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**Partial Differential Equations.** – *On the symmetric rearrangement of the gradient of a Sobolev function*, by Vincenzo Amato and Andrea Gentile, communicated on 10 February 2023.

Abstract. – In this paper, we generalize a classical comparison result for solutions to Hamilton–Jacobi equations with Dirichlet boundary conditions, to solutions to Hamilton–Jacobi equations with non-zero boundary trace.

As a consequence, we prove the isoperimetric inequality for the torsional rigidity (with Robin boundary conditions) and for other functionals involving such boundary conditions.

Keywords. – Rearrangements, Robin boundary conditions.

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## 1. Introduction

Let $\Omega$ be a bounded, open, and Lipschitz set and let $u \in W^{1,p}(\Omega)$, for some $p \geq 1$, be a non-negative function.

In this paper, we deal with the problem of comparing a function $u \in W^{1,p}(\Omega)$ with a radial function having the modulus of the gradient equi-rearranged with $|\nabla u|$. Hence, we aim to extend the results presented by Giarrusso and Nunziante in [11] to a more general setting.

Throughout this article, $|\cdot|$ will denote both the $n$-dimensional Lebesgue measure and the $(n-1)$-dimensional Hausdorff measure; the meaning will be clear by the context.

If $A$ is a bounded and open set with the same measure as $\Omega$, we say that a function $f^\star \in L^p(A)$ is equi-rearranged to $f \in L^p(\Omega)$ if they have the same distribution function; that is clear by the following definition.

Definition 1.1. Let $f : \Omega \to \mathbb{R}$ be a measurable function; the *distribution function* of $f$ is the function

$$\mu_f : [0, +\infty[ \to [0, +\infty[$$

defined by

$$\mu_f(t) = \left|\left\{x \in \Omega : |f(x)| > t\right\}\right|.$$

In order to state our results, we recall some definitions.

DEFINITION 1.2. Let $f:\Omega\to\mathbb{R}$ be a measurable function:

- the *decreasing rearrangement* of $f$, denoted by $f^*$, is the distribution function of $\mu_f$. Moreover, we can write

$$f^*(s)=\inf\{t\geq 0\mid \mu_f(t)<s\};$$

- the *increasing rearrangement* of $f$ is defined as

$$f_*(s)=f^*\big(|\Omega|-s\big);$$

- the *spherically symmetric decreasing rearrangement* of $f$, defined in $\Omega^\sharp$, i.e. the ball centered at the origin with the same measure as $\Omega$, is the function

$$f^\sharp(x)=f^*\big(\omega_n|x|^n\big),$$

  where $\omega_n$ is the measure of the $n$-dimensional unit-ball of $\mathbb{R}^n$;
- the *spherically symmetric increasing rearrangement* of $f$, defined in $\Omega^\sharp$, is

$$f_\sharp(x)=f_*\big(\omega_n|x|^n\big).$$

Clearly, we can construct several rearrangements of a given function $f$, but the one we will refer to is the spherically symmetric increasing rearrangement defined in $\Omega^\sharp$.

The starting point of our work, and many others, is [11, Theorem 2.2].

THEOREM 1.1. *Let $p\geq 1$, $f:\Omega\to\mathbb{R}$, $H:\mathbb{R}^n\to\mathbb{R}$ be measurable non-negative functions and let $K:[0,+\infty)\to[0,+\infty)$ be a strictly increasing real-valued function such that*

$$0\leq K\big(|y|\big)\leq H(y)\quad \forall y\in\mathbb{R}^n \text{ and } K^{-1}(f)\in L^p(\Omega).$$

*Let $v\in W_0^{1,p}(\Omega)$ be a function that satisfies*

$$\begin{cases} H(\nabla v)=f(x) & \text{a.e. in } \Omega,\\ v=0 & \text{on } \partial\Omega.\end{cases}$$

*Then, denoting by $\bar v$ the unique decreasing spherically symmetric solution to*

$$\begin{cases} K\big(|\nabla \bar v|\big)=f_\sharp(x) & \text{a.e. in } \Omega^\sharp,\\ \bar v=0 & \text{on } \partial\Omega^\sharp,\end{cases}$$

*it holds that*

$$\|v\|_{L^1(\Omega)}\leq\|\bar v\|_{L^1(\Omega^\sharp)}. \tag{1.1}$$

They give also a similar result for the spherically symmetric decreasing rearrangement of the gradient, with an $L^\infty$ comparison.

In recent decades, many authors studied this kind of problems, in particular, in [4] Alvino, Lions, and Trombetti proved the existence of a spherically symmetric rearrangement of the gradient of $v$ which gives an $L^q$ comparison as in (1.1) for a fixed $q$.

Moreover, Cianchi in [8] gives a characterization of such rearrangement; clearly, the rearrangement found by Cianchi is different both from the spherically symmetric increasing and decreasing rearrangement if $q \in (1, \infty)$.

Furthermore, in [9, 10] the authors studied the optimization of the norm of a Sobolev function in the class of functions with fixed rearrangement of the gradient.

Incidentally, let us mention that the case where the $L^{q,1}$ Lorentz norm (see Section 2 for its definition) takes the place of the $L^q$ norm in (1.1) has been studied in [15]. In particular, he stated the following theorem.

THEOREM 1.2. *Let $u$ be a real-valued function defined in $\mathbb{R}^n$. Suppose that $u$ is nice enough – e.g. Lipschitz continuous – and the support of $u$ has finite measure. Let $M$ and $V$ denote the distribution function of $|\nabla u|$ and the measure of the support of $u$, respectively.*

*Let $v$ be the real-valued function defined in $\mathbb{R}^n$ that satisfies the following conditions:*

(1) *$|\nabla v|$ is a rearrangement of $|\nabla u|$;*

(2) *the support of $v$ has the same measure of the support of $u$;*

(3) *$v$ is radially decreasing and $|\nabla v|$ is radially increasing.*

*Then,*

$$\|u\|_{L^{p,1}(\Omega)} \le \|v\|_{L^{p,1}(\Omega^\sharp)} \quad \text{if } n = 1 \text{ or } 0 < p \le \frac{n}{n-1};$$

*furthermore,*

$$\|v\|_{L^{p,1}(\Omega^\sharp)} = \frac{p^2}{\omega_n^{\frac{1}{n}}(n+p)} \int_0^\infty \left[V^{\frac{1}{p}+\frac{1}{n}} - \left(V - M(t)\right)^{\frac{1}{p}+\frac{1}{n}}\right] dt.$$

On the other hand, the problem of studying the rearrangement of the Laplacian has been widely studied by several authors. The bibliography is extensive; for the sake of completeness, let us recall some of the works: [14] for the Dirichlet boundary conditions, [1, 2, 5] for the Robin conditions.

As we already said, we focus on the case in which the functions do not vanish on the boundary. Our main theorem is the following.

THEOREM 1.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded, open, and Lipschitz set and let $u \in W^{1,p}(\Omega)$ be a non-negative function. If we denote by $\Omega^\sharp$ the ball centered at the origin with the* *same measure as* $\Omega$*, then there exists a non-negative function* $u^\star \in W^{1,p}(\Omega^\sharp)$ *that satisfies*

$$\begin{cases} |\nabla u^\star| = |\nabla u|_\sharp(x) & \text{a.e. in } \Omega^\sharp, \\ u^\star = \dfrac{\int_{\partial\Omega} u \, d\mathcal{H}^{n-1}}{|\partial\Omega^\sharp|} & \text{on } \partial\Omega^\sharp \end{cases} \tag{1.2}$$

*and such that*

$$\|u\|_{L^1(\Omega)} \leq \|u^\star\|_{L^1(\Omega^\sharp)}. \tag{1.3}$$

Remark 1.4. By the explicit expression of $u^*$ on the boundary and the Hölder inequality, we can estimate the $L^p$ norm of the trace:

$$\begin{aligned} &|\partial\Omega^\sharp|^{p-1} \int_{\partial\Omega^\sharp} (u^\star)^p \, d\mathcal{H}^{n-1} \\ &\quad = \left( \int_{\partial\Omega} u \, d\mathcal{H}^{n-1} \right)^p \leq |\partial\Omega|^{p-1} \int_{\partial\Omega} u^p \, d\mathcal{H}^{n-1} \quad \forall p \geq 1. \end{aligned} \tag{1.4}$$

This result allows us to compare solutions to PDE with Robin boundary conditions with solutions to their symmetrized.

Precisely, we are able to compare solutions to

$$\begin{cases} -\Delta u = 1 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} + \beta|\partial\Omega| u = 0 & \text{on } \partial\Omega \end{cases}$$

with the solution to

$$\begin{cases} -\Delta v = 1 & \text{in } \Omega^\sharp, \\ \dfrac{\partial v}{\partial \nu} + \beta|\partial\Omega^\sharp| v = 0 & \text{on } \partial\Omega^\sharp. \end{cases}$$

In particular, we get the following.

Corollary 1.5. *Let* $\beta > 0$ *and let* $\Omega \subset \mathbb{R}^n$ *be a bounded, open, and Lipschitz set. If we denote by* $\Omega^\sharp$ *the ball centered at the origin with the same measure as* $\Omega$*, it holds that*

$$T(\Omega, \beta) \geq T(\Omega^\sharp, \beta),$$

*where*

$$T(\Omega, \beta) = \inf_{w \in W^{1,2}(\Omega)} \frac{\int_\Omega |\nabla w|^2 \, dx + \beta|\partial\Omega| \int_{\partial\Omega} w^2 \, d\mathcal{H}^{n-1}}{(\int_\Omega w \, dx)^2} \quad \text{for } w \in W^{1,2}(\Omega).$$

The paper is organized as follows. In Section 2, we recall some basic notions, definitions, and classical results and we prove Theorem 1.3. Eventually, Section 3 is dedicated to the application to the Robin torsional rigidity and in Section 4 we get a comparison between Lorentz norm of $u$ and $u^\star$.

## 2. Notations, preliminaries, and proof of the main result

Observe that obviously $\forall\, p \geq 1$

$$\begin{aligned}\|f\|_{L^p(\Omega)} &= \|f^*\|_{L^p([0,|\Omega|])} = \|f^\sharp\|_{L^p(\Omega^\sharp)} \\ &= \|f_*\|_{L^p([0,|\Omega|])} = \|f_\sharp\|_{L^p(\Omega^\sharp)};\end{aligned}$$

moreover, the Hardy–Littlewood inequalities hold true:

$$\int_\Omega |f(x)g(x)|\,dx \leq \int_0^{|\Omega|} f^*(s)g^*(s)\,ds = \int_{\Omega^\sharp} f^\sharp(x)g^\sharp(x)\,dx,$$

$$\int_{\Omega^\sharp} f^\sharp(x)g_\sharp(x)\,dx = \int_0^{|\Omega|} f^*(s)g_*(s)\,ds \leq \int_\Omega \big|f(x)g(x)\big|\,dx.$$

Finally, the operator which assigns to a function its symmetric decreasing rearrangement is a contraction in $L^p$ (see [7]) i.e.

$$\|f^* - g^*\|_{L^p([0,|\Omega|])} \leq \|f - g\|_{L^p(\Omega)}. \tag{2.1}$$

One can find more results and details about rearrangements for instance in [13] and in [15].

Other powerful tools are the pseudo-rearrangements. Let $u \in W^{1,p}(\Omega)$ and let $f \in L^1(\Omega)$, as in [3] $\forall s \in [0, |\Omega|]$, there exists a subset $D(s) \subseteq \Omega$ such that

(1) $|D(s)| = s$;

(2) $D(s_1) \subseteq D(s_2)$ if $s_1 < s_2$;

(3) $D(s) = \{x \in \Omega \mid |u(x)| > t\}$ if $s = \mu(t)$.

So the function

$$\int_{D(s)} f(x)\,dx$$

is absolutely continuous, therefore there exists a function $F$ such that

$$\int_0^s F(t)\,dt = \int_{D(s)} f(x)\,dx. \tag{2.2}$$

We will use the following property [3, Lemma 2.2].

LEMMA 2.1. *Let $f \in L^p$ for $p > 1$ and let $D(s)$ be a family described above. If $F$ is defined as in (2.2), then there exists a sequence $\{F_k\}$ such that $F_k$ has the same rearrangement as $f$ and*

$$F_k \rightharpoonup F \quad \text{in } L^p\big([0, |\Omega|]\big).$$

*If $f \in L^1$, it follows that*

$$\lim_k \int_0^{|\Omega|} F_k(s) g(s)\, ds = \int_0^{|\Omega|} F(s) g(s)\, ds$$

*for each function $g \in \mathrm{BV}([0, |\Omega|])$.*

Moreover, for sake of completeness, we will recall the definition of the Lorentz norm.

DEFINITION 2.1. Let $\Omega \subseteq \mathbb{R}^n$ be a measurable set, $0 < p < +\infty$, and $0 < q < +\infty$. Then, a function $g$ belongs to the Lorentz space $L^{p,q}(\Omega)$ if

$$\|g\|_{L^{p,q}(\Omega)} = \left(\int_0^{+\infty} \left[t^{\frac{1}{p}} g^*(t)\right]^q \frac{dt}{t}\right)^{\frac{1}{q}} < +\infty.$$

Let us notice that for $p = q$ the Lorentz space $L^{p,p}(\Omega)$ coincides with the Lebesgue space $L^p(\Omega)$ by Cavalieri's principle.

Let us now prove the main theorem.

PROOF OF THEOREM 1.3. Let us consider $\varepsilon$, $\delta := \delta_\varepsilon$, and the sets

$$\begin{aligned} \Omega_\varepsilon &= \{x \in \mathbb{R}^n | d(x, \Omega) < \varepsilon\} & \Sigma_\varepsilon &= \Omega_\varepsilon \setminus \Omega, \\ \Omega_\varepsilon^\sharp &= \{x \in \mathbb{R}^n | d(x, \Omega^\sharp) < \delta\} & \Sigma_\varepsilon^\sharp &= \Omega_\varepsilon^\sharp \setminus \Omega^\sharp, \\ |\Omega_\varepsilon| &= |\Omega_\varepsilon^\sharp| & |\Sigma_\varepsilon| &= |\Sigma_\varepsilon^\sharp|, \end{aligned}$$

where, since $|\Sigma_\varepsilon|/\varepsilon \to |\partial\Omega|$ and $|\Sigma_\varepsilon^\sharp|/\delta \to |\partial\Omega^\sharp|$ as $\varepsilon \to 0$, we have

$$\lim_{\varepsilon \to 0} \frac{\delta}{\varepsilon} = \frac{|\partial\Omega|}{|\partial\Omega^\sharp|}.$$

Let $d(\cdot, \Omega)$ be defined as follows:

$$d(x, \Omega) := \inf_{y \in \Omega} |x - y|.$$

Then, we divide the proof into four steps.

Step 1. First of all, we assume $\Omega$ with $C^{1,\alpha}$ boundary, $u \in W^{1,\infty}(\Omega)$, and $u \geq \sigma > 0$ in $\Omega$.

So we can consider the following "linear" extension of $u$, $u_\varepsilon$ in $\Omega_\varepsilon$:

$$u_\varepsilon(x) = u\big(p(x)\big)\left(1 - \frac{d(x,\partial\Omega)}{\varepsilon}\right) \quad \forall x \in \Omega_\varepsilon \setminus \Omega,$$

where $p(x)$ is the projection of $x$ on $\partial\Omega$ (for $\varepsilon$ sufficiently small, this definition is well posed since $\Omega$ is smooth; see [12]). The function $u_\varepsilon$ has the following properties:

(a) $u_\varepsilon|_\Omega = u$,

(b) $u_\varepsilon = 0$ on $\partial\Omega_\varepsilon$,

(c) $\|\nabla u_\varepsilon\|_{L^\infty(\Omega)} \le |\nabla u_\varepsilon|(y)\ \forall y \in \Sigma_\varepsilon$ for $\varepsilon$ sufficiently small,

(d) $\lim_{\varepsilon\to 0^+} \int_{\Sigma_\varepsilon} |\nabla u_\varepsilon|\, dx = \int_{\partial\Omega} u\, d\mathcal{H}^{n-1}$.

Properties (a) and (b) follow immediately by the definition of $u_\varepsilon$, while (c) is a consequence of the regularity of $u$. Property (d) can be obtained by an easy calculation; indeed

$$\nabla u_\varepsilon(x) = \nabla\big(u\big(p(x)\big)\big)\left[1 - \frac{d(x,\partial\Omega)}{\varepsilon}\right] - u\big(p(x)\big)\frac{\nabla d(x,\partial\Omega)}{\varepsilon}.$$

For the first term, we can notice that

$$\int_{\Sigma_\varepsilon} \big|\nabla\big(u\big(p(x)\big)\big)\big|\left[1 - \frac{d(x,\partial\Omega)}{\varepsilon}\right] dx \le L \int_{\Sigma_\varepsilon} dx = L|\Sigma_\varepsilon|,$$

where $L$ is the $L^\infty$ norm of $\nabla u(p(x))$. Now we deal with the second term and, keeping in mind that $|\nabla d| = 1$ and using coarea formula, we have

$$\lim_{\varepsilon\to 0^+} \int_{\Sigma_\varepsilon} |\nabla u_\varepsilon|\, dx = \lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon} \int_{\Sigma_\varepsilon} u\big(p(x)\big)\, dx = \lim_{\varepsilon\to 0^+} \int_0^\varepsilon dt \int_{\Gamma_t} (u \circ p)\, d\mathcal{H}^{n-1},$$

where $\Gamma_t = \{x \in \Sigma_\varepsilon \mid d(x,\partial\Omega) = \varepsilon\}$. By continuity of $u$ and Lebesgue differentiation theorem, we get

$$\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon} \int_0^\varepsilon dt \int_{\Gamma_t} u \circ p\, d\mathcal{H}^{n-1} = \int_{\Gamma_0} (u \circ p)\, d\mathcal{H}^{n-1} = \int_{\partial\Omega} u\, d\mathcal{H}^{n-1}$$

that proves property (d).

For every $\varepsilon > 0$, we consider the problem

$$\begin{cases} |\nabla v_\varepsilon|(x) = |\nabla u_\varepsilon|_\sharp(x) & \text{in } \Omega_\varepsilon^\sharp, \\ v_\varepsilon = 0 & \text{on } \partial\Omega_\varepsilon^\sharp, \end{cases} \tag{2.3}$$

and by Theorem 1.1 it holds that

$$\|u_\varepsilon\|_{L^1(\Omega_\varepsilon)} \le \|v_\varepsilon\|_{L^1(\Omega_\varepsilon^\sharp)}. \tag{2.4}$$

Moreover, there exists $\overline{\varepsilon}$ such that for every $\varepsilon \leq \overline{\varepsilon}$

$$|\nabla v_\varepsilon|(x) = |\nabla u_\varepsilon|_\sharp(x) = |\nabla u|_\sharp(x) \quad \forall x \in \Omega^\sharp. \tag{2.5}$$

We can see $u_\varepsilon$ as a $W^{1,1}(\Omega_{\overline{\varepsilon}})$ function and we have

$$\int_{\Omega^\sharp_{\overline{\varepsilon}}} |\nabla v_\varepsilon| = \int_{\Omega_{\overline{\varepsilon}}} |\nabla u_\varepsilon| = \int_\Omega |\nabla u| + \int_{\Sigma_\varepsilon} |\nabla u_\varepsilon| \leq \|\nabla u\|_{L^1(\Omega)} + 2\|u\|_{L^1(\partial\Omega)} \tag{2.6}$$

by property (d).

Finally, by Poincarè and (2.6), there exists a constant $0 < C = C(n, \Omega)$ such that

$$\|v_\varepsilon\|_{W^{1,1}(\Omega^\sharp_{\overline{\varepsilon}})} \leq C\|\nabla v_\varepsilon\|_{L^1(\Omega_{\overline{\varepsilon}})} \leq C(n, \Omega)\|u\|_{W^{1,1}(\Omega)}.$$

Therefore, up to a subsequence, there exists a limit function $u^\star \in \mathrm{BV}(\Omega^\sharp_{\overline{\varepsilon}})$ such that [6, Proposition 3.13]

$$v_\varepsilon \to u^\star \text{ in } L^1(\Omega^\sharp_{\overline{\varepsilon}}) \quad \nabla v_\varepsilon \overset{*}{\rightharpoonup} \nabla u^\star \text{ in } \Omega;$$

namely,

$$\lim_{\varepsilon\to 0} \int_{\Omega^\sharp_{\overline{\varepsilon}}} \varphi \, d\nabla v_\varepsilon = \int_{\Omega^\sharp_{\overline{\varepsilon}}} \varphi \, d\nabla u^\star \quad \forall \varphi \in C_0(\Omega, \mathbb{R}^n).$$

Our aim is to show that $u^\star$ satisfies properties (1.2), (1.3), and (1.4).

Concerning (1.2), then $|\nabla u^\star| = |\nabla u|_\sharp$ follows from (2.5).

To find the value of $u^\star$ at the boundary, we observe that, from (2.3) and (2.5), we have

$$\int_{\Sigma_\varepsilon} |\nabla u_\varepsilon| = \int_{\Sigma^\sharp_\varepsilon} |\nabla v_\varepsilon|.$$

Now, for $t > 0$ setting $\Gamma_t = \{d(x, \Omega) = t\}$, $\Gamma^\sharp_t = \{d(x, \Omega^\sharp) = t\}$, $r = (\frac{|\Omega|}{\omega_n})^{\frac{1}{n}}$, and recalling that $v_\varepsilon$ is radially symmetric, we have

$$\int_{\Sigma^\sharp_\varepsilon} |\nabla v_\varepsilon| = \int_r^{r+\delta} \int_{\Gamma^\sharp_t} |\nabla v_\varepsilon| \, d\mathcal{H}^{n-1} \, dt = |\Gamma^\sharp_t| \int_r^{r+\delta} -v'_\varepsilon |\Gamma^\sharp_t| \, dt = |\Gamma^\sharp_t| \, v_\varepsilon(r).$$

Therefore, by monotonicity of $|\Gamma^\sharp_t|$ we have

$$|\Gamma^\sharp_r| v_\varepsilon(r) \leq \int_r^{r+\delta} \left(-v'_\varepsilon(t)|\Gamma^\sharp_t|\right) dt \leq |\Gamma^\sharp_{r+\delta}| v_\varepsilon(r),$$

and since

$$|\Gamma^\sharp_r| v_\varepsilon(r) = \int_{\partial\Omega^\sharp} v_\varepsilon \, d\mathcal{H}^{n-1}$$

using the fact that $v_\varepsilon \to v$ in $L^1(\Omega)$, $\nabla v_\varepsilon = \nabla u$ in $\Omega$ and the continuity embedding of $W^{1,1}(\Omega)$ in $L^1(\Omega)$, in the end we have

$$\int_{\Sigma_\varepsilon^\sharp} |\nabla v_\varepsilon| \to \int_{\partial\Omega^\sharp} u^\star \, d\mathcal{H}^{n-1}.$$

Using property (d), we obtain

$$\int_{\partial\Omega} u \, d\mathcal{H}^{n-1} = \int_{\partial\Omega^\sharp} u^\star \, d\mathcal{H}^{n-1}.$$

In the end, we have that for $u^\star$ it holds that

$$\begin{cases} |\nabla u^\star| = |\nabla u|_\sharp & \text{in } \Omega^\sharp, \\ u^\star = \dfrac{\int_{\partial\Omega} u \, d\mathcal{H}^{n-1}}{|\partial\Omega^\sharp|} & \text{on } \partial\Omega^\sharp \end{cases}$$

that proves (1.2).

Furthermore, by

$$\|u_\varepsilon\|_{L^1(D)} \to \|u\|_{L^1(D)} \quad \text{and} \quad \|v_\varepsilon\|_{L^1(D^\sharp)} \to \|u^\star\|_{L^1(D^\sharp)},$$

we can pass to the limit $\varepsilon \to 0$ in (2.4) and we get

$$\|u\|_{L^1(\Omega)} \leq \|u^\star\|_{L^1(\Omega^\sharp)}$$

that proves (1.3).

Step 2. Now, we remove the extra-assumption $u \geq \delta > 0$ defining

$$u_\sigma := u + \sigma.$$

Then, $u_\sigma$ is strictly positive in $\Omega$ and we can apply the previous result: there exists a function $v_\sigma$ in $\Omega^\sharp$ such that

$$\begin{cases} |\nabla v_\sigma| = |\nabla u_\sigma|_\sharp = |\nabla u|_\sharp & \text{a.e. in } \Omega^\sharp, \\ v_\sigma = \dfrac{\int_{\partial\Omega} u_\sigma \, d\mathcal{H}^{n-1}}{|\partial\Omega^\sharp|} = \dfrac{\int_{\partial\Omega} u \, d\mathcal{H}^{n-1}}{|\partial\Omega^\sharp|} + \sigma \dfrac{|\partial\Omega|}{|\partial\Omega^\sharp|} & \text{on } \partial\Omega^\sharp, \end{cases}$$

and

$$\|u_\sigma\|_{L^1(\Omega)} \leq \|v_\sigma\|_{L^1(\Omega^\sharp)}. \tag{2.7}$$

If we define

$$u^\star := v_\sigma - \sigma \frac{|\partial\Omega|}{|\partial\Omega^\sharp|},$$

then $u^\star$ solves

$$\begin{cases} |\nabla u^\star| = |\nabla u|_\sharp & \text{in } \Omega^\sharp, \\ u^\star = \dfrac{\int_{\partial\Omega} u \, d\mathcal{H}^{n-1}}{|\Omega^\sharp|} & \text{on } \partial\Omega^\sharp. \end{cases}$$

Sending $\sigma \to 0$ in (2.7), we have

$$\|u\|_{L^1(\Omega)} \leq \|u^\star\|_{L^1(\Omega^\sharp)}.$$

Step 3. Now we remove the assumption on the regularity of $\Omega$.

Let $\Omega$ be a bounded, open, and Lipschitz set and let $u \in W^{1,\infty}(\Omega)$. Then, there exists a sequence $\{\Omega_k\} \subset \mathbb{R}^n$ of open set with $C^2$ boundary such that $\Omega \subset \Omega_k$, $\forall k \in \mathbb{N}$ (for instance you can mollify $\chi_\Omega$ and take a suitable superlevel set) and

$$|\Omega_k \triangle \Omega| \to 0, \quad \mathcal{H}^{n-1}(\partial\Omega_k) \to \mathcal{H}^{n-1}(\partial\Omega) \quad \text{for } k \to +\infty.$$

Let $\tilde{u}$ be an extension of $u$ in $\mathbb{R}^n$ such that

$$\tilde{u}|_\Omega \equiv u, \quad \|\tilde{u}\|_{W^{1,\infty}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,\infty}(\Omega)}.$$

We define

$$u_k = \tilde{u}\chi_{\Omega_k}$$

and clearly $u_k = u$ in $\Omega$. By the previous step, we can construct $u_k^\star \in W^{1,\infty}(\Omega_k^\sharp)$ such that it is radial, $|\nabla u_k|_* = |\nabla u_k^\star|_*$, and

$$\|u_k\|_{L^1(\Omega_k)} \leq \|u_k^\star\|_{L^1(\Omega_k^\sharp)}, \tag{2.8}$$

$$\int_{\partial\Omega_k} u_k \, d\mathcal{H}^{n-1} = \int_{\partial\Omega_k^\sharp} u_k^\star \, d\mathcal{H}^{n-1}. \tag{2.9}$$

Therefore, since $\|u_k\|_{W^{1,p}(\Omega_k)} \leq M$, for all $p$, the sequence $\{u_k^\star\}$ is equibounded in $W^{1,p}(\Omega^\sharp)$ and it has a subsequence which converges strongly in $L^p$ and weakly in $W^{1,p}$ to a function $w$.

Let us prove that $|\nabla u|$ and $|\nabla w|$ have the same rearrangement:

$$\limsup_k \left\| |\nabla u_k^\star| - |\nabla u|_\sharp \right\|_{L^p(\Omega^\sharp)} \leq \lim_k \left\| (f_k)_\sharp - f_\sharp \right\|_{L^p(\mathbb{R}^n)},$$

where

$$f(x) = \begin{cases} |\nabla \tilde{u}| & \text{in } \Omega, \\ \|\nabla \tilde{u}\|_{L^\infty(\mathbb{R}^n)} & \text{in } \mathbb{R}^n \setminus \Omega \end{cases} \quad \text{and} \quad f_k = \begin{cases} |\nabla u_k| & \text{in } \Omega_k, \\ \|\nabla \tilde{u}\|_{L^\infty(\mathbb{R}^n)} & \text{in } \mathbb{R}^n \setminus \Omega_k. \end{cases}$$

So using (2.1), we have

$$\|(f_k)_\sharp - f_\sharp\|_{L^p(\mathbb{R}^n)} \le \|f_k - f\|_{L^p(\mathbb{R}^n)} = \|f_k - f\|_{L^p(\Omega_k\setminus\Omega)}$$
$$\le 2\|\nabla\tilde{u}\|_{L^\infty(\mathbb{R}^n)}|\Omega_k \setminus \Omega|$$

that tends to 0 as $k \to +\infty$ by the fact that $|\Omega_k \Delta \Omega| \to 0$.

Hence, the functions $\nabla w$ and $\nabla u$ have the same rearrangement by the uniqueness of the weak limit in $\Omega^\sharp$.

In the end, passing to limit $k \to +\infty$ in (2.8) and (2.9), we have

$$\|u\|_{L^1(\Omega)} \le \|w\|_{L^1(\Omega^\sharp)},$$
$$\int_{\partial\Omega} u \, d\mathcal{H}^{n-1} = \int_{\partial\Omega^\sharp} w \, d\mathcal{H}^{n-1}.$$

Hence, $w = u^\star$.

Step 4. Finally, we proceed by removing the assumption $u \in W^{1,\infty}(\Omega)$.

If $u \in W^{1,p}(\Omega)$, by Meyers–Serrin theorem, there exists a sequence $\{u_k\} \subset C^\infty(\Omega) \cap W^{1,p}(\Omega)$ such that $u_k \to u$ in $W^{1,p}(\Omega)$. We can apply a previous step to obtain $u_k^\star \in W^{1,\infty}(\Omega^\sharp)$ such that $|\nabla u_k|$ and $|\nabla u_k^\star|$ are equally distributed and

$$\|u_k\|_{L^1(\Omega)} \le \|u_k^\star\|_{L^1(\Omega^\sharp)} \qquad \forall k \in \mathbb{N}, \tag{2.10}$$

$$\int_{\partial\Omega} u_k \, d\mathcal{H}^{n-1} = \int_{\partial\Omega^\sharp} u_k^\star \, d\mathcal{H}^{n-1} \quad \forall k \in \mathbb{N}. \tag{2.11}$$

Arguing as the previous step, there exists a function $w$ such that up to a subsequence

$$u_k^\star \to w \text{ in } L^p(\Omega), \quad \nabla u_k^\star \rightharpoonup \nabla w \text{ in } L^p(\Omega;\mathbb{R}^n),$$

and $|\nabla w|$ has the same rearrangement as $|\nabla u|$.

Finally, sending $k \to +\infty$ in (2.10) and (2.11), we have

$$\|u\|_{L^1(\Omega)} \le \|w\|_{L^1(\Omega^\sharp)},$$
$$\int_{\partial\Omega} u \, d\mathcal{H}^{n-1} = \int_{\partial\Omega^\sharp} w \, d\mathcal{H}^{n-1}.$$

Hence, $w = u^\star$. ■

## 3. An application to torsional rigidity

Let $\beta > 0$, let $\Omega \subset \mathbb{R}^n$ be a bounded and open set with Lipschitz boundary, and let us consider the functional

$$\mathcal{F}_\beta(\Omega, w) = \frac{\int_\Omega |\nabla w|^2 \, dx + \beta|\partial\Omega| \int_{\partial\Omega} w^2 \, d\mathcal{H}^{n-1}}{(\int_\Omega w \, dx)^2} \qquad w \in W^{1,2}(\Omega)$$

and the associate minimum problem

$$T(\Omega,\beta) = \min_{w\in W^{1,2}(\Omega)} \mathcal{F}_\beta(w).$$

The minimum $u$ is a weak solution to

$$\begin{cases} -\Delta u = 1 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} + \beta|\partial\Omega| u = 0 & \text{on } \partial\Omega. \end{cases}$$

Our aim is to compare $T(\Omega,\beta)$ with

$$\begin{aligned} T(\Omega^\sharp,\beta) &:= \min_{v\in W^{1,2}(\Omega)} \mathcal{F}_{\Omega,\beta}(v) \\ &= \min_{v\in W^{1,2}(\Omega)} \frac{\int_{\Omega^\sharp} |\nabla v|^2\,dx + \beta|\partial\Omega^\sharp| \int_{\partial\Omega^\sharp} v^2\,d\mathcal{H}^{n-1}}{(\int_{\Omega^\sharp} v\,dx)^2}, \end{aligned}$$

where the minimum is a weak solution to

$$\begin{cases} -\Delta z = 1 & \text{in } \Omega^\sharp, \\ \dfrac{\partial z}{\partial \nu} + \beta|\partial\Omega^\sharp|\, z = 0 & \text{on } \partial\Omega^\sharp. \end{cases}$$

PROOF OF COROLLARY 1.5. Let $w \in W^{1,p}(\Omega)$. By Theorem 1.3 and Remark 1.4, there exists $w^\star \in W^{1,\infty}(\Omega^\sharp)$ radial such that

$$\int_\Omega |\nabla w|^2\,dx = \int_{\Omega^\sharp} |\nabla w^\star|^2\,dx,$$
$$\int_\Omega |w|\,dx \le \int_{\Omega^\sharp} |w^\star|\,dx,$$
$$|\partial\Omega^\sharp| \int_{\partial\Omega^\sharp} (w^\star)^2 \le |\partial\Omega| \int_{\partial\Omega} w^2.$$

Therefore,

$$\mathcal{F}_\beta(w) \ge \mathcal{F}_\beta(w^\star).$$

Passing to the infimum on the right-hand side and successively to the left-hand side, we obtain

$$T(\Omega,\beta) \ge T(\Omega^\sharp,\beta). \qquad \blacksquare$$

REMARK 3.1. We highlight that all the arguments work also in the non-linear case, where the functional

$$\mathcal{F}_{\beta,p}(w) = \frac{\int_\Omega |\nabla w|^p\,dx + \beta|\partial\Omega|^{p-1} \int_{\partial\Omega} w^p\,d\mathcal{H}^{n-1}}{(\int_\Omega w\,dx)^p} \quad \text{for } w \in W^{1,p}(\Omega)$$

is considered.

## 4. A WEIGHTED $L^1$ COMPARISON

Let us check how to extend the result by [15] to the case of function non-vanishing on the boundary.

THEOREM 4.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded, open, and Lipschitz set. Let $f \in L^\infty(\Omega)$ be a function such that*

$$f^*(t) \geq \left(1 - \frac{1}{n}\right)\frac{1}{t}\int_0^t f^*(s)\,ds \quad \forall t \in \big[0, |\Omega|\big]. \tag{4.1}$$

*If $u \in W^{1,p}(\Omega)$ and $u^\star$ is the function given by Theorem 1.3, then*

$$\int_\Omega f(x)u(x)\,dx \leq \int_{\Omega^\sharp} f^\sharp(x)u^\star(x)\,dx. \tag{4.2}$$

PROOF. If $u \in W_0^{1,p}(\Omega)$, the result is contained in [15]. We recall it for the sake of completeness.

By [11, (2.7)], it is known that

$$u^*(s) \leq \frac{1}{n\omega_n^{\frac{1}{n}}}\int_s^{|\Omega|}\frac{F(t)}{t^{1-\frac{1}{n}}}\,dt, \tag{4.3}$$

where $F$ is a function such that

$$\int_0^s F(t)\,dt = \int_{D(s)} |\nabla u|_*(s)\,ds$$

with $D(s)$ defined in Section 2.

Setting $g(t) := \frac{1}{t^{1-\frac{1}{n}}}\int_0^t f^*(s)\,ds$, multiplying both terms of (4.3) for $f^*(s)$, integrating from 0 to $|\Omega|$, and using Fubini's theorem, we get

$$\begin{aligned}\int_0^{|\Omega|} f^*(s)u^*(s)\,ds &\leq \frac{1}{n\omega_n^{\frac{1}{n}}}\int_0^{|\Omega|} f^*(s)\left(\int_s^{|\Omega|}\frac{F(t)}{t^{1-\frac{1}{n}}}\,dt\right)ds \\ &= \frac{1}{n\omega_n^{\frac{1}{n}}}\int_0^{|\Omega|} F(t)g(t)\,dt.\end{aligned} \tag{4.4}$$

Let us suppose that $g(t)$ is non-decreasing, so $g_*(s) = g(s)$ and by Lemma 2.1 there exists a sequence $\{F_k\}$ such that $(F_k)_* = (\nabla u)_*$ and $F_k \rightharpoonup F$ in BV. Therefore,

$$\int_0^{|\Omega|} F(t)g(t)\,dt = \lim_k \int_0^{|\Omega|} F_k(t)g(t)\,dt.$$

Using Hardy–Littlewood's inequality, we have

$$\lim_k \int_0^{|\Omega|} F_k(t)g(t)\,dt \le \int_0^{|\Omega|} |\nabla u|_*(t)g_*(t)\,dt = \int_0^{|\Omega|} |\nabla u|_*(t)g(t)\,dt.$$

Hence, by (4.4) and Fubini's theorem, we obtain

$$\begin{aligned}\int_0^{|\Omega|} f^*(t)u^*(t)\,dt &\le \frac{1}{n\omega_n^{\frac{1}{n}}}\int_0^{|\Omega|} |\nabla u|_*(t)\,g(t)\,dt\\ &= \frac{1}{n\omega_n^{\frac{1}{n}}}\int_0^{|\Omega|} |\nabla u|_*(t)\left(\frac{1}{t^{1-\frac{1}{n}}}\int_0^t f^*(s)\,ds\right)dt\\ &= \int_0^{|\Omega|} f^*(s)\left(\frac{1}{n\omega_n^{\frac{1}{n}}}\int_s^{|\Omega|}\frac{|\nabla u|_*(t)}{t^{1-\frac{1}{n}}}\,dt\right)ds\\ &= \int_0^{|\Omega|} f^*(s)(u^\star)^*(s)\,ds.\end{aligned}$$

Therefore, by Hardy–Littlewood's inequality, we have

$$\begin{aligned}\int_\Omega f(x)u(x)\,dx \le \int_0^{|\Omega|} f^*(t)u^*(t) &\le \int_0^{|\Omega|} f^*(s)(u^\star)^*(s)\,ds\\ &= \int_{\Omega^\sharp} f^\sharp(x)\,u^\star(x)\,dx.\end{aligned}$$

But we have to deal with the assumption that $g$ is non-decreasing; that is

$$\begin{aligned}g'(t)\ge 0 \iff \frac{d}{dt}\left(\frac{1}{t^{1-\frac{1}{n}}}\int_0^t f^*(s)\,ds\right) &= -\frac{n-1}{n}\frac{1}{t^{2-\frac{1}{n}}}\left(\int_0^t f^*(s)\,ds\right)\\ &\quad+\frac{1}{t^{1-\frac{1}{n}}}f^*(t)\ge 0\end{aligned}$$

if and only if

$$f^*(t)\ge\left(1-\frac{1}{n}\right)\frac{1}{t}\int_0^t f^*(s)\,ds.$$

Now let us deal with $u\notin W_0^{1,p}(\Omega)$. Suppose that $u\in C^2(\Omega)$ is a non-negative function, that $\Omega$ has $C^2$ boundary, and that $f$ satisfies (4.1). Proceeding as in Step 1 of Theorem 1.3, for every $\varepsilon>0$ we can construct $u_\varepsilon$ that coincides with $u$ in $\Omega$ and is zero on $\partial\Omega_\varepsilon$. Moreover, we can extend $f$ to $\Omega_\varepsilon$ simply defining

$$f_\varepsilon(t)=\begin{cases} f(x) & \text{in } \Omega,\\ f^*\big(|\Omega|\big) & \text{in } \Omega_\varepsilon\setminus\Omega.\end{cases}$$

The rearrangement, for every $\varepsilon > 0$, is

$$f_\varepsilon^*(t) = \begin{cases} f^*(t) & \text{in } \big[0, |\Omega|\big], \\ f^*\big(|\Omega|\big) & \text{in } \big[|\Omega|, |\Omega_\varepsilon|\big], \end{cases}$$

so we just have to check (4.1) for $t \in [|\Omega|, |\Omega_\varepsilon|]$; namely,

$$f_\varepsilon^*(t) \geq \left(\frac{n-1}{n}\right)\frac{1}{t}\int_0^t f_\varepsilon^*(s)\,ds. \tag{4.5}$$

Keeping in mind that $f$ verifies (4.1), we have

$$f_\varepsilon^*(t) = f^*(|\Omega|) \geq \left(\frac{n-1}{n}\right)\frac{1}{|\Omega|}\int_0^{|\Omega|} f^*(s)\,ds.$$

If we show that

$$\begin{aligned}\frac{1}{|\Omega|}\int_0^{|\Omega|} f^*(s)\,ds &\geq \left[\frac{1}{t}\int_0^{|\Omega|} f^*(s)\,ds + \frac{t-|\Omega|}{t} f^*\big(|\Omega|\big)\right] \\ &= \frac{1}{t}\int_0^t f_\varepsilon^*(s)\,ds,\end{aligned}$$

then (4.5) is true. By direct calculations,

$$\frac{t-|\Omega|}{t|\Omega|}\int_0^{|\Omega|} f^*(s)\,ds \geq \frac{t-|\Omega|}{t} f^*\big(|\Omega|\big) \iff \frac{1}{|\Omega|}\int_0^{|\Omega|} f^*(s)\,ds \geq f^*\big(|\Omega|\big);$$

that is true of the fact that $f^*$ is decreasing.

So, $\forall\varepsilon > 0$ we can apply the first part of the theorem obtaining

$$\int_{\Omega_\varepsilon} u_\varepsilon f_\varepsilon\,dx \leq \int_{\Omega_\varepsilon^\sharp} v_\varepsilon f_\varepsilon^\sharp\,dx.$$

Sending $\varepsilon \to 0$, we get

$$\int_\Omega uf\,dx \leq \int_{\Omega_\sharp} u^\star f^\sharp\,dx.$$

Arguing as in Theorem 1.3, we get (4.2). ∎

REMARK 4.2. Condition (4.1) implies that the $f$ is strictly positive. Moreover, if the essential oscillation of $f$ is bounded

$$\operatorname{ess\,osc}|f| := \frac{\operatorname{ess\,sup}_{x\in\Omega}\big|f(x)\big|}{\operatorname{ess\,inf}_{x\in\Omega}\big|f(x)\big|} \leq \frac{n}{n-1},$$

then (4.1) is satisfied.

Theorem 4.1 allows us to compare the minimum of

$$T_{\beta,f}(\Omega) := \min_{w\in W^{1,2}(\Omega)} \left\{ \frac{1}{2}\int_\Omega |\nabla w|^2\,dx + \frac{\beta|\partial\Omega|}{2}\int_{\partial\Omega} w^2\,d\mathcal{H}^{n-1} - \int_\Omega wf\,dx \right\}$$

with the one of

$$T_{\beta,f}(\Omega^\sharp) := \min_{v\in W^{1,2}(\Omega^\sharp)} \left\{ \frac{1}{2}\int_{\Omega^\sharp} |\nabla v|^2\,dx + \frac{\beta|\partial\Omega^\sharp|}{2}\int_{\partial\Omega^\sharp} v^2\,d\mathcal{H}^{n-1} - \int_{\Omega^\sharp} vf^\sharp\,dx \right\}.$$

COROLLARY 4.3. *Let $\beta > 0$ and let $\Omega \subset \mathbb{R}^n$ be a bounded, open, and Lipschitz set. If $f$ satisfies (4.1), then denoting by $\Omega^\sharp$ the ball centered at the origin with the same measure as $\Omega$, it holds that*

$$T_{\beta,f}(\Omega) \geq T_{\beta,f^\sharp}(\Omega^\sharp).$$

Moreover, we can use Theorem 4.1 to get a comparison between Lorentz norm of $u$ and $u^\star$.

COROLLARY 4.4. *Let $1 \leq p \leq \frac{n}{n-1}$. Under the assumption of Theorem 1.3, it holds that*

$$\|u\|_{L^{p,1}(\Omega)} \leq \|u^\star\|_{L^{p,1}(\Omega^\sharp)}, \tag{4.6}$$

*where $u^\star$ is the function given by Theorem 1.3.*

PROOF. Let us explicit the $L^{p,1}$ norm of $u$:

$$\|u\|_{L^{p,1}(\Omega)} = \int_0^{+\infty} t^{\frac{1}{p}-1} u^*(t)\,dt = \int_0^{+\infty} t^{-\frac{1}{p'}} u^*(t)\,dt.$$

Hence by Theorem 4.1, it is sufficient that

$$t^{-\frac{1}{p'}} - \frac{n-1}{n}\frac{1}{t}\int_0^t s^{-\frac{1}{p'}}\,ds \geq 0. \tag{4.7}$$

If we compute

$$\frac{1}{t}\int_0^t s^{-\frac{1}{p'}}\,ds = \frac{1}{t}p\,t^{-\frac{1}{p'}+1} = p\,t^{-\frac{1}{p'}},$$

then we have

$$t^{-\frac{1}{p'}} - \frac{n-1}{n}\frac{1}{t}\int_0^t s^{-\frac{1}{p'}}\,ds = t^{-\frac{1}{p'}}\left(1 - \frac{n-1}{n}p\right) \geq 0 \iff p \leq \frac{n}{n-1},$$

so (4.7) is true and we can apply Theorem 4.1 obtaining

$$\int_0^{+\infty} t^{-\frac{1}{p'}} u^*(t)\,dt \leq \int_0^{+\infty} t^{-\frac{1}{p'}} u^\star(t)\,dt$$

that is (4.6). ∎

Remark 4.5. We emphasize that the bound $p \leq \frac{n}{n-1}$ is the best we can hope for Lorentz norm $L^{q,1}$. Indeed, if by absurd (4.6) holds for $p > \frac{n}{n-1}$, then by the embedding of $L^{p,q}$ spaces, $L^{q,1}(\Omega) \subseteq L^{q,q}(\Omega) = L^q(\Omega)$, which gives a contradiction.

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