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**Algebraic Geometry.** – *Fano's last Fano*, by Marco Andreatta and Roberto Pignatelli, communicated on 16 December 2022.

Abstract. – In 1949, Fano published his last paper on 3-folds with canonical sectional curves. There, he constructed and described a 3-fold of degree 22 in a projective space of dimension 13 with canonical curve section, which we like to call *Fano's last Fano*. We report on Fano's construction and we provide various (in our opinion missing) proofs in modern language. Moreover, we try to use results and techniques available at that time. After that we construct Fano's last Fano with modern tools, in particular, via the Hilbert scheme of zero cycles on a rational surface; as a consequence we easily point out the corresponding example in the Mori–Mukai classification.

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## 1. Introduction

In the early 1900s, Gino Fano started a systematic study of projective varieties of dimension 3. His pioneering work was remarkably original and deep, although at that time the necessary mathematical tools, especially in the field of Algebra, were not well developed. It is generally accepted that his proofs are not enough rigorous for the modern standard; on the other hand, they contain many intuitions on the geometry of projective varieties, which turned out to be correct and fundamental.

We consider smooth projective varieties $X$ defined over $\mathbb{C}$; if $n$ is the dimension of $X$, we sometimes call $X$ an *n-fold*. We denote by $K_X$ the *canonical sheaf* of $X$.

Fano studied projective 3-folds $X \subset \mathbb{P}^N$ such that for general hyperplanes $H_1, H_2$ the curve $\Gamma := X \cap H_1 \cap H_2$ is canonically embedded into $H_1 \cap H_2$ (i.e., $K_\Gamma$ embeds $\Gamma$). Fano called them *Varietà algebriche a tre dimensioni a curve sezioni canoniche* (Algebraic varieties of dimension 3 with canonical curve section) [4–8].

He considered this class of varieties to provide a counterexample to a Castelnuovo-type rationality criteria for 3-folds and to the Lüroth problem. Although some of these varieties have all plurigenera and irregularity equal to zero, he understood that some of them should not be rational. It is generically accepted that none of Fano's attempts to prove nonrationality should be considered rigorous. The first modern and accepted proof of the nonrationality of quartic 3-folds in $\mathbb{P}^4$ is the celebrated Iskovskikh and

Manin's [14], whereas the nonrationality of the cubic 3-fold in $\mathbb{P}^4$ was proved by Clemens and Griffiths [3]. B. Segre constructed some unirational quartic 3-fold in $\mathbb{P}^4$ [22]; therefore these unirational but nonrational 3-fold represent counterexamples to the Lüroth problem in dimension 3.

Fano started also a biregular classification of his varieties in the case of dimension 3. Starting from Fano's results, a large number of mathematicians have constructed clever theories in the last 50 years, which are among the most spectacular achievements of contemporary mathematics. Initially, Fano's legacy has been taken into account in V. Iskovskikh's work, as well as in V. Shokurov's work, and soon after in that of S. Mori and S. Mukai. The theory of minimal models developed by S. Mori gave an enormous impulse; on the one hand, the minimal model program changed the approach to the classification of projective varieties and, on the other hand, the varieties studied by Fano had a central role in the classification.

The following proposition is a well-known result; we provide a proof for the reader's convenience.

PROPOSITION 1.1. *Let $X \subset \mathbb{P}^N$ be a projective $n$-fold and let $H := \mathcal{O}_{\mathbb{P}^N}(1)_{|X}$ be the hyperplane bundle on $X$. Assume for general hyperplanes $H_1, H_2, \dots, H_{n-1} \in |H|$ the curve $\Gamma := H_1 \cap H_2 \cap \dots \cap H_{n-1}$ is a canonically embedded curve of genus $g$. Then $-K_X = (n-2)H$.*

*In particular, if $n = 3$, the linear system $|-K_X]$ embeds $X$ as a 3-fold of degree $2g-2$ into projective space of dimension $g+1$; i.e. $X := X_3^{2g-2} \subset \mathbb{P}^{g+1}$.*

PROOF. Let $S := H_1 \cap H_2 \cap \dots \cap H_{n-2}$ be a general surface section. Denote by $\Gamma = S \cap H_{n-1}$ a curve section. For $m \geq 0$, consider the exact sequence

$$0 \to \mathcal{O}_S(m-1) \to \mathcal{O}_S(m) \to \mathcal{O}_\Gamma(m) \to 0$$

and the corresponding long exact cohomology sequence

$$\begin{aligned} 0 \to H^0\big(\mathcal{O}_S(m-1)\big) \to H^0\big(\mathcal{O}_S(m)\big) \xrightarrow{\alpha} H^0\big(\mathcal{O}_\Gamma(m)\big) \to H^1\big(\mathcal{O}_S(m-1)\big) \\ \xrightarrow{\beta} H^1\big(\mathcal{O}_S(m)\big) \to H^1\big(\mathcal{O}_\Gamma(m)\big) \to H^2\big(\mathcal{O}_S(m-1)\big) \to H^2\big(\mathcal{O}_S(m)\big) \to 0. \end{aligned}$$

For $m \geq 1$, the map $\alpha$ is onto since a canonically embedded curve, $\Gamma$, is projectively normal (this is a classical result attributed to Noether and Enriques–Petri). Therefore, $\beta$ is injective and by decreasing induction on $m$ we have

$$H^1\big(S, \mathcal{O}_S(m)\big) = 0, \quad m \geq 0.$$

For $m > 1$, we have, by Serre duality, $H^1(\Gamma, \mathcal{O}_\Gamma(m)) = H^0(\Gamma, \mathcal{O}_\Gamma(1-m))^\vee = 0$; again by decreasing induction, we get

$$H^2\big(S, \mathcal{O}_S(m)\big) = 0, \quad m > 0.$$

For $m = 1$, we have $\dim H^1(\Gamma, \mathcal{O}_\Gamma(1)) = 1$, which gives

$$\dim H^2(S, \mathcal{O}_S) = 1.$$

Since $\dim H^2(S, \mathcal{O}_S) = 1$, by Serre duality $K_S$ is effective or trivial. By adjunction formula, we have

$$(K_S + \Gamma)\cdot\Gamma = K_\Gamma = \Gamma\cdot\Gamma;$$

hence $K_S\cdot\Gamma = 0$. By Kleiman's criterium for the ampleness (of $\Gamma$), this implies that $K_S = 0$. Since $H^1(S, \mathcal{O}_S) = 0$, $S$ is a K3 surface.

Adjunction formula on $X$ implies the following:

$$\big(K_X + (n-2)H\big)\cdot S = K_S = 0.$$

By Weil's equivalence criterium [26, Théorème 2, p. 111], this implies $-K_X = (n-2)H$.

The computations for the case of 3-folds are quite straightforward. ■

Nowadays, we define a Fano manifold as follows.

Definition 1.2. A smooth projective variety $X$ is called a *Fano manifold* if $-K_X$ is ample.

The *index of* $X$ is defined as the greatest integer $r$ such that $-K_X = -rL$ for a line bundle $L$. The ample line bundle $L$ which achieves the maximum is called the *fundamental line bundle (or divisor)*.

If $\mathrm{Pic}(X) = \mathbb{Z}$, then $X$ is called a *Fano manifold of the first species* or a *prime Fano manifold*.

Remark 1.3. By Proposition 1.1, the varieties considered by Fano are Fano manifolds with $-K_X$ very ample and index $n-2$. It is straightforward to check that Fano manifolds with $-K_X$ very ample and index $n-2$ have canonical curve section.

The slightly more general definition with $-K_X$ simply ample was not on hand at that time and it is absolutely appropriate for modern taste and techniques.

V. A. Iskovskikh [12, 13] has taken up the classification. By modern tools, he has been able to justify and generalize Fano's work, thus obtaining a complete classification of *prime Fano* 3*-folds*. If $g$ is the genus of the curve section, he proved that $3 \leq g \leq 12$ and $g \neq 11$. For every such $g$, he gave a satisfactory description of the associated Fano variety. He used Fano's method of double projection from a line. In particular, he needed the existence of a line and the existence of smooth divisor in the linear system $|-K_X|$. These are delicate results proved later by Shokurov in [24, 25].

Among his results, a nice one is the construction of a prime Fano manifold $X_3^{22} \subset \mathbb{P}^{13}$, which apparently was omitted by Fano. Some years later, S. Mukai gave a new method to classify prime Fano–Iskovskikh 3-folds based on vector bundle constructions [18], providing a new description of $X_3^{22} \subset \mathbb{P}^{13}$ (see also [19]).

In the same period, S. Mori and S. Mukai [17] gave a classification of all Fano 3-folds with Picard number greater than or equal to 2 (i.e., not prime), and they finished the classification of Fano 3-fold. Their classification is based on the Iskovskikh's and on the Mori theory of extremal rays, via the so-called "two rays game".

Fano manifolds of any dimension $n$ and index $r \geq (n-2)$ were classified by Kobayashi and Ochiai [15, index $n$], T. Fujita [11, index $(n-1)$], and S. Mukai; the latter classified all Fano manifolds of index $(n-2)$ under the assumption that the fundamental divisor has an effective smooth member [18]. Later on, M. Mella proved that this assumption is always satisfied [16].

Let us now briefly describe the purpose and the content of our paper. In 1949, Fano published in *Rendiconti dell'Accademia dei Lincei* his last paper on 3-folds with canonical sectional curves, under the title *Su una particolare varietà a tre dimensioni a curve-sezioni canoniche*[1] [8]. At that time, he was 78 years old and he died three years afterwards. In the paper, he constructed and described a 3-fold of the type $X_3^{22} \subset \mathbb{P}^{13}$ with canonical curve section, which we like to call Fano's last Fano. At first, we even thought that this was the variety which was missing in his classification, as claimed by Iskovskikh; very soon we realized that this variety is not *prime*; i.e. it has Picard rank 2. Therefore, it is not isomorphic to either the Iskovskikh nor to the Mukai example and it should be searched in the Mori–Mukai classification.

Fano's paper was almost never quoted after its publication and it has been long ignored by most modern mathematicians. Very likely, this is due to the fact that L. Roth cited the paper on page 93 of his book *Algebraic Threefolds* [21]. He wrote that Fano examined *a particular fourfold of the third species* . . . ; probably Roth read the paper too quickly and did not realize that Fano was actually searching for a 3-fold and not (only) for a 4-fold.

In Section 2, we will report on Fano's construction, using his own words in Italian, with our translation in English. His arguments are correct but very often without a complete proof. We provide diverse proofs in modern language and try to use results and techniques available at that time in order to support Fano's correctness. We hope that the reader might enjoy, as we did, the beauty as well as the elegance and simplicity of Fano's example.

In Section 3, we construct Fano's with modern tools, in particular, via the Hilbert scheme of zero cycles on a rational surface. As a consequence, we can easily point out the corresponding example in the Mori–Mukai list. Within the modern description,

(1) On a special 3-fold with canonical curve section.

the reader can easily derive the properties of the example studied in the second part of Fano's paper, for instance its rationality and the description of ruled sub-surfaces.

## 2. Fano's construction of a $X_3^{22} \subset \mathbb{P}^{13}$

### 2.1. *Construction and smoothness*

We report and comment on the first section of Fano's paper of 1949 [8] using *verbatim* Fano's words in Italian, providing an English translation in the footnotes.

*Ho incontrato recentemente una varietà a tre dimensioni a curve-sezioni canoniche, che naturalmente appartiene alla serie delle $M_3^{2p-2}$ di $S_{p+1}$ (qui $p = 12$), oggetto di mie ricerche in quest'ultimo periodo, ma non ha finora richiamata particolare attenzione. Ne darò qui un breve cenno.*[2]

Let us explain Fano's notation. $S_n$ is what we denote now by $P_{\mathbb{C}}^n$, whereas $M_m^d$ is a subvariety of $P_{\mathbb{C}}^n$ of dimension $m$ and degree $d$ (note that in what follows he uses the world *ordine* for degree). Therefore, in the notation of the previous section, he discovered an $X_3^{22} \subset \mathbb{P}^{13}$ of dimension 3 with canonical sectional curves. We proceed with his notation.

*Consideriamo nello spazio $S_5$ una rigata razionale normale $R^4$ (non cono), che per semplicità supponiamo del tipo più generale, cioè con $\infty^1$ coniche direttrici irriducibili; e con essa la varietà $\infty^4$ delle sue corde. Quale ne è l'immagine $M_4$ nella Grassmanniana $M_8^{14}$ di $S_{14}$ delle rette di $S_5$?*[3]

The ruled surface $R^4$ is viewed as the image of $\mathbb{P}^1 \times \mathbb{P}^1$ embedded in $\mathbb{P}^5$ by the complete linear system $|(1,2)|$. It is rational and it has degree 4. It is normal because the general hyperplane section is a normal rational curve of degree 4 in $\mathbb{P}^4$, the image of general element of $|(1,2)|$. Thus, the surface $R^4$ has two rulings: one is given by the *lines* contained in $R^4$, the images of the divisors in the complete linear system $|(0,1)|$, and the other is given by the *conics* mentioned by Fano, which are all irreducible conics in $R^4$, the images of the divisors in the complete linear system $|(1,0)|$.

Consider the Grassmannian of the lines in $\mathbb{P}^5$ embedded via the Plücker embedding and following Fano; denote it by $M_8^{14} \subset S_{14}$. To explain Fano's notation, note that it is

(2) Recently, I discovered a 3-dimensional variety with canonical sectional curves, which naturally belongs to the collection of $M_3^{2p-2}$ of $S_{p+1}$ (here $p = 12$), which was the topic of my research in this last period, but which up to now has not drawn special attention. I will give here a brief mention.

(3) Let us consider in the space $S_5$ a normal rational ruled surface $R^4$ (not a cone), which for simplicity we suppose of general type, that is, with $\infty^1$ irreducible conics as ruling; with it we consider the variety $\infty^4$ of its chords. What is the image of $M_4$ in the Grassmannian $M_8^{14}$ in $S_{14}$ of lines in $S_5$?

a compact complex manifold of dimension 8 which is embedded in a projective space of dimension $\frac{5\cdot 6}{2} - 1 = 14$. Its degree as a subvariety can be computed by standard Schubert calculus; namely it is equal to $\sigma_1^8 = 1 + 3^2 + 2^2 = 14$.

The variety $M_4$ is defined by Fano as the subset of the Grassmannian of lines in $\mathbb{P}^5$ given by the *chords* of $R^4$. Since we are looking for a complete variety, we need to interpret the word "corde" in a broad sense, that is secant and tangent lines.

At this point, it seems to us that Fano gives for granted that $M_4$ is a smooth irreducible variety of dimension 4. This is in our opinion not obvious, in particular its smoothness. For these purposes, we formulate the proposition below and dedicate some pages at its proof. In the next section, we provide a second, more geometric, proof.

Proposition 2.1. *$M_4 \subset M_8^{14}$ is an irreducible smooth variety of dimension* 4.

For the proof we need to set up some notation and preliminaries. We take homogeneous coordinates $([x_0, x_1], [y_0, y_1])$ on $\mathbb{P}^1 \times \mathbb{P}^1$ and we denote by $\pi_x, \pi_y : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ the natural projections; namely,

$$\pi_x\big([x_0, x_1], [y_0, y_1]\big) = [x_0, x_1],$$
$$\pi_y\big([x_0, x_1], [y_0, y_1]\big) = [y_0, y_1].$$

We take coordinates $[z_{ij}]$ in $\mathbb{P}^5$ so that the embedding $\mathbb{P}^1 \times \mathbb{P}^1 \to R^4 \subset \mathbb{P}^5$ is given by identifying $z_{ij} = x_i y_0^{2-j} y_1^j$; that is,

$$\begin{aligned} z_{00} &= x_0 y_0^2, & z_{01} &= x_0 y_0 y_1, & z_{02} &= x_0 y_1^2, \\ z_{10} &= x_1 y_0^2, & z_{11} &= x_1 y_0 y_1, & z_{12} &= x_1 y_1^2. \end{aligned} \tag{2.1.1}$$

$R^4$ is not the Veronese surface, therefore secant and tangent lines cover all $\mathbb{P}^5$ (Severi [23] proved that the Veronese surface, that is $\mathbb{P}^2$ embedded by $\mathcal{O}(2)$, is the only surface in $\mathbb{P}^5$ whose secant and tangent lines do not cover $\mathbb{P}^5$).

Consider the 3-fold $G_3^3 \subset \mathbb{P}^5$ defined by the condition

$$rk \begin{pmatrix} z_{00} & z_{01} & z_{02} \\ z_{10} & z_{11} & z_{12} \end{pmatrix} = 1.$$

We claim that $G_3^3$ contains $R^4$. More precisely, if we consider the ruling by conics of $R^4$, $G_3^3$ is the union of the planes containing these conics. It is obvious that for each point in $G_3^3$ there are at least two (in fact infinitely many) secants to $R^4$: the point, say $p$, is contained in a plane, say $\pi$, in $G_3^3$ containing a conic of the ruling by conics of $R^4$ and thus all the lines through $p$ contained in $\pi$ are secants to $R^4$. We show that the converse is also true.

LEMMA 2.2. *For each point $p \notin G_3^3$, there is a unique secant of $R^4$ through it.*

*In particular, for each point on a general line $l \subset \mathbb{P}^5$ (i.e., disjoint from $G_3^3$), there is a unique secant to $R^4$ through it.*

PROOF. The existence follows from the above quoted result of Severi [23] since $R^4$ is not the Veronese surface.

To prove uniqueness, we argue by contradiction. Assume that there are two distinct secants lines through $p$, say $s_1 \neq s_2$. Let $\pi$ be the plane spanned by $s_1$ and $s_2$. The hyperplanes containing $\pi$ define a codimension 3 subsystem $\Gamma$ of $|\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,2)|$.

$\Gamma$ does not have fixed components: assume by contradiction that $\Gamma$ has a fixed component (with reduced part) $F$. By dimension reasons, $F$ has to be contained in $|\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,0)|$ or in $|\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(0,1)|$, since the movable part has dimension 2. $F$ cannot be in $|\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,0)|$, since the plane $\pi$ coincides with the intersection of all hyperplanes through an irreducible conic in $R^4$, so that $p$ is contained in $G_3^3$, i.e. a contradiction. $F$ cannot be in $|\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(0,1)|$, because the plane $\pi$ contains a line of $R^4$, say $r$. By assumption, $r$ is neither $s_1$ nor $s_2$, so each $s_j$ intersects $R^4$ in at least one point $p_i$ out of $r$. On the other hand, there is no plane in $\mathbb{P}^5$ containing a line and two more distinct points all contained in $R^4$, so $p_1 = p_2$. Then both $p_i$s coincide with $p$ contradicting the assumption $p \notin R^4$.

Since $\Gamma$ has no fixed components, there is a hyperplane containing both $s_1$ and $s_2$ that cuts an irreducible curve of $R^4$; however any irreducible element of $|\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,2)|$ is smooth, so this is then a smooth rational normal quartic $C^4$. Both $s_1$ and $s_2$ are secants to it, cutting, respectively, subschemes $\delta_1$ and $\delta_2$ of length at least two on $C^4$. Since $p \notin R^4$, then $\delta_1 \cap \delta_2 = \emptyset$ and therefore the plane $\pi$ gives a pencil in

$$|\mathcal{O}_{C^4}(1,2) \cong \mathcal{O}_{\mathbb{P}^1}(4)|$$

with fixed locus of degree at least 4, a contradiction. ∎

PROOF OF PROPOSITION 2.1. Consider the rational map

$$(\mathbb{P}^1 \times \mathbb{P}^1)^2 \dashrightarrow M_4 \subset M_8^{14}$$

associating, to each pair $(p, q)$ of distinct points $p \neq q \subset \mathbb{P}^1 \times \mathbb{P}^1$, the unique line $\overline{pq}$ through them.

This is a dominant map, hence $M_4$ is irreducible.

The claim about the dimension $\dim M_4 = \dim(\mathbb{P}^1 \times \mathbb{P}^1)^2 = 4$ follows since this map is generically finite. To show this take a secant line $r$ of $R^4$ intersecting $R^4$ transversally; for example the line $z_{01} = z_{02} = z_{10} = z_{11}$ which intersects $R^4$ transversally in $[1,0,0,0,0,0]$ and $[0,0,0,0,0,1]$, the images of the points $([1,0],[1,0])$ and $([0,1],[0,1])$ of $\mathbb{P}^1 \times \mathbb{P}^1$. Then $r \cap R^4$ is a finite set $p_1, \dots, p_l$ of $l \geq 2$ points and the pre-image of $r$ in $(\mathbb{P}^1 \times \mathbb{P}^1)^2$ is the set of pairs $(p_i, p_j)$ with $i \neq j$, finite as well.

From now on we prove smoothness. We first show that the action (of possibly a subgroup) of the automorphism group of the variety $M_4$ splits it in finitely many orbits; this reduces our claim to finitely many local computations.

Consider the standard action of $\mathbb{P}\,\mathrm{GL}_2(\mathbb{C})$ on $\mathbb{P}^1$ and its standard *linearization*, the action of $\mathrm{GL}_2(\mathbb{C})$ on the coordinate ring of $\mathbb{P}^1$, $\mathbb{C}[x_0, x_1] = \bigoplus_d H^0(\mathcal{O}_{\mathbb{P}^1}(d))$. This associates to each matrix a ring homomorphism as follows:

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} : \alpha_0 x_0 + \alpha_1 x_1 \mapsto (a_{11}\alpha_0 + a_{12}\alpha_1)x_0 + (a_{21}\alpha_0 + a_{22}\alpha_1)x_1.$$

Taking two copies of this action, we get an action of $\mathrm{GL}_2(\mathbb{C})^2$ on $\mathbb{P}^1 \times \mathbb{P}^1$ and a linearization of it to each $H^0(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(d_1, d_2))$; in particular on $H^0(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1, 2))$, which we identify via (2.1.1) to $H^0(\mathcal{O}_{\mathbb{P}^5}(1))$. The naturally induced action of $\mathrm{GL}_2(\mathbb{C})^2$ on $\mathbb{P}^5$ preserves $R^4$, inducing on it the action (on $\mathbb{P}^1 \times \mathbb{P}^1$) we started with. Finally, the action on $\mathbb{P}^5$ induces an action on the Grassmannian of the lines in $\mathbb{P}^5$ that preserves $M_4$. Therefore, we can define an action of $\mathrm{GL}_2(\mathbb{C})^2$ on $M_4$.

We show now that this action splits $M_4$ in exactly 5 orbits. We say that two distinct points $p \neq q$ of $R^4$ are *in general position with respect to the rulings* if $p$ and $q$ belong to two different lines and two different irreducible conics in $R^4$.

For every element $g \in \mathrm{GL}_2(\mathbb{C})^2$, if two points $p$ and $q$ are in general position with respect to the rulings, then their images $gp$ and $gq$ are in general position with respect to the rulings as well. In particular, the orbit of $\overline{pq} \in M_4$ is made by secants on two distinct points in general position with respect to the rulings. Since the action of $\mathrm{GL}_2(\mathbb{C})$ on $\mathbb{P}^1$ is 2-transitive,[4] we conclude that the Zariski open subsets of $M_4$ of the secants on two points in $R^4$ in general position with respect to the rulings form an orbit.

A similar argument shows that the secants through two distinct points belonging to the same irreducible conic of $R^4$ form a second orbit of dimension 3. A third orbit is obtained by considering the secants through two distinct points belonging to the same line of $R^4$: these are the lines contained in $R^4$, forming an orbit of dimension 1.

Considering the lines that are tangent to $R^4$, we get similarly three orbits. However, the tangent to a point in the direction of the line through it coincides with the line itself, thus giving an orbit that has been already considered.

Summing up, $\mathrm{GL}_2(\mathbb{C})^2$ decomposes $M_4$ in 5 orbits as follows, the notation for each stratum $r_d$ is settled so that $d$ is its dimension:

$0_4$ the secants through two points in general position with respect to the rulings;

$0_3$ the tangents to a point $p$ in a direction different from both the directions of the line and of the conic through it;

(4) In fact, it is 3-transitive, but we only need 2-transitivity here.

$2_3$ the secants through two points $p \neq q$ that belong to the same irreducible conic;

$2_2$ the lines tangent to any irreducible conic contained in $R^4$;

$1_1$ the lines contained in $R^4$.

It is enough to show the smoothness of $M_4$ in one point for each orbit. Moreover, since the smooth locus of a variety is a Zariski open subset, it is enough to show the smoothness at a point of $2_2$ and at a point of $1_1$: in fact a neighborhood of them intersects all other orbits.

Let us start with a point of $1_1$. We consider a "standard" chart for the Grassmannian as follows. To each rank 2 matrix

$$\begin{pmatrix} a_{00} & a_{01} & a_{02} & a_{10} & a_{11} & a_{12} \\ b_{00} & b_{01} & b_{02} & b_{10} & b_{11} & b_{12} \end{pmatrix}$$

we associate the line through the points $a, b \in \mathbb{P}^5$ given by its rows. In other words, the coordinate $z_{ij}$ evaluated in $a$ and $b$ gives $a_{ij}$ and $b_{ij}$, respectively.

The matrices

$$\begin{pmatrix} 1 & a_{01} & a_{02} & 0 & a_{11} & a_{12} \\ 0 & b_{01} & b_{02} & 1 & b_{11} & b_{12} \end{pmatrix}$$

give a standard chart of the Grassmannian of the lines in $\mathbb{P}^5$, an open subset parametrized by the affine coordinates $a_{01}, a_{02}, a_{11}, a_{12}, b_{01}, b_{02}, b_{11}, b_{12}$. Note that its origin

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

corresponds to the line $z_{01} = z_{02} = z_{11} = z_{12} = 0$ in $R^4$, which by (2.1.1) originates from the line $y_1 = 0$ in $\mathbb{P}^1 \times \mathbb{P}^1$. So the origin is a point in the orbit $1_1$.

A point in this chart corresponds to the line in $\mathbb{P}^5$ given by parametric equations:

$$\left(\alpha, \quad \alpha a_{01} + \beta b_{01}, \quad \alpha a_{02} + \beta b_{02}, \quad \beta \quad \alpha a_{11} + \beta b_{11}, \quad \alpha a_{12} + \beta b_{12}\right)$$

with parameters $\alpha, \beta$. The rational surface $R^4$ is contained in the following three hyper-quadrics in $\mathbb{P}^5$:

$$z_{00}z_{11} = z_{01}z_{10}, \quad z_{00}z_{02} = z_{01}^2, \quad z_{10}z_{12} = z_{11}^2.$$

Therefore, if a point in the above parametrized line belongs to $R^4$, it will a fortiori belong to each of these hyper-quadrics; that is

$$\alpha(\alpha a_{11} + \beta b_{11}) = \beta(\alpha a_{01} + \beta b_{01}), \tag{2.1.2}$$

$$\alpha(\alpha a_{02} + \beta b_{02}) = (\alpha a_{01} + \beta b_{01})^2, \tag{2.1.3}$$

$$\beta(\alpha a_{12} + \beta b_{12}) = (\alpha a_{11} + \beta b_{11})^2. \tag{2.1.4}$$

These are three homogeneous equations of degree 2 in the variables $\alpha, \beta$. If the line is secant (or tangent), then these three homogeneous equations define a scheme of $\mathbb{P}^1$ of length at least 2. Therefore, the equations must be pairwise proportional. This will allow to express the four variables $a_{ij}$ as holomorphic functions in terms of the other four as follows.

Assume that $a_{11} \neq 0$, $b_{01} \neq 0$. The coefficient of $\alpha^2$ in (2.1.4) is equal to $a_{11}^2$. Therefore, in order to have (2.1.2) proportional to (2.1.4), we have to multiply it by $-a_{11}$. Comparing the other coefficients, we get the following two conditions:

$$2a_{11}b_{11} - a_{12} = a_{11}(b_{11} - a_{01}), \quad b_{12} - b_{11}^2 = a_{11}b_{01}.$$

Similarly, looking at the coefficient of $\beta^2$, we see that (2.1.3) equals (2.1.2) multiplied by $b_{01}$. Comparing the other coefficients, we get

$$2a_{01}b_{01} - b_{02} = b_{01}(a_{01} - b_{11}), \quad a_{02} - a_{01}^2 = a_{11}b_{01}.$$

We use these four equations to express the variables $a_{02}, b_{02}, a_{12}, b_{12}$ as holomorphic functions of the others. Hence we obtain a smooth parametrization given by the matrices of the form

$$\begin{pmatrix} 1 & a_{01} & a_{01}^2 + a_{11}b_{01} & 0 & a_{11} & a_{11}(a_{01} + b_{11}) \\ 0 & b_{01} & b_{01}(a_{01} + b_{11}) & 1 & b_{11} & a_{11}b_{01} + b_{11}^2 \end{pmatrix}.$$

Note that the parametrization above is an embedding of $\mathbb{C}^4$ in $M_8^{14}$, thus giving a smooth 4-dimensional manifold. By construction, this manifold contains the intersection of $M_4$ with our chart.[5] Since $M_4$ is irreducible of dimension 4, they coincide. This proves the smoothness of $M_4$ at a point of $1_1$, the origin of this chart.

Finally, consider a point in $2_2$, a line tangent to a conic in $R^4$, for example the line $z_{02} = z_{10} = z_{11} = z_{12} = 0$. By the same techniques used in the previous case, the reader can show that a neighborhood of this point in $M_4$ is the smooth manifold given by the matrices

$$\begin{pmatrix} 1 & 0 & a_{02} & a_{10} & a_{02}b_{10} & a_{02}(a_{10} + b_{10}b_{02}) \\ 0 & 1 & b_{02} & b_{10} & a_{10} + b_{02}b_{10} & a_{10}b_{02} + b_{10}a_{02} + b_{10}b_{02}^2 \end{pmatrix}. \qquad \blacksquare$$

### 2.2. *Computing the degree*

Fano proceeds to compute the degree of $M_4$ with respect to the Plücker embedding of the Grassmannian.

([5]) We have parametrized the secants to the complete intersection of three quadrics, which is a variety that contains $R^4$ but does not coincide with it. Hence the parametrized locus contains the intersection of $M_4$ with the chosen chart.

*Determiniamo anzitutto l'ordine di questa $M_4$, ad esempio l'ordine della superficie sua intersezione con un $S_{12}$, vale a dire della $\infty^2$ di rette comune alla $\infty^4$ suddetta e a due complessi lineari. Valendoci di due complessi costituiti risp. dalle rette incidenti a due $S_3$, questi ultimi contenuti in un $S_4 \equiv \sigma$ e aventi perciò a comune un piano $\pi$, la $\infty^2$ di rette in parola si spezzerà nei due sistemi delle corde di $R$ contenute in $\sigma$ e di quelle incidenti al piano $\pi$.*[6]

Fano's strategy is to compute the degree of $M_4$ as the degree of a surface obtained cutting $M_4$ by two hyperplane sections in $M_8^{14}$. He chooses two very special hyperplanes, given by the lines intersecting two linear subspaces of dimension 3 in $\mathbb{P}^5$ which are in "special position". Namely, these two $\mathbb{P}^3$ in $\mathbb{P}^5$ intersect along a plane $\pi$; equivalently both of them are contained in a hyperplane $\sigma \subset \mathbb{P}^5$.

He notices that the lines in the intersection of the two hyperplanes in $M_8^{14}$ are exactly the lines contained in $\sigma$ and the lines intersecting $\pi$. In fact, on one side a line intersecting $\pi$ in a point $p$ intersects both $\mathbb{P}^3$s in $p$; and also a line contained in $\sigma$ must intersect both $\mathbb{P}^3$s, which are in $\sigma$. On the other side, if a line intersects both $\mathbb{P}^3$s and does not intersect $\pi$, it intersects them in distinct points; the line, therefore, contains two distinct points of $\sigma$, thus it is contained in $\sigma$.

Let us denote with $S^\sigma$ the subvariety of $M_4$ of the lines contained in $\sigma$ and with $S_\pi$ the subvariety of lines intersecting $\pi$. Fano uses the following formula, which we like to justify with a proof.

Lemma 2.3.

$$\deg M_4 = \deg S^\sigma + \deg S_\pi .$$

Proof. The degree of a variety equals the degree of any hyperplane section; in particular, the degree of $M_4$ is equal to the degree of its intersection with the above two (special) hyperplanes. So far, we have only proved that this intersection coincides with $S^\sigma \cup S_\pi$ set-theoretically. We need to prove that the intersection is reduced or, equivalently, that there is a choice of the two $\mathbb{P}^3$s such that the linear section is smooth in (at least) a point of $S_\pi$ and a point of $S^\sigma$.

Let us prove it in coordinates. Take the two hyperplane sections giving the secant lines which intersect $\{z_{01} - z_{10} = z_{00} = 0\}$ and $\{z_{01} - z_{10} = z_{11} = 0\}$, respectively, so that $\sigma = \{z_{01} - z_{10} = 0\}$ and $\pi = \{z_{01} - z_{10} = z_{11} = z_{00} = 0\}$.

([6]) Let's first determine the order of this $M_4$, for instance the order of the surface which is the intersection with an $S_{12}$, that is the order of the $\infty^2$ common lines of the above $\infty^4$ and two linear complexes. Making use of two constituted complexes resp. from the incident lines to two $S_3$, both contained in an $S_4 \equiv \sigma$ and having therefore a common plane $\pi$, the $\infty^2$ of straight lines in question will break in the two systems of the chords of $R^4$ contained in $\sigma$ and of those incident to the plane $\pi$.

In the chart near a point of type $1_1$ studied in the proof of Proposition 2.1, the two hyperplane sections are defined, respectively, by $b_{01} = 1$ and $a_{01}b_{11} = a_{11}(b_{01} - 1)$. Their intersection is the locus $b_{01} - 1 = a_{01}b_{11} = 0$. It is smooth at the general point of both components, namely $S^\sigma$, which is $b_{01} - 1 = a_{01} = 0$, and $S_\pi$, which is $b_{01} - 1 = b_{11} = 0$. ∎

Fano computes first the degree of $S^\sigma$.

*Le prime sono le $\infty^2$ corde di una $C^4$ razionale normale, e nella Grassmanniana delle rette di $\sigma$ hanno per immagine una superficie $\varphi^9$ di $S_9$ di del Pezzo.*[7]

If $\sigma$ is general, $R^4 \cap \sigma$ is a rational normal curve of degree 4 in $\sigma = \mathbb{P}^4$; we denote it by $C^4$. The lines contained in a hyperplane $\sigma \subset \mathbb{P}^5$ are mapped by the Plücker embedding into a linear $\mathbb{P}^9 \subset \mathbb{P}^{14}$. Since $S^\sigma$ is the subvariety of $M_4$ of the lines contained in $\sigma$, it is contained in that $\mathbb{P}^9$. Fano's claim can be formulated in the following way.

Lemma 2.4. *The surface $S^\sigma$ is embedded in $\mathbb{P}^9$ as a del Pezzo surface of degree* 9.

Proof. The subvariety $S^\sigma$ in $\mathbb{P}^9$ is by construction the image of the map from $C^4 \times C^4 \to \mathbb{P}^9$ which maps a pair $(p, q)$ on the point of the Grassmannian of the lines in $\mathbb{P}^4$ corresponding to the secant $\overline{pq}$, embedded in $\mathbb{P}^9$ by the standard Plücker embedding.

It factors through the second symmetric product of $C^4 \cong \mathbb{P}^1$, which is isomorphic to $\mathbb{P}^2$ as shown by the degree 2 map $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^2$

$$\big([x_0, x_1], [y_0, y_1]\big) \mapsto [x_0y_0, x_0y_1 + x_1y_0, x_1y_1].$$

A hyperplane section in $\mathbb{P}^9$ pulls back on $\mathbb{P}^2$ to a divisor in $|\mathcal{O}_{\mathbb{P}^2}(d)|$ and on $\mathbb{P}^1 \times \mathbb{P}^1$ to $|\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(d, d)|$ for some positive integer $d$. In order to compute $d$, consider the hyperplane section given by the secants of $C^4$ intersecting a fixed general plane $\pi \subset \sigma$ and its pullback $H$ to $\mathbb{P}^1 \times \mathbb{P}^1$. The intersection of $H$ with $\{p\} \times C^4 \cong C^4$ is the set of the points $q \in C^4$ such that $\overline{pq}$ is a secant to $C^4$ intersecting $\pi$. Since $H \in |\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(d, d)|$, $d$ equals the number of secant lines through a general point $p \in C^4$ intersecting $\pi$. Choose $p \notin \pi$ and take the projection $f_p\colon \sigma \dashrightarrow \mathbb{P}^3$. The secant lines through $p$ intersecting $\pi$ are projected to the points of the plane $f_p(\pi)$ intersecting the rational normal cubic $f_p(C^4)$, so there are exactly 3 of them; i.e. $d = 3$.

We proved that $S^\sigma$ is the image of a map from $\mathbb{P}^2$ on $\mathbb{P}^9$ given by cubics. Note that $S^\sigma$ is not contained in any hyperplane, otherwise, by contradiction, this would imply that $C^4$ is contained in a hyperplane of $\mathbb{P}^4$. Therefore, $S^\sigma$ is the image by the full linear system $|\mathcal{O}_{\mathbb{P}^2}(3)|$, defining the del Pezzo surface of degree 9. ∎

(7) The first are the $\infty^2$ chords of a normal rational $C^4$, and in the Grassmannian of the lines of $\sigma$ they have as image a del Pezzo surface $\varphi^9$ of $S_9$.

Let us now interpret Fano's argument to compute the degree of $S_\pi$.

*Della seconda $\infty^2$ prendiamo l'intersezione con un ulteriore complesso lineare, anche con un $S_3 \equiv \tau$ direttore incontrante $\pi$ in una retta. Si ha una rigata composta di una parte luogo delle corde di $R$ contenute nello spazio $S_4 \equiv \tau\pi$ e incidenti a $\pi$, la cui immagine è sezione iperpiana di altra $\varphi_9$ di del Pezzo; e di una seconda parte luogo delle corde incidenti alla retta $\tau\pi$.*[8]

Fano considers a special codimension 2 subspace $\tau \subset \mathbb{P}^5$, special in the sense that $\tau$ intersects $\pi$ in a line. He takes then a special hyperplane section of $S_\pi$, the one given by the lines that meet $\tau$. By the same argument used above, this curve has two irreducible components: the secant lines in $S_\pi$ contained in the unique $\mathbb{P}^4$ generated by $\tau$ and $\pi$ and intersecting $\pi$, call it $C_\pi^{\langle\tau,\pi\rangle}$, and those intersecting the line $\tau \cap \pi$, call it $C_{\tau\cap\pi}$.

Lemma 2.5. *For a general choice of $\pi, \tau$, the following holds:*

$$\deg S_\pi = \deg C_\pi^{\langle\tau,\pi\rangle} + C_{\tau\cap\pi}.$$

Proof. The argument, with coordinates, is the same as that used in the proof of Lemma 2.3. We suppose again that $\pi = \{z_{01} - z_{10} = z_{00} = z_{11} = 0\}$ and we work in the same chart, where we proved that $S_\pi$ is defined by $b_{11} = b_{01} - 1 = 0$; i.e. it is the parametrized surface

$$\begin{pmatrix} 1 & a_{01} & a_{01}^2 + a_{11} & 0 & a_{11} & a_{11}a_{01} \\ 0 & 1 & a_{01} & 1 & 0 & a_{11} \end{pmatrix}.$$

Choosing $\tau = \{z_{01} - z_{10} = z_{12}\}$, the corresponding hyperplane section is

$$\det \begin{pmatrix} a_{01} - 0 & a_{11}a_{01} \\ 1 - 1 & a_{11} \end{pmatrix} = 0,$$

which is smooth at the general point of both components, $C_\pi^{\langle\tau,\pi\rangle}$ $(a_{01} = 0)$ and $C_{\tau\cap\pi}$ $(a_{11} = 0)$. ■

Fano claims that $C_\pi^{\langle\tau,\pi\rangle}$ is a hyperplane section of a del Pezzo surface of degree 9 and, therefore, that

$$\deg C_\pi^{\langle\tau,\pi\rangle} = 9. \tag{2.2.1}$$

In fact, for a general choice, the hyperplane containing $\tau$ and $\pi$ cuts on $R^4$ a rational normal curve $C^4$ of degree 4. As proved above (see Lemma 2.4) $C_\pi^{\langle\tau,\pi\rangle}$ is contained

(8) Of the second $\infty^2$, we take the intersection with a further linear complex, also with $S_3 \equiv \tau$ as director, meeting $\pi$ in a line. We get a ruled surface composed of a part which is the locus of the chords of $R_4$ contained in the space $S_4 \equiv \tau\pi$ and incident to $\pi$, whose image is a hyperplane section of another del Pezzo $\varphi_9$; and of a second part, the locus of the chords incident to the line $\tau\pi$.

in the degree 9 del Pezzo surface of its secants: more precisely it is given by those intersecting the codimension 2 subspace $\pi$, therefore it is a hyperplane of it.

To study the curve $C_{\tau\cap\pi}$, Fano considers the union of the lines defined by it as a ruled surface $F \subset \mathbb{P}^5$. More precisely, as we have proved above (see Lemma 2.2), for each point on a general line $l \subset \mathbb{P}^5$, therefore also for $\tau \cap \pi$, there is a unique secant to $R^4$ through it. Mapping each secant to its unique intersection point with $l$, one obtains a ruling of the surface $F$ to $l$.

*Quest'ultima rigata è di $4^o$ ordine, avendo la retta $\tau\pi$ come direttrice semplice, e 3 generatrici in ogni $S_4$ per essa (poiché la proiezione della rigata dalla retta $\tau\pi$ ha una cubica doppia).*[9]

Here Fano claims that $\deg F = 4$. We interpret Fano's computation of the degree of this surface as follows.

Project the surface $F$ from the line $l = \tau \cap \pi$ and obtain a curve $C \subset \mathbb{P}^3$, as each line in the ruling maps to a point. Then consider the same projection from $l$ this time restricted to $R^4$, call it $\varphi_l : R^4 \to \mathbb{P}^3$. Since $l$ is general, it does not intersect $R^4$ and therefore $\varphi_l$ is a morphism. Namely, a finite morphism, as every plane through $l$ intersects $R^4$ in finitely many points, or we would contradict $l \cap R^4 = \emptyset$.

We claim that $\varphi_l$ is generically injective and $C$, defined as the projection of $F$, equals the singular locus of $\varphi_l(R^4)$. In fact, we will see that $C$ is a double curve of $\varphi_l(R^4)$.

Since $R^4$ is defined as a codimension 3 subvariety in $\mathbb{P}^5$, a general plane through each point $p \in R^4$ intersects $R^4$ transversally only at the point $p$. Thus, for a line $l$ in this plane not passing through $p$, the projection $\varphi_l$ separates $p$ from any other point of $R^4$; this proves that $\varphi_l$ is generically injective.

Since $\varphi_l$ is a birational morphism, the image of $R^4$ is a quartic curve. Since such a curve is contained in the 3-dimensional space parametrizing the planes containing $l$, its canonical system is trivial. The singular locus of $\varphi_l(R^4)$ is the set of the planes containing a subscheme of $R^4$ of length at least 2. Any such plane contains a secant line, the unique line contained the given length 2-scheme. Since this secant intersects $l$ (they are two lines in the same plane), it is a curve in the ruling of $F$. This shows that

$$\operatorname{Sing}\left(\varphi_l(R^4)\right) \subset C.$$

Conversely, every secant line $r$ to $R^4$ intersecting $l$ is contained in a unique plane, i.e. the plane spanned by $r$ and $l$, cutting the corresponding scheme of length 2 on $R^4$. This shows that the singular locus of $\varphi_l(R^4)$ is $C$, that is in fact a double curve.

(9) This last ruled surface is of 4th order, having the line $\tau\pi$ as a simple directrix, and 3 generators in each $S_4$ for it (since the projection of the ruled surface from the line $\tau\pi$ has a double cubic).

Finally, since the canonical class of $\varphi_l(R^4)$ is trivial, by adjunction the reduced transform of the double curve is an anticanonical divisor. Therefore, it is an element in $|\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(2,2)|$. The intersection computation

$$(2,2)(1,2) = 4 + 2 = 6$$

shows that its image has degree $\frac{6}{2} = 3 = \deg C$.

Therefore, we can conclude, as Fano did, that the ruled surface has degree 4. More precisely, choose a general hyperplane $H$ containing the line $l$; its image via $\varphi_l$ in $\mathbb{P}^3$ is a hyperplane that intersects the curve $C$ transversally in three points. As a consequence, the intersection of the hyperplane $H$ with the ruled surfaces $F$ is the union of three secants passing from the three points above and the line $l$. All together, they are 4 lines and therefore the degree of $F$, which is equal to $\deg(F \cap H)$, is 4.

Moreover, this will imply that

$$\deg C_{\tau\cap\pi} = 4. \tag{2.2.2}$$

Indeed, the degree of this curve is the number of secants in $C_{\tau\cap\pi}$ in a transversal hyperplane section as, for example, the one given by the secant lines intersecting a general $\mathbb{P}^3$. As proved before, such a general projective space $\mathbb{P}^3$ intersects $F$ in $\deg F = 4$ points, each belonging to one of these secants, the line in the ruling of $F$ containing it. So $\deg C_{\tau\cap\pi} = \deg F = 4$.

Fano summarizes his computation in this way:

*Complessivamente la superficie immagine delle corde di $R^4$ appoggiate a un piano è dunque di ordine $9 + 4 = 13$; e la $M_4$ immagine del sistema di tutte le corde di $R$ è di ordine $9 + 13 = 22$.*[10]

In other words, he claims the following proposition.

Proposition 2.6. *$M_4$ has degree 22.*

Proof. The statement follows from Lemmas 2.3, 2.4, and 2.5 as well as formulas (2.2.1) and (2.2.2). ∎

### 2.3. *$M_4$ and its general hyperplane sections are Fano*

Fano shows that the general curve section of $M_4^{22}$ is a canonical curve of genus 12.

*Le due superficie $\varphi^9$ e $F^{13}$, costituenti insieme una sezione superficiale della $M_4^{22}$, hanno a comune una curva sezione iperpiana della $\varphi^9$ (collo spazio $\sigma$), perciò*

(10) Overall, the image surface of the chords of $R^4$ intersecting a plane is therefore of order $9 + 4 = 13$; and the $M_4$ image of the system of all the chords of $R^4$ is of order $9 + 13 = 22$.

*ellittica, di ordine* 9*; la* $M_4^{22}$ *ha quindi superficie-sezioni di genere uno, e curve-sezioni canoniche di genere* 12 *(appunto* $= 1 + 3 + 9 - 1$*).*[11]

Fano picks a hyperplane section of the surface section he had considered, namely of $S^\sigma \cup S_\pi$. Recall that in our notation $S^\sigma = \varphi^9$ and $S_\pi = F^{13}$.

Since $S^\sigma$ is the del Pezzo surface of degree 9, its general hyperplane section is a smooth plane cubic, which has genus 1.

We know a reducible hyperplane section of $S_\pi$, namely $C_\pi^{\langle\tau,\pi\rangle} \cup C_{\tau\cap\pi}$. Our discussion shows that $C_\pi^{\langle\tau,\pi\rangle}$ is a general hyperplane section of a del Pezzo surface of degree 9 too, so a smooth curve of genus 1. On the other hand, $C_{\tau\cap\pi}$ is smooth and rational, being isomorphic to $l$ by mapping each point of $l$ in the unique secant to $R^4$ through it. The intersection $C_\pi^{\langle\tau,\pi\rangle} \cap C_{\tau\cap\pi}$ is given by the secant lines contained in the hyperplane $\langle\tau,\pi\rangle$ intersecting the line $l = \tau \cap \pi$. These are the secant lines to a general hyperplane section $R^4 \cap \langle\tau,\pi\rangle$, a rational normal curve of degree 4, whose secants form a cubic surface. Intersecting it with a codimension 2 linear subspace, we obtain 3 points, and the 3 secants through them are the intersection points of $C_\pi^{\langle\tau,\pi\rangle}$ and $C_{\tau\cap\pi}$. Summing up, we have hyperplane sections of $S_\pi$ formed by 2 smooth curves of genus 0 and 1, respectively, which intersect in 3 points. Hence the general hyperplane section is a smooth curve of genus $0 + 1 + 3 - 1 = 3$.

The intersection of $S^\sigma$ and $S_\pi$ is a hyperplane section of $S^\sigma$, a curve of degree 9. The two curves obtained cutting $S^\sigma$ and $S_\pi$ with a general hyperplane intersect then in 9 points. We conclude that the sectional genus of $M_4^{22}$, which is the genus of a hyperplane section of $S^\sigma \cup S_\pi$, is equal to $1 + 3 + 9 - 1 = 12$.

A general curve section of $M_4^{22}$ is, therefore, a non-degenerate smooth curve of genus 12 in $\mathbb{P}^{14-3=11}$ of degree 22; by Riemann–Roch, this general curve section is a canonical curve; i.e. it is embedded by its complete canonical system.

To recap, we have the following proposition.

PROPOSITION 2.7. *The* 4*-fold* $M_4 = M_4^{22} \subset M_8^{14}$ *is an irreducible smooth variety of dimension* 4 *with canonical sectional curves.*

*In particular, by Proposition* 1.1 *it is a Fano* 4*-fold of index* 2*, i.e.* $-K_{M_4} = 2H$*, where* $H$ *is the hyperplane bundle of the Grassmannian* $M_8^{14}$.

Fano concludes by taking a very general hyperplane section $M_3$ of $M_4$. By Bertini's theorem, this section is a smooth 3-fold whose curve section, by construction, is canonical. Again by Proposition 1.1, this is a smooth Fano 3-fold of degree 22.

(11) The two surfaces $\varphi^9$ and $F^{13}$, which together constitute a surface section of the $M_4^{22}$, have a common hyperplane section curve of the $\varphi^9$ (with the space $\sigma$), therefore elliptic, of order 9; thus the $M_4^{22}$ has surface-sections of genus one, and canonical curves-sections of genus 12 (precisely $= 1 + 3 + 9 - 1$).

*Le sezioni iperpiane della $M_4^{22}$ sono pertanto $M_3^{22}$ di $S_{13}$, corrispondenti al tipo generale $M_3^{2p-2}$ di $S_{p+1}$, per $p = 12$, e razionali (come risulterà pure dai sistemi lineari di superficie che vi sono contenuti). Indicheremo d'ora in poi questa varietà con $\mu_3^{22}$, o semplicemente $\mu$; essa è l'immagine del sistema $\infty^3$ di rette $\Sigma$ intersezione della $\infty^4$ delle corde di $R$ con un complesso lineare $K$ (che si supporrà per ora del tipo più generale, e in posizione generica rispetto a $R^4$).*[12]

## 3. What is nowadays Fano's last Fano?

Where can we find Fano's last Fano in modern literature? In order to answer this question, we reformulate Fano's construction via a modern and rather subtle tool, namely the Hilbert scheme.

Let $S$ be a smooth projective surface and consider the Hilbert scheme which parametrizes its zero dimensional subschemes of length 2. By Grothendieck's theory, this is a projective scheme of dimension 4 which is usually denoted by $\mathrm{Hilb}^2(S)$, or simply $S^{[2]}$. One can also consider the Chow scheme of the set of two points on $S$, namely $S^{(2)} := (S \times S)/\sigma_2$, where $\sigma_2$ is the symmetric group of permutations of two elements. Consider then the natural morphism Hilb to Chow, $S^{[2]} \to S^{(2)}$.

A line bundle $L$ on $S$ induces the $\sigma_2$-equivariant line bundle $L^{\boxtimes 2}$ on $S \times S$, which descends to a line bundle $L^{(2)}$ on $S^{(2)}$, which in turn can be pulled back via the Hilb to Chow morphism to the line bundle $L^{[2]}$ on $S^{(2)}$.

For simplicity, we assume that $S$ has irregularity zero; we will use the following very well-known theorems by Fogarty [9, 10].

Theorem 3.1. *Let $S$ be a smooth projective surface with $q = h^1(\mathcal{O}_S) = 0$.*

(a) $\mathrm{Hilb}^2(S) = S^{[2]}$ *is a smooth projective variety of dimension* 4 *which resolves the singularities of the Chow scheme via the Hilb to Chow morphism $S^{[2]} \to S^{(2)}$.*

(b) $\mathrm{Pic}(S^{[2]}) = \mathrm{Pic}(S) \oplus \mathbb{Z}(B/2)$*, where* $\mathrm{Pic}(S)$ *is embedded in* $\mathrm{Pic}(S^{[2]})$ *via the above-described map $L \to L^{[2]}$ and $B$ is the locus of non-reduced schemes, i.e. the exceptional divisor of the Hilb to Chow map* [10, Corollary 6.3].

In the next proposition, we specialize the results of Fogarty to the case $S = \mathbb{P}^1 \times \mathbb{P}^1$ and we add, for this case, the description of the Nef (or Mori) Cone of $(\mathbb{P}^1 \times \mathbb{P}^1)^{[2]}$; for a proof we refer, for instance, to [2, Theorem 2.4].

(12) The hyperplane sections of the $M_4^{22}$ are, therefore, $M_3^{22}$ in $S_{13}$, corresponding to the general type $M_3^{2p-2}$ of $S_{p+1}$, for $p = 12$, and rational (as well as the linear systems of surfaces contained in it). From now on, we will denote this variety with $\mu_3^{22}$, or simply $\mu$; it is the image of the system $\infty^3$ of lines $\Sigma$ intersection of the $\infty^4$ of chords of $R^4$ with a linear complex (which, for now, we will suppose to be of very general type, and in general position with respect to $R^4$).

PROPOSITION 3.2. *Let $S = \mathbb{P}^1 \times \mathbb{P}^1$. Denote by $\pi_i$ the two projections and $H_i := \pi_i^*(\mathcal{O}_{\mathbb{P}^1}(1))$ (the line bundles associated to the two fibers).*

*For brevity, we denote by $\mathcal{H}$ the Hilbert scheme $(\mathbb{P}^1 \times \mathbb{P}^1)^{[2]}$.*

(a) *$\mathcal{H}$ is a smooth projective variety of dimension 4 and $\mathrm{Pic}(\mathcal{H}) = \mathbb{Z}(H_1^{[2]}) \oplus Z(H_2^{[2]}) \oplus \mathbb{Z}(B/2)$.*

(b) *The Nef Cone of $\mathcal{H}$ is the simplicial cone spanned by $H_1^{[2]}$, $H_2^{[2]}$, and $H_1^{[2]} + H_2^{[2]} - (B/2)$.*

Let us describe the maps associated to the nef bundles which span the Nef Cone. We will get six maps from $\mathcal{H}$, one for each of the three extremal rays, and one for each of the three extremal faces.

The map associated with a sufficiently high multiple of $H_1$ (respectively, $H_2$), call it $\psi_1 \colon \mathcal{H} \to M'$ (respectively, $\psi_2 \colon \mathcal{H} \to M''$), is a birational map which contracts the divisor $D_2 \subset \mathcal{H}$ consisting of the zero cycles supported on the fibers $f_2$ of the second ruling (respectively, on the fibers $f_1$ of the first ruling) of $\mathbb{P}^1 \times \mathbb{P}^1$. More precisely, the map contracts all zero cycles on a fiber to a point. This divisor is clearly isomorphic to $\mathbb{P}^1 \times (\mathbb{P}^1)^{[2]} = \mathbb{P}^1 \times \mathbb{P}^2$ and it is contracted to a rational curve $\mathbb{P}^1$ by contracting each $\mathbb{P}^2$ to a point. The two divisors are disjoint (a zero cycle of length two can be contained in at most one fiber) and they can be contracted simultaneously; this contraction corresponds to the face of the Nef Cone joining the two rays, call it $\psi : \mathcal{H} \to Q$.

The map associated to $H_1^{[2]} + H_2^{[2]} - (B/2)$ is the Hilb to Chow map,

$$(\mathbb{P}^1 \times \mathbb{P}^1)^{[2]} \to (\mathbb{P}^1 \times \mathbb{P}^1)^{(2)},$$

which contracts the divisor $B$. We have, moreover, two natural maps of fiber type,

$$(\mathbb{P}^1 \times \mathbb{P}^1)^{(2)} \to (\mathbb{P}^1)^{(2)} = \mathbb{P}^2,$$

and finally other two maps $\phi_1 : M' \to \mathbb{P}^2$, $\phi_2 : M'' \to \mathbb{P}^2$ making the following diagram commutative:

$$\begin{tikzcd}
 & \mathbb{P}^2 & & M' \arrow[ll, "\phi_1"'] \arrow[dr] & \\
(\mathbb{P}^1 \times \mathbb{P}^1)^{(2)} \arrow[ur] \arrow[dr] & & \mathcal{H} \arrow[ll] \arrow[ul] \arrow[dl] \arrow[ur, "\psi_1"] \arrow[dr, "\psi_2"] \arrow[rr, "\psi"] & & Q. \\
 & \mathbb{P}^2 & & M'' \arrow[ll, "\phi_2"] \arrow[ur] &
\end{tikzcd}$$

Now we would like to give a concrete projective description of the above abstract varieties. In order to do this, we take a smooth projective surface $S$ and we choose an embedding $S \hookrightarrow \mathbb{P}^N$. Note that to a subscheme of length 2 on $S$ we can now associate the unique line containing its image in $\mathbb{P}^N$. In this way, we can think at $\mathrm{Hilb}^2(S)$ as "the variety $\infty^4$ of its (i.e., of $S \hookrightarrow \mathbb{P}^N$) chords" used by Fano.

Thus we have a natural map from $\mathrm{Hilb}^2(S)$ to the Grassmannian of lines in $\mathbb{P}^N$, whose image is the variety of the lines that are secants or tangents to $S \subset \mathbb{P}^N$; we further compose with the Plücker embedding of the Grassmannian, $\mathrm{Gr}(1, N) \to \mathbb{P}^{\frac{(N+1)N}{2}-1}$.

If $S \subset \mathbb{P}^N$ with $N \geq \dim S + 2$, then the generic secant of $S$ is not 3-secant. This follows easily, cutting with general hyperplanes, by the classical so-called trisecant lemma, which states that a nonsingular nondegenerate curve $C \subset \mathbb{P}^r$, $r \geq 3$, admits only $\infty^1$ trisecant lines (see, for instance, [20, Chapter 7B]).

This implies that, if $N \geq 4$, the total map $\mathrm{Hilb}^2(S) \to \mathbb{P}^{\frac{(N+1)N}{2}-1}$ is a birational map (onto its image).

If we embed $S = \mathbb{P}^1 \times \mathbb{P}^1$ via the linear system $H_1 \otimes H_2^{\otimes 2}$, we get the normal rational scroll of degree 4, $R^4 \subset \mathbb{P}^5$, and *the map* $\mathrm{Hilb}^2(\mathbb{P}^1 \times \mathbb{P}^1) \to M_4 \subset \mathbb{P}^{14}$ *is exactly the one in Fano's paper*.

Let us consider first the special case $S = \mathbb{P}^1 \times \mathbb{P}^1$ and the embedding given by the complete linear system $H_1 \otimes H_2$; that is we embed $\mathbb{P}^1 \times \mathbb{P}^1$ as a smooth quadric surface $Q_2 \subset \mathbb{P}^3$. Note that the secant lines fill up the whole Grassmannian $G(1, 3)$, since every line in $\mathbb{P}^3$ is secant to any quadric surface. The Plücker embedding maps $G(1, 3)$ into a (Klein) quadric 4-fold $Q_4$ in $\mathbb{P}^5$.

Therefore, we have a birational surjective map $\mathcal{H} \to Q_4 \subset \mathbb{P}^5$. We show that it is the map $\psi$ in the above diagram.

Indeed, it contracts the two divisors $D_1$ and $D_2$ to two curves, $C_1, C_2 \subset Q_4$, which describe in the Grassmannian the lines in the ruling. These curves are conics. In fact, their degree is the number of lines in a ruling that meet a fixed general line: since a general line intersect a quadric in two points, it will meet exactly two fibers for each ruling.

As said before, all non-zero dimensional fibers of $\psi$ are isomorphic to $\mathbb{P}^2$; in particular, they all have the same dimension. By a general result, see [1, Corollary 4.11], $\psi : \mathrm{Hilb}^2(\mathbb{P}^1 \times \mathbb{P}^1) \to Q \subset \mathbb{P}^5$ is the blow-up of the quadric $Q_4 \subset \mathbb{P}^5$ along two disjoint smooth conics: $C_1, C_2$.

Let us now go back to the Fano case; i.e. we embed $S = \mathbb{P}^1 \times \mathbb{P}^1$ via the linear system $H_1 \otimes H_2^{\otimes 2}$ as the normal rational scroll of degree 4, $R^4 \subset \mathbb{P}^5$. The birational surjective map $\mathcal{H} \to M_4 \subset \mathbb{P}^{14}$ is the one studied by Fano. In this case, the map contracts only one of the two above-mentioned divisors, namely the one corresponding to the ruling in lines of $R^4$ which we denote as above by $D_1$. In other words, the map is the map $\psi_1$ in the above diagram.

Thus $M_4$ is smooth and $\psi_1$ is the smooth blow-up along the transform $\widetilde{C}_2 \subset M_4$ of $C_2$. This gives a different, more geometric, proof of the smoothness of $M_4$.

The other divisor $E := D_1$ remains isomorphically equal in $M_4$ and it can be contracted as a smooth blow-down to the curve $C_1 \subset Q_4 \subset \mathbb{P}^5$, $\nu\colon M_4 \to Q_4$. We proved that $C_1$ is a smooth conic, here we can add the fact that it is not contained in any plane $\mathbb{P}^2 \subset Q_4 \subset \mathbb{P}^5$. In fact, if this were the case, all lines in the ruling of lines of $R^4$, which are parametrized by $C_1$, would be contained in a $\mathbb{P}^4 \subset \mathbb{P}^5$, and the same for $R^4$, a contradiction.

Let $H$ be the hyperplane bundle in $\mathbb{P}^5$; the formula for the canonical bundle of the blow-up gives

$$-K_{M_4} = \nu^*(4H) - 2E = 2\big(\nu^*(2H) - E\big).$$

The line bundle $\mathcal{L} := \nu^*(2H) - E$ is very ample; it embeds $M_4$ into $\mathbb{P}^{14}$ (the space of quadrics in $\mathbb{P}^5$ containing a conic has dimension 15) as a Fano manifold of index 2 and genus 12.

Since $M_4$ is the blow-up of a quadric, we have $\mathrm{Pic}(M_4) = \mathbb{Z}^2$; that is $M_4$ is not "prime".

On the other hand, the line bundle $\nu^*(H) - E$ is nef and it gives a map

$$\phi_1\colon M_4 \to \mathbb{P}^2$$

which is a quadric bundle fibration over $\mathbb{P}^2$.

Looking at the classification obtained by Mukai [18] of Fano 4-folds of index 2 (coindex 3 in Mukai's notation) one can find $M_4$, given as the blow-up of a 4-dimensional quadric along a conic, as the only one of genus 12 [18, Example 2]; see also the paper [27] with more detailed proofs. The classification was based on Conjecture (ES) which was later proved in [16].

Since $-K_{M_4} = 2\mathcal{L}$, a general hyperplane section in $\mathcal{L} = \nu^*(2H) - E$ is a Fano 3-fold, which we denote as Fano did by $M_3^{22}$. $\mathcal{L}$ embeds $M_4$ as the image of $Q_4$ by the rational map given by the quadric hypersurfaces through a general (= not contained in a plane) conic in $Q_4$, therefore the hyperplane section $M_3^{22}$ is obtained blowing up the conic in the intersection of $Q_4$ with another quadric containing the conic. This proves that the $M_3^{22}$, Fano's last Fano, is the number 16 in the Mori–Mukai list of Fano 3-folds with Picard number 2; see [17, Table 2]. In fact, they describe this case as the blow-up of a prime Fano 3-fold of degree 4 in $\mathbb{P}^5$ along a conic; a prime Fano 3-fold of degree 4 in $\mathbb{P}^5$, according to Iskovskikh, is a complete intersection of two quadrics in $\mathbb{P}^5$.

We summarize the results of the section in the following.

PROPOSITION 3.3. *The projective variety $M_4^{22} \subset \mathbb{P}^{14}$ constructed by Fano is a smooth Fano 4-fold of index 2 which can be described also as the blow-up of a smooth hyperquadric $Q_4 \subset \mathbb{P}^5$ along a smooth conic not contained in any $\mathbb{P}^2$ of the hyperquadric. This is Example 2 ($5 \cdot (g = 12)$) in Mukai's classification* [18].

*A general hyperplane section of $M_4^{22}$ is a smooth (non-prime) Fano 3-fold, denoted by Fano as $\mu := M_3^{22} \subset \mathbb{P}^{13}$, which can be constructed as the blow-up of a complete intersection of two quadrics in $\mathbb{P}^5$ along a conic. This is number* 16 *in Mori–Mukai classification; see* [17, Table 2].

REMARK 3.4. One can embed $S = \mathbb{P}^1 \times \mathbb{P}^1$ via the linear system $H_1 \otimes H_2^{\otimes m}$, with $m \geq 2$, to get the normal rational scroll $R^{2m}$ in $\mathbb{P}^{2m+1}$. In this case, the birational map $\mathrm{Hilb}^2(\mathbb{P}^1 \times \mathbb{P}^1) \to M_4 \subset \mathbb{P}^{(m+1)(2m+1)-1}$ is given by the contraction of one divisor, corresponding to the ruling in lines of $R^{2m}$. The immersion $M_4 \subset \mathbb{P}^{(m+1)(2m+1)-1}$ is not given by the fundamental line bundle.

If we embed $S = \mathbb{P}^1 \times \mathbb{P}^1$ via the linear system $H_1^{\otimes l} \otimes H_2^{\otimes m}$, with $l, m \geq 2$, the above construction gives simply different embeddings of $\mathrm{Hilb}^2(\mathbb{P}^1 \times \mathbb{P}^1)$.

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