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Calculus of Variations. – *Regularity improvement for the minimizers of the two-dimensional Griffith energy*, by Camille Labourie and Antoine Lemenant, communicated on 10 November 2022.

Abstract. – In this paper, we prove that the singular set of connected minimizers of the planar Griffith functional has Hausdorff dimension strictly less than one, together with the higher integrability of the symmetrized gradient.

Keywords. – Free discontinuity problems, Griffith functional, ε-regularity.

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1. Introduction

In a planar elasticity setting, the Griffith energy is defined by

$$\mathcal{G}(u, K) := \int_{\Omega \setminus K} \mathbf{A}e(u) : e(u)\,\mathrm{d}x + \mathcal{H}^1(K),$$

where $\Omega \subset \mathbb{R}^2$, which is bounded and open, stands for the reference configuration of a linearized elastic body, and

$$\mathbf{A}\xi = \lambda(\operatorname{tr}\xi)I + 2\mu\xi \quad \text{for all } \xi \in \mathbb{M}^{2\times 2}_{\mathrm{sym}},$$

where λ and μ are the Lamé coefficients satisfying $\mu > 0$ and $\lambda + \mu > 0$. Here, $e(u) = (Du + Du^T)/2$ is the symmetric gradient of the displacement $u : \Omega \setminus K \to \mathbb{R}^2$ which is defined outside the crack $K \subset \overline{\Omega}$.

This energy functional is defined on pairs function/set

$$(u, K) \in \mathcal{A}(\Omega) := \big\{K \subset \overline{\Omega} \text{ is closed and } u \in LD(\Omega' \setminus K)\big\},$$

where $\Omega' \supset \overline{\Omega}$ is a bounded open set and LD is the space of functions of Lebesgue deformation for which $e(u) \in L^2$. Note that, by definition, K is a compact subset of $\overline{\Omega}$ and u is defined in $\Omega' \setminus K$, but we work in the ambient space Ω so as to build local competitors.

We say that (u, K) is a minimizer for the Griffith energy if it is a solution to the problem

$$\inf\left\{\int_\Omega \mathbf{A}e(v):e(v)\,\mathrm{d}x+\mathcal{H}^1(K):(v,K)\in\mathcal{A}(\Omega),\ v=\psi \text{ a.e. in } \Omega'\setminus\overline{\Omega}\right\}$$

for some datum $\psi \in W^{1,\infty}$.

A lot of attention has been given on the Griffith functional these last years (see [4–10,17]), and, in particular, it has been proved that a global minimizer $(u, K) \in \mathcal{A}(\Omega)$ (with a prescribed Dirichlet boundary condition) does exist and that the crack set K is $\mathcal{H}^1$-rectifiable and locally Ahlfors-regular in Ω. The latter means that there exists $C_0 \geq 1$ (depending on $\mathbf{A}$) such that for all $x \in K$ and all $r > 0$ with $B(x,r) \subset \Omega$,

$$C_0^{-1}r \leq \mathcal{H}^1\big(K\cap B(x,r)\big) \leq C_0 r.$$

In [4], it was proved that any isolated connected component of the singular set K of a Griffith minimizer is $C^{1,\alpha}$ a.e. It also applies to a connected minimizer K (for, e.g., minimizer with connected constraints). In this paper, we slightly improve the Hausdorff dimension of the singular set. We also prove some higher integrability property on the symmetrized gradient.

The main results of this paper are the following.

THEOREM 1.1. *Let $(u, K) \in \mathcal{A}(\Omega)$ be a minimizer of the Griffith energy with K connected. Then the following hold.*

(1) *There exist $\alpha \in (0,1)$ and a relatively closed set $\Sigma \subset K \cap \Omega$ with $\dim_{\mathcal{H}}(\Sigma) < 1$ such that $K \cap \Omega \setminus \Sigma$ is locally a $\mathcal{C}^{1,\alpha}$ curve.*

(2) *There exist $C \geq 1$ and $p > 1$ (depending on $\mathbf{A}$) such that for all $x \in \Omega$ and $r > 0$ such that $B(x,r) \subset \Omega$,*

$$\int_{B(x,r/2)} \big|e(u)\big|^{2p}\,\mathrm{d}x \leq C r^{2-p}.$$

The proof of our main theorem follows from standard techniques that were already used in the scalar context of the Mumford–Shah functional, but adapted to the vectorial Griffith functional in a non-trivial manner. In particular, for (1) we follow the approach of David [11] and Rigot [19], based on uniform rectifiability of the singular set and Carleson measure estimates (see also [1,14] for an alternative approach). The idea is to estimate to number of balls in which one can apply the ε-regularity theorem contained in [4]. But the latter needs a topological separating property that one has to control in any initialized balls which is one of the main issues of the present work (Lemma 3.1). We also need to control the 2-energy by a p-energy (Corollary 5.1) which also uses a topological argument (Lemma 5.3). The proof of (2) is based on a strategy similar to what was first introduced by De Philippis and Figalli in [15] and also used in [18], which easily follows from the porosity of the singular set together with elliptic estimates. Since the elliptic estimates needed relatively for the Lamé system are not easy to find in the literature, we have developed an appendix containing the precise results.

Let us stress that the famous Cracktip function, that arises as blow-up limits of Mumford–Shah minimizers at the tip of the crack, has a vectorial analogue. This was the purpose of the work in [3]. Since the vectorial Cracktip is homogeneous of degree 1/2 (see [3, Theorem 6.4]), it is natural to conjecture that, akin to the standard Mumford–Shah functional, the integrability exponent of $|e(u)|$ should reach every $p < 4$, as asked by De Giorgi for the Mumford–Shah functional.

2. Preliminaries

Notation

The Lebesgue measure in $\mathbb{R}^n$ is denoted by $\mathcal{L}^n$, and the k-dimensional Hausdorff measure by $\mathcal{H}^k$. If E is a measurable set, we will sometimes write $|E|$ instead of $\mathcal{L}^n(E)$. If a and $b \in \mathbb{R}^n$, we write $a \cdot b = \sum_{i=1}^n a_i b_i$ for the Euclidean scalar product, and we denote the norm by $|a| = \sqrt{a \cdot a}$. The open (resp., closed) ball of center x and radius r is denoted by $B(x,r)$ (resp., $\overline{B}(x,r)$).

We write $\mathbb{M}^{2\times 2}$ for the set of real 2×2 matrices, and $\mathbb{M}^{2\times 2}_{\mathrm{sym}}$ for that of all real symmetric 2×2 matrices. Given two matrices $A, B \in \mathbb{M}^{2\times 2}$, we recall the Frobenius inner product $A : B = \mathrm{tr}({}^t A B)$ and the corresponding norm $|A| = \sqrt{\mathrm{tr}(A^T A)}$.

Functions of Lebesgue deformation

Given a weakly differentiable vector field u, the symmetrized gradient of u is denoted by

$$e(u) := \frac{Du + Du^T}{2}.$$

The p-normalized energy

Let $(u, K) \in \mathcal{A}(\Omega)$. Then for any $x_0 \in \Omega$ and $r > 0$ such that $B(x_0, r) \subset \Omega$, we define the *normalized elastic energy* by

$$\omega_p(x_0, r) := r^{1-\frac{4}{p}} \left(\int_{B(x_0,r)\setminus K} |e(u)|^p \,\mathrm{d}x \right)^{\frac{2}{p}}.$$

The flatness

Let K be a relatively closed subset of Ω. For any $x_0 \in K$ and $r > 0$ such that $B(x_0, r) \subset \Omega$, we define the *(bilateral) flatness* by

$$\beta_K(x_0, r) := \frac{1}{r} \inf_L \max \Big\{ \sup_{y \in K \cap \bar{B}(x_0,r)} \operatorname{dist}(y, L), \sup_{y \in L \cap \bar{B}(x_0,r)} \operatorname{dist}(y, K) \Big\},$$

where L belongs to the set of lines passing through x_0. When a minimizer $(u, K) \in \mathcal{A}(\Omega)$ is given, we write simply $\beta(x_0, r)$ for $\beta_K(x_0, r)$.

REMARK 2.1. *The flatness $\beta_K(x_0, r)$ only depends on the set $K \cap B(x_0, 2r)$. We have that for all $0 < t \leq r$,*

$$\beta_K(x_0, t) \leq \frac{r}{t} \beta_K(x_0, r),$$

and for $y_0 \in K \cap B(x_0, r/2)$ and $0 < t \leq r/2$

$$\beta_K(y_0, t) \leq \frac{2r}{t} \beta_K(x_0, r).$$

If $K' = \frac{1}{r}(K - x_0)$, then

$$\beta_K(x_0, r) = \beta_{K'}(0, 1).$$

In the sequel, we will consider the situation where $x_0 \in K$, $r > 0$ are such that $B(x_0, r) \subset \Omega$ and

$$\beta_K(x_0, r) \leq \varepsilon,$$

for $\varepsilon > 0$ small. This implies, in particular, that $K \cap B(x_0, r)$ is contained in a narrow strip of thickness εr passing through the center of the ball. Let $L(x_0, r)$ be a line passing through x_0 and satisfying

$$K \cap B(x_0, r) \subset \big\{ y \in B(x_0, r) \mid \operatorname{dist}(y, L) \leq r \beta_K(x_0, r) \big\}. \tag{2.1}$$

We will often use a local basis (depending on x_0 and r) denoted by (e_1, e_2), where e_1 is a tangent vector to the line $L(x_0, r)$, while e_2 is an orthogonal vector to $L(x_0, r)$. The coordinates of a point y in that basis will be denoted by (y_1, y_2).

Provided (2.1) is satisfied with $\beta_K(x_0, r) \leq 1/2$, we can define two discs $D^+(x_0, r)$ and $D^-(x_0, r)$ of radius $r/4$ and such that $D^\pm(x_0, r) \subset B(x_0, r) \setminus K$. Indeed, using the notation introduced above, setting $x_0^\pm := x_0 \pm \frac{3}{4} r e_2$, we can check that $D^\pm(x_0, r) := B(x_0^\pm, r/4)$ satisfy the above requirements.

A property that will be fundamental in our analysis is the separation in a closed ball.

DEFINITION 2.1. *Let K be a closed set of $\mathbb{R}^2$, let $x_0 \in K$, and let $r > 0$ be such that $\beta_K(x_0, r) \leq 1/2$. We say that K* separates *in $B(x_0, r)$ if the balls $D^{\pm}(x_0, r)$ are contained into two different connected components of $B(x_0, r) \setminus K$.*

The following lemma guarantees that when passing from a ball $B(x_0, r)$ to a smaller one $B(x_0, t)$, and provided that $\beta_K(x, r)$ is relatively small, the property of separating is preserved for t varying in a range depending on $\beta_K(x, r)$.

LEMMA 2.1 ([4, Lemma 3.1]). *Let $\tau \in (0, 1/16)$, let $K \subset \mathbb{R}^2$ be a closed set, let $x_0 \in K$, and let $r > 0$ be such that $B(x_0, r) \subset \Omega$ and $\beta_K(x_0, r) \leq \tau$. If K separates in $B(x_0, r)$, then for all $t \in (16\tau r, r)$, we have $\beta_K(x_0, t) \leq 1/2$ and K still separates in $B(x_0, t)$.*

3. LOCAL SEPARATION IN MANY BALLS IN A CONNECTED UNIFORMLY RECTIFIABLE SET

The purpose of this section is the following general result on compact connected sets which are locally Ahlfors-regular.

LEMMA 3.1. *Let $K \subset \overline{\Omega}$ be a compact connected set which is locally Ahlfors-regular in Ω; that is, there exists $C_0 \geq 1$ such that for all $x \in K$ and for all $r > 0$ with $B(x, r) \subset \Omega$,*

$$C_0^{-1} r \leq \mathcal{H}^1\big(K \cap B(x, r)\big) \leq C_0 r.$$

Then, for every $0 < \varepsilon \leq \frac{1}{2}$, there exists $a \in (0, 1/2)$ small enough (depending on C_0 and ε) such that for all $x \in K$ and $r > 0$ with $B(x, r) \subset \Omega$, one can find $y \in B(x, r/2)$ and $t \in (ar, r/2)$ satisfying

(3.1) $\beta_K(y, t) \leq \varepsilon$ *and K separates in $B(y, t)$ in the sense of Definition 2.1.*

PROOF. The letter C is a constant ≥ 1 that depends on C_0 and whose value might increase from one line to another but a finite number of times. Let K be as in the statement of the lemma. For every $\varepsilon > 0$, we denote by $\mathcal{B}(\varepsilon)$ the bad set where β_K is large:

$$\mathcal{B}(\varepsilon) := \big\{(z, s) \mid z \in K,\ s > 0 \text{ and } \beta_K(z, s) > \varepsilon\big\}.$$

Let $x \in K$ and let $r > 0$ be such that $B(x, r) \subset \Omega$. It is more convenient to assume that $B(x, 4r) \subset \Omega$ and $r \leq \operatorname{diam}(K)/4$ and we are going to justify that we can make this assumption without loss of generality. First, we draw from the local Ahlfors-regularity that there exists a constant $C_1 \geq 1$ (depending on C_0) such $\operatorname{diam}(K) \geq C_1^{-1} r$. Let us consider $\kappa := 4C_1$,

$$r_1 := \kappa^{-1} r \leq \min\big(r/4, \operatorname{diam}(K)/4\big),$$

and some $a \in (0, 1/2)$. If we solve the problem in the ball $B(x, r_1)$, that is, if we find $y \in B(x, r_1/2)$ and $t \in (ar_1, r_1/2)$ such that (3.1) holds true, then we have solved the problem in $B(x, r)$ as well because $y \in B(x, r/2)$ and $t \in (br, r/2)$, where $b = a\kappa^{-1}$. This shows that it suffices to solve the problem in the ball $B(x, r_1)$ which has all the desired properties. To simplify the notations, we directly assume that $B(x, 4r) \subset \Omega$ and $r \le \mathrm{diam}(K)/6$.

In the sequel, we want to apply the results of [13], which work with sets of infinite diameter. This explains why we need to slightly modify our set K to fit in the definition of [13]. Precisely, given an arbitrary line L passing through x, one can check that the set

$$E = \big(K \cap B(x, 3r)\big) \cup \partial B(x, 3r) \cup \big(L \setminus B(x, 3r)\big) \tag{3.2}$$

is connected and Ahlfors-regular in the exact sense of [13, Definition 1.13]; that is, E is closed and there exists $C \ge 1$ (that depends on C_0 as usual) such that for all $y \in E$ and for all $\rho > 0$,

$$C^{-1}\rho \le \mathcal{H}^1\big(E \cap B(y, \rho)\big) \le C\rho.$$

As a consequence, it is contained in an (Ahlfors)-regular curve (see [13, (1.63)] and the discussion below) and thus is uniformly rectifiable with a constant $C \ge 1$ that depends on C_0 [13, Theorem 1.57 and Definition 1.65]. In particular, it satisfies a geometric characterization of uniform rectifiability called *bilateral weak geometric lemma* [13, Definition 2.2 and Theorem 2.4]. It means that, for all $\varepsilon > 0$, there exists $C(\varepsilon) \ge 1$ (depending on C_0 and ε) such that for all $y \in E$ and all $\rho > 0$,

$$\int_{z \in E \cap B(y,\rho)} \int_0^\rho \mathbf{1}_{\mathscr{C}(\varepsilon)}(z, s) \frac{\mathrm{d}s}{s} \, \mathrm{d}\mathcal{H}^1(z) \le C(\varepsilon)\rho,$$

where

$$\mathscr{C}(\varepsilon) := \big\{(z, s) \mid z \in E,\ s > 0,\ \text{and } \beta_E(z, s) > \varepsilon\big\}.$$

We are going to apply this property with $y := x$ and $\rho := r$. We observe that for all $z \in K \cap B(x, r)$ and for all $0 < s < r$, we have $K \cap B(z, 2s) = E \cap B(z, 2s)$ and hence $\beta_K(z, s) = \beta_E(z, s)$. Thus, (3.2) simplifies to

$$\int_{z \in K \cap B(x,r)} \int_0^r \mathbf{1}_{\mathscr{B}(\varepsilon)}(z, s) \frac{\mathrm{d}s}{s} \, \mathrm{d}\mathcal{H}^1(z) \le C(\varepsilon)r. \tag{3.3}$$

We now return to the statement of the lemma. We fix $\varepsilon > 0$. Let $a \in (0, 1)$ be a parameter that will be fixed later. Assume by contradiction that for all $y \in K \cap B(x, r)$ and $t \in (ar, r)$, we have $\beta_K(y, t) > \varepsilon$. This means that for such pairs (y, t), we have

$\mathbf{1}_{\mathcal{B}(\varepsilon)}(y,t) = 1$. Moreover, we have by local Ahlfors-regularity that $\mathcal{H}^1(K \cap B(x,r)) \geq C_0^{-1} r$, so

$$\int_{y \in K \cap B(x,r)} \int_{0<t<r} \mathbf{1}_{\mathcal{B}(\varepsilon)}(y,t) \frac{\mathrm{d}t}{t} \, \mathrm{d}\mathcal{H}^1(z) \\ \geq \mathcal{H}^1\big(K \cap B(x,r)\big) \int_{ar}^{r} \frac{dt}{t} \geq \mathcal{H}^1\big(K \cap B(x,r)\big) \ln\left(\frac{1}{a}\right) \geq C_0^{-1} r \ln\left(\frac{1}{a}\right).$$

Using now (3.3), we arrive at a contradiction, provided that a is small enough (depending on C_0 and ε).

At this stage, we have proved that, for every $x \in K$ and $r > 0$ with $B(x,r) \subset \Omega$, there exist $y \in B(x, r/2)$ and $t \in (ar, r/2)$ such that $t \leq \operatorname{diam}(K)/4$ and

$$\beta_K(y,t) \leq \varepsilon.$$

It remains to deal with the separation property. For that purpose, we will use the fact that a compact connected set with finite length is arcwise connected (see [13, Theorem 1.8]). Let us fix the coordinate system such that $y = (0,0)$ and the line L that realizes the infimum in the definition of $\beta_K(y,t)$ is the x axis: $L = \mathbb{R} \times \{0\}$. This means that

$$K \cap B(y,t) \subset \big\{(z_1, z_2) \mid |z_2| \leq \varepsilon t\big\}.$$

Since $t \leq \operatorname{diam}(K)/4$, there exist a point $z \in K \setminus B(y,t)$ and a curve $\Gamma \subset K$ from y to z. This curve touches $\partial B(y,t)$ at some point z'. Let $\Gamma' \subset \Gamma$ be the piece of curve from y to z'. The point z' must lie either on $\partial B(y,t) \cap \{(z_1, z_1) \mid z_1 > 0\}$ or $\partial B(y,t) \cap \{(z_1, z_1) \mid z_1 < 0\}$. Let us assume that the first case occurs (for the second case we can argue similarly). The curve Γ' stays inside the strip $\{(z_1, z_2) \mid |z_2| \leq \varepsilon t\}$, and runs from y (the center of the ball) to z' (on the boundary). Let $y' := y + \frac{t}{2}\mathbf{e}_1$. Then we have $\beta_K(y', \frac{1}{4}t) \leq 4\varepsilon$ and, assuming that $\varepsilon \leq 1/8$, the curve Γ' separates in the ball $B(y', \frac{1}{4}t)$ (as in Definition 2.1), and so does K. This achieves the proof of the proposition. ∎

4. Carleson measure estimates on $\omega_p(x,r)$

$$\Delta := \big\{(x,r) \mid x \in K,\ r > 0 \text{ and } B(x,r) \subset \Omega\big\}.$$

The purpose of this section is to state the following fact.

Proposition 4.1. *Let $(u,K) \in \mathcal{A}(\Omega)$ be a minimizer of the Griffith functional. For all $p \in [1,2)$, there exists $C_p \geq 1$ (depending on p and* **A***) such that*

$$\int_{y \in K \cap B(x,r)} \int_{0<t<r} \omega_p(y,t) \frac{\mathrm{d}t}{t} \, \mathrm{d}\mathcal{H}^1(y) \leq C_p r,$$

for all $(x, 2r) \in \Delta$.

PROOF. The proof was originally performed by David and Semmes in the scalar context of Mumford–Shah minimizers (see [12, Section 23]). It relies on the local Ahlfors-regularity of Griffith minimizers; that is, there exists $C_0 \geq 1$ (depending on $\mathbf{A}$) such that for all $(x, r) \in \Delta$,

$$\int_{B(x,r)} |e(u)|^2 \,\mathrm{d}x + \mathcal{H}^1\big(K \cap B(x,r)\big) \leq C_0 r \tag{4.1}$$

and

$$\mathcal{H}^1\big(K \cap B(x,r)\big) \geq C_0^{-1} r.$$

The inequality (4.1) directly follows by taking $(K \setminus B(x,r)) \cup \partial B(x,r)$ and $u\mathbf{1}_{\Omega \setminus B(x,r)}$ as a competitor, and the ellipticity of $\mathbf{A}$. The proof in [12, Section 23] on Mumford–Shah minimizers can be followed verbatim so we prefer to omit the details and refer directly to [12]. ■

5. CONTROL OF ω_2 BY ω_p

The main ε-regularity theorem uses an assumption on the smallness of ω_2. Unfortunately, what we can really control in many balls (thanks to Proposition 4.1) is ω_p for $p < 2$, which is weaker. This is why in this section we prove that ω_2 can be estimated from ω_p, for a minimizer. This strategy was already used in [12, 19] for the Mumford–Shah functional. The adaptation for the Griffith energy is not straightforward, but can be done by following a similar approach as the one already used in [4, Section 4.1], generalized with ω_p instead of only ω_2. Some estimates from the book [12] were also useful.

LEMMA 5.1 (Harmonic extension in a ball from an arc of circle). *Let $p \in (1, 2]$, $0 < \delta \leq 1/2$, $x_0 \in \mathbb{R}^2$, and $r > 0$. Let $\mathcal{C}_\delta \subset \partial B(x_0, r)$ be the arc of circle defined by*

$$\mathcal{C}_\delta := \big\{(x_1, x_2) \in \partial B(x_0, r) : (x - x_0)_2 > \delta r\big\}.$$

Then, there exists a constant $C > 0$ (independent of δ, x_0, and r) such that every function $u \in W^{1,p}(\mathcal{C}_\delta; \mathbb{R}^2)$ extends to a function $g \in W^{1,2}(B(x_0, r); \mathbb{R}^2)$ with $g = u$ on $\mathcal{C}_\delta$ and

$$\int_{B(x_0,r)} |\nabla g|^2 \,\mathrm{d}x \leq C r^{2-\frac{2}{p}} \left(\int_{\mathcal{C}_\delta} |\partial_\tau u|^p \,\mathrm{d}\mathcal{H}^1 \right)^{\frac{2}{p}},$$

where $C = C(p)$.

PROOF. Let $\Phi : \mathcal{C}_\delta \to \mathcal{C}_0$ be a bi-Lipschitz mapping with Lipschitz constants independent of $\delta \in (0, 1/2]$, x_0, and $r > 0$. Since $u \circ \Phi^{-1} \in W^{1,p}(\mathcal{C}_0; \mathbb{R}^2)$, we can define the extension by reflection $\tilde{u} \in W^{1,p}(\partial B(x_0, r); \mathbb{R}^2)$ on the whole circle $\partial B(x_0, r)$, that satisfies

$$\int_{\partial B(x_0,r)} |\partial_\tau \tilde{u}|^p \,\mathrm{d}\mathcal{H}^1 \leq C \int_{\mathcal{C}_\delta} |\partial_\tau u|^p \,\mathrm{d}\mathcal{H}^1,$$

where $C > 0$ is a constant which is independent of δ. We next define g as the harmonic extension of $\tilde{u}$ in $B(x_0, r)$. Using [12, Lemma 22.16], we obtain

$$\int_{B(x_0,r)} |\nabla g|^2 \, dx \leq C r^{2-\frac{2}{p}} \left(\int_{\partial B(x_0,r)} |\partial_\tau \tilde{u}|^p \, d\mathcal{H}^1 \right)^{\frac{2}{p}}$$
$$\leq C r^{2-\frac{2}{p}} \left(\int_{\mathcal{C}_\delta} |\partial_\tau u|^2 \, d\mathcal{H}^1 \right)^{\frac{2}{p}},$$

which completes the proof. ∎

LEMMA 5.2. *Let $(u, K) \in \mathcal{A}(\Omega)$ be a minimizer of the Griffith functional, and let $x_0 \in K$ and $r > 0$ be such that $B(x_0, r) \subset \Omega$ and $\beta(x_0, r) \leq 1/2$. Let S be the strip defined by*

$$S := \{ y \in B(x_0, r) \mid \operatorname{dist}(y, L) \leq r\beta(x_0, r) \},$$

where L is the line passing through x_0 which achieves the infimum in $\beta_K(x_0, r)$. Then there exist a universal constant $C > 0$, $\rho \in (r/2, r)$, and $v^\pm \in H^1(B(x_0, \rho); \mathbb{R}^2)$, such that $v^\pm = u$ on $\mathcal{C}^\pm$, $\mathcal{C}^\pm$ being the connected components of $\partial B(x_0, \rho) \setminus S$, and

$$\int_{B(x_0,\rho)} \left| e(v^\pm) \right|^2 dx \leq C r^{2-\frac{4}{p}} \left(\int_{B(x_0,r)\setminus K} \left| e(u) \right|^p dx \right)^{\frac{2}{p}}.$$

PROOF. Let $A^\pm$ be the connected components of $B(x_0, r) \setminus S$. Since $K \cap A^\pm = \emptyset$, by Korn inequality there exist two skew-symmetric matrices $R^\pm$ such that the functions $x \mapsto u(x) - R^\pm x$ belong to $W^{1,p}(A^\pm; \mathbb{R}^2)$ and

$$\int_{A^\pm} |\nabla u - R^\pm|^p \, dx \leq C \int_{A^\pm} \left| e(u) \right|^p dx,$$

where the constant $C > 0$ is universal since the domains $A^\pm$ are all uniformly Lipschitz for all possible values of $\beta(x_0, r) \leq 1/2$. Using the change of variables in polar coordinates, we infer that

$$\int_{A^\pm} |\nabla u - R^\pm|^p \, dx = \int_0^r \left(\int_{\partial B(x_0,\rho)\cap A^\pm} |\nabla u - R^\pm|^p \, d\mathcal{H}^1 \right) d\rho$$

which allows us to choose a radius $\rho \in (r/2, r)$ satisfying

$$\int_{\partial B(x_0,\rho)\cap A^+} |\nabla u - R^+|^p \, d\mathcal{H}^1 + \int_{\partial B(x_0,\rho)\cap A^-} |\nabla u - R^-|^p \, d\mathcal{H}^1$$
$$\leq \frac{2}{r} \int_{A^+} |\nabla u - R^+|^p \, dx + \frac{2}{r} \int_{A^-} |\nabla u - R^-|^p \, dx \leq \frac{C}{r} \int_{B(x_0,r)\setminus K} \left| e(u) \right|^p dx.$$

Setting $\mathcal{C}^{\pm} := \partial A^{\pm} \cap \partial B(x_0, \rho)$, in view of Lemma 5.1 applied to the functions $u^{\pm} : x \mapsto u(x) - R^{\pm}x$, which belong to $W^{1,p}(\mathcal{C}^{\pm}; \mathbb{R}^2)$ since they are regular, for $\delta = \beta(x_0, r)$ we get two functions $g^{\pm} \in W^{1,2}(B(x_0, \rho); \mathbb{R}^2)$ satisfying $g^{\pm}(x) = u(x) - R^{\pm}x$ for $\mathcal{H}^1$-a.e. $x \in \mathcal{C}^{\pm}$ and

$$\int_{B(x_0,\rho)} |\nabla g^{\pm}|^2 \,\mathrm{d}x \le C\rho^{2-\frac{2}{p}} \left(\int_{\mathcal{C}^{\pm}} |\partial_\tau u^{\pm}|^p \,\mathrm{d}\mathcal{H}^1 \right)^{\frac{2}{p}}$$
$$\le C r^{2-\frac{4}{p}} \left(\int_{B(x_0,r)\setminus K} \left|e(u)\right|^p \,\mathrm{d}x \right)^{\frac{2}{p}}.$$

Finally, the functions $x \mapsto v^{\pm}(x) := g^{\pm}(x) + R^{\pm}x$ satisfy the required properties. ∎

Using the competitor above, we can obtain the following.

Proposition 5.1. *Let $(u, K) \in \mathcal{A}(\Omega)$ be a minimizer of the Griffith functional, and let $x_0 \in K$ and $r > 0$ be such that $B(x_0, r) \subset \Omega$ and $\beta(x_0, r) \le 1/2$. Then there exist a universal constant $C > 0$ and a radius $\rho \in (r/2, r)$ such that*

$$\int_{B(x_0,\rho)\setminus K} \mathbf{A}e(u) : e(u) \,\mathrm{d}x + \mathcal{H}^1\big(K \cap B(x_0, \rho)\big) \le 2\rho + C\rho\big(\omega_p(x_0, r) + \beta(x_0, r)\big).$$

Proof. We keep using the same notation as used in the proof of Lemma 5.2. Let $\rho \in (r/2, r)$ and $v^{\pm} \in H^1(B(x_0, \rho); \mathbb{R}^2)$ be given by the conclusion of Lemma 5.2. We now construct a competitor in $B(x_0, \rho)$ as follows. First, we consider a "wall" set $Z \subset \partial B(x_0, \rho)$ defined by

$$Z := \big\{ y \in \partial B(x_0, \rho) \mid \operatorname{dist}\big(y, L(x_0, r)\big) \le r\beta(x_0, r) \big\}.$$

Note that $K \cap \partial B(x_0, \rho) \subset Z$,

$$\partial B(x_0, \rho) = \big[\partial A^+ \cap \partial B(x_0, \rho)\big] \cup \big[\partial A^- \cap \partial B(x_0, \rho)\big] \cup Z = \mathcal{C}^+ \cup \mathcal{C}^- \cup Z,$$

and that

$$\mathcal{H}^1(Z) = 4\rho \arcsin\left(\frac{r\beta(x_0, r)}{\rho} \right) \le 2\pi r\beta(x_0, r).$$

We are now ready to define the competitor (v, K') by setting

$$K' := \big[K \setminus B(x_0, \rho)\big] \cup Z \cup \big[L(x_0, r) \cap B(x_0, \rho)\big],$$

and, denoting by $V^{\pm}$ the connected components of $B(x_0, \rho) \setminus L(x_0, r)$ which intersect $A^{\pm}$,

$$v := \begin{cases} v^{\pm} & \text{in } V^{\pm}, \\ u & \text{otherwise.} \end{cases}$$

Since $\mathcal{H}^1(K' \cap \overline{B}(x_0,\rho)) \le 2\rho + 2\pi r\beta(x_0,r)$, we deduce that

$$\begin{aligned}
&\int_{B(x_0,\rho)\setminus K} \mathbf{A}e(u) : e(u)\,\mathrm{d}x + \mathcal{H}^1\big(K \cap \overline{B}(x_0,\rho)\big)\\
&\quad\le \int_{B(x_0,\rho)\setminus K} \mathbf{A}e(v) : e(v)\,\mathrm{d}x + \mathcal{H}^1\big(K' \cap \overline{B}(x_0,\rho)\big)\\
&\quad\le C r^{2-\frac{4}{p}}\bigg(\int_{B(x_0,r)\setminus K} |e(u)|^p\,\mathrm{d}x\bigg)^{\frac{2}{p}} + \rho\big(2 + C\beta(x_0,r)\big)\\
&\quad\le 2\rho + C\rho\big(\omega_p(x_0,r) + \beta(x_0,r)\big),
\end{aligned}$$

and the proposition follows. ■

The next lemma is of purely topological nature.

Lemma 5.3. *Let $K \subset \mathbb{R}^2$ be a compact connected set with $\mathcal{H}^1(K) < +\infty$. Assume that, for some $x \in K$ and $r \in (0, \operatorname{diam}(K))$, we have $\beta_K(x,r) \le 1/2$. Then*

$$\mathcal{H}^1\big(K \cap B(x,r)\big) \ge 2r - 3r\beta_K(x,r).$$

Proof. Let $\varepsilon := \beta_K(x,r)$. We can assume that $x = (0,0)$ and that $L := L(x,r) = \mathbb{R} \times \{0\}$ so that $K \cap B(x,r)$ is contained in the strip S defined by

$$S := B(x,r) \cap \big\{(z_1,z_2) \in \mathbb{R}^2 \mid |z_2| \le \varepsilon r\big\}.$$

Let $\pi_1 : \mathbb{R}^2 \to L$ be the projection defined by $\pi_1(z_1,z_2) = (z_1,0)$. As π_1 is 1-Lipschitz, we know that

$$\mathcal{H}^1\big(K \cap B(x,r)\big) \ge \mathcal{H}^1\big(\pi_1\big(K \cap B(x,r)\big)\big).$$

Let us denote $E = \pi_1(K \cap B(x,r))$. Now we define the constant

$$c_\varepsilon := \sqrt{1-\varepsilon^2},$$

and we use that K is connected to claim that $L \cap [-rc_\varepsilon, rc_\varepsilon] \setminus E$ is an interval (we identify L with the real axis). Indeed, notice that even if K is connected, it may be that $K \cap B(x,r)$ is not. However, for each $a \in E$, there exists $|t| \le \varepsilon r$ such that $z_0 := (a,t) \in K$, and since $r < \operatorname{diam}(K)$ there exists a curve Γ that connects z_0 to some point $z_1 \in K \setminus B(x,r)$. But then E has to contain $\pi_1(\Gamma)$, and since $\beta_K(x,r) \le 1/10$, it means that either $[a, rc_\varepsilon] \subset E$ or $[-rc_\varepsilon, a] \subset E$. Indeed, the curve Γ is contained in the strip S and has to "escape the ball" $B(x,r)$ either from the right or from the left. The projection with minimal length would be when Γ escapes exactly at the corner of $S \cap B(x,r)$ which gives the definition of c_ε (see Figure 1).

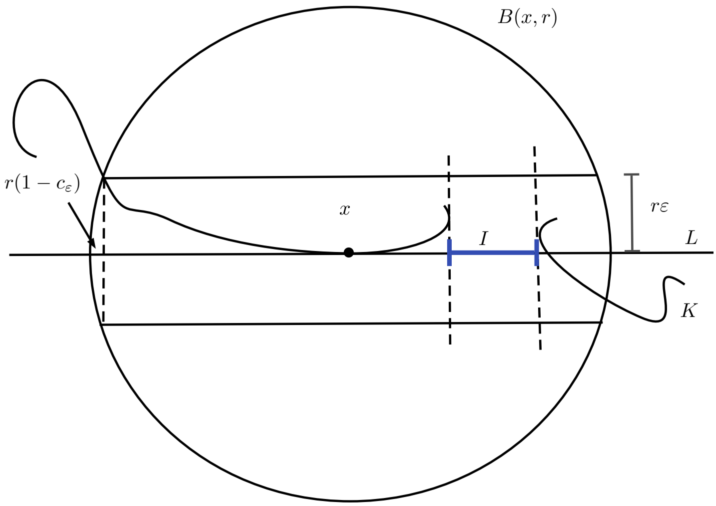

FIGURE 1. Estimating the length of $K \cap B(x,r)$.

This holds true for all $a \in E$, which necessarily imply that $[-c_\varepsilon r, c_\varepsilon r] \setminus E$ is an interval, that we denote by I. As $(I \times [-\varepsilon r, \varepsilon r]) \cap K = \emptyset$, we must have $|I| \leq 2\varepsilon r$, otherwise $\beta_K(x,r) > \varepsilon$. All in all, we have proved that

$$\mathcal{H}^1\big(K \cap B(x,r)\big) \geq \mathcal{H}^1\big(\pi_1\big(K \cap B(x,r)\big)\big) \geq 2rc_\varepsilon - 2\varepsilon r.$$

Now, we estimate $2c_\varepsilon r - 2\varepsilon r$. We have

$$1 - \sqrt{1-\varepsilon^2} = \frac{\varepsilon^2}{1+\sqrt{1-\varepsilon^2}} \leq \varepsilon^2 \leq \frac{1}{2}\varepsilon,$$

whence $2c_\varepsilon \geq 2 - \varepsilon$ and the result follows. ∎

We now come to the interesting “reverse Hölder” type estimate that will be needed later.

COROLLARY 5.1. *Let $(u,K) \in \mathcal{A}(\Omega)$ be a minimizer of the Griffith functional, and let $x_0 \in K$ and $r \in (0, \operatorname{diam}(K))$ be such that $B(x_0,r) \subset \Omega$ and $\beta(x_0,r) \leq 1/2$. Then there exist a universal constant $C > 0$ and a radius $\rho \in (r/2, r)$ such that*

$$\omega_2(x_0,\rho) \leq C\big(\omega_p(x_0,r) + \beta(x_0,r)\big).$$

PROOF. By Proposition 5.1, we already know that there exist a universal constant $C > 0$ and a radius $\rho \in (r/2, r)$ such that

$$\int_{B(x_0,\rho)\setminus K} \mathbf{A}e(u) : e(u)\,\mathrm{d}x + \mathcal{H}^1\big(K \cap B(x_0,\rho)\big) \leq 2\rho + C\rho\big(\omega_p(x_0,r) + \beta(x_0,r)\big).$$

Now noticing that $\beta(x,\rho) \le 2\beta(x,r) \le \frac{1}{5}$, we can use Lemma 5.3 in $B(x_0,\rho)$ which yields

$$\mathcal{H}^1\big(K \cap B(x_0,\rho)\big) \ge 2\rho - 3\beta(x_0, r\rho),$$

hence,

$$\int_{B(x_0,\rho)\setminus K} \mathbf{A}e(u) : e(u)\,\mathrm{d}x \le C\rho\big(\omega_p(x_0,r) + \beta(x_0,r)\big).$$

Finally, by ellipticity of A we get

$$\int_{B(x_0,\rho)\setminus K} \mathbf{A}e(u) : e(u)\,\mathrm{d}x \ge r\omega_2(x_0,\rho),$$

which finishes the proof. ∎

6. Porosity of the bad set

Given $0<\alpha<1$, $x_0 \in K$, and $r>0$ such that $B(x,r)\subset\Omega$, we say that the crack-set K is $C^{1,\alpha}$-regular in the ball $B(x_0,r)$ if it is the graph of a $C^{1,\alpha}$ function f such that, in a convenient coordinate system, it holds that $f(0)=x_0$, $f'(0)=0$ and $r^\alpha\|f'\|_{C^\alpha}\le 1/4$.

We recall the following ε-regularity theorem coming from [4].

Theorem 6.1 ([4]). *Let $(u,K)\in\mathcal{A}(\Omega)$ be a minimizer of the Griffith functional with K connected and $\mathcal{H}^1(K)>0$. There exist constants $a,\alpha,\varepsilon_2\in(0,1)$ (depending on $\mathbf{A}$) such that the following property holds true. Let $x_0\in K$ and $r>0$ be such that $B(x_0,r)\subset\Omega$ and*

$$\omega_2(x_0,r) + \beta(x_0,r) \le \varepsilon_2,$$

and K separates in $B(x_0,r)$. Then K is $C^{1,\alpha}$-regular in $B(x_0,ar)$.

Proof. Unfortunately, the above statement is not explicitly stated in [4], but it directly follows from the proof of [4, Proposition 3.4]. Indeed, in the latter proof, some explicit thresholds $\delta_1>0$ and $\delta_2>0$ and an exponent $\alpha\in(0,1)$ are given so that, provided that

$$\omega_2(x_0,r)\le\delta_2, \quad \beta(x_0,r)\le\delta_1,$$

and K separates in $B(x_0,r)$, then

$$\beta(y,tr)\le Ct^\alpha$$

for all $y\in B(x_0,r/2)$ and $t\in(0,1/2)$. It implies that there exists $a\in(0,1)$ (which depends on C and α) such that $B(x_0,ar)$ is a $C^{1,\alpha}$ curve as well as a 10^{-2}-Lipschitz graph (thanks to [4, Lemma 6.4]). In addition, the graph is $C^{1,\alpha}$ with $\|f\|_\infty\le 10^{-2}r$ and [4, the estimate (6.8)] says moreover that $\|f'\|_{C^{0,\alpha}}\le C$, from which we easily get

$(ar)^\alpha \|f'\|_{C^{0,\alpha}} \le C \le 1/4$ up to take a smaller radius r. The fact that α and a depend only on $\mathbf{A}$ follows from a careful inspection of the proof in [4]. ■

We are now in a position to prove the following, which says that the singular set is porus in K.

Proposition 6.1. *Let $(u, K) \in \mathcal{A}(\Omega)$ be a minimizer of the Griffith functional with K connected and $\mathcal{H}^1(K) > 0$. There exist constants $a \in (0, 1/2)$ (depending on $\mathbf{A}$) such that the following property holds true. For all $x_0 \in K$ and $r > 0$ such that $B(x_0, r) \subset \Omega$, there exists $y \in K \cap B(x_0, r/2)$ such that $K \cap B(y, ar)$ is $C^{1,\alpha}$-regular (where α is the constant of Theorem 6.1).*

Proof. In view of Theorem 6.1, it is enough to prove the following fact: there exists $a \in (0, 1/2)$ such that for all $x \in K$ and $r > 0$ with $B(x, r) \subset \Omega$, there exist $y \in K \cap B(x, r/2)$ and $ar < s < r/2$ such that

$$\omega_2(y, s) + \beta(y, s) \le \varepsilon_2$$

and K separates in $B(y, s)$, where ε_2 is the constant of Theorem 6.1. We already know from Lemma 3.1 how to control β and the separation. We, therefore, need to add a control on ω_2, and this will be done by applying successively Proposition 4.1 and Corollary 5.1, but we need to fix carefully the constants so that it compiles well.

Let us pick any $p \in (1, 2)$ and let C_p be the constant of Proposition 4.1, and let C_0 be the constant of Corollary 5.1. Then we define

$$b := \frac{1}{4} e^{-\frac{8C_p C_0}{\varepsilon_2}} \quad \text{and} \quad \varepsilon_0 := \frac{b\varepsilon_2}{8C_0} \wedge \frac{1}{2}.$$

We fix $x \in K$ and $r > 0$ such that $B(x, r) \subset \Omega$. As noticed at the beginning of the proof of Lemma 3.1, we can assume without loss of generality that $B(x, 2r) \subset \Omega$ and $r \le \operatorname{diam}(K)/4$. We apply Lemma 3.1 with the previous definition of ε_0 and we get that for some $a \in (0, 1/2)$ (depending on $\mathbf{A}$), there exist $y \in B(x, r/2)$ and $t \in (ar, r/2)$ satisfying

$$\beta(y, t) \le \varepsilon_0 \text{ and } K \text{ separates in } B(y, t) \text{ as in Definition 2.1.}$$

Then we apply Proposition 4.1 in $B(y, t)$ which yields

$$\int_{z \in K \cap B(y,t)} \int_{0<s<t} \omega_p(z, s) \frac{\mathrm{d}s}{s} \, \mathrm{d}\mathcal{H}^1(z) \le C_p t. \tag{6.1}$$

From this estimate, we claim that we obtain the following fact:

$$\text{there exist } z \in B(y, t/2) \text{ and } s \in (bt, t/2) \text{ such that } \omega_p(z, s) \le \frac{\varepsilon_2}{4C_0}. \tag{6.2}$$

Indeed, remember that K is connected, $y \in K$, and $r \le \operatorname{diam}(K)/4$; thus there exist $z \in K \setminus B(y, t/2)$ and $\mathcal{H}^1(K \cap B(y, t/2)) \ge \mathcal{H}^1([y, z]) \ge t/2$. Therefore, if the claim in (6.2) is not true, then

$$\int_{z \in K \cap B(y,t/2)} \int_{bt<s<t/2} \omega_p(z, t) \frac{\mathrm{d}s}{s} \, \mathrm{d}\mathcal{H}^1(z)$$
$$\ge \frac{\varepsilon_2}{4C_0} \mathcal{H}^1\big(K \cap B(y, t/2)\big) \int_{bt}^{t/2} \frac{\mathrm{d}t}{t} \ge \frac{t\varepsilon_2}{8C_0} \ln\left(\frac{1}{2b}\right).$$

Returning back to (6.1), we get

$$\ln\left(\frac{1}{2b}\right) \le \frac{8C_0}{\varepsilon_2} C_p$$

which contradicts our definition of b. The claim is now proved.

Now let us check what we have got in the ball $B(z, s)$. We already know that $s \ge cr$ for some constant $c \in (0, 1/2)$ (which depends on $\mathbf{A}$) and $\omega_p(z, s) \le \frac{\varepsilon_2}{4C_0}$. Concerning β, we have

$$\beta(z, s) \le \frac{2}{b} \beta(y, t) \le \frac{2}{b} \varepsilon_0 \le \frac{\varepsilon_2}{4C_0}.$$

Next, we apply Corollary 5.1 which says that there exists $s' \in (s/2, s)$ such that

$$\omega_2(z, s') \le C_0\big(\omega_p(z, s) + \beta(z, s)\big) \le \frac{\varepsilon_2}{2}.$$

The ball $B(z, s')$ satisfies all the required properties because $\beta(z, s') \le 2\beta(z, s) \le \frac{\varepsilon_2}{2}$ so that

$$\omega_2(z, s') + \beta(z, s') \le \varepsilon_2,$$

as required.

It remains to see that K still separates in the ball $B(z, s')$. But once $\beta(z, s')$ is controlled and knowing that K already separates in $B(y, t)$, it follows from Lemma 2.1. ■

We are now ready to state one of our main results about the Hausdorff dimension of the singular set.

COROLLARY 6.1. *Let $(u, K) \in \mathcal{A}(\Omega)$ be a minimizer of the Griffith functional with K connected. Then there exists a closed set $\Sigma \subset K$ such that $\dim_{\mathcal{H}}(\Sigma) < 1$ and $K \setminus \Sigma$ is locally a $C^{1,\alpha}$ curve.*

PROOF. The proof is standard now that Proposition 6.1 is established. Indeed, we can argue exactly as Rigot in [19, Remark 3.29] which we refer to for more details. ■

7. Higher integrability of $e(u)$

Theorem 7.1. *Let $(u, K) \in \mathcal{A}(\Omega)$ be a minimizer of the Griffith functional with K connected and $\mathcal{H}^1(K) > 0$. There exist $C \geq 1$ and $p > 1$ (depending on* **A***) such that the following property holds true. For all $x \in \Omega$ and all $r > 0$ such that $B(x,r) \subset \Omega$,*

$$\int_{B(x,r/2)} \left|e(u)\right|^{2p} \mathrm{d}x \leq C r^{2-p}.$$

We rely on a higher integrability lemma [18, Lemma 4.2] which is inspired by the technique of [15]. We recall that given $0 < \alpha < 1$, a closed set K, $x_0 \in K$, and $r > 0$, we say that K is $C^{1,\alpha}$-regular in the ball $B(x_0, r)$ if it is the graph of a $C^{1,\alpha}$ function f such that, in a convenient coordinate system, it holds that $f(0) = x_0$, $\nabla f(0) = 0$, and $r^\alpha \|\nabla f\|_{C^\alpha} \leq 1/4$. We take the convention that the $C^{1,\alpha}$ norm is small enough because we do not want it to interfere with the boundary gradient estimates for the Lamé equations. It is also required by the covering lemma [18, Lemma 4.3] on which [18, Lemma 4.2] is based.

Lemma 7.1. *We fix a radius $R > 0$ and an open ball B_R of radius R in $\mathbb{R}^n$. Let K be a closed subset of B_R and $v\colon B_R \to \mathbb{R}^+$ be a non-negative Borel function. We assume that there exist $C_0 \geq 1$ and $0 < \alpha \leq 1$ such that the following holds true.*

(i) *For all ball $B(x,r) \subset B_R$ centered in K,*

$$C_0 r^{n-1} \leq \mathcal{H}^{n-1}\big(K \cap B(x,r)\big) \leq C_0 r^{n-1}.$$

(ii) *For all ball $B(x,r) \subset B_R$ centered in K, there exists a smaller ball $B(y, C_0^{-1} r) \subset B(x,r)$ in which K is $C^{1,\alpha}$-regular.*

(iii) *For all ball $B(x,r) \subset B_R$ such that K is disjoint from $B(x,r)$ or K is $C^{1,\alpha}$-regular in $B(x,r)$, we have*

$$\sup_{B(x,r/2)} v(x) \leq C_0 \left(\frac{R}{r}\right).$$

Then there exist $p > 1$ and $C \geq 1$ (depending on n, C_0) such that

$$\fint_{\frac{1}{2}B_R} v^p \leq C.$$

Proof of Theorem 7.1. We apply Lemma 7.1. More precisely, for all $x \in \Omega$ and all $R > 0$ such that $B(x,R) \subset \Omega$, one can apply [18, Lemma 4.2] in the ball $B(x,R)$ to the function $v := R|e(u)|^2$. Assumption (i) follows from the local Ahlfors-regularity of K. Assumption (ii) follows from the porosity (Proposition 6.1). Assumption (iii) follows from interior/boundary gradient estimates for the Lamé equations and from the local Ahlfors-regularity. In particular, the boundary estimate is detailed in Lemma A.1. ■

Lamé's equations

We work in the Euclidean space $\mathbb{R}^n$ ($n > 1$). For $r > 0$, B_r denotes the ball of radius r and centered at 0. We fix a radius $0 < R \leq 1$, an exponent $0 < \alpha \leq 1$, a constant $A > 0$, and a $C^{1,\alpha}$ function $f\colon \mathbb{R}^{n-1} \cap B_R \to \mathbb{R}$ such that $f(0) = 0$, $\nabla f(0) = 0$, and $R^\alpha [\nabla f]_\alpha \leq A$. We introduce

$$V_R := \{x \in B_R \mid x_n > f(x')\},$$
$$\Gamma_R := \{x \in B_R \mid x_n = f(x')\}.$$

We denote by ν the normal vector field to Γ_R going upward. For $0 < t \leq 1$, we write tV_R for $V_R \cap B_t$ and $t\Gamma_R$ for $\Gamma_R \cap B_t$. For $u \in W^{1,2}(V_R; \mathbb{R}^n)$, we denote by u^* the trace of u in $L^2(\partial V_R; \mathbb{R}^n)$. For a function $\xi\colon V_R \to \mathbb{R}^{n \times n}$, we define (formally) $\mathrm{div}(\xi)$ as the vector field whose ith coordinate is given by $\mathrm{div}(\xi)_i = \sum_j \partial_j \xi_{ij}$. We also recall the notation for the linear strain tensor

$$e(u) = \frac{Du + Du^T}{2}$$

and the stress tensor

$$\mathbf{A}e(u) = \lambda \,\mathrm{div}(u) I_n + 2\mu e(u),$$

where λ and μ are the Lamé coefficients satisfying $\mu > 0$ and $\lambda + \mu > 0$. We denote by $W_0^{1,2}(V_R \cup \Gamma_R; \mathbb{R}^n)$ the space of functions $v \in W^{1,2}(V_R; \mathbb{R}^n)$ such that $v^* = 0$ on $\partial V_R \setminus \Gamma_R$.

Our object of study are the functions $u \in W^{1,2}(V_R) \cap L^\infty(V_R)$ which are weak solutions of

$$\begin{cases} \mathrm{div}\left(\mathbf{A}e(u)\right) = 0 & \text{in } V_R, \\ \mathbf{A}e(u) \cdot \nu = 0 & \text{on } \Gamma_R, \end{cases} \tag{A.1}$$

that is for all $v \in W_0^{1,2}(V_R \cup \Gamma_R; \mathbb{R}^n)$,

$$\int_{V_R} \mathbf{A}e(u) : Dv \,\mathrm{d}x = 0.$$

Remark A.1. *As*

$$\mathbf{A}e(u) = (\lambda + \mu) Du^T + \mu Du + \lambda\big(\mathrm{div}(u) I_n - Du^T\big)$$

and the part $\mathrm{div}(u) I_n - Du^T$ *is divergence free, we can also write formally*

$$\mathrm{div}\left(\mathbf{A}e(u)\right) = (\lambda + \mu) \nabla \,\mathrm{div}(Du) + \mu \Delta u.$$

We are going to justify the following estimate.

Lemma A.1. *Let us assume that $n = 2$. There exists $C \geq 1$ (depending on α, A, λ, μ) such that*

$$\sup_{\frac{1}{2}V_R} \left|e(u)\right| \leq C \left(\fint_{V_R} \left|e(u)\right|^2 \,\mathrm{d}x \right)^{\frac{1}{2}}.$$

Proof. It suffices to prove that for all solutions of (A.1), we have

$$\sup_{\frac{1}{2}V_R} |Du| \leq C \left(\fint_{V_R} |Du|^2 \,\mathrm{d}x \right)^{\frac{1}{2}}. \tag{A.2}$$

Indeed, we observe first that $|e(u)| \leq |Du|$, so (A.2) implies that

$$\sup_{\frac{1}{2}V_R} \left|e(u)\right| \leq C \left(\fint_{V_R} |Du|^2 \,\mathrm{d}x \right)^{\frac{1}{2}}. \tag{A.3}$$

By Korn inequality, there exists a skew-symmetric matrix R such that

$$\int_{V_R} |Du - R|^2 \,\mathrm{d}x \leq \int_{V_R} \left|e(u)\right|^2 \mathrm{d}x,$$

so it is left to apply (A.3) to $x \mapsto u(x) - Rx$, which also solves Lamé's equations.

From now on, we deal with (A.2). The letter C plays the role of a constant ≥ 1 that depends on λ, μ and α, A. We refer to the proof of [16, Theorem 3.18] which itself refers to the proof of [2, Theorem 7.53]. We straighten the boundary Γ_R via the $C^{1,\alpha}$ diffeomorphism $\phi\colon x \mapsto x' + (x_n - f(x'))e_n$. We observe that $\phi(\overline{V_R})$ contains a half-ball $B^+ = \overline{B}(0, C_0^{-1}R)^+$, where $C_0 \geq 1$ is a constant that depends on λ, μ, α. The Neumann problem satisfied by u in V_R is transformed into a Neumann problem satisfied by a function v in $\overline{B}(0, C_0^{-1}R)^+$. Then we symmetrize the elliptic system to the whole ball $B = B(0, C_0^{-1}R)$ as in [16, Theorem 3.18]. Following the proof of [2, Theorem 7.53] (in the special case where the right-hand side h is zero), we arrive to the fact that there exists $q > n = 2$ (depending on λ, μ, α) such that for all $x_0 \in \frac{1}{2}B$ and $0 < \rho \leq r \leq C^{-1}R$

$$\begin{aligned}\int_{B_\rho(x_0)} &\left|\nabla v - (\nabla v)_{x_0,\rho}\right|^2 \mathrm{d}x \\ &\leq C\left(\frac{\rho}{r}\right)^q \int_{B_r(x_0)} \left|\nabla v - (\nabla v)_{x_0,r}\right|^2 \mathrm{d}x + C\rho^q \int_B |\nabla v|^2 \,\mathrm{d}x.\end{aligned}$$

In particular, by Poincaré–Sobolev inequality,

$$\int_{B_\rho(x_0)} \left|\nabla v - (\nabla v)_{x_0,\rho}\right|^2 \mathrm{d}x \leq C\left(\frac{\rho}{r}\right)^q \int_B |\nabla v|^2 \,\mathrm{d}x.$$

According to the Campanato characterization of Hölder spaces,

$$[\nabla v]_{C^{0,\sigma}(\frac{1}{2}B)} \leq C\left(R^{-(2+2\sigma)}\int_B |\nabla v|^2\,\mathrm{d}x\right)^{\frac{1}{2}},$$

where $\sigma = \frac{q-2}{2}$, and this implies that

$$\sup_{\frac{1}{2}B}|\nabla v| \leq C\left(\fint_B |\nabla v|^2\,\mathrm{d}x\right)^2.$$

This property is inherited by u via the diffeomorphism ϕ,

$$\sup_{C^{-1}V_R}|\nabla u| \leq C\left(\fint_{V_R} |\nabla u|^2\,\mathrm{d}x\right)^2.$$

We can finally bound the supremum of $|\nabla u|$ on $\frac{1}{2}V_R$ by a covering argument. ∎

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References

[1] L. Ambrosio – N. Fusco – J. E. Hutchinson, Higher integrability of the gradient and dimension of the singular set for minimisers of the Mumford–Shah functional. *Calc. Var. Partial Differential Equations* **16** (2003), no. 2, 187–215. Zbl 1047.49015 MR 1956854

[2] L. Ambrosio – N. Fusco – D. Pallara, *Functions of bounded variation and free discontinuity problems*. Oxford Math. Monogr., Oxford University Press, New York, 2000. Zbl 0957.49001 MR 1857292

[3] J.-F. Babadjian – A. Chambolle – A. Lemenant, Energy release rate for non-smooth cracks in planar elasticity. *J. Éc. polytech. Math.* **2** (2015), 117–152. Zbl 1325.74126 MR 3366673

[4] J.-F. Babadjian – F. Iurlano – A. Lemenant, Partial regularity for the crack set minimizing the two-dimensional Griffith energy. *J. Eur. Math. Soc. (JEMS)* **24** (2022), no. 7, 2443–2492. Zbl 1501.49024 MR 4413770

[5] A. Chambolle – S. Conti – G. Francfort, Korn-Poincaré inequalities for functions with a small jump set. *Indiana Univ. Math. J.* **65** (2016), no. 4, 1373–1399. Zbl 1357.49151 MR 3549205

[6] A. Chambolle – S. Conti – F. Iurlano, Approximation of functions with small jump sets and existence of strong minimizers of Griffith's energy. *J. Math. Pures Appl. (9)* **128** (2019), 119–139. Zbl 1419.49054 MR 3980846

[7] A. Chambolle – V. Crismale, Existence of strong solutions to the Dirichlet problem for the Griffith energy. *Calc. Var. Partial Differential Equations* **58** (2019), no. 4, article no. 136. Zbl 1419.49055 MR 3984096

[8] A. Chambolle – V. Crismale, Compactness and lower semicontinuity in $GSBD$. *J. Eur. Math. Soc. (JEMS)* **23** (2021), no. 3, 701–719. Zbl 1475.49018 MR 4210722

[9] S. Conti – M. Focardi – F. Iurlano, Existence of strong minimizers for the Griffith static fracture model in dimension two. *Ann. Inst. H. Poincaré C Anal. Non Linéaire* **36** (2019), no. 2, 455–474. Zbl 1458.74126 MR 3913194

[10] S. Conti – M. Focardi – F. Iurlano, A note on the Hausdorff dimension of the singular set of solutions to elasticity type systems. *Commun. Contemp. Math.* **21** (2019), no. 6, article no. 1950026. Zbl 1432.35079 MR 3996978

[11] G. David, C^1-arcs for minimizers of the Mumford-Shah functional. *SIAM J. Appl. Math.* **56** (1996), no. 3, 783–888. Zbl 0870.49020 MR 1389754

[12] G. David, *Singular sets of minimizers for the Mumford-Shah functional*. Progr. Math. 233, Birkhäuser, Basel, 2005. Zbl 1086.49030 MR 2129693

[13] G. David – S. Semmes, *Analysis of and on uniformly rectifiable sets*. Math. Surveys Monogr. 38, American Mathematical Society, Providence, RI, 1993. Zbl 0832.42008 MR 1251061

[14] C. De Lellis – M. Focardi – B. Ruffini, A note on the Hausdorff dimension of the singular set for minimizers of the Mumford–Shah energy. *Adv. Calc. Var.* **7** (2014), no. 4, 539–545. Zbl 1304.49091 MR 3276120

[15] G. De Philippis – A. Figalli, Higher integrability for minimizers of the Mumford–Shah functional. *Arch. Ration. Mech. Anal.* **213** (2014), no. 2, 491–502. Zbl 1316.49044 MR 3211857

[16] I. Fonseca – N. Fusco – G. Leoni – M. Morini, Equilibrium configurations of epitaxially strained crystalline films: existence and regularity results. *Arch. Ration. Mech. Anal.* **186** (2007), no. 3, 477–537. Zbl 1126.74029 MR 2350364

[17] M. Friedrich – F. Solombrino, Quasistatic crack growth in 2d-linearized elasticity. *Ann. Inst. H. Poincaré C Anal. Non Linéaire* **35** (2018), no. 1, 27–64. Zbl 1386.74124 MR 3739927

[18] C. Labourie – E. Milakis, Higher integrability of the gradient for the thermal insulation problem. *Interfaces Free Bound.* **25** (2023), no. 2, 193–216. Zbl 07721321 MR 4595295

[19] S. Rigot, Big pieces of $C^{1,\alpha}$-graphs for minimizers of the Mumford-Shah functional. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **29** (2000), no. 2, 329–349. Zbl 0960.49024 MR 1784178

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