As the indeterminacy set is a cluster, any point on $\mathcal{E}^k$ is not an indeterminacy point of f^{-1}. Moreover, since π is a proper blowup for f, there is a point $p' \in \mathcal{E}^k$ blown up by π if and only if there is a point $p'' \in \mathcal{E}^{k-1}$ blown up by π such that $f_0(p'') = p'$, which shows that F sends the irreducible components of the exceptional divisor over $f^{k-1}(p)$ to those over $f^k(p)$. Therefore, F^* sends the exceptional divisor over $f^k(p)$ to that over $f^{k-1}(p)$. ∎

EXAMPLE 2.6. We consider a quadratic birational map on $\mathbb{P}^2$. It is known that the inverse of any quadratic birational map is also quadratic, and the indeterminacy set of a quadratic birational map consists of exactly three non-collinear (possibly infinitely near) points. Let $f : \mathbb{P}^2 \to \mathbb{P}^2$ be a quadratic birational map in $\mathcal{B}(C)$, and put

$$I(f^{\pm 1}) = \{p_1^{\pm}, p_2^{\pm}, p_3^{\pm}\} \subset C^*.$$

Then, f lifts to an automorphism if and only if $f^k(p_i^-) \notin I(f)$ for $0 \leq k < n_i$ and $f^{n_i}(p_i^-) = p_{\sigma(i)}^+$ for any $i \in \{1,2,3\}$, where $n_1, n_2, n_3 \geq 0$ are integers and $\sigma : \{1,2,3\} \to \{1,2,3\}$ is a permutation. Let π_0 be the proper blowup for f, which lifts f to an automorphism $F_0 : X_0 \to X_0$. With a suitable matching of the indices between forward and backward indeterminacies, the action $F_0^* : H^2(X_0;\mathbb{Z}) \to H^2(X_0;\mathbb{Z})$ is expressed as

$$\begin{cases} [H] \mapsto 2[H] - [E_1^{n_1}] - [E_2^{n_2}] - [E_3^{n_3}], & \\ [E_i^0] \mapsto [H] - [E_{\sigma(j)}^{n_j}] - [E_{\sigma(k)}^{n_k}] & \big(\{i,j,k\} = \{1,2,3\}\big), \\ [E_l^m] \mapsto [E_l^{m-1}] & \big(l \in \{1,2,3\},\ m \geq 1\big), \end{cases}$$

where E_l^m is the exceptional divisor over $f^m(p_l^-)$ (see [5, 12]).

As is mentioned in Proposition 2.2, we assume that the points $(p_1, \dots, p_\rho)$ blown up by $\pi : X \to \mathbb{P}^2$ lie on the smooth locus C^* of the cubic curve C, and we also assume that π lifts a birational map $f : \mathbb{P}^2 \to \mathbb{P}^2$ in $\mathcal{B}(C)$ to an automorphism $F : X \to X$. Since f preserves C, the automorphism F also preserves the strict transform Y of C, which is the closure of $\pi^{-1}(C \setminus \{p_1, \dots, p_\rho\})$. Moreover, as the points p_i lie on C^*, the curve Y is isomorphic to C and anticanonical on X; namely, $[Y] = -K_X$, where $K_X := -3[H] + \sum_{i=1}^{\rho}[E_i]$. Under the above notation, we have the following proposition.

Proposition 2.7. *Assume that* $\mathrm{Pic}^0(C) \cong \mathbb{C}$, *and also assume that*

(1) $\#\{1 \le i \le \rho \mid p_i \in V_j\} \ge \deg V_j$ *for any irreducible component* V_j *of* C,

(2) $\kappa(p_i) \ne 0$ *for some* $1 \le i \le \rho$, *where* $\kappa : V_j \cap C^* \to \mathrm{Pic}^0(C) \cong \mathbb{C}$ *is given by* $\kappa(p) = [p] - [0_j]$.

Then, the determinant $\delta(f)$ *is an eigenvalue of* $F^* : H^2(X;\mathbb{Z}) \to H^2(X;\mathbb{Z})$.

Proof. Let $r \in \{1,2,3\}$ be the number of irreducible components of C. From assumption (1), we may assume that $\#\{1 \le i \le 3 \mid p_i \in V_j\} = \deg V_j$ for $1 \le j \le r$, after reordering (p_i) if necessary, and also choose $\sigma : \{1,\dots,\rho\} \to \{1,\dots,r\}$ so that $p_i \in V_{\sigma(i)}$ for $1 \le i \le \rho$. Let us consider the restriction map $u : H^2(X;\mathbb{Z}) \cong \mathrm{Pic}(X) \to \mathrm{Pic}(Y) \cong \mathrm{Pic}(C)$, explicitly given by

$$u[H] = \sum_{i=1}^{r} \deg V_i \cdot [0_i], \quad u[E_i] = [p_i] \quad (i = 1,\dots,\rho).$$

Then, the following diagram commutes:

$$\begin{array}{ccc} H^2(X;\mathbb{Z}) & \xrightarrow{F_*} & H^2(X;\mathbb{Z}) \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle u} \\ \mathrm{Pic}(C) & \xrightarrow{f_*} & \mathrm{Pic}(C). \end{array}$$

For simplicity, we denote by the same notation V_i the strict transform of V_i. Since F^* preserves the intersection form and permutes the curves $\{V_1,\dots,V_r\}$, it preserves the orthogonal complement $\mathcal{H}_X := \{[V_1],\dots,[V_r]\}^{\perp} \subset H^2(X;\mathbb{Z})$, generated by $(B_0, B_{r+1},\dots,B_\rho)$ with

$$B_0 := [H] - [E_1] - [E_2] - [E_3], \quad B_i := [E_i] - [E_{\sigma(i)}] \quad (i = r+1,\dots,\rho).$$

We notice that the image of u restricted to $\mathcal{H}_X$ is contained in $\mathrm{Pic}^0(C)$.

Now, let us fix a vector $\xi \in H^2(X;\mathbb{C}) = H^2(X;\mathbb{Z}) \otimes \mathbb{C}$ satisfying

$$\kappa(p_i) = -\big(\xi, [H]/3 - [E_i]\big) \in \mathrm{Pic}^0(C) \cong \mathbb{C}.$$

Note that under assumption (2), the vector ξ is nonzero and unique in $H^2(X;\mathbb{C})/\mathbb{C}[K_X]$. Then, we have

$$\begin{aligned} u(B_0) &= \sum_{i=1}^{r} \deg V_i \cdot [0_i] - \sum_{i=1}^{3} [p_i] = \sum_{i=1}^{3} \{[0_{\sigma(i)}] - [p_i]\} \\ &= -\sum_{i=1}^{3} \kappa(p_i) = \sum_{i=1}^{3} \big(\xi, [H]/3 - [E_i]\big) = (\xi, B_0). \end{aligned}$$

In a similar manner, it follows that $u(B_i) = (\xi, B_i)$ and thus $u(D) = (\xi, D)$ for any $D \in \mathcal{H}_X$. Note that the action f_* on $\mathrm{Pic}^0(C) \cong \mathbb{C}$ is given by $f_*(t) = \delta(f)t$ for $t \in \mathbb{C}$. Therefore, for any $D \in \mathcal{H}_X$, we have

$$u(F_*D) = (\xi, F_*D) = (F^*\xi, D) = f_*u(D) = \delta(f)(\xi, D) = \big(\delta(f)\xi, D\big),$$

which yields $F^*\xi = \delta(f)\xi + \sum_{i=1}^r c_i[V_i]$ for some $c_i \in \mathbb{C}$. Since F^* preserves $\{[V_1], \dots, [V_r]\}$, $\delta(f)$ is an eigenvalue of F^*. The proposition is established. ∎

Now, in addition to the assumptions in Proposition 2.7, we also assume that C is a cuspidal cubic curve and the determinant $\delta(f)$ is not a root of unity. Then, $\delta(f)$ is a root of the Salem polynomial $S_F(t) = 0$ by Proposition 2.7, and the entropy of F is positive: $h_{\mathrm{top}}(F) = \log \lambda(F^*) > 0$. In this case, the birational morphism $\nu : X \to X_0$ mentioned in Proposition 2.2 is expressed as follows. Let $q \in Y^*$ be a fixed point on the smooth locus $Y^* \cong \mathbb{C}$ of the anticanonical curve Y, which uniquely exists as F has the determinant $\delta(f) \neq 1$. A result in [11] says that if ν is not an isomorphism, then there is a unique (-1)-curve passing through q, which is contracted by ν and is preserved by F. Through the contraction of the (-1)-curve, F descends to an automorphism. Repeating this argument, we can consider the decomposition

$$\nu : X = X_m \xrightarrow{\nu_m} X_{m-1} \xrightarrow{\nu_{m-1}} \cdots \xrightarrow{\nu_2} X_1 \xrightarrow{\nu_1} X_0, \tag{1}$$

where $\nu_i : X_i \to X_{i-1}$ is the contraction of a (-1)-curve through p_i to p_{i-1} with $p_m := q$. Then, F descends to an automorphism $F_0 : X_0 \to X_0$.

Let $\mathcal{N}_i \subset X$ be the strict transform of the exceptional curve of ν_i under $\nu_{i+1} \circ \cdots \circ \nu_m$. As $\mathcal{N}_i$ is isomorphic to $\mathbb{P}^1$ and is preserved by F, we inductively let q_i be the unique fixed point on $\mathcal{N}_i \setminus \{q_{i+1}\}$ of F with $q_{m+1} := q$. In particular, $(q_1, \dots, q_m, q)$ are all of the fixed points lying on the exceptional divisors of ν. Moreover, let $p \in C$ be the singular point of C, which is also a fixed point of F.

Proposition 2.8 ([11]). *Under the above assumptions, we have the following.*

(1) *The eigenvalues of DF at p are $1/\delta(f)^2$ and $1/\delta(f)^3$.*

(2) *The eigenvalues of DF at q are $\delta(f)$ and $1/\delta(f)^{N-4}$, where $N = \mathrm{rank}\,\mathrm{Pic}(X)$.*

(3) *The eigenvalues of DF at q_i for $1 \le i \le m$ are $\delta(f)^{N-m+i-4}$ and $1/\delta(f)^{N-m+i-5}$.*

In particular, F has no Siegel disk centered at any fixed point on the anticanonical curve Y and the exceptional divisors of ν.

Next, we give an estimate of the number of isolated fixed points of an automorphism.

Proposition 2.9. *Assume that an automorphism $F : X \to X$ on a rational surface X has positive entropy, and the derivative $DF(x)$ of F on any fixed point x has an eigenvalue different from 1. Then, F has at most $\mathrm{Tr}(F^*|_{H^2(X;\mathbb{Z})}) + 2$ isolated fixed points.*

We postpone its proof to Section 5. The following two propositions are applications of Proposition 2.9.

Proposition 2.10. *Let $C \subset \mathbb{P}^2$ be a reduced cubic curve with* $\mathrm{Pic}^0(C) \cong \mathbb{C}$, *and let $F : X \to X$ be an automorphism with positive entropy such that F is obtained from a birational map $f \in \mathcal{B}(C)$ by the blowup $\pi : X \to \mathbb{P}^2$ of points on C^*. Assume that $\delta(f)$ is not a root of unity. Then, F has at most* $\mathrm{Tr}(F^*|_{H^2(X;\mathbb{Z})}) + 2$ *isolated fixed points.*

Proof. First, we notice that our assumption says that for any fixed point x on the strict transform Y of C, which is an anticanonical curve on X, the derivative $DF(x)$ of F on x has an eigenvalue different from 1. Indeed, if x lies on the smooth locus Y^*, then $DF(x)$ has $\delta(f)$ as an eigenvalue. On the other hand, if x is a singular point of Y, then $DF(x)$ has eigenvalues of the form $\epsilon\delta(f)^{-m}$, where ϵ is a root of unity and $m \in \mathbb{Z}_{>0}$ is a positive integer (see [8, Section 9]).

This remains true for any fixed point x outside Y since $\mathrm{Det}\, DF(x) = \delta(f) \neq 1$ from the existence of a nowhere vanishing meromorphic 2-form $\eta_X = \pi^*\eta$ on X with $(\eta_X) = -Y$ and $F^*\eta_X = \delta(f)\eta_X$. Hence, the proposition follows from Proposition 2.9. ∎

Proposition 2.11. *For a cuspidal cubic curve C, let $f \in \mathcal{B}(C)$ be a quadratic birational map with $\delta(f)$ being not a root of unity such that f is lifted to an automorphism $F : X \to X$ by the blowup $\pi : X \to \mathbb{P}^2$ of points on C^*. Then, F has at most two fixed points at which Siegel disks are centered.*

Proof. Note that π satisfies the assumptions in Proposition 2.7. Indeed, assumption (1) holds as it follows from Proposition 2.2 (2) that three indeterminacy points $\{p_1^+, p_2^+, p_3^+\}$ of f are blown up by π. Moreover, assumption (2) also holds as the points $\{p_1^+, p_2^+, p_3^+\}$ are not collinear. Hence, Proposition 2.2 (2) and the above argument show that the blowup π can be decomposed as $\pi = \pi_0 \circ \nu$, where $\pi_0 : X_0 \to \mathbb{P}^2$ is the proper blowup for f, which lifts f to an automorphism $F_0 : X_0 \to X_0$, and $\nu : X \to X_0$ is expressed as the decomposition (1). The cohomological action

$$F_0^* : H^2(X_0;\mathbb{Z}) \to H^2(X_0;\mathbb{Z})$$

is given in Example 2.6, which means that $\mathrm{Tr}(F_0^*|_{H^2(X_0;\mathbb{Z})}) \le 2$. Hence, F_0 has at most 4 isolated fixed points by Proposition 2.10 since $h_{\mathrm{top}}(F_0) = h_{\mathrm{top}}(F) > 0$. Among the fixed points, two fixed points lie on the anticanonical curve $Y_0 = \pi_0^{-1}(C)$ of X_0, at which no Siegel disks are centered from Proposition 2.8. On the other hand, Proposition 2.8 also shows that at none of the fixed points of F on the exceptional divisors of ν, a Siegel disk is centered. Since each fixed point of F either is identified with that of F_0 or lies on the exceptional divisors of ν (see also Remark 2.4), F has at most two fixed points at which Siegel disks are centered. ∎

We conclude this section by stating a result for a class of birational maps with algebraic coefficients that we will treat in the following sections. To this end, for a reduced cubic curve $C \subset \mathbb{P}^2$ and a birational map $f : \mathbb{P}^2 \to \mathbb{P}^2$ in $\mathcal{B}(C)$ with $\delta = \delta(f)$, we assume that C is expressed as

$$C = \{x = [x_1 : x_2 : x_3] \in \mathbb{P}^2 \mid g(x_1 : x_2 : x_3) = 0\},$$

where g is a homogeneous polynomial in $\mathbb{Z}[\delta][x_1, x_2, x_3]$, and that $f = f_\delta$ is also expressed as

$$f(x) = f_\delta(x) = \big[f_1(x_1 : x_2 : x_3) : f_2(x_1 : x_2 : x_3) : f_3(x_1 : x_2 : x_3)\big] \in \mathbb{P}^2,$$

where f_i are homogeneous polynomials in $\mathbb{Z}[\delta][x_1, x_2, x_3]$ with $\deg_x f_1 = \deg_x f_2 = \deg_x f_3$. Note that if $\delta \in \mathbb{C}^*$ is an algebraic number, then so is any fixed point w of f, which enables us to consider the Galois conjugates of δ and w, and also the eigenvalues (μ, ν) of $Df(w)$ are algebraic.

Proposition 2.12. *Under the above assumptions, let $\delta \in \mathbb{C}^*$ be an algebraic number with $|\delta| = 1$ that is not a root of unity, and let $w \in \mathbb{P}^2 \setminus C$ be a fixed point of f_δ outside C. Moreover, assume that there are Galois conjugates (δ_*, w_*) of (δ, w) with $|\delta_*| = 1$ and $f_{\delta_*}(w_*) = w_*$ such that*

$$\{\mathrm{Tr}\, Df_\delta(w)\}^2 / \mathrm{Det}\, Df_\delta(w) \in [0, 4],$$
$$\{\mathrm{Tr}\, Df_{\delta_*}(w_*)\}^2 / \mathrm{Det}\, Df_{\delta_*}(w_*) \notin [0, 4].$$

Then, $f = f_\delta$ has a Siegel disk centered at w.

Proof (See [8]). Let (μ_*, ν_*) be the eigenvalues of $Df_{\delta_*}(w_*)$, which are Galois conjugates of the eigenvalues (μ, ν) of $Df_\delta(w)$. Note that $\mu_* \nu_* = \mathrm{Det}\, Df_{\delta_*}(w_*) = \delta_*$, as w_* also lies outside C. Moreover, it should be noted that

$$\{\mathrm{Tr}\, Df_\delta(w)\}^2 / \mathrm{Det}\, Df_\delta(w) = \frac{(\mu + \nu)^2}{\mu\nu} = \frac{\mu}{\nu} + \frac{\nu}{\mu} + 2$$

and that a complex number $z \in \mathbb{C}$ satisfies $z + z^{-1} + 2 \in [0, 4]$ if and only if $|z| = 1$. Hence, it follows from our assumption that $|\mu/\nu| = 1$ and $|\mu_*/\nu_*| \neq 1$. Since $|\mu\nu| = |\delta| = 1$, we have $(\mu, \nu) \in (S^1)^2$. Now, assume that $\mu^k \nu^l = 1$ for $(k, l) \in \mathbb{Z}^2$. Since (μ_*, ν_*) are Galois conjugates of (μ, ν), one has $1 = \mu_*^k \nu_*^l = (\delta_*)^{(k+l)/2} (\mu_*/\nu_*)^{(k-l)/2}$ and thus $k = l$ as $|\delta_*| = 1$ and $|\mu_*/\nu_*| \neq 1$. Since $1 = \mu_*^k \nu_*^k = \delta_*^k$ and δ_* is not a root of unity, we have $k = 0$; namely, $(k, l) = (0, 0)$. Therefore, $Df(w)$ is an irrational rotation with the algebraic eigenvalues (μ, ν), which shows that f has a Siegel disk centered at w. ∎

3. Birational maps preserving a Cuspidal curve

In this section, we consider a class of quadratic birational maps preserving a cuspidal cubic curve. For a parameter $\delta \in \mathbb{C} \setminus \{0, 1\}$, let us consider a quadratic map $f = f_\delta : \mathbb{P}^2 \to \mathbb{P}^2$, which is explicitly given by $f[x : y : z] = [f_x : f_y : f_z]$ in homogeneous coordinates, where

$$\tag{2} \begin{cases} f_x[x : y : z] = \delta \cdot (xy - 2dyz + 2d^3xz - d^4z^2), \\ f_y[x : y : z] = \delta^3 \cdot (y^2 - 3d^2xy + 3d^4x^2 - d^6z^2), \\ f_z[x : y : z] = yz - 3dx^2 + 3d^2xz - d^3z^2 \end{cases}$$

with $d := (3\delta)^{-1}(1 - \delta)$. Then, f is a birational map preserving the cubic curve $C := \{yz^2 = x^3\} \subset \mathbb{P}^2$ with a cusp located at $[0 : 1 : 0]$ and also preserving its smooth locus $C^* = C \setminus \{[0 : 1 : 0]\}$. Indeed, with the parametrization $p : \mathbb{C} \to C^*$ given by $p(t) = [t : t^3 : 1]$, the restriction of f to C^* is expressed as $f|_{C^*} : \mathbb{C} \ni t \mapsto \delta \cdot (t + d) \in \mathbb{C}$. The indeterminacy sets of $f^{\pm 1}$ are given by $I(f^{\pm 1}) = \{p_1^\pm, p_2^\pm, p_3^\pm\}$, where $p_1^+ := p(d) \in C^*$ and $p_1^- := p(-\delta \cdot d) \in C^*$. Moreover, for $i = 1, 2$, the point $p_{i+1}^\pm$ is defined by the property $\{p_{i+1}^\pm\} = C_i^\pm \cap \mathcal{E}_i^\pm$, where $C_0^\pm := C^*$ and $C_i^\pm$ is inductively given by the strict transform $(\pi_i^\pm)^{-1}(C_{i-1}^\pm)$ under the blowup $\pi_i^\pm$ of $p_i^\pm$ with exceptional curve $\mathcal{E}_i^\pm$. In this case, we write $p_1^\pm < p_2^\pm < p_3^\pm$. Hence, by permitting infinitely near points, we conclude that $I(f^{\pm 1})$ are contained in C^* and that f is a quadratic birational map in $\mathcal{B}(C)$ with $\delta(f) = \delta$ from the expression for $f|_{C^*}$. Conversely, if a quadratic map $f \in \mathcal{B}(C)$ with $I(f) = \{p_1^+, p_2^+, p_3^+\}$ satisfies $\delta(f) = \delta$ and $p_1^+ = p(d) < p_2^+ < p_3^+$, then $f = f_\delta$ is given by (2) (see [11, 12]).

There are exactly two fixed points $\{w_1, w_2\}$ of f outside the curve C, and each point is expressed as $w_i = [x_i : r_\tau(x_i) : 1]$, where

$$r_\tau(x) := \frac{\tau - 2}{3(\tau + 1)} x - \frac{(\tau - 2)^2}{27(\tau + 1)}$$

with $\tau := \delta + 1/\delta$, and x_i is a root of the quadratic equation

$$Q_\tau(x) := 27x^2 - 9(\tau - 2)x + (\tau - 1)(\tau - 2) = 0.$$

Moreover, we have

$$\frac{\{\mathrm{Tr}\, Df(w_i)\}^2}{\mathrm{Det}\, Df(w_i)} = s(\tau, x_i) := \frac{1}{\tau + 2}\{9(\tau - 1)x_i - (\tau^2 - 4\tau + 6)\}^2.$$

Now, in order to construct an automorphism on a rational surface, we consider the case where the orbit $p_i^k := f^k(p_i^-)$ of each backward indeterminacy point p_i^- reaches the forward indeterminacy point p_i^+; namely, $p_i^n = p_i^+$ for some $n \geq 1$. If such an

$n \geq 1$ exists, then Proposition 2.2 shows that the proper blowup $\pi : X \to \mathbb{P}^2$ for f lifts f to an automorphism $F : X \to X$.

From now on, we assume $n = 8$. As $p_1^k = p(-\delta^{k+1} \cdot d + (1 - \delta^k)/3)$, it follows from the relation $p(-\delta^9 \cdot d + (1 - \delta^8)/3) = p(d)$ that δ is a root of $(\delta + 1)S(\delta) = 0$, where

$$S(\delta) = \delta^8 - 2\delta^7 + \delta^6 - 2\delta^5 + \delta^4 - 2\delta^3 + \delta^2 - 2\delta + 1$$

is a Salem polynomial. Conversely, for any root δ of $S(\delta) = 0$, the birational map $f = f_\delta$ satisfies $p_i^8 = p_i^+$ for any $i \in \{1, 2, 3\}$, as $p_1^k < p_2^k < p_3^k$ for any $0 \leq k \leq 8$, and hence lifts to the automorphism $F = F_\delta : X \to X$. The roots of $S(\delta) = 0$ on the real line are $\delta \approx 1.9940, 0.5015$, and the other roots lie on the unit circle, given by $\delta \approx 0.6098 \pm 0.7925i$, $-0.1098 \pm 0.9939i$, $-0.7478 \pm 0.6640i$, which yields $\tau \approx 1.2197, -0.2197, -1.4955$. By virtue of Proposition 2.7 (see also the proof of Proposition 2.11), $\lambda \approx 1.9940$ is an eigenvalue of $F^* : H^2(X; \mathbb{Z}) \to H^2(X; \mathbb{Z})$ and thus the spectral radius of F^*, which means that F has positive entropy

$$h_{\mathrm{top}}(F) = \log \lambda \approx 0.6901 > 0.$$

Now, we put

$$(\delta_0, \tau_0) \approx (0.6098 + 0.7925i, 1.2197),$$
$$(\delta_*, \tau_*) \approx (-0.7478 + 0.6640i, -1.4955).$$

LEMMA 3.1. *We have $s(\tau_0, x_i) \in [0, 4]$ for any root x_i of $Q_{\tau_0}(x) = 0$ and $s(\tau_*, x_*) \notin [0, 4]$ for some root x_* of $Q_{\tau_*}(x) = 0$.*

PROOF. It should be noted that $\tau_0 \in I_0 := [1.219, 1.220]$ and $\tau_* \in I_* := [-1.496, -1.495]$. Moreover, the roots x_i of $Q_{\tau_0}(x) = 0$ satisfy either $x_i \in I_1 := [0.022, 0.023]$ or $x_i \in I_2 := [-0.283, -0.282]$ as $Q_\tau(0.022) < 0$, $Q_\tau(0.023) > 0$, $Q_\tau(-0.283) > 0$, $Q_\tau(-0.282) < 0$ for any $\tau \in I_0$, and a root x_* of $Q_{\tau_*}(x) = 0$ satisfies $x_* \in I_{**} := [-0.711, -0.710]$ as $Q_\tau(-0.711) > 0$, $Q_\tau(-0.710) < 0$ for any $\tau \in I_*$. In particular, we have $s(\tau_0, x_i) \geq 0$ and $s(\tau_*, x_*) \geq 0$. A little calculation shows that

$$s(\tau, x) \leq s(1.219, 0.022) < 2.05 < 4 \quad \text{for any } (\tau, x) \in I_0 \times I_1,$$
$$s(\tau, x) \leq s(1.220, -0.283) < 3.12 < 4 \quad \text{for any } (\tau, x) \in I_0 \times I_2,$$
$$s(\tau, x) \geq s(-1.495, -0.710) > 5.91 > 4 \quad \text{for any } (\tau, x) \in I_* \times I_{**}.$$

Hence, the lemma is established. ■

Note that $Q_{\tau_0}(x)$ is irreducible over $\mathbb{Q}[\tau_0]$, and thus both (δ_0, w_1) and (δ_0, w_2) are Galois conjugates of (δ_*, w_*). Proposition 2.12 yields the following (see also Remark 2.4).

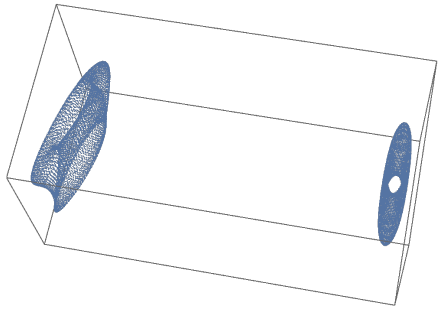

FIGURE 1. Two Siegel disks for automorphism F.

PROPOSITION 3.2. *The automorphism $F = F_{\delta_0} \in \mathcal{QF}(C)$ has Siegel disks centered at w_1, w_2.*

PROOF OF THEOREM 1.2. As C is reduced irreducible, C is either smooth or a nodal cubic or a cuspidal cubic. A result of Diller [5] says that there is no automorphism $F \in \mathcal{QF}(C)$ when C is a nodal cubic. On the other hand, when C is smooth, the determinant $\delta(F)$ of any automorphism $F \in \mathcal{QF}(C)$ is a root of unity. Hence, for the fixed point x, the derivative $DF(x)$ has an eigenvalue $\delta(F)$ if $x \in C$ and has the determinant Det $DF(x) = \delta(F)$ if $x \notin C$. In either case, the eigenvalues of $DF(x)$ are not multiplicatively independent, which means that F has no Siegel disk. Therefore, if C is irreducible and $F \in \mathcal{QF}(C)$ has a Siegel disk, then C is a cuspidal cubic curve. Moreover, if C is a cuspidal cubic, then F admits at most two Siegel disks by Proposition 2.11. Finally, Proposition 3.2 guarantees the existence of the automorphism $F \in \mathcal{QF}(C)$ admitting exactly two Siegel disks. ■

In Figure 1, we describe two Siegel disks for the automorphism F with the help of Mathematica.

4. BIRATIONAL MAPS PRESERVING THREE LINES

In this section, we consider birational maps preserving three lines meeting at a single point. To this end, for parameters $\delta \in \mathbb{C}^*$, $a = (a_i)_{i=1}^m \in (\mathbb{C}^*)^m$, $b = (b_j)_{j=1}^n \in (\mathbb{C}^*)^n$, let $f = f_{\delta,a,b} : \mathbb{C}^2 \to \mathbb{C}^2$ be a birational map given by

$$f(x,y) = \big(f_1(x,y), f_2(x,y)\big) = \left(y, \frac{g_1(y)(x+\delta y)}{\delta\big\{\big(g_2(y) - g_1(y)\big)\frac{x}{y} - \delta g_1(y)\big\}}\right), \tag{3}$$

where $g_1(y) = \prod_{i=1}^m (1 - y/a_i)$ and $g_2(y) = \prod_{j=1}^n (1 - y/b_j)$. The map f preserves the three lines

$$C = L_1 \cup L_2 \cup L_3,$$

where $L_1 = \{x = 0\}$, $L_2 = \{x + \delta y = 0\}$, $L_3 = \{y = 0\}$, and sends these lines as

$$
\begin{aligned}
&f|_{L_1}(0, y) = \left(y, -\frac{y}{\delta}\right) \in L_2, \quad f|_{L_2}(-\delta y, y) = (y, 0) \in L_3,\\
&f|_{L_3}(x, 0) = \left(0, \frac{-x}{\delta^2 + \delta c x}\right) \in L_1.
\end{aligned} \tag{4}
$$

Here and hereafter, we use the following notations:

$$
\begin{aligned}
\alpha &= \sum_{i=1}^{m} \frac{1}{a_i}, \quad \beta = \sum_{j=1}^{n} \frac{1}{b_j},\\
\alpha_0 &:= \prod_{i=1}^{m} \frac{1}{a_i}, \quad \beta_0 := \prod_{j=1}^{n} \frac{1}{b_j},\\
c &= \beta - \alpha.
\end{aligned} \tag{5}
$$

Note that the map (3) is derived under a certain assumption as in the following lemma.

LEMMA 4.1. *Assume that a birational map $h : \mathbb{C}^2 \to \mathbb{C}^2$ of the form $h(x, y) = (y, h_2(x, y))$ satisfies $h(L_i) = L_{i+1}$ for $i = 1, 2, 3 \pmod 3$. Then, we have $h = f_{\delta,a,b}$ for some δ, $a = (a_i)$ and $b = (b_j)$.*

PROOF. Since h is a birational map, for a generic $(x_0, y_0) \in \mathbb{C}^2$, the equation $h(x, y) = (y, h_2(x, y)) = (x_0, y_0)$, or $h_2(x, x_0) = y_0$ has a unique root for x. Hence, $h_2(x, y)$ is a rational function of degree 1 with respect to x. As $h_2(-\delta y, y) = 0$, $h_2(0, y) = -y/\delta$, and $h_2(x, 0) \neq 0$, h_2 has the form $h_2(x, y) = g_1(y)(x + \delta y)/(g_3(y)x - \delta^2 g_1(y))$ with $g_1(0) \neq 0$. By multiplying the denominator and numerator by a common constant if necessary, one can put $g_1(y) = \prod_{i=1}^{m}(1 - y/a_i)$ and then $g_2(y) = g_1(y) + y g_3(y)/\delta = \prod_{j=1}^{n}(1 - y/b_j)$, which yields the lemma. ∎

From now on, we assume the following.

ASSUMPTION 1. $m = n = N$.

With the embedding $\mathbb{C}^2 \ni (x, y) \hookrightarrow [x : y : 1] \in \mathbb{P}^2$, we will regard the birational map f and the lines C as those on $\mathbb{P}^2$. Then, the indeterminacy sets of $f^{\pm 1}$ are given by $I(f^{\pm 1}) = \{p^{\pm}_{a,i}\}_{i=1}^{N} \cup \{p^{\pm}_{b,j}\}_{j=1}^{N} \cup \{p^{\pm}_0\}$, where

$$
\begin{aligned}
p^{+}_{a,i} &= [0 : a_i : 1], \quad p^{+}_{b,j} = [-b_j\delta : b_j : 1], \quad p^{+}_0 = [1 : 0 : 0],\\
p^{-}_{a,i} &= [a_i : 0 : 1], \quad p^{-}_{b,j} = [b_j : -b_j/\delta : 1], \quad p^{-}_0 = [0 : 1 : 0].
\end{aligned}
$$

Since any indeterminacy point of $f^{\pm 1}$ lies on the smooth locus C^* of the three lines C, we can conclude that $f \in \mathcal{B}(C)$. Moreover, it follows from (4) that $\delta = \delta(f)$ is the determinant of f.

REMARK 4.2. The birational map f contracts curves to indeterminacy points as follows:

$$\begin{aligned} L_i^a &:= \{[x : a_i : 1] \mid x \in \mathbb{P}^1\} \to p_{a,i}^-, \\ L_j^b &:= \{[x : b_j : 1] \mid x \in \mathbb{P}^1\} \to p_{b,j}^-, \\ D &:= \{[x : y : 1] \mid (g_2(y) - g_1(y))x/y - \delta g_1(y) = 0\} \to p_0^-. \end{aligned}$$

The curves L_i^a and L_j^b are lines passing through $\{p_{a,i}^+, p_0^+\}$ and $\{p_{b,j}^+, p_0^+\}$ respectively, and D is a curve of degree N passing through $I(f)$ with multiplicities

$$\mathrm{mult}_{p_{a,i}^+} D = \mathrm{mult}_{p_{b,j}^+} D = 1 \quad \text{and} \quad \mathrm{mult}_{p_0^+} D = N - 1.$$

A straightforward calculation shows that the blowup of p_0^- lifts f to a birational map whose restriction to D is an isomorphism to the exceptional curve of the blowup. Similarly, if $a_i \neq a_k$ for any $k \neq i$, then the blowup of $p_{a,i}^-$ lifts f to a birational map whose restriction to L_i^a is an isomorphism to the exceptional curve, and also if $b_j \neq b_k$ for any $k \neq j$, then the blowup of $p_{b,j}^-$ lifts f to a birational map whose restriction to L_j^b is an isomorphism to the exceptional curve. Moreover, the pullback of a generic line by f is a curve $\mathcal{D}$ of degree $N + 1$ passing through $I(f)$ with multiplicities $\mathrm{mult}_{p_{a,i}^+} \mathcal{D} = \mathrm{mult}_{p_{b,j}^+} \mathcal{D} = 1$ and $\mathrm{mult}_{p_0^+} \mathcal{D} = N$.

Next, we determine the fixed points of $f : \mathbb{P}^2 \to \mathbb{P}^2$. The fixed points of f on $\mathbb{C}^2$ are given by the singular point $(0, 0)$ of C, and $(x_l, x_l) \in \mathbb{C}^2$, where x_l are the roots of the equation

$$\frac{(1+\delta)^2}{\delta} \prod_{i=1}^{N} \left(1 - \frac{x_l}{a_i}\right) = \prod_{j=1}^{N} \left(1 - \frac{x_l}{b_j}\right). \tag{6}$$

Moreover, under Assumption 1, the birational map $f : \mathbb{P}^2 \to \mathbb{P}^2$ preserves the line $L = \{[x : y : z] \mid z = 0\}$ at infinity, and the restriction $f|_L$ is expressed as

$$f[x : y : 0] = \left[\delta(\beta_0 - \alpha_0)x - \delta^2 \alpha_0 y : \alpha_0(x + \delta y) : 0\right], \tag{7}$$

where α_0, β_0 are given in (5). Hence, the fixed points of $f : \mathbb{P}^2 \to \mathbb{P}^2$ lying on L are given by $[x_l : 1 : 0]$, where x_l are the roots of the equation

$$\alpha_0 x_l^2 + \delta(2\alpha_0 - \beta_0)x_l + \alpha_0 \delta^2 = 0. \tag{8}$$

Consequently, we have the following proposition.

PROPOSITION 4.3. *The fixed points of $f : \mathbb{P}^2 \to \mathbb{P}^2$ are given by $w_0 = [0 : 0 : 1] \in C$, $w_l = [x_l : x_l : 1] \in \mathbb{C}^2$ for $l \in \{1, \ldots, N\}$, where x_l are the roots of (6), and $w_l = [x_l : 1 : 0] \in L$ for $l \in \{N + 1, N + 2\}$, where x_l are the roots of (8). Moreover, when $l \in \{1, \ldots, N + 2\}$, the fixed point w_l lies outside C and hence satisfies* $\mathrm{Det}\, Df(w_l) = \delta$.

REMARK 4.4. It is straightforward to calculate that the eigenvalues of $Df(w_0)$ at the singular point w_0 of C are given by $(\omega\delta^{-1}, \omega^{-1}\delta^{-1})$, where ω is a primitive cube root of unity. Therefore, a Siegel disk is never centered at w_0, as $(\omega\delta^{-1}, \omega^{-1}\delta^{-1})$ are not multiplicatively independent.

Now, for $\mathbb{R}^* := \mathbb{R} \setminus \{0\}$, we put

$$A := \left\{ c = (\delta, a, b) \in S^1 \times (\mathbb{R}^*)^N_{\neq} \times (\mathbb{R}^*)^N_{\neq} \mid \sum_{j=1}^{N} \frac{1}{b_j} - \sum_{i=1}^{N} \frac{1}{a_i} = 1 \right\},$$

where $(\mathbb{R}^*)^N_{\neq} := \{a \in (\mathbb{R}^*)^N \deg a_i \neq a_j (i \neq j)\}$, and for $c_0 = (\delta_0, a_0, b_0) \in A$ and $\varepsilon > 0$, put

$$A(c_0; \varepsilon) := \left\{ (\delta, a, b) \in A \mid |\delta - \delta_0| < \varepsilon,\ |a - a_0| < \varepsilon,\ |b - b_0| < \varepsilon \right\}.$$

Then, we have the following proposition, whose proof is given in Section 6.

PROPOSITION 4.5. *Under the above notations, there exists $\varepsilon > 0$ and $c_0, c_* \in A$ such that*

(1) $\frac{\{\mathrm{Tr}\, Df(w_l)\}^2}{\mathrm{Det}\, Df(w_l)} \in [0, 4]$ *for* $l \in \{1, \dots, N+2\}$ *if* $(\delta, a, b) \in A(c_0; \varepsilon)$,

(2) $\frac{\{\mathrm{Tr}\, Df(w_l)\}^2}{\mathrm{Det}\, Df(w_l)} \notin [0, 4]$ *for* $l \in \{1, \dots, N+2\}$ *if* $(\delta, a, b) \in A(c_*; \varepsilon)$.

It should be noted that the indeterminacy point $p_0^- \in I(f^{-1})$ satisfies

$$f^2(p_0^-) = p_0^+ \in I(f).$$

Furthermore, we assume the following.

ASSUMPTION 2. For given parameters $m = (m_i)_{i=1}^N, n = (n_j)_{j=1}^N \in \mathbb{N}^N$ except for $(m, n) = ((1), (1)) \in (\mathbb{N}^1)^2$, the map $f : \mathbb{P}^2 \to \mathbb{P}^2$ satisfies

$$\begin{aligned} f^{3m_i-2}(p_{a,i}^-) &= p_{a,i}^+ \quad (i = 1, \dots, N), \\ f^{3n_j}(p_{b,j}^-) &= p_{b,j}^+ \quad (j = 1, \dots, N). \end{aligned} \tag{9}$$

LEMMA 4.6. *Under Assumption 2, we have*

$$\frac{1}{a_i} = -\frac{\delta(\delta^{3m_i} - 1)}{(\delta^3 - 1)(\delta^{3m_i - 1} + 1)} c, \qquad \frac{1}{b_j} = \frac{\delta^2(\delta^{3n_j} - 1)}{(\delta^3 - 1)(\delta^{3n_j + 1} + 1)} c,$$

where $c = \beta - \alpha$ is given in (5). In particular, if $c \neq 0$, then δ satisfies the equation

$$\chi_{m,n}(\delta) := \sum_{j=1}^{N} \frac{\delta^2(\delta^{3n_j} - 1)}{(\delta^3 - 1)(\delta^{3n_j+1} + 1)} + \sum_{i=1}^{N} \frac{\delta(\delta^{3m_i} - 1)}{(\delta^3 - 1)(\delta^{3m_i - 1} + 1)} = 1. \tag{10}$$

PROOF. It follows from (4) that

$$f^3(0,y) = \big(0, h_1(y)\big), \quad f^3(x, -x/\delta) = \big(h_1(x), -h_1(x)/\delta\big)$$

and hence

$$f^{3k}(0,y) = \big(0, h_k(y)\big), \quad f^{3k}(x, -x/\delta) = \big(h_k(x), -h_k(x)/\delta\big),$$

where

$$h_k(x) := \frac{1}{\delta^{3k}(\frac{1}{x} - p) + p}, \quad p := \frac{\delta c}{(\delta^3 - 1)}.$$

Since

$$f(a_i, 0) = \big(0, -a_i\{\delta(\delta + ca_i)\}^{-1}\big),$$

assumption (9) is equivalent to $h_{m_i-1}(-a_i\{\delta(\delta + ca_i)\}^{-1}) = a_i$ and $h_{n_j}(b_j) = -b_j\delta$, which yield the desired expressions for $1/a_i$ and $1/b_j$. Finally, the relation (10) follows from

$$c = \beta - \alpha = \sum_{j=1}^{N} 1/b_j - \sum_{i=1}^{N} 1/a_i. \qquad \blacksquare$$

Conversely, for given $m = (m_i), n = (n_j) \in \mathbb{N}^N$, let $\delta \in \mathbb{C}^*$ be any root of (10), and let $a = (a_i), b = (b_j)$ be parameters given by $a_i = a_{m_i}(\delta), b_j = b_{n_j}(\delta)$, where

$$a_k(\delta) := -\frac{(\delta^3 - 1)(\delta^{3k-1} + 1)}{\delta(\delta^{3k} - 1)}, \quad b_k(\delta) := \frac{(\delta^3 - 1)(\delta^{3k+1} + 1)}{\delta^2(\delta^{3k} - 1)}. \tag{11}$$

Then, the birational map $f = f_{\delta,a,b}$ satisfies the condition (9). Proposition 2.2 shows that there is a proper blowup $\pi : X \to \mathbb{P}^2$ for f, and π lifts $f : \mathbb{P}^2 \to \mathbb{P}^2$ to an automorphism $F_{m,n} : X \to X$. Note that the points blown up by π satisfy the assumptions in Proposition 2.7. Thus, the root δ of the equation (10), which is the determinant of f, is an eigenvalue of $F_{m,n}^* : H^2(X;\mathbb{Z}) \to H^2(X;\mathbb{Z})$. On the other hand, under Assumption 2, there exists $\lambda > 1$ so that $\chi_{m,n}(\lambda) = 1$ since $\chi_{m,n}(1) > 1$ and $\lim_{\delta\to\infty} \chi_{m,n}(\delta) = 0$. Hence, $\lambda = \lambda_{m,n} := \lambda(F_{m,n}^*) > 1$ is the spectral radius, which is a root of the Salem polynomial $S_{m,n}(t) := S_{F_{m,n}}(t) = 0$. As $S_{m,n}(t)$ is irreducible, any root of $S_{m,n}(t) = 0$ is a root of $\chi_{m,n}(t) = 1$. Therefore, we have the following corollary.

COROLLARY 4.7. *Under the assumption that $(m,n) \neq ((1),(1))$, any root δ of $S_{m,n}(t) = 0$ satisfies $\chi_{m,n}(\delta) = 1$. Moreover, the birational map $f = f_{\delta,(a_{m_i}(\delta)),(b_{n_j}(\delta))}$ lifts to the automorphism $F_{m,n}$, having positive entropy $h_{\mathrm{top}}(F_{m,n}) = \log\lambda_{m,n} > 0$ with the spectral radius $\lambda_{m,n} = \lambda(F_{m,n}^*) > 1$.*

LEMMA 4.8. *If $\delta \in S^1$ is given by $\delta = \exp(2\pi i \nu)$ with an irrational real number ν, then $\{a_k(\delta)\}_{k\in\mathbb{N}}$ and $\{b_k(\delta)\}_{k\in\mathbb{N}}$ are sequences of real numbers and dense in $\mathbb{R}$.*

PROOF. First we notice that

$$
\begin{aligned}
-\frac{(\delta^3-1)(\delta^{3k-1}+1)}{\delta(\delta^{3k}-1)} &= -\frac{(\delta^{3/2}-\delta^{-3/2})(\delta^{(3k-1)/2}+\delta^{-(3k-1)/2})}{(\delta^{3k/2}-\delta^{-3k/2})} \\
&= -2\frac{\sin(3\pi\nu)\cos\left\{(3k-1)\pi\nu\right\}}{\sin(3k\pi\nu)} \\
&= -2\sin(3\pi\nu)\left\{\frac{\cos(\pi\nu)}{\tan(3k\pi\nu)}+\sin(\pi\nu)\right\}, \\
\frac{(\delta^3-1)(\delta^{3k+1}+1)}{\delta^2(\delta^{3k}-1)} &= \frac{(\delta^{3/2}-\delta^{-3/2})(\delta^{(3k+1)/2}+\delta^{-(3k+1)/2})}{(\delta^{3k/2}-\delta^{-3k/2})} \\
&= 2\frac{\sin(3\pi\nu)\cos\left\{(3k+1)\pi\nu\right\}}{\sin(3k\pi\nu)} \\
&= 2\sin(3\pi\nu)\left\{\frac{\cos(\pi\nu)}{\tan(3k\pi\nu)}-\sin(\pi\nu)\right\}
\end{aligned}
$$

are real numbers. Since $\{3k\pi\nu\}_{k=1}^{\infty} \subset (-\pi/2, \pi/2) \pmod{\pi}$ is dense from the irrationality of ν, so is $\{\tan(3k\pi\nu)\}_{k=1}^{\infty} \subset \mathbb{R}$, which establishes the lemma as

$$\sin(3\pi\nu)\cos(\pi\nu) \neq 0. \qquad \blacksquare$$

PROPOSITION 4.9. *The roots of $S_{m,n}(t) = 0$ other than $\lambda_{m,n}^{\pm 1}$ are equidistributed on S^1 as either $m_i \to \infty$ for some i or $n_j \to \infty$ for some j.*

PROOF. A result of Bilu (see [4, 8]) says that if $\{\rho_k\}_{k\in\mathbb{N}}$ is a sequence of algebraic units with $\lim_{k\to\infty} \deg(\rho_k) = \infty$, then $\{\overline{\delta}_{\rho_k}\}$ weakly converges to the normalized Haar measure on S^1. Here, for an algebraic number $\rho \neq 0$, we put

$$\overline{\delta}_\rho := \frac{1}{\deg(\rho)} \sum_{\rho' \underset{\text{conj.}}{\sim} \rho} \delta_{\rho'}$$

with the Dirac measure $\delta_{\rho'}$ at ρ'. Since $\lambda_{m,n}$ satisfies $\lambda_{m,n} \to \lambda < \infty$ as $m_i \to \infty$ or $n_j \to \infty$ and λ is not a Salem number, we have $\deg(\lambda_{m,n}) \to \infty$. As $\lambda_{m,n}$ is an algebraic unit, the proposition is established. $\blacksquare$

PROPOSITION 4.10. *Let $\varepsilon > 0$ and $c_0, c_* \in A$ be given in Proposition 4.5, and let $a_k(\delta)$, $b_k(\delta)$ be given in (11). Then, there exist $m, n \in \mathbb{N}^N$ and $\delta_0, \delta_* \in S^1$ such that*

(1) $S_{m,n}(\delta_0) = S_{m,n}(\delta_*) = 0$,

(2) $(\delta_0, (a_{m_i}(\delta_0)), (b_{n_j}(\delta_0))) \in A(c_0; \varepsilon)$, $(\delta_*, (a_{m_i}(\delta_*)), (b_{n_j}(\delta_*))) \in A(c_*; \varepsilon)$.

PROOF. We put $c_0 = (d_0, (a_i^0), (b_j^0))$, $c_* = (d_*, (a_i^*), (b_j^*))$, and without loss of generality, we may assume that d_0 and d_* are multiplicatively independent. Then, from

Lemma 4.8, one can fix $(m_i)_{i=1}^{N-1}$ and $(n_j)_{j=1}^{N}$ so that $a_i^0 \approx a_{m_i}(d_0)$, $a_i^* \approx a_{m_i}(d_*)$ for $i \in \{1, \dots, N-1\}$ and $b_j^0 \approx b_{n_j}(d_0)$, $b_j^* \approx b_{n_j}(d_*)$ for $j \in \{1, \dots, N\}$. By Proposition 4.9, there exists $m_N \gg 1$ such that roots $\delta_0, \delta_* \in S^1$ of $S_{m,n}(t) = 0$ satisfy $\delta_0 \approx d_0$, $\delta_* \approx d_*$ and hence $a_i^0 \approx a_{m_i}(\delta_0)$, $a_i^* \approx a_{m_i}(\delta_*)$ for $i \in \{1, \dots, N-1\}$ and $b_j^0 \approx b_{n_j}(\delta_0)$, $b_j^* \approx b_{n_j}(\delta_*)$ for $j \in \{1, \dots, N\}$. As

$$\sum_{j=1}^{N} \frac{1}{b_j} - \sum_{i=1}^{N} \frac{1}{a_i} = \chi_{m,n}(\delta_0) = \chi_{m,n}(\delta_*) = 1$$

from Corollary 4.7, we have $a_N^0 \approx a_{m_N}(\delta_0)$, $a_N^* \approx a_{m_N}(\delta_*)$ so that condition (2) holds. ∎

For the parameters given in Proposition 4.10, fix the birational maps

$$f_0 = f_{\delta_0,(a_{m_i}(\delta_0)),(b_{n_j}(\delta_0))} \quad \text{and} \quad f_* = f_{\delta_*,(a_{m_i}(\delta_*)),(b_{n_j}(\delta_*))}.$$

As f_0 and f_* are Galois conjugate and each fixed point of f_0 outside C is a Galois conjugate of a fixed point of f_* outside C, Propositions 2.12, 4.5, and 4.10 yield the following corollary.

Corollary 4.11. *The map f_0 has $N+2$ fixed points $w_1, \dots, w_{N+2}$ at which Siegel disks are centered.*

Proposition 4.12. *Let $F : X \to X$ be the automorphism that is the lift of f_0 by the proper blowup $\pi : X \to \mathbb{P}^2$ for f_0. Then, F has positive entropy $h_{\mathrm{top}}(F) = \log \lambda_{m,n} > 0$ and has exactly $N+3$ isolated fixed points $w_0, \dots, w_{N+2}$ (see also Remark 2.4).*

Proof. Corollary 4.7 says that $F = F_{m,n}$ has positive entropy $h_{\mathrm{top}}(F_{m,n}) = \log \lambda_{m,n} > 0$. Now, note that the indeterminacy points $I(f^{\pm 1})$ are blown up by π. Remark 4.2 says that F^* sends curves as

$$\begin{aligned}
[H] &\mapsto (N+1)[H] - N[E_0^+] - \sum_{i=1}^{N}[E_{a,i}^+] - \sum_{j=1}^{N}[E_{b,j}^+], \\
[E_0^-] &\mapsto N[H] - (N-1)[E_0^+] - \sum_{i=1}^{N}[E_{a,i}^+] - \sum_{j=1}^{N}[E_{b,j}^+], \\
[E_{a,i}^-] &\mapsto [H] - [E_0^+] - [E_{a,i}^+], \\
[E_{b,j}^-] &\mapsto [H] - [E_0^+] - [E_{b,j}^+],
\end{aligned}$$

where $E_0^\pm$, $E_{a,i}^\pm$, $E_{b,j}^\pm$ are the exceptional divisors over $p_0^\pm$, $p_{a,i}^\pm$, $p_{b,j}^\pm$, respectively. It follows from Lemma 2.5 that any exceptional divisor over the point outside $I(f^{-1})$ is sent to another exceptional one by F^*. Hence, we have $\mathrm{Tr}(F^*|_{H^2(X;\mathbb{Z})}) \leq N+1$.

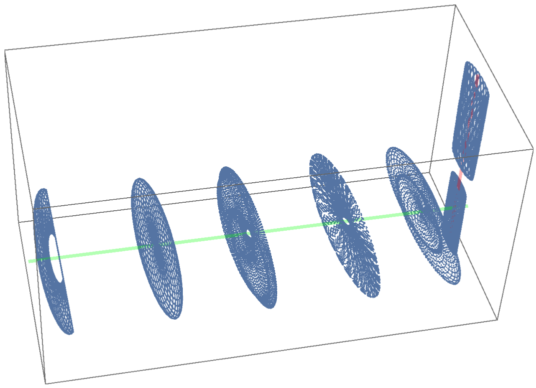

FIGURE 2. Siegel disks for an automorphism ($N = 5$).

Proposition 2.10 says that there are at most $N+3$ isolated fixed points for F, and the existence of the fixed points $w_0, \dots, w_{N+2}$ given in Proposition 4.3 says that there are exactly $N+3$ isolated fixed points for F. ■

PROOF OF THEOREM 1.1. First, assume $k \geq 3$ and put $N = k-2$. The automorphism F mentioned in Proposition 4.12 has positive entropy and has exactly $k+1$ fixed points $w_0, \dots, w_k$. Among the fixed points, no Siegel disk is centered at w_0 from Remark 4.4, and Siegel disks are centered at $w_1, \dots, w_k$ from Corollary 4.11. Therefore, F is a desired automorphism satisfying the condition mentioned in Theorem 1.1.

When $k = 0, 1$, McMullen [8] and Bedford–Kim [2] showed the existence of an automorphism F satisfying the condition. The automorphism F realizes the so-called Coxeter element and is obtained from a birational map $f : \mathbb{P}^2 \to \mathbb{P}^2$ of degree 2 by blowing up points on the smooth locus of a cubic curve C. Moreover, C is a cuspidal cubic if $k = 0$, and C is either a conic with a tangent line or three lines through a point if $k = 1$. Finally, when $k = 2$, the existence is shown in Theorem 1.2. The theorem is established. ■

With the help of Mathematica, we describe Siegel disks of an automorphism for the parameters $N = 5, m = (280, 104, 54, 36, 27), n = (205, 381, 432, 450, 459)$, and $\delta \approx -0.5037 + 0.8639i$ in Figure 2.

5. PROOF OF PROPOSITION 2.9

This section is devoted to the proof of Proposition 2.9. Since automorphisms may fix a curve pointwise, we use S. Saito's fixed point formula instead of a classical fixed point one (see [7, 10]). Let X be a smooth projective surface and $f : X \to X$ an automorphism different from the identity. Then, the idea of Saito is to divide the set

$X_1(f)$ of irreducible curves fixed pointwise by f into the curves of type I and those of type II:

$$X_1(f) = X_I(f) \amalg X_{II}(f),$$

and to contribute different types of curves to the formula in different ways. Namely, the formula says that the Lefschetz number

$$L(f) := \sum_i (-1)^i \operatorname{Tr}\left[f^* : H^i(X;\mathbb{Z}) \to H^i(X;\mathbb{Z})\right]$$

of the automorphism f is expressed as

$$L(f) = \sum_{x \in X_0(f)} \nu_x(f) + \sum_{C \in X_I(f)} \chi_C \cdot \nu_C(f) + \sum_{C \in X_{II}(f)} \tau_C \cdot \nu_C(f),$$

where $X_0(f)$ is the set of fixed points of f, χ_C is the Euler characteristic of the normalization of $C \in X_I(f)$, and τ_C is the self-intersection number of $C \in X_{II}(f)$. We shall omit the precise definitions of the indices $\nu_x(f)$ and $\nu_C(f)$. However, it is known that $\nu_C(f)$ is a positive integer, and $\nu_x(f)$ is a nonnegative integer, which is positive if $x \in X_0(f)$ is an isolated fixed point. On the other hand, the types of fixed curves are defined by using the action of f on the completion A_x of the local ring of X at x, which is isomorphic to the formal power series ring $\mathbb{C}[\![z_1, z_2]\!]$, as X is assumed to be a smooth surface. Now, given a fixed curve $C \in X_1(f)$, we take a smooth point x of C and identify A_x with $\mathbb{C}[\![z_1, z_2]\!]$ in such a manner that C has the local defining equation $z_1 = 0$ near x. Then, the induced automorphism $f_x^* : A_x \to A_x$ can be expressed as

$$\begin{cases} f_x^*(z_1) = z_1 + z_1^k \cdot f_1, \\ f_x^*(z_2) = z_2 + z_1^l \cdot f_2 \end{cases} \tag{12}$$

for some $k, l \in \mathbb{N} \cup \{\infty\}$ and some $f_i \in A_x$ such that $f_i(0, z_2)$ is a nonzero element of $\mathbb{C}[\![z_2]\!]$. Here, we put $z_1^\infty := 0$ by convention. Then, it turns out (see [7, Lemma 6.1]) that $\nu_C(f) = \min\{k, l\}$ and $C \in X_I(f)$ if and only if $k \le l$, which is independent of the choice of the smooth point x on C and the coordinates z_1, z_2. Note that if the derivative $Df(x)$ has an eigenvalue different from 1, then the relation (12) yields $k = 1$ and $f_1(0, 0) \neq 0$. In particular, the fixed curve C must be of type I.

Proof of Proposition 2.9. Now, if X is a rational surface, then the cohomology group of X is expressed as

$$H^i(X;\mathbb{Z}) \cong \begin{cases} \mathbb{Z}^{\rho+1} & (i = 2) \\ \mathbb{Z} & (i = 0, 4) \\ 0 & (i \neq 0, 2, 4) \end{cases}$$

for some $\rho \geq 0$. Moreover, if F is an automorphism on X, then the action F^* on $H^i(X;\mathbb{Z})$ is trivial for $i = 0, 4$, which shows that

$$L(F) = \mathrm{Tr}(F^*|_{H^2(X;\mathbb{Z})}) + 2.$$

On the other hand, the above argument says that any fixed curve is of type I. Furthermore, if F has positive entropy, then it is known (see [6]) that any fixed curve C has nonnegative Euler characteristic $\chi_C \geq 0$. Hence, the fixed point formula says that F has at most $L(F) = \mathrm{Tr}(F^*|_{H^2(X;\mathbb{Z})}) + 2$ isolated fixed points. ■

6. Proof of Proposition 4.5

In this section, we will prove Proposition 4.5. To this end, we need some auxiliary lemmas. Let f be the birational map given by (3) with $m = n = N$.

Lemma 6.1. *For any fixed point $(x_l, x_l) \in \mathbb{C}^2$ with x_l satisfying (6), we have*

$$\mathrm{Tr}\, Df(x_l, x_l) = \frac{\partial f_2}{\partial y}(x_l, x_l) = (\delta + 1)\left\{1 - \sum_{i=1}^{N} \frac{1}{1 - x_l/a_i} + \sum_{j=1}^{N} \frac{1}{1 - x_l/b_j}\right\}.$$

Proof. First, it follows from $(f_1)_x = 0$ that $\mathrm{Tr}\, Df(x_l, x_l) = (f_2)_y(x_l, x_l)$. Moreover, by the relation

$$g_2(x_l) = g_1(x_l)(1 + \delta)^2/\delta,$$

one has

$$\frac{\partial f_2}{\partial y}(x_l, x_l) = \frac{g_1(x_l)(1 + \delta)^2 + x_l g_1'(x_l)(1 + \delta)^2 - x_l g_2'(x_l)\delta}{g_1(x_l)(1 + \delta)}.$$

Therefore by combining the relations

$$x_l g_1'(x_l) = g_1(x_l) \sum_{i=1}^{N} \frac{-x_l/a_i}{1 - x_l/a_i} = g_1(x_l)\left\{N - \sum_{i=1}^{N} \frac{1}{1 - x_l/a_i}\right\},$$

$$x_l g_2'(x_l)\delta = g_2(x_l)\delta \sum_{j=1}^{N} \frac{-x_l/b_j}{1 - x_l/b_j} = g_1(x_l)(1 + \delta)^2\left\{N - \sum_{j=1}^{N} \frac{1}{1 - x_l/b_j}\right\},$$

we obtain the desired form. ■

Lemma 6.2. *Assume $\delta \in S^1$. For any fixed point $w_l = [x_l : 1 : 0] \in L$ with x_l satisfying (8), we have*

$$\frac{\{\mathrm{Tr}\, Df(w_l)\}^2}{\mathrm{Det}\, Df(w_l)} \in [0, 4] \iff \frac{\beta_0}{\alpha_0} \in [0, 4].$$

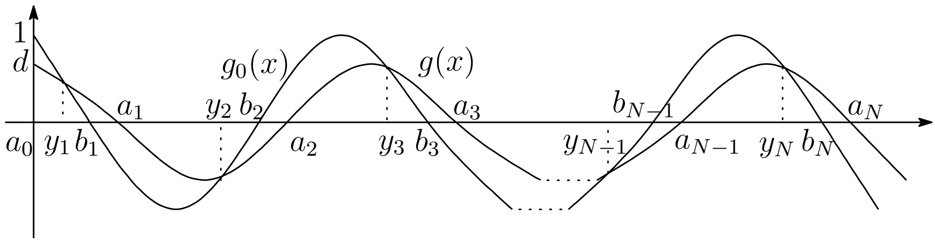

FIGURE 3. Two functions $g(x)$ and $g_0(x)$.

PROOF. We use the fact that the eigenvalues of Df at $w_l = [x_l : 1 : 0]$ for $l \in \{N+1, N+2\}$ are given by $(\delta x_l^{-1}, x_l)$. It follows from the equation (8) that $t := \delta^{-1} x_l$ satisfies

$$t = \frac{1}{2}\left\{\frac{\beta_0}{\alpha_0} - 2 \pm \sqrt{\frac{\beta_0}{\alpha_0}\left(\frac{\beta_0}{\alpha_0} - 4\right)}\right\}.$$

Moreover, one has $\{\mathrm{Tr}\, Df(w_l)\}^2/\,\mathrm{Det}\, Df(w_l) = 2 + \delta t^2 + (\delta t^2)^{-1}$. As $\delta \in S^1$, it turns out that $\{\mathrm{Tr}\, Df(w_l)\}^2/\,\mathrm{Det}\, Df(w_l) \in [0, 4]$ if and only if $t \in S^1$, or in other words, $\beta_0/\alpha_0 \in [0, 4]$. ■

Now, we show the existence of the parameters $c, c_* \in A$ mentioned in Proposition 4.5. Note that any birational map $f_{\delta,(a_i),(b_j)}$ is conjugate to $f_{\delta,(a_i/c),(b_j/c)}$ for any $c \in \mathbb{C}^*$ via the linear map $[x : y : z] \mapsto [cx : cy : z]$. Hence, it is enough to show the existence of $(\delta, (a_i), (b_j))$ with $\sum_{i=1}^N 1/a_i - \sum_{j=1}^N 1/b_j \neq 0$ instead of $\sum_{i=1}^N 1/a_i - \sum_{j=1}^N 1/b_j = 1$.

For given real numbers $0 = a_0 < a_1 < a_2 < \cdots < a_N$ and an N-tuple $b = (b_i) \in (\mathbb{R}^*)^N$ with $a_{i-1} < b_i < a_i$, put

$$g(x) = d\prod_{i=1}^N\left(1 - \frac{x}{a_i}\right), \quad g_0(x) = \prod_{i=1}^N\left(1 - \frac{x}{b_i}\right),$$

where $d = (1+\delta)^2/\delta \in [0, 4]$ with $\delta \in S^1$. Moreover, we assume that $0 < d < 1$. Since $g(x)$ and $g_0(x)$ are polynomials of degree n satisfying the relations $g(a_0) = d < 1 = g_0(a_0)$, $g(a_i) = 0 < (-1)^i g_0(a_i)$, and $(-1)^i g(b_i) < 0 = g_0(b_i)$ for $i \geq 1$, there is a unique real number $y_i \in (a_{i-1}, b_i)$ such that $g(y_i) = g_0(y_i)$ for $i \in \{1, \ldots, N\}$ (see Figure 3). It is seen that $y_i = y_i(b)$ is continuous as a function of $b = (b_i) \in \prod_{i=1}^N (a_{i-1}, a_i)$.

LEMMA 6.3. *Assume that* $0 \ll d < 1$. *Then, there exists* $b = (b_i) \in \prod_{i=1}^N (a_{i-1}, a_i)$ *such that* $y_i(b) = x_i$ *for any* $i \in \{1, \ldots, N\}$, *where* $x_i = (a_{i-1} + a_i)/2$. *Moreover, each component* b_i *satisfies* $\lim_{d \nearrow 1} b_i = a_i$.

PROOF. For $i \in \{1, \dots, N\}$, we put

$$s_i(x) = d \prod_{j=1}^{i} \Bigl(1 - \frac{x}{a_j}\Bigr), \quad t_i(x) = \prod_{j=1}^{i} \Bigl(1 - \frac{x}{a_j - \varepsilon_j}\Bigr),$$

where ε_i is inductively determined by the relation $s_i(x_i) = t_i(x_i)$ (see also the following). We claim that $\varepsilon_i > 0$ and $\varepsilon_i \searrow 0$ as $d \nearrow 1$. Indeed, if $i = 1$, then the relation $s_1(x_1) = t_1(x_1)$ yields $\varepsilon_1 = a_1(1-d)(a_1 - x_1)/\{a_1 - d(a_1 - x_1)\} > 0$, and $\varepsilon_1 \searrow 0$ as $d \nearrow 1$. Note that $d_2 := s_1(x_2)/t_1(x_2)$ satisfies $0 < d_2 < 1$ since $x_2 > a_1$, and $d_2 \nearrow 1$ as $d \nearrow 1$. Moreover, for $i \geq 2$, assume that $d_i := s_{i-1}(x_i)/t_{i-1}(x_i)$ satisfies $0 < d_i < 1$, and $d_i \nearrow 1$ as $d \nearrow 1$. The relation

$$d_i \Bigl(1 - \frac{x_i}{a_i}\Bigr) = \frac{s_i(x_i)}{t_{i-1}(x_i)} = \frac{t_i(x_i)}{t_{i-1}(x_i)} = \Bigl(1 - \frac{x_i}{a_i - \varepsilon_i}\Bigr)$$

yields $\varepsilon_i = a_i(1-d_i)(a_i - x_i)/\{a_i - d_i(a_i - x_i)\} > 0$, and $\varepsilon_i \searrow 0$ as $d \nearrow 1$. Similarly, $d_{i+1} = s_i(x_{i+1})/t_i(x_{i+1})$ satisfies $0 < d_{i+1} < 1$, and $d_{i+1} \nearrow 1$ as $d \nearrow 1$. Our claim is proved.

Assume $0 \ll d < 1$ so that $\varepsilon_i < a_i - x_i$. Regarding $y_i = y_i(b)$ as a function of $b = (b_i)$, we claim that $y_i(b_1, \dots, b_{i-1}, a_i - \varepsilon_i, b_{i+1}, \dots, b_N) < x_i$ for any $i \in \{1, \dots, N\}$ and $(b_1, \dots, b_{i-1}, b_{i+1}, \dots, b_N)$ with $(a_{j-1} < x_j <) a_j - \varepsilon_j < b_j < a_j$. Indeed, by putting

$$g_i(x) := \Bigl(1 - \frac{x}{a_i - \varepsilon_i}\Bigr) \prod_{j \neq i} \Bigl(1 - \frac{x}{b_j}\Bigr),$$

one has

$$\begin{aligned}
(-1)^{i-1} g_i(x_i) &= \Bigl(1 - \frac{x_i}{a_i - \varepsilon_i}\Bigr) \prod_{j<i} \Bigl(\frac{x_i}{b_j} - 1\Bigr) \prod_{j>i} \Bigl(1 - \frac{x_i}{b_j}\Bigr) \\
&< \Bigl(1 - \frac{x_i}{a_i - \varepsilon_i}\Bigr) \prod_{j<i} \Bigl(\frac{x_i}{a_j - \varepsilon_j} - 1\Bigr) \prod_{j>i} \Bigl(1 - \frac{x_i}{a_j}\Bigr) \\
&= d \Bigl(1 - \frac{x_i}{a_i}\Bigr) \prod_{j<i} \Bigl(\frac{x_i}{a_j} - 1\Bigr) \prod_{j>i} \Bigl(1 - \frac{x_i}{a_j}\Bigr) = (-1)^{i-1} g(x_i)
\end{aligned}$$

and $(-1)^{i-1} g_i(a_{i-1}) > 0 = (-1)^{i-1} g(a_{i-1})$, which yield the claim.

Finally, we prove the existence of b with $y_i(b) = x_i$. To this end, note that there is a root z_i of $g(x) = g_0(x)$ such that $z_i \nearrow a_i$ as $b_i \nearrow a_i$. For $i = N$, the root z_N must satisfy $z_N = y_N$ since $y_j \leq a_{N-1}$ for $j \leq N-1$. The above claim says that $y_N(b_1, \dots, b_{N-1}, a_N - \varepsilon_N) < x_N$, which means that there exists

$$b_N = b_N(b_1, \dots, b_{N-1}) \in (a_N - \varepsilon_N, a_N),$$

depending continuously on $(b_j)_{j=1}^{N-1}$, such that

$$y_N\bigl(b_1,\dots,b_{N-1},b_N(b_1,\dots,b_{N-1})\bigr)=x_N.$$

Put $y_j(b_1,\dots,b_{N-1})=y_j(b_1,\dots,b_{N-1},b_N(b_1,\dots,b_{N-1}))$, which is continuous with respect to $(b_j)_{j=1}^{N-1}$. Moreover, for $i\le N-1$, we assume that $y_j=y_j(b_1,\dots,b_i)$ satisfies $y_j=x_j$ for $j\ge i+1$. Similarly, z_i must satisfy $z_i=y_i$ since $y_j\ge x_{i+1}$ for $j\ge i+1$ and $y_j\le a_{i-1}$ for $j\le i-1$. The above claim says that

$$y_i(b_1,\dots,b_{i-1},a_i-\varepsilon_i)<x_i,$$

which means that there is a continuous function $b_i=b_i(b_1,\dots,b_{i-1})\in(a_i-\varepsilon_i,a_i)$ with $y_i(b_1,\dots,b_{i-1},b_i(b_1,\dots,b_{i-1}))=x_i$. Defining a continuous function

$$y_j(b_1,\dots,b_{i-1})=y_j\bigl(b_1,\dots,b_{i-1},b_i(b_1,\dots,b_{i-1})\bigr),$$

we can continue the induction.

Hence, there is $b=(b_i)\in\prod_{i=1}^{N}(a_i-\varepsilon_i,a_i)$ such that $y_i(b)=x_i$ for any $i\in\{1,\dots,N\}$. Since $\varepsilon_i\searrow 0$ as $d\nearrow 1$, we establish the lemma. ∎

Lemma 6.4. *There exists $c_0\in A$ such that the birational map f determined by c_0 satisfies*

$$\bigl\{\operatorname{Tr}Df(w_l)\bigr\}^2/\operatorname{Det}Df(w_l)\in(0,4)\quad\textit{for any } l\in\{1,\dots,N+2\}.$$

Proof. Under the notations mentioned in Lemma 6.3, we can choose $0\ll d<1$ and $0<b_1<a_1<b_2<\cdots<b_N<a_N$ so that

$$\left|\frac{1}{1-x_l/b_i}-\frac{1}{1-x_l/a_i}\right|=\left|\frac{(a_i-b_i)x_l}{(a_i-x_l)(b_i-x_l)}\right|<\frac{1}{N}\qquad\bigl(l\in\{1,\dots,N\}\bigr)$$

and $1<a_i/b_i<2^{1/N}$ for any $i\in\{1,\dots,N\}$. Then, from Lemma 6.1 and the fact that $\operatorname{Det}Df(w_l)=\delta$, we have

$$\frac{\bigl\{\operatorname{Tr}Df(w_l)\bigr\}^2}{\operatorname{Det}Df(w_l)}=d\left\{1+\sum_{i=1}^{N}\left(\frac{1}{1-x_l/b_i}-\frac{1}{1-x_l/a_i}\right)\right\}^2\in(0,4)$$

for any $l\in\{1,\dots,N\}$. Choose $\delta\in S^1$ so that $d=(1+\delta)^2/\delta$. It follows from Lemma 6.2 and the fact that $1<\beta_0/\alpha_0=\prod_{i=1}^{N}a_i/b_i<2$ that $\{\operatorname{Tr}Df(w_l)\}^2/\operatorname{Det}Df(w_l)\in[0,4]$ and then $\{\operatorname{Tr}Df(w_l)\}^2/\operatorname{Det}Df(w_l)\in(0,4)$ for $l\in\{N+1,N+2\}$ by slightly modifying the parameters if necessary. Thus, we have the desired parameters $c_0=(\delta,(a_i),(b_i))$. ∎

Next, we consider the case $0 < b_0 := b_1 = \cdots = b_N < a_0 := a_1 = \cdots = a_N$. Then, the fixed points $w_l = [x_l : x_l : 1]$ for $l \in \{1, \ldots, N\}$ are given by the roots of $d(1 - x_l/a_0)^N = (1 - x_l/b_0)^N$ with $d = (1+\delta)^2/\delta \in [0, 4]$, which yields

$$x_l = \frac{a_0 b_0 (1 - \lambda_N \epsilon_N^l)}{a_0 - b_0 \lambda_N \epsilon_N^l},$$

where $\lambda_N := d^{1/N} \geq 0$ and $\epsilon_N := \cos(2\pi/N) + i \sin(2\pi/N)$ is a primitive N-th root of unity. Thus, it follows from Lemma 6.1 and the fact that $\mathrm{Det}\, Df(w_l) = \delta$ that

$$\frac{\{\mathrm{Tr}\, Df(w_l)\}^2}{\mathrm{Det}\, Df(w_l)} = d\left\{1 - \frac{N}{a_0 - b_0}(a_0 + b_0 - a_0 \lambda_N^{-1} \epsilon_N^{-l} - b_0 \lambda_N \epsilon_N^l)\right\}^2 = d\Delta_l^2 \tag{13}$$

for $l \in \{1, \ldots, N\}$, where

$$\begin{aligned}\Delta_l = &\left(1 - N\frac{a_0/b_0 + 1}{a_0/b_0 - 1} + N\frac{\lambda_N^{-1} a_0/b_0 + \lambda_N}{a_0/b_0 - 1}\cos\frac{2\pi l}{N}\right)\\ &- iN\frac{\lambda_N^{-1} a_0/b_0 - \lambda_N}{a_0/b_0 - 1}\sin\frac{2\pi l}{N}.\end{aligned}$$

Moreover, from Lemma 6.2, one has

$$\frac{\{\mathrm{Tr}\, Df(w_l)\}^2}{\mathrm{Det}\, Df(w_l)} \notin [0, 4] \iff \left(\frac{a_0}{b_0}\right)^N \notin [0, 4]$$

for $l \in \{N+1, N+2\}$.

Lemma 6.5. *There exists $c_* \in A$ such that the birational map f determined by c_* satisfies* $\{\mathrm{Tr}\, Df(w_l)\}^2/\,\mathrm{Det}\, Df(w_l) \notin [0, 4]$ *for any* $l \in \{1, \ldots, N+2\}$.

Proof. First, we assume that $d = 1/4^2$ and $a_0/b_0 = 4^{1/N}$ in the above notations. If $l \in \{1, \ldots, N\}$, then the only possibilities for $\{\mathrm{Tr}\, Df(w_l)\}^2/\,\mathrm{Det}\, Df(w_l)$ to become a nonnegative real number occur when $(\cos 2\pi l/N, \sin 2\pi l/N) = (\pm 1, 0)$ in (13). On the other hand, in the case $(\cos 2\pi l/N, \sin 2\pi l/N) = (\pm 1, 0)$, it is seen that $\{\mathrm{Tr}\, Df(w_l)\}^2/\,\mathrm{Det}\, Df(w_l) > 4$. Indeed, when $(\cos 2\pi l/N, \sin 2\pi l/N) = (1, 0)$, one has

$$\frac{\{\mathrm{Tr}\, Df(w_l)\}^2}{\mathrm{Det}\, Df(w_l)} = \frac{1}{4^2}\left(1 - N\frac{4^{1/N} + 1}{4^{1/N} - 1} + N\frac{4^{3/N} + 4^{-2/N}}{4^{1/N} - 1}\right)^2 = \left(2 + \frac{N}{4}g(N)\right)^2$$

with $g(N) := (4^{2/N} - 4^{-2/N}) + (4^{1/N} - 4^{-1/N}) - 7/N$. The function $g(N)$ satisfies $g(N) > 0$ for any $N \geq 1$, as $g(N)$ is monotone decreasing in N and $\lim_{N\to\infty} g(N) = 0$. Thus, we have $\{\mathrm{Tr}\, Df(w_l)\}^2/\,\mathrm{Det}\, Df(w_l) > 4$. The case $(\cos 2\pi l/N, \sin 2\pi l/N) =$

$(-1, 0)$ can be treated in a similar manner. Thus, the condition

$$\left\{\operatorname{Tr} Df(w_l)\right\}^2 / \operatorname{Det} Df(w_l) \notin [0, 4]$$

holds for any $l \in \{1, \ldots, N\}$.

Now, since $\{\operatorname{Tr} Df(w_l)\}^2 / \operatorname{Det} Df(w_l)$ continuously depends on the parameters $(\delta, a, b) \in A$, with the above condition, we slightly modify the parameters so that $0 < b_1 < \cdots < b_N < b_0 < a_0 < a_1 < \cdots < a_N$, which means that $\beta_0/\alpha_0 > (a_0/b_0)^N = 4$ and thus yields $\{\operatorname{Tr} Df(w_l)\}^2 / \operatorname{Det} Df(w_l) \notin [0, 4]$ for any $l \in \{1, \ldots, N + 2\}$. By fixing $\delta \in S^1$ with $d = (1 + \delta)^2/\delta$, we show the existence of $c_* = (\delta, a, b) \in A$. ■

Proof of Proposition 4.5. Note that $\{\operatorname{Tr} Df(w_l)\}^2 / \operatorname{Det} Df(w_l)$ continuously depends on the parameters $(\delta, a, b) \in A$. Hence, the proposition is the consequence of Lemmas 6.4 and 6.5. ■

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Takato Uehara
Graduate School of Natural Science and Technology, Okayama University,
3-1-1 Tsushimanaka, Kita-ku, Okayama 700-8530, Japan
takaue@okayama-u.ac.jp