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Calculus of Variations. – *Asymptotic behavior of the capacity in two-dimensional heterogeneous media*, by ANDREA BRAIDES and GIUSEPPE COSMA BRUSCA, communicated on 16 December 2022.

Dedicated to Irene Fonseca on the occasion of her 65th birthday.

ABSTRACT. – We describe the asymptotic behavior of the minimal inhomogeneous two-capacity of small sets in the plane with respect to a fixed open set Ω. This problem is governed by two small parameters: ε, the size of the inclusion (which is not restrictive to assume to be a ball), and δ, the period of the inhomogeneity modeled by oscillating coefficients. We show that this capacity behaves as $C|\log\varepsilon|^{-1}$. The coefficient C is explicitly computed from the minimum of the oscillating coefficient and the determinant of the corresponding homogenized matrix, through a harmonic mean with a proportion depending on the asymptotic behavior of $|\log\delta|/|\log\varepsilon|$.

KEYWORDS. – Concentration, capacity, Γ-convergence, homogenization.

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1. INTRODUCTION

Scaling-invariant variational problems present challenging technical issues and intriguing properties due to the concentration of minimal energy configurations on several scales. The paradigmatic example highlighting the effects of scaling invariance can be considered that of Ginzburg–Landau functionals, whose minimizers may generate vortices triggered by topological incompatibilities (see, e.g., [4]). A recent paper [1] investigates the minimal energy of Ginzburg–Landau vortices in heterogeneous media in dimension two, showing the different type of interactions between the Ginzburg–Landau energy and heterogeneities at different scales. An even simpler scaling-invariant functional is the Dirichlet integral in dimension two, which entails singular properties of the two-capacity (more in general, this holds for the d-capacity in dimension d), and for which in this paper we examine a problem analogous to that of [1], highlighting the corresponding analogies and differences.

We consider the problem of the minimal capacity of a small set in a highly heterogeneous media at the critical scaling. We restrict to the planar case in which we may provide a clearer explanation of the asymptotic analysis confining to quadratic functionals. It is not restrictive to suppose that the inclusion is a ball, for which the problem can be stated as the study of the asymptotic behavior of minima:

$$m_{\varepsilon,\delta} = \min\Big\{\int_\Omega a\Big(\frac{x}{\delta}\Big)|\nabla u|^2\,dx : u \in H^1_0(\Omega),\ u = 1 \text{ on } B_\varepsilon(z),\ z \in \Omega\Big\}, \tag{1}$$

with Ω a fixed bounded open set of $\mathbb{R}^2$. Here, a is a 1-periodic function representing the geometry of heterogeneities, δ is the scale of oscillations of the heterogeneities, and $B_\varepsilon(x)$ is the small inclusion, a ball of center x and radius ε. Since the asymptotic result is independent of the particular Ω, we omit the dependence on the set in the notation in (1).

Note that, if a in (1) is a constant c, the value of $m_\varepsilon = m_{\varepsilon,\delta}$ can be easily computed recalling that for $R > r$ we have

$$\min\Big\{\int_{B_R} |\nabla u|^2\,dx : u \in H^1_0(B_R),\ u = 1 \text{ in } B_r\Big\} = 2\pi\frac{1}{\log(R/r)}, \tag{2}$$

where $B_R = B_R(0)$ and $B_r = B_r(0)$. This computation is straightforward since the minimizer is an explicit logarithmic function and implies that

$$\lim_{\varepsilon\to 0} |\log\varepsilon| m_\varepsilon = 2\pi c. \tag{3}$$

We now turn to a non-constant coefficient a. We set $\alpha = \text{ess-}\inf a > 0$ and let $A_{\rm hom}$ denote the constant-coefficient matrix of the corresponding homogenized quadratic form. We consider $\delta = \delta_\varepsilon$ and study the asymptotic behavior of $m_{\varepsilon,\delta}$ as $\varepsilon, \delta \to 0$. We will prove that the limit analysis is determined by the parameter $\lambda \in [0, 1]$ defined by

$$\lambda = \lim_{\varepsilon\to 0}\frac{|\log\delta|}{|\log\varepsilon|} \wedge 1 \tag{4}$$

(a limit which we may suppose exists up to subsequences), and that the behavior of $m_{\varepsilon,\delta}$ is logarithmic, more precisely, that we have

$$\lim_{\varepsilon\to 0} |\log\varepsilon| m_{\varepsilon,\delta} = 2\pi\frac{\alpha\sqrt{\det A_{\rm hom}}}{\lambda\alpha + (1-\lambda)\sqrt{\det A_{\rm hom}}}. \tag{5}$$

Recalling (3), we remark that (5) mixes the two cases $c = \alpha$ (minimal value) and $c = \sqrt{\det A_{\rm hom}}$ (homogenized value) through their *harmonic mean*.

We briefly explain formula (5) by describing an approximate minimizer u^ε for $m_{\varepsilon,\delta}$ for ε small. For simplicity, we suppose that $0 < \lambda < 1$ and a is continuous. We choose the center of the ball $z = z^\varepsilon$ to be a minimum point for $a(x/\delta)$, and we fix any $\lambda' < \lambda < \lambda''$; note that

$$\varepsilon^{\lambda''} =: \delta'' \ll \delta \ll \delta' := \varepsilon^{\lambda'}.$$

We fix a constant $\bar u$ (to be optimized later) and take u^ε on $B'' = B_{\delta''}(z^\varepsilon)$ defined as the minimizer of

$$\min\Big\{\int_{B''} \alpha|\nabla u|^2\,dx : u=\bar u \text{ on } \partial B'', u=1 \text{ in } B_\varepsilon(z^\varepsilon)\Big\},$$

whose value is

$$2\pi\alpha\frac{1}{|\log\varepsilon|(1-\lambda'')}(1-\bar u)^2$$

by (2). Note that in B'' we have $a(x/\delta)\sim\alpha$. We may also take $u^\varepsilon=\bar u$ on $B'\setminus B''$, where $B'=B_{\delta'}(x^\varepsilon)$. Indeed, letting $\lambda''-\lambda'\to 0$, we see that the contribution of the Dirichlet integral on $B'\setminus B''$ is negligible. We may suppose that $B_1(z^\varepsilon)\subset\Omega$, so that we can take u^ε on $B_1(z^\varepsilon)\setminus\overline{B'}$ as the function minimizing

$$\min\Big\{\int_{B_1(z^\varepsilon)} a\Big(\frac{x}{\delta}\Big)|\nabla u|^2\,dx : u\in H^1_0\big(B_1(x^\varepsilon)\big),\ u=\bar u \text{ in } B'\Big\}.$$

By the scale-invariance properties of the Dirichlet integral in dimension two, we can scale the problem to

$$\min\Big\{\int_{B_{\frac{1}{\delta'}}(\frac{z^\varepsilon}{\delta'})} a\Big(\frac{x}{\delta/\delta'}\Big)|\nabla u|^2\,dx : u\in H^1_0\Big(B_{\frac{1}{\delta'}}\Big(\frac{z^\varepsilon}{\delta'}\Big)\Big),\ u=\bar u \text{ in } B_1\Big(\frac{z^\varepsilon}{\delta'}\Big)\Big\}.$$

Since $\delta/\delta'\ll 1$, this problem can be approximated by its homogenized one

$$\min\Big\{\int_{B_{\frac{1}{\delta'}}(\frac{z^\varepsilon}{\delta'})} \langle A_{\rm hom}\nabla u,\nabla u\rangle\,dx : u\in H^1_0\Big(B_{\frac{1}{\delta'}}\Big(\frac{z^\varepsilon}{\delta'}\Big)\Big),\ u=\bar u \text{ in } B_1\Big(\frac{z^\varepsilon}{\delta'}\Big)\Big\}$$

whose value can be approximated by

$$\frac{2\pi\sqrt{\det A_{\rm hom}}}{|\log\varepsilon|\lambda'}\bar u^2$$

by (2) and a change of variables. Summing up these two values and letting $\lambda',\lambda''\to\lambda$, we have the estimate

$$\limsup_{\varepsilon\to 0} m_{\varepsilon,\delta}|\log\varepsilon| \le 2\pi\Big(\sqrt{\det A_{\rm hom}}\frac{1}{\lambda}\bar u^2+\alpha\frac{1}{1-\lambda}(1-\bar u)^2\Big).$$

Taking $\bar u$ minimizing the last expression, we obtain (5). Note that in the argument above some care must be used in the passage to the limit for the homogenization since it is performed on varying domains. The optimality of this construction is rather technical and involves a lemma that allows to decompose minimizations for $m_{\varepsilon,\delta}$ into separate minimizations on annuli of given ratio between inner and outer radii.

We can compare the result above with the one in [1], regarding the asymptotic analysis of the heterogeneous Ginzburg–Landau energy

$$\int_\Omega a\Big(\frac{x}{\delta}\Big)|\nabla u|^2\,dx + \frac{1}{\varepsilon^2}\int_\Omega \big(|u|^2-1\big)^2\,dx \tag{6}$$

at the vortex scaling, where $u:\Omega\to\mathbb{R}^2$. Such an analysis can be formalized as the study of the asymptotic behavior of the *hard-core energy*

$$m^{\rm hc}_{\varepsilon,\delta} = \min\Big\{\int_\Omega a\Big(\frac{x}{\delta}\Big)|\nabla u|^2\,dx : u\in H^1(\Omega;S^1),\ \deg u = 1 \text{ on } \partial B_\varepsilon(z),\ z\in\Omega\Big\},$$

leading to an analogy with problem (1), with a condition on the degree of the test functions in the place of fixing boundary values. The behavior is determined by the same parameter λ in (4), but in this case we have

$$\lim_{\varepsilon\to0}\frac{1}{|\log\varepsilon|}m^{\rm hc}_{\varepsilon,\delta} = 2\pi\big(\lambda\sqrt{\det A_{\rm hom}} + (1-\lambda)\alpha\big); \tag{7}$$

that is, 2π times the *arithmetic mean* of $\sqrt{\det A_{\rm hom}}$ and α with proportion λ. Again the minimization can be split into two regions: between $B_\varepsilon(z^\varepsilon)$ and B'' and between B' and $B_1(z^\varepsilon)$, with a condition on the degree. Minimization in the first region considers for a only the value α, and minimization in the second region can be approximated by substituting the homogenized quadratic form $\langle A_{\rm hom}\xi,\xi\rangle$ to the oscillating quadratic form $a(x/\delta)|\xi|^2$. Both problems behave logarithmically in complete analogy with (2). Contrary to the capacitary problem, here there is no further optimization in the boundary condition, and the two minima can be simply added, to obtain (7). Note, moreover, the different logarithmic prefactor due to the different scaling of boundary conditions.

Note that in both problems we have "well-separated" extreme regimes when

$$\begin{aligned}\lim_{\varepsilon\to0}|\log\varepsilon|m_{\varepsilon,\delta} &= \lim_{\varepsilon\to0}\frac{1}{|\log\varepsilon|}m^{\rm hc}_{\varepsilon,\delta}\\ &= \begin{cases}2\pi\alpha & \text{if } |\log\delta|\ll|\log\varepsilon|,\\ 2\pi\sqrt{\det A_{\rm hom}} & \text{if } \frac{|\log\delta|}{|\log\varepsilon|}\geq 1+o(1).\end{cases}\end{aligned}$$

In these regimes, we have a "separation of scales": we can formally optimize the capacity (or compute the vortex energy) at fixed δ or apply the homogenization procedure first, respectively, while in all other regimes the two effects both appear. This is in contrast with problems in the gradient theory of phase transitions, with energies formally similar to (6) but with scalar u, where separation of scales occurs for all regimes of ε and δ except when $\varepsilon\sim\delta$ (see [3, 8, 11]).

2. Asymptotic analysis

We assume that $\delta = \delta_\varepsilon$, meaning that δ is a function of ε defined on $(0, 1)$, which vanishes as ε tends to 0. Up to subsequences, we may suppose that there exists the limit $\lim_{\varepsilon\to 0} \frac{|\log\delta|}{|\log\varepsilon|}$, and we define

$$\lambda = \min\left\{1, \lim_{\varepsilon\to 0} \frac{|\log\delta|}{|\log\varepsilon|}\right\} \in [0, 1] \tag{8}$$

as in (4). Note that, if $\delta \le C\varepsilon$, then $\lambda = 1$, while if $\delta = \varepsilon^\eta$ with $\eta \in (0, 1)$, then $\lambda = \eta$. The cases $\lambda = 0$ and $\lambda = 1$ include also very slowly converging δ, e.g. $\delta = |\log\varepsilon|^{-1}$, and almost linear δ, e.g. $\delta = \varepsilon|\log\varepsilon|$, respectively.

We will simplify our exposition by choosing a special form of a; namely, we let $a : \mathbb{R}^2 \to \mathbb{R}$ be a 1-periodic checkerboard function with values α, β, such that $0 < \alpha < \beta$, defined on $[0, 1)^2$ as

$$a(x) := \begin{cases} \alpha & \text{if } x \in \big[0, \frac{1}{2}\big)^2 \cup \big[\frac{1}{2}, 1\big)^2, \\ \beta & \text{if } x \in \big[\frac{1}{2}, 1\big) \times \big[0, \frac{1}{2}\big) \cup \big[0, \frac{1}{2}\big) \times \big[\frac{1}{2}, 1\big). \end{cases}$$

The analysis of this particular inhomogeneity will contain all the main features of the general case, with some simplifications in the proofs. We will prove the following result describing the *asymptotic behavior of the capacity in a heterogeneous medium.*

Theorem 2.1. *Let Ω be a bounded open set in $\mathbb{R}^2$, and, for $\varepsilon, \delta > 0$, let $m_{\varepsilon,\delta}$ be defined by*

$$m_{\varepsilon,\delta} = \min\left\{\int_\Omega a\Big(\frac{y}{\delta}\Big)|\nabla u|^2\,dy : u \in H^1_0(\Omega),\ u = 1 \text{ on } B_\varepsilon(x),\ x \in \Omega\right\}.$$

If $\delta = \delta_\varepsilon$ is infinitesimal as $\varepsilon \to 0$, and λ is defined as in (8), then we have

$$\lim_{\varepsilon\to 0} |\log\varepsilon| m_{\varepsilon,\delta} = 2\pi \frac{\alpha\sqrt{\alpha\beta}}{\lambda\alpha + (1-\lambda)\sqrt{\alpha\beta}}. \tag{9}$$

The proof of this theorem will be split between a lower and an upper bound in the following sections.

The appearance of the term $\sqrt{\alpha\beta}$ is a consequence of a classical homogenization result (see [6, 10]). For convenience in the proofs of the following sections, we state this result below with some translations τ_η. To formally derive this modified statement from the usual one without translations, it suffices first to note that by the periodicity of a we can suppose that $\|\tau_\eta\|_\infty \le 1$, so that, if $u_\eta \to u$ in $L^2(\mathbb{R}^2)$, then, defining $\bar u_\eta(x) = u_\eta(x - \eta\tau_\eta)$, we still have $\bar u_\eta \to u$ in $L^2(\mathbb{R}^2)$. This change of variables allows to deduce the modified statement in the following form.

THEOREM 2.2 (Homogenization of the square checkerboard). *Let A be a bounded open subset of $\mathbb{R}^2$ with Lipschitz boundary. For $\eta > 0$, let $\tau_\eta \in \mathbb{R}^2$ and let*

$$F_\eta(u) = \int_A a\Big(\frac{x}{\eta} + \tau_\eta\Big)|\nabla u|^2\,dx, \quad u \in H^1(A).$$

Then F_η Γ-converge to $\sqrt{\alpha\beta}\int_A |\nabla u|^2\,dx$ with respect to the $L^2(\Omega)$-convergence. In particular, for all $\phi \in H^1(A)$ we have

$$\lim_{\eta\to 0}\min\big\{F_\eta(u) : u = \phi \text{ on } \partial A\big\} = \sqrt{\alpha\beta}\min\Big\{\int_A |\nabla u|^2\,dx : u = \phi \text{ on } \partial A\Big\}.$$

Furthermore, this convergence is uniform in τ_η.

A fundamental tool in the proof of Theorem 2.1 is a variant elaborated in [2] (see also [12]) of a method by De Giorgi to vary boundary conditions on converging sequences (see [5,9]). In our context, it can be stated as follows.

LEMMA 2.3 (Reduction to constants on a suitable circumference). *Let $\gamma : \mathbb{R}^2 \to \mathbb{R}$ be a measurable function and let α, β be real constants such that $0 < \alpha \le \gamma(x) \le \beta$ for every $x \in \mathbb{R}^2$. Let $z \in \mathbb{R}^2$, $R > 0$ and define*

$$F(u, A) := \int_A \gamma(x)\big|\nabla u(x)\big|^2\,dx$$

for every $u \in H^1(B_R(z))$ and measurable subset $A \subseteq B_R(z)$.

We fix $\eta \in (0,1)$ and set $S := \max\{s \in \mathbb{N} : \eta 2^s \le R\}$. We assume that $S \ge 3$. Let N be a natural number such that $2 \le N < S$ and let r be a positive real number such that $r \le \eta 2^{S-N}$.

Then there exists a function v with the following properties:

(i) $v \in H^1(B_R(z) \setminus \overline{B}_r(z))$,

(ii) *there exists $j \in \{1, \dots, N-1\}$ such that*

$$v = u \quad \text{on } \big(B_{\eta 2^{S-j-1}}(z) \setminus \overline{B}_R(z)\big) \cup \big(B_R(z) \setminus \overline{B}_{\eta 2^{S-j+1}}(z)\big),$$

(iii) *for the same j, the function v is constant on $\partial B_{\eta 2^{S-j}}(z)$,*

(iv) *there exists a positive constant C depending only on α and β such that*

$$F\big(v, B_R(z) \setminus \overline{B}_r(z)\big) \le \Big(1 + \frac{C}{N-1}\Big)F\big(u, B_R(z) \setminus \overline{B}_r(z)\big).$$

This lemma states that, up to a change in the energy, which by (iv) is small if N is large, by (iii) we can suppose that functions u have a constant value on the boundary of some ball, which is of relative radius close to R if N is small compared to S. In the following, this will be applied with fixed N and diverging S. The proof of this lemma is obtained by fixing j, modifying u only on the annulus $B_{\eta 2^{S-j+1}}(z) \setminus B_{\eta 2^{S-j-1}}(z)$ in such a way that (iii) holds. This is done by a cut-off joining u and its average on an annulus, and estimating the change of energy using a Poincaré inequality. By the scaling properties of the Poincaré inequality, this estimate can be shown to involve a constant C depending only on α and β since the sets in which it is applied are all homothetic. Eventually, De Giorgi's trick consists in choosing an optimal j in (ii), which gives estimate (iv). We refer to [7] for an explicit construction.

2.1. *Lower bound*

Let R_Ω denote the maximum between the diameter of Ω and 1 (this condition is just to have $\log R_\Omega$ non-negative in the following), so that for all $z \in \Omega$ the extension of any $u \in H^1_0(\Omega)$ belongs to $H^1_0(B_{R_\Omega}(z))$. By the estimate $a(x) \geq \alpha$, for all x we have

$$\begin{aligned} m_{\varepsilon,\delta} &\geq \alpha \min\left\{\int_{B_{R_\Omega}(z)} |\nabla u|^2\,dx : u \in H^1_0\big(B_{R_\Omega}(z)\big),\ u = 1 \text{ on } B_\varepsilon(z),\ z \in \Omega\right\} \\ &= \alpha \min\left\{\int_{B_{R_\Omega}} |\nabla u|^2\,dx : u \in H^1_0(B_{R_\Omega}),\ u = 1 \text{ on } B_\varepsilon\right\}. \end{aligned}$$

Using (2), we then get

$$\liminf_{\varepsilon\to 0} |\log \varepsilon| m_{\varepsilon,\delta} \geq 2\pi\alpha,$$

which is the desired lower bound for $\lambda = 0$.

If $\lambda \in (0, 1]$, we choose $0 < \lambda_1 < \lambda$ such that

$$\lim_{\varepsilon\to 0} \frac{\delta}{\varepsilon^{\lambda_1}} = 0. \tag{10}$$

We consider $z \in \Omega$ and $u \in H^1_0(\Omega)$ with $u = 1$ on $B_\varepsilon(z)$, extended to 0 on $\mathbb{R}^2 \setminus \Omega$. We now construct a family of concentric annuli centered at z, to each of which we will apply Lemma 2.3. We first set

$$T = \max\{n \in \mathbb{N} : \varepsilon^{\lambda_1} 2^n \leq R_\Omega\}. \tag{11}$$

Note that the set of such n is not empty as soon as ε is small enough. Moreover, we can explicitly write

$$T = \left\lfloor \frac{\lambda_1 |\log \varepsilon| + \log R_\Omega}{\log 2} \right\rfloor.$$

In the constructions below, we suppose ε small enough so that $T \geq 4$. We fix naturals $M \in (2, T)$ and $N \in \mathbb{N} \cap (0, M)$, and apply Lemma 2.3 $\lfloor T/M \rfloor$ times, with

$$\gamma(x) = a\left(\frac{x}{\delta}\right), \quad \eta = \varepsilon^{\lambda_1}, \quad R = \varepsilon^{\lambda_1} 2^{kM}, \quad r = \varepsilon^{\lambda_1} 2^{(k-1)M}, \quad \text{and} \quad S = kM,$$

obtaining functions $v^k \in H^1(B_{\varepsilon^{\lambda_1}2^{kM}}(z) \setminus \overline{B}_{\varepsilon^{\lambda_1}2^{(k-1)M}}(z))$. Note that we do not apply the lemma to the larger annulus, where we already have the boundary condition $u = 0$ on $\partial B_{\varepsilon^{\lambda_1}2^{(\lfloor T/M \rfloor+1)M}}(z)$.

Since $v^k = u$ on $\partial(B_{\varepsilon^{\lambda_1}2^{kM}}(z) \setminus B_{\varepsilon^{\lambda_1}2^{(k-1)M}}(z))$, we deduce that the function v, defined on Ω by

$$v := \begin{cases} v^k & \text{on } B_{\varepsilon^{\lambda_1}2^{kM}}(z) \setminus \overline{B}_{\varepsilon^{\lambda_1}2^{(k-1)M}}(z),\ k \in \{1, \dots, \lfloor T/M \rfloor\}, \\ u & \text{otherwise,} \end{cases}$$

belongs to $H^1_0(\Omega)$ and $v = 1$ on $B_\varepsilon(z)$.

With a slight abuse of notation with respect to that in Theorem 2.2, we will write

$$F_\delta(u, A) = \int_A a\Big(\frac{x}{\delta}\Big)|\nabla u|^2\,dx$$

for $u \in H^1_0(\Omega)$ and $A \subset \Omega$.

After noting that $\varepsilon^{\lambda_1}2^{(\lfloor T/M \rfloor+1)M} \geq R_\Omega$, we can write

$$\begin{aligned}
F_\delta(v, \Omega) &= F_\delta\big(v, B_{\varepsilon^{\lambda_1}2^{(\lfloor T/M \rfloor+1)M}}(z)\big) \\
&= F_\delta\big(v, B_{\varepsilon^{\lambda_1}}(z) \setminus B_\varepsilon(z)\big) + \sum_{k=1}^{\lfloor T/M \rfloor+1} F_\delta\big(v, B_{\varepsilon^{\lambda_1}2^{kM}}(z) \setminus B_{\varepsilon^{\lambda_1}2^{(k-1)M}}(z)\big) \\
&= F_\delta\big(u, B_{\varepsilon^{\lambda_1}}(z) \setminus B_\varepsilon(z)\big) + \sum_{k=1}^{\lfloor T/M \rfloor} F_\delta\big(v^k, B_{\varepsilon^{\lambda_1}2^{kM}}(z) \setminus B_{\varepsilon^{\lambda_1}2^{(k-1)M}}(z)\big) \\
&\quad + F_\delta\big(u, B_{\varepsilon^{\lambda_1}2^{(\lfloor T/M \rfloor+1)M}}(z) \setminus B_{\varepsilon^{\lambda_1}2^{\lfloor T/M \rfloor M}}(z)\big),
\end{aligned}$$

so that, by claim (iv) of Lemma 2.3, we get

$$F_\delta(v, \Omega) \leq \Big(1 + \frac{C}{N-1}\Big) F_\delta(u, \Omega).$$

Now we make use of the fact that v is a modification of u with the property of being the constant $c_k = c_k^\varepsilon$ on spheres centered in z of radius $\varepsilon^{\lambda_1}2^{kM-j_k}$, where $j_k \in \{1, \dots, N-1\}$ for every $k = 1, \dots, \lfloor T/M \rfloor$ (claim (ii) of Lemma 2.3). We will omit the dependence of c_k on ε as long as this is kept fixed. Moreover, v preserves the conditions $v = u = 1$ on $B_\varepsilon(z)$ and $v = u = 0$ on $\partial B_{\varepsilon^{\lambda_1}2^{(\lfloor T/M \rfloor+1)M}}(z)$. Thus we write

$$\begin{aligned}
(12)\quad & F_\delta\big(v, B_{\varepsilon^{\lambda_1}2^{(\lfloor T/M \rfloor+1)M}}(z)\big) \\
&= F_\delta\big(v, B_{\varepsilon^{\lambda_1}2^{M-j_1}}(z)\big) + \sum_{k=2}^{\lfloor T/M \rfloor} F_\delta\big(v, B_{\varepsilon^{\lambda_1}2^{kM-j_k}}(z) \setminus B_{\varepsilon^{\lambda_1}2^{(k-1)M-j_{k-1}}}(z)\big) \\
&\quad + F_\delta\big(v, B_{\varepsilon^{\lambda_1}2^{(\lfloor T/M \rfloor+1)M}}(z) \setminus B_{\varepsilon^{\lambda_1}2^{\lfloor T/M \rfloor M - j_{\lfloor T/M \rfloor}}}(z)\big).
\end{aligned}$$

We now define functions w^k as follows. The function $w^1 \in H^1(B_{\varepsilon\lambda_1 2^M}(z))$ is defined as

$$w^1(x) = \begin{cases} v(x) & \text{if } x \in B_{\varepsilon\lambda_1 2^{M-j_1}}(z), \\ c_1 & \text{otherwise.} \end{cases}$$

In this way, we get

$$F_\delta\big(w^1, B_{\varepsilon\lambda_1 2^M}(z)\big) = F_\delta\big(v, B_{\varepsilon\lambda_1 2^{M-j_1}}(z)\big).$$

For $k \in \{2, \dots, \lfloor T/M \rfloor + 1\}$, we set

$$A^N_{M,k} := B_{\varepsilon\lambda_1 2^{kM}}(z) \setminus \overline{B}_{\varepsilon\lambda_1 2^{(k-1)M-N}}(z).$$

If $k \in \{2, \dots, \lfloor T/M \rfloor\}$, we define $w^k \in H^1(A^N_{M,k})$ by setting

$$w^k(x) = \begin{cases} c_{k-1} & \text{if } x \in B_{\varepsilon\lambda_1 2^{(k-1)M-j_{k-1}}}(z) \setminus \overline{B}_{\varepsilon\lambda_1 2^{(k-1)M-N}}(z), \\ v(x) & \text{if } x \in B_{\varepsilon\lambda_1 2^{kM-j_k}}(z) \setminus \overline{B}_{\varepsilon\lambda_1 2^{(k-1)M-j_{k-1}}}(z), \\ c_k & \text{if } x \in B_{\varepsilon\lambda_1 2^{kM}}(z) \setminus \overline{B}_{\varepsilon\lambda_1 2^{kM-j_k}}(z). \end{cases}$$

For $k = \lfloor T/M \rfloor + 1$, we define

$$w^{\lfloor T/M \rfloor+1}(x) = \begin{cases} c_{\lfloor T/M \rfloor} & \text{if } x \in B_{\varepsilon\lambda_1 2^{\lfloor T/M \rfloor M-j_{k-1}}}(z) \setminus \overline{B}_{\varepsilon\lambda_1 2^{(k-1)M-N}}(z), \\ v(x) & \text{otherwise.} \end{cases}$$

In this way, we have

$$F_\delta(w^k, A^N_{M,k}) = F_\delta\big(v, B_{\varepsilon\lambda_1 2^{kM-j_k}}(z) \setminus B_{\varepsilon\lambda_1 2^{(k-1)M-j_{k-1}}}(z)\big)$$

for all $k \in \{2, \dots, \lfloor T/M \rfloor + 1\}$.

If we set $A^N_{M,1} = B_{\varepsilon\lambda_1 2^M}(z)$, then we can rewrite (12) as

$$F_\delta\big(v, B_{\varepsilon\lambda_1 2^{(\lfloor T/M \rfloor+1)M}}(z)\big) = \sum_{k=1}^{\lfloor T/M \rfloor+1} F_\delta(w^k, A^N_{M,k}).$$

By the arbitrariness of $z \in \Omega$ and $u \in H^1_0(\Omega)$, after setting $c_{\lfloor T/M \rfloor+1} := 0$ it follows that

$$\begin{aligned} &\Big(1 + \frac{C}{N-1}\Big) m_{\varepsilon,\delta} \\ &\geq \Big(1 + \frac{C}{N-1}\Big) \min\Big\{F_\delta\big(u, B_{\varepsilon\lambda_1 2^{(\lfloor T/M \rfloor+1)M}}(z)\big) : \\ &\qquad\qquad u \in H^1_0\big(B_{\varepsilon\lambda_1 2^{(\lfloor T/M \rfloor+1)M}}(z)\big),\ u = 1 \text{ on } B_\varepsilon(z)\Big\} \end{aligned}$$

$$\geq \min\big\{F_\delta\big(v, B_{\varepsilon^{\lambda_1}2^M}(z)\big) : v \in H^1\big(B_{\varepsilon^{\lambda_1}2^M}(z)\big),\\ v = 1 \text{ on } B_\varepsilon(z),\ v = c_1 \text{ on } \partial B_{\varepsilon^{\lambda_1}2^M}(z)\big\}\\ + \sum_{k=2}^{\lfloor T/M\rfloor+1} \min\big\{F_\delta(v, A^N_{M,k}) : v \in H^1(A^N_{M,k}),\\ v = c_{k-1} \text{ on } \partial B_{\varepsilon^{\lambda_1}2^{(k-1)M-N}}(z),\ v = c_k \text{ on } \partial B_{\varepsilon^{\lambda_1}2^{kM}}(z)\big\}$$

$$= (1-c_1)^2 \min\big\{F_\delta\big(v, B_{\varepsilon^{\lambda_1}2^M}(z)\big) : v \in H^1_0\big(B_{\varepsilon^{\lambda_1}2^M}(z)\big),\ v=1 \text{ on } B_\varepsilon(z)\big\} \tag{13}$$

$$+ \sum_{k=2}^{\lfloor T/M\rfloor+1} (c_{k-1}-c_k)^2 \min\big\{F_\delta(v, A^N_{M,k}) : v \in H^1(A^N_{M,k}),\\ v = 1 \text{ on } \partial B_{\varepsilon^{\lambda_1}2^{(k-1)M-N}}(z),\ v = 0 \text{ on } \partial B_{\varepsilon^{\lambda_1}2^{kM}}(z)\big\}, \tag{14}$$

where we have used the 2-homogeneity of minimum problems in the last equality.

The minimum in (13) can be simply estimated by

$$\alpha \min\bigg\{\int_{B_{\varepsilon^{\lambda_1}2^M}(z)} |\nabla v|^2\,dx : v \in H^1_0\big(B_{\varepsilon^{\lambda_1}2^M}(z)\big),\ v = 1 \text{ on } B_\varepsilon(z)\bigg\}\\ = \alpha\frac{2\pi}{\log(\varepsilon^{(\lambda_1-1)}2^M)} = \frac{2\pi\alpha}{(1-\lambda_1)|\log\varepsilon| + M\log 2}.$$

As for the minima in (14), with fixed $k \in \{2, \dots, \lfloor T/M\rfloor + 1\}$ the change of variable $w(x) = v(z + x\,\varepsilon^{\lambda_1}2^{(k-1)M})$ leads to rewriting the corresponding minimum as

$$\min\big\{F_\delta(v, A^N_{M,k}) : v \in H^1(A^N_{M,k}),\\ v = 1 \text{ on } \partial B_{\varepsilon^{\lambda_1}2^{(k-1)M-N}}(z),\ v = 0 \text{ on } \partial B_{\varepsilon^{\lambda_1}2^{kM}}(z)\big\}\\ = \min\bigg\{\int_{B_{2^M}\setminus B_{2^{-N}}} a\Big(\frac{x}{\delta}\varepsilon^{\lambda_1}2^{(k-1)M} + \tau^\varepsilon_k\Big)\big|\nabla w(x)\big|^2\,dx :\\ w \in H^1(B_{2^M}\setminus \overline{B}_{2^{-N}}),\ w = 1 \text{ on } \partial B_{2^{-N}},\ w = 0 \text{ on } \partial B_{2^M}\bigg\},$$

where $\tau^\varepsilon_k = -\frac{z}{\delta}\varepsilon^{\lambda_1}2^{(k-1)M}$. We can use Theorem 2.2 with $A = B_{2^M}\setminus\overline{B}_{2^{-N}}$, $\eta = \frac{\delta}{\varepsilon^{\lambda_1}2^{(k-1)M}}$, noting that $\eta\to 0$ by (10), and ϕ is any function in $H^1(B_{2^M}\setminus\overline{B}_{2^{-N}})$ such that $\phi = 1$ on $\partial B_{2^{-N}}$ and $\phi = 0$ on ∂B_{2^M}, so that this last minimum is estimated by

$$\sqrt{\alpha\beta}\min\bigg\{\int_{B_{2^M}\setminus B_{2^{-N}}} \big|\nabla w(x)\big|^2\,dx : w \in H^1(B_{2^M}\setminus\overline{B}_{2^{-N}}) :\\ w = 1 \text{ on } \partial B_{2^{-N}},\ v = 0 \text{ on } \partial B_{2^M}\bigg\}\\ = 2\pi\sqrt{\alpha\beta}\frac{1}{(M+N)\log 2}$$

up to an infinitesimal term as $\varepsilon \to 0$. This term can be chosen independent of k since $\eta \leq \frac{\delta}{\varepsilon^{\lambda_1}}2^M$ for all k and $\frac{\delta}{\varepsilon^{\lambda_1}}2^M$ is infinitesimal as $\varepsilon \to 0$.

By the convexity inequality for the square and the fact that

$$\sum_{k=2}^{\lfloor T/M\rfloor+1}(c_{k-1}-c_k)=c_1,$$

we have

$$\sum_{k=2}^{\lfloor T/M\rfloor+1}(c_{k-1}-c_k)^2\geq\frac{c_1^2}{\lfloor T/M\rfloor-1}\geq\frac{Mc_1^2}{T}\geq\frac{c_1^2M\log 2}{\lambda_1|\log\varepsilon|+\log 2},$$

having taken (11) into account.

Gathering the estimates above, we obtain

$$\Big(1+\frac{C}{N-1}\Big)m_{\varepsilon,\delta}$$
$$\geq\frac{2\pi\alpha(1-c_1)^2}{(1-\lambda_1)|\log\varepsilon|+M\log 2}+\Big(\frac{2\pi\sqrt{\alpha\beta}}{(M+N)\log 2}+o_\varepsilon(1)\Big)\frac{c_1^2M\log 2}{\lambda_1|\log\varepsilon|+\log 2}.$$

Reinstating the dependence of c_1 on ε in the notation, we deduce that

$$\Big(1+\frac{C}{N-1}\Big)|\log\varepsilon|m_{\varepsilon,\delta}$$
$$\geq\frac{2\pi\alpha(1-c_1^\varepsilon)^2|\log\varepsilon|}{(1-\lambda_1)|\log\varepsilon|+M\log 2}\Big[\frac{2\pi\sqrt{\alpha\beta}}{(M+N)\log 2}+o_\varepsilon(1)\Big]\frac{(c_1^\varepsilon)^2M\log 2|\log\varepsilon|}{\lambda_1|\log\varepsilon|+\log 2}$$

for every $N<M<T$. We can assume that $c_1^\varepsilon\to c_1$ as $\varepsilon\to 0$. Since $T\to+\infty$ as $\varepsilon\to 0$, we get

$$\Big(1+\frac{C}{N-1}\Big)\liminf_{\varepsilon\to 0}|\log\varepsilon|m_{\varepsilon,\delta}\geq\frac{2\pi\alpha(1-c_1)^2}{(1-\lambda_1)}+\frac{2\pi\sqrt{\alpha\beta}\,c_1^2}{\lambda_1}\frac{M}{M+N},$$

for every N, M natural numbers such that $N<M$. We successively let $M\to+\infty$ and then $N\to+\infty$ obtaining

$$\liminf_{\varepsilon\to 0}|\log\varepsilon|m_{\varepsilon,\delta}\geq\frac{2\pi\alpha(1-c_1)^2}{(1-\lambda_1)}+\frac{2\pi\sqrt{\alpha\beta}\,c_1^2}{\lambda_1}.$$

This expression in c_1 is minimized for

$$c_1=\frac{\alpha\lambda_1}{\alpha\lambda_1+\sqrt{\alpha\beta}(1-\lambda_1)},\tag{15}$$

which gives the minimal value

$$2\pi\frac{\alpha\sqrt{\alpha\beta}}{\lambda_1\alpha+(1-\lambda_1)\sqrt{\alpha\beta}}.\tag{16}$$

We finally observe that our estimate holds true for every $\lambda_1 < \lambda$, thus

$$\liminf_{\varepsilon\to 0} |\log\varepsilon| m_{\varepsilon,\delta} \geq \sup_{0<\lambda_1<\lambda} 2\pi \frac{\alpha\sqrt{\alpha\beta}}{\lambda_1\alpha + (1-\lambda_1)\sqrt{\alpha\beta}} = 2\pi \frac{\alpha\sqrt{\alpha\beta}}{\lambda\alpha + (1-\lambda)\sqrt{\alpha\beta}}, \tag{17}$$

which is the desired lower bound.

2.2. *Construction of optimal capacitary profiles*

We now construct functions (almost) realizing the lower bound (17). The corresponding small balls will be centered in a point where the value of a is α uniformly distant from the boundary of Ω. To that end, for every $\varepsilon > 0$ we consider $z_\varepsilon \in \Omega$ as the center of a square Q_ε of side length $\delta/2$ in which $a(x/\delta) = \alpha$ and such that there exists $R_0 < 1$ independent of ε such that $B_{R_0}(z_\varepsilon) \subset \Omega$. We now subdivide the construction in the case $\lambda = 0$, in which it is trivial, and $\lambda \in (0, 1]$.

If $\lambda = 0$, then we choose

$$u_\varepsilon(x) = \begin{cases} 1 & \text{if } |x - z_\varepsilon| \leq \varepsilon, \\ 1 - \frac{1}{\log\delta - \log\varepsilon} \log\frac{|x-z_\varepsilon|}{\varepsilon} & \text{if } \varepsilon < |x - z_\varepsilon| < \delta, \\ 0 & \text{otherwise.} \end{cases}$$

A direct computation shows that

$$F_\delta(u_\varepsilon, \Omega) = 2\pi\alpha \frac{1}{|\log\varepsilon - \log\delta|},$$

which implies that $\limsup_{\varepsilon\to 0} |\log\varepsilon| m_{\varepsilon,\delta} \leq 2\pi\alpha$, and the claim.

Let $\lambda \in (0, 1]$ and let

$$c := c(\lambda) = \frac{\alpha\lambda}{\alpha\lambda + \sqrt{\alpha\beta}(1-\lambda)}.$$

If $\lambda \in (0, 1)$, this is the optimal value in (15), while for $\lambda = 1$ we have $c = 1$. We consider λ_1 with $0 < \lambda_1 < \lambda$, so that $\delta \ll \varepsilon^{\lambda_1}$ as $\varepsilon \to 0$, and let $u^0_\varepsilon \in H^1(B_{\varepsilon^{\lambda_1}}(z_\varepsilon))$ be defined by

$$u^0_\varepsilon(x) = \begin{cases} 1 & \text{if } |x - z_\varepsilon| \leq \varepsilon, \\ 1 - \frac{1-c}{(1-\lambda_1)|\log\varepsilon|} \log\frac{|x-z_\varepsilon|}{\varepsilon} & \text{if } \varepsilon < |x - z_\varepsilon| \leq \varepsilon^{\lambda_1}. \end{cases}$$

Note that u_ε^0 is the solution of

$$\min\Big\{\int_{B_{\varepsilon^{\lambda_1}}(z_\varepsilon)} \big|\nabla u(x)\big|^2\,dx : u\in H^1\big(B_{\varepsilon^{\lambda_1}}(z_\varepsilon)\big),\\ u=1 \text{ on } B_\varepsilon(z_\varepsilon),\ u=c \text{ on } \partial B_{\varepsilon^{\lambda_1}}(z_\varepsilon)\Big\}.$$

We now define an optimal capacitary profile outside $B_{\varepsilon^{\lambda_1}}(z_\varepsilon)$ using a construction on concentric annuli homothetic to $B_1\setminus \overline{B}_{1/2}$. To that end, for all $\eta>0$, τ_η and F_η as in Theorem 2.2 with $A=B_1\setminus\overline{B}_{1/2}$, we consider the solution w_η of the problem

$$m_\delta=\min\big\{F_\eta(w): w=1 \text{ on } \partial B_{1/2},\ w=0 \text{ on } \partial B_1\big\}.$$

Note that by Theorem 2.2 there exists $\omega:[0,+\infty)\to[0,+\infty)$ increasing and with $\omega(0)=0$ such that

$$\left|m_\delta-\frac{2\pi\sqrt{\alpha\beta}}{\log 2}\right|\le\omega(\eta).$$

We now let $T:=\max\{n\in\mathbb{N}:\varepsilon^{\lambda_1}2^n\le R_0\}$. For $\varepsilon>0$ small enough, T is well defined and

$$T=\left\lfloor\frac{\lambda_1|\log\varepsilon|+\log R_0}{\log 2}\right\rfloor,$$

so that we can assume that $T\ge 2$. We set

$$A_k=B_{\varepsilon^{\lambda_1}2^k}(z_\varepsilon)\setminus\overline{B}_{\varepsilon^{\lambda_1}2^{k-1}}(z_\varepsilon)$$

for all $1\le k\le T$.

For k fixed, let $\eta=\frac{\delta}{\varepsilon^{\lambda_1}2^k}$, let w_η be defined above, and let $u_\varepsilon^k\in H^1(A_k)$ be defined by

$$u_\varepsilon^k(x)=\frac{T-k}{T}c+\frac{c}{T}w_\eta\big((x-z)\varepsilon^{-\lambda_1}2^{-k}\big),$$

so that u_ε^k is the solution of the minimum problem

$$m_\varepsilon^k=\min\Big\{F_\delta(v,B_k): v\in H^1(A_k),\ v=\frac{T-k+1}{T}c \text{ on } \partial B_{\varepsilon^{\lambda_1}2^{k-1}}(z_\varepsilon),\\ v=\frac{T-k}{T}c \text{ on } \partial B_{\varepsilon^{\lambda_1}2^k}(z_\varepsilon)\Big\}=\frac{c^2}{T^2}m_\eta,$$

so that

$$\left|m_\eta^k-\frac{c^2}{T^2}\frac{2\pi\sqrt{\alpha\beta}}{\log 2}\right|\le\frac{c^2}{T^2}\omega(\eta)\le\frac{c^2}{T^2}\omega\Big(\frac{\delta}{\varepsilon^{\lambda_1}}\Big).$$

Define

$$u_\varepsilon := \begin{cases} u_\varepsilon^0 & \text{on } B_{\varepsilon^{\lambda_1}}(z_\varepsilon), \\ u_\varepsilon^k & \text{on } A_k,\ k \in \{1, \dots, T\}, \\ 0 & \text{on } \Omega \setminus B_{\varepsilon^{\lambda_1} 2^T}(z_\varepsilon), \end{cases}$$

which is an admissible function for the problem defining $m_{\varepsilon,\delta}$.

We now proceed with the estimates of $F_\delta(u_\varepsilon, \Omega)$. If $\lambda = 1$, then

$$c = 1 \quad \text{and} \quad F_\delta\big(u_\varepsilon, B_{\varepsilon^{\lambda_1}}(z_\varepsilon)\big) = 0.$$

If $\lambda \in (0, 1)$, let λ_2 with $\lambda < \lambda_2 < 1$. This condition ensures that $\varepsilon^{\lambda_2} \ll \delta \ll \varepsilon^{\lambda_1}$ as $\varepsilon \to 0$, so that we deduce the inclusions

$$B_{\varepsilon^{\lambda_2}}(z_\varepsilon) \subseteq Q_\varepsilon \subseteq B_{\varepsilon^{\lambda_1}}(z_\varepsilon).$$

Hence, we have

$$\begin{aligned} F_\delta\big(u_\varepsilon, B_{\varepsilon^{\lambda_1}}(z_\varepsilon)\big) &= \int_{B_{\varepsilon^{\lambda_2}}(z_\varepsilon)} a\Big(\frac{x}{\delta}\Big)|\nabla u_\varepsilon^0|^2\,dx \\ &\quad + \int_{B_{\varepsilon^{\lambda_1}}(z_\varepsilon)\setminus B_{\varepsilon^{\lambda_2}}(z_\varepsilon)} a\Big(\frac{x}{\delta}\Big)|\nabla u_\varepsilon^0|^2\,dx \\ &\le \alpha \int_{B_{\varepsilon^{\lambda_2}}(z_\varepsilon)} |\nabla u_\varepsilon^0|^2\,dx + \beta \int_{B_{\varepsilon^{\lambda_1}}(z_\varepsilon)\setminus B_{\varepsilon^{\lambda_2}}(z_\varepsilon)} |\nabla u_\varepsilon^0|^2\,dx \\ &= \frac{2\pi(1-c)^2}{|\log\varepsilon|}\Big(\alpha\frac{1-\lambda_2}{(1-\lambda_1)^2} + \beta\frac{\lambda_2-\lambda_1}{(1-\lambda_1)^2}\Big). \end{aligned} \tag{18}$$

As for the remaining part, we can write it as

$$\begin{aligned} \sum_{k=1}^T F_\delta(u_\varepsilon, B_k) &= \Big(\sqrt{\alpha\beta}\frac{2\pi}{\log 2} + o_\varepsilon(1)\Big)\frac{c^2}{T} \\ &\le \Big(\sqrt{\alpha\beta}\frac{2\pi}{\log 2} + o_\varepsilon(1)\Big)\frac{c^2\log 2}{\lambda_1|\log\varepsilon| + \log R_0}. \end{aligned} \tag{19}$$

Combining (18) and (19), we get

$$\begin{aligned} |\log\varepsilon| m_{\varepsilon,\delta} &\le 2\pi(1-c)^2\Big(\alpha\frac{1-\lambda_2}{(1-\lambda_1)^2} + \beta\frac{\lambda_2-\lambda_1}{(1-\lambda_1)^2}\Big) \\ &\quad + \Big(\sqrt{\alpha\beta}\frac{2\pi}{\log 2} + o_\varepsilon(1)\Big)\frac{c^2\log 2|\log\varepsilon|}{\lambda_1|\log\varepsilon| + \log R_0}. \end{aligned}$$

Letting $\varepsilon \to 0$, we then obtain

$$\limsup_{\varepsilon\to 0} |\log\varepsilon| m_{\varepsilon,\delta} \le 2\pi(1-c)^2\Big(\alpha\frac{1-\lambda_2}{(1-\lambda_1)^2} + \beta\frac{\lambda_2-\lambda_1}{(1-\lambda_1)^2}\Big) + 2\pi\sqrt{\alpha\beta}\frac{c^2}{\lambda_1}.$$

If $\lambda = 1$, then $c = 1$, so that the first term in the right-hand side term is 0 and, after letting $\lambda_1 \to 1$, we have

$$\limsup_{\varepsilon\to 0} |\log \varepsilon| m_{\varepsilon,\delta} \leq 2\pi\sqrt{\alpha\beta}.$$

If $\lambda \in (0,1)$, then we let $\lambda_1, \lambda_2 \to \lambda$, so that we get

$$\limsup_{\varepsilon\to 0} |\log \varepsilon| m_{\varepsilon,\delta} \leq 2\pi(1-c)^2\alpha\frac{1}{1-\lambda} + 2\pi\sqrt{\alpha\beta}\frac{c^2}{\lambda}.$$

Recalling (16), we get

$$\limsup_{\varepsilon\to 0} |\log \varepsilon| m_{\varepsilon,\delta} \leq 2\pi\frac{\alpha\sqrt{\alpha\beta}}{\lambda\alpha + (1-\lambda)\sqrt{\alpha\beta}},$$

and the claim.

2.3. *Remarks*

We briefly comment on the case of general quadratic forms, when

$$m_{\varepsilon,\delta} = \min\left\{\int_\Omega \left\langle A\left(\frac{x}{\delta}\right)\nabla u, \nabla u\right\rangle dx : u \in H^1_0(\Omega),\ u = 1 \text{ on } B_\varepsilon(z),\ z \in \Omega\right\},$$

where A is a periodic matrix, which we assume as usual uniformly elliptic and bounded. If A is continuous, then formula (9) rereads

$$\lim_{\varepsilon\to 0} |\log \varepsilon| m_{\varepsilon,\delta} = 2\pi\frac{\alpha'\beta'}{\lambda\alpha' + (1-\lambda)\beta'},$$

where $\alpha' = \min\sqrt{\det A}$ and $\beta' = \sqrt{\det A_{\rm hom}}$, $\langle A_{\rm hom}\xi, \xi\rangle$ being the integrand of the Γ-limit as in Theorem 2.2. The proof is the same, with small changes in the computation, taking balls centered in z_ε minimizing $\sqrt{\det A}$ for the construction and using Theorem 2.2. A minor issue is the fact that, if A or $A_{\rm hom}$ is not a multiple of the identity, the computation in (2) is not exact but must be slightly adapted (e.g., using ellipses in the place of circles).

If A is only measurable, then the formula above holds with α' the essential minimum of $\sqrt{\det A}$, and some extra care must be used as in the analog result in [1]. We may recover the checkerboard case above noting that $A_{\rm hom}$ is equal to $\sqrt{\alpha\beta}$ times the identity, so that $\det A_{\rm hom} = \alpha\beta$. For an extension to any dimension or to general integrands, we cannot use the special structure of quadratic forms and some issues are more technical. We refer to [7] for an asymptotic formula and applications in that case.

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