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Calculus of Variations. – *The Sobolev class where a weak solution is a local minimizer*, by Filomena De Filippis, Francesco Leonetti, Paolo Marcellini and Elvira Mascolo, communicated on 10 February 2023.

Abstract. – The aim of this paper is to propose some results which we hope could contribute to understand better *Lavrentiev's phenomenon* for energy integrals as in (1.1) under some p,q-growth conditions as in (1.2); in fact, we expect that Lavrentiev's phenomenon does not occur if the quotient q/p is not too large in dependence of n, for instance, as in the cases – either scalar or vectorial ones – that we consider in this manuscript.

Keywords. – Lavrentiev's phenomenon, calculus of variations, local minimizer, p,q-growth, elliptic equations, elliptic systems.

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1. Introduction

We consider the following non-autonomous energy integral:

$$F(u,\Omega) = \int_\Omega \{f(x,Du) + \langle b(x), u\rangle\}\,dx, \tag{1.1}$$

where $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ is a vector-valued map defined in a bounded open set $\Omega \subset \mathbb{R}^n$, $n \geq 2$, with values in $\mathbb{R}^m$, $m \geq 1$, and Du is its gradient $m \times n$ matrix. The function $f : \Omega \times \mathbb{R}^{m\times n} \to \mathbb{R}$, $f = f(x,z)$, is measurable with respect to $x \in \Omega \subset \mathbb{R}^n$ and it is convex and C^1 with respect to $z \in \mathbb{R}^{m\times n}$. Moreover, $\langle b(x), u\rangle = \sum_{\alpha=1}^m b^\alpha(x) u^\alpha(x)$ represents the scalar product by $b(x) = (b^\alpha(x))_{\alpha=1,2,\ldots,m}$ and $u = u(x) = (u^\alpha(x))_{\alpha=1,2,\ldots,m}$.

We assume the following p,q-*growth conditions* for some p, q exponents, with $1 < p \leq q$:

$$c_1|z|^p - c_2 \leq f(x,z) \leq c_3|z|^q + c_4, \tag{1.2}$$

for some positive constants $c_1,\ldots,c_4$; we also assume the *summability condition* on $b = (b^\alpha)_{\alpha=1,2,\ldots,m}$:

$$b \in L^{\frac{p}{p-1}}(\Omega,\mathbb{R}^m). \tag{1.3}$$

The *coercivity condition* at the left-hand side of (1.2) ensures the existence of global minimizers $u \in W^{1,p}(\Omega, \mathbb{R}^m)$ of F when suitable boundary values have been fixed. Let us recall the difference between *global* and *local* minimizers. A mapping $u \in W^{1,p}(\Omega, \mathbb{R}^m)$ is a *global* minimizer when $x \mapsto f(x, Du(x)) \in L^1(\Omega)$ and

$$F(u, \Omega) \leq F(u + \varphi, \Omega),$$

for every $\varphi \in W_0^{1,p}(\Omega, \mathbb{R}^m)$. On the other hand, a mapping $u \in W_{\mathrm{loc}}^{1,p}(\Omega, \mathbb{R}^m)$ is a *local* minimizer when $x \to f(x, Du(x)) \in L_{\mathrm{loc}}^1(\Omega)$ and

$$F(u, \operatorname{supp} \varphi) \leq F(u + \varphi, \operatorname{supp} \varphi), \tag{1.4}$$

for every $\varphi \in W^{1,p}(\Omega, \mathbb{R}^m)$ with $\operatorname{supp} \varphi \Subset \Omega$. Note that a *global* minimizer is also a *local* minimizer.

The p, q*-growth* was introduced and firstly studied in [32] within the field of the theory of regularity. More recent related researches are described and quoted below in this manuscript. Nowadays, the mathematical literature on regularity under p, q and general growth conditions is very large; for instance, see the extensive list of references in [34–38].

In the study of elliptic-convex problems with p, q-growth, if $p \neq q$, the coercivity and the boundedness condition in (1.2) hold with different growth with respect to the gradient variable $|z| = |Du|$; then, the loss of the regularity may occur. In addition, some other unexpected properties of solutions may occur, as highlighted by Zhikov in his pioneer paper [40]. We refer in particular to the *Lavrentiev phenomenon*, that is, the feature of some energy integrals to have different infima if considered either on the complete class U of admissible functions or on a smaller class (dense and) contained in U. In particular, for the energy integral in (1.1), under p, q-growth (1.2) with $p < q$, the definition (i.e. the precise meaning) of the integral $F(u, \Omega)$ is not ambiguous if $u \in W_{\mathrm{loc}}^{1,q}(\Omega)$. On the contrary, a priori, it is not uniquely defined if $u \in W^{1,p}(\Omega) \setminus W_{\mathrm{loc}}^{1,q}(\Omega)$; in fact, a further definition is, for every $u \in W^{1,p}(\Omega)$,

$$\overline{F}(u, \Omega) = \inf_{\{u_j\}} \left\{ \liminf_{j \to +\infty} F(u_j, \Omega) : u_j \in W_{\mathrm{loc}}^{1,q}(\Omega) \; \forall j \in \mathbb{N}, \; u_j \to u \text{ in } W^{1,p}(\Omega) \right\}.$$

Here, we used the symbol $\overline{F}$ to emphasize the fact that a priori the functional $\overline{F}$ is different from F, and it is also an *extension by lower semicontinuity* (a *closure*) of F (see, for instance, [19, 21, 31]). Then, we cannot a priori exclude the Lavrentiev phenomenon in this context, in particular in the case of x-dependence. For the gap in the Lavrentiev phenomenon, we refer to [9].

The main aim of this paper is to give conditions to rule out the Lavrentiev phenomenon in the general case $p \leq q$. More precisely, under growth conditions on p and q, we propose cases with

$$\inf\{F(v) : v \in u + W_0^{1,p}(\Omega')\} = \inf\{F(v) : v \in u + W_0^{1,q}(\Omega')\},$$

for every $\Omega' \Subset \Omega$, where u is a local minimizer of F; see Theorem 2.2 for vector-valued integrals and Theorem 3.2 in the scalar case.

In order to give some details, we start to consider the standard growth $p = q$, and we remark that since $z \mapsto f(x,z)$ is convex and C^1, growth conditions (1.2) and [32, Lemma 2.1] imply

$$\left|\frac{\partial f}{\partial z_i^\alpha}(x,z)\right| \le c_5\left(1 + |z|^{q-1}\right); \tag{1.5}$$

see also [31, Step 2, Section 2]. When $p = q$, (1.5) implies

$$x \mapsto \frac{\partial f}{\partial z_i^\alpha}\big(x, Du(x)\big) \in L^{\frac{p}{p-1}}(\Omega); \tag{1.6}$$

then, we can write the *Euler equation in weak form*, which in fact is a *system of m differential equations* in divergence form, when we see each equation corresponding to each separate component of the test vector-valued map $\varphi(x) = (\varphi^\alpha(x))_{\alpha=1,2,\dots,m}$. For every $\varphi \in W_0^{1,p}(\Omega, \mathbb{R}^m)$, we have

$$\int_\Omega \sum_{\alpha=1}^m \sum_{i=1}^n \frac{\partial f}{\partial z_i^\alpha}\big(x, Du(x)\big) D_i\varphi^\alpha(x)\,dx + \int_\Omega \sum_{\alpha=1}^m b^\alpha(x)\varphi^\alpha(x)\,dx = 0. \tag{1.7}$$

Note also that when $u \in W^{1,p}(\Omega, \mathbb{R}^m)$ and $p = q$, then (1.2) implies that $x \to f(x, Du(x)) \in L^1(\Omega)$. Moreover, the convexity of $z \mapsto f(x,z)$ guarantees that if $u \in W^{1,p}(\Omega, \mathbb{R}^m)$, then

$$u \text{ globally minimizes } F \iff u \text{ solves the Euler equation.}$$

The case $p < q$ is more delicate. Indeed, in such a situation, if $u \in W^{1,p}(\Omega, \mathbb{R}^m)$, (1.6) changes into

$$x \mapsto \frac{\partial f}{\partial z_i^\alpha}\big(x, Du(x)\big) \in L^{\frac{p}{q-1}}(\Omega).$$

Since $\frac{p}{q-1} < \frac{p}{p-1}$, then we cannot test (1.7) with $\varphi \in W_0^{1,p}(\Omega, \mathbb{R}^m)$ any longer. On the contrary, if $u \in W^{1,q}_{\mathrm{loc}}(\Omega; \mathbb{R}^m)$, then (1.5) implies

$$x \mapsto \frac{\partial f}{\partial z_i^\alpha}\big(x, Du(x)\big) \in L^{\frac{q}{q-1}}_{\mathrm{loc}}(\Omega)$$

and, being q and $\frac{q}{q-1}$ conjugate exponents, (1.7) can be tested by any $\varphi \in W^{1,q}(\Omega, \mathbb{R}^m)$ with compact support in Ω. In this note, we use this approach to obtain the existence (and regularity) of minimizers. In fact, the idea is to achieve existence in a Sobolev class by means of regularity results, precisely by means of the higher integrability for the gradient Du of solutions of the Euler equation or system. A weak solution of (1.7), with a high degree of integrability, can be found, and such a weak solution turns out to be a local minimizer of F also in a larger Sobolev class. Main steps in the proofs are the results due by Cupini, Leonetti, and Mascolo [14] in the vector-valued case $m \geq 1$ and, in the more strict scalar case $m = 1$ but with better exponents, by Marcellini [33, Section 4] and Cupini, Marcellini, and Mascolo [16].

2. Vectorial case

Consider the Dirichlet problem in $\Omega \subset \mathbb{R}^n$:

$$\begin{cases} \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\big(A_i^\alpha(x, Du(x))\big) = b^\alpha(x) & \text{in } \Omega,\ \alpha = 1, \dots, m, \\ u(x) = \tilde{u}(x) & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

The following theorem holds.

THEOREM 2.1. *Let Ω be a bounded open subset of $\mathbb{R}^n$, $n \geq 2$, and let $A_i^\alpha : \Omega \times \mathbb{R}^{nm} \to \mathbb{R}$, $i = 1, \dots, n$, $\alpha = 1, \dots, m$, be continuous with respect to z. We assume that there exists $0 < \gamma \leq 1$ and $\nu, L, H > 0$ such that*

$$\nu\big(|z|^2 + |\tilde{z}|^2\big)^{\frac{p-2}{2}} |z - \tilde{z}|^2 \leq \sum_{\alpha=1}^{m} \sum_{i=1}^{n} \big[A_i^\alpha(x, z) - A_i^\alpha(x, \tilde{z})\big][z_i^\alpha - \tilde{z}_i^\alpha], \quad z, \tilde{z} \in \mathbb{R}^{nm}, \tag{2.2}$$

$$\big|A_i^\alpha(x, z)\big| \leq L\big(1 + |z|\big)^{q-1}, \tag{2.3}$$

$$\sum_{\alpha=1}^{m} \sum_{i=1}^{n} \big|A_i^\alpha(x, z) - A_i^\alpha(\tilde{x}, z)\big| \leq H|x - \tilde{x}|^\gamma \big(1 + |z|\big)^{q-1}, \quad x, \tilde{x} \in \Omega, \tag{2.4}$$

with p and q satisfying

$$2 \leq p \leq q < p\frac{n + \gamma}{n}. \tag{2.5}$$

Let b be a function in $L^{p/(p-1)}(\Omega, \mathbb{R}^m)$. Then, for all $\tilde{u} \in W^{1,p\frac{q-1}{p-1}}(\Omega, \mathbb{R}^m)$, there exists a weak solution $u \in (\tilde{u} + W_0^{1,p}(\Omega, \mathbb{R}^m)) \cap W_{\mathrm{loc}}^{1,s}(\Omega, \mathbb{R}^m)$ of the Dirichlet problem (2.1), for all $q \leq s < p\frac{n}{n-\gamma}$.

The proof of Theorem 2.1 proceeds basically in the same way as the one of [14, Theorem 1.1] where $b(x) \equiv 0$.

Under the previous assumptions, a vector-valued map $u \in W^{1,q}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$ is a weak solution to the system if

$$\int_\Omega \sum_{\alpha=1}^m \sum_{i=1}^n A_i^\alpha\big(x, Du(x)\big) D_i\varphi^\alpha(x)\,dx + \int_\Omega \sum_{\alpha=1}^m b^\alpha(x)\varphi^\alpha(x)\,dx = 0, \tag{2.6}$$

for all $\varphi \in W^{1,q}(\Omega, \mathbb{R}^m)$ with $\operatorname{supp}\varphi \Subset \Omega$.

We focus our attention on growth condition (2.3). First, we observe that since $|Du|^{q-1} \in L^{\frac{s}{q-1}}_{\mathrm{loc}}$, we obtain $A(x, Du(x)) \in L^{\frac{s}{q-1}}_{\mathrm{loc}}$, for s such that $q \le s < p\frac{n}{n-\gamma}$.

The question is now the following: if we assume $\varphi \in W^{1,p}(\Omega, \mathbb{R}^m)$ with $\operatorname{supp}\varphi \Subset \Omega$, do we have that $A(x, Du(x)) \in L^{\frac{p}{p-1}}(\operatorname{supp}\varphi)$?

If $q < \frac{np-\gamma}{n-\gamma}$, the answer is affirmative; indeed, in this case, it is easy to check that

$$q < \frac{np-\gamma}{n-\gamma} \Rightarrow \exists s \quad \text{such that} \quad q < s < \frac{np}{n-\gamma}, \qquad \frac{p}{p-1} < \frac{s}{q-1}.$$

Moreover, when $p \le \frac{n}{\gamma}$, we have that

$$\frac{np-\gamma}{n-\gamma} \le p\frac{n+\gamma}{n} < p\frac{n}{n-\gamma}.$$

Let us assume $q < \frac{np-\gamma}{n-\gamma}$; therefore, the weak solution u given by Theorem 2.1 satisfies (2.6) for all $\varphi \in W^{1,p}(\Omega, \mathbb{R}^m)$ with $\operatorname{supp}\varphi \Subset \Omega$.

Let us consider functional F in (1.1) under the p, q-growth (1.2) and assumption (1.3) on b. We assume that

$$A_i^\alpha(x, z) = \frac{\partial f}{\partial z_i^\alpha}(x, z)$$

satisfy (2.2), (2.3), (2.4), and (2.5); we assume also that

$$q < \frac{np-\gamma}{n-\gamma} \tag{2.7}$$

and $\tilde{u} \in W^{1,p\frac{q-1}{p-1}}(\Omega, \mathbb{R}^m)$.

By Theorem 2.1, we have that there exists $u \in (\tilde{u} + W_0^{1,p}(\Omega, \mathbb{R}^m)) \cap W^{1,s}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$ satisfying the Euler equation (1.7) of functional F, i.e., for every test function

$$\varphi \in W^{1,p}(\Omega, \mathbb{R}^m)$$

with $\operatorname{supp}\varphi \Subset \Omega$.

On the other hand, since $z \mapsto f(x,z)$ is convex, we have

$$f\big(x, Du(x) + D\varphi(x)\big) \geq f\big(x, Du(x)\big) + \sum_{\alpha=1}^{m}\sum_{i=1}^{n} \frac{\partial f}{\partial z_i^\alpha}\big(x, Du(x)\big) D_i \varphi^\alpha(x).$$

Consequently by (2.6), for all $\varphi \in W^{1,p}(\Omega, \mathbb{R}^m)$ with $\operatorname{supp}\varphi \Subset \Omega$, we have

$$\begin{aligned} (2.8)\quad & \int_{\operatorname{supp}\varphi} f\big(x, Du(x) + D\varphi(x)\big)\,dx \\ & \geq \int_{\operatorname{supp}\varphi} f\big(x, Du(x)\big)\,dx + \int_{\operatorname{supp}\varphi} \sum_{\alpha=1}^{m}\sum_{i=1}^{n} \frac{\partial f}{\partial z_i^\alpha}\big(x, Du(x)\big) D_i\varphi^\alpha(x)\,dx \\ & = \int_{\operatorname{supp}\varphi} \Big[f\big(x, Du(x)\big) - \sum_{\alpha=1}^{m} b^\alpha(x)\varphi^\alpha(x) \Big]\,dx. \end{aligned}$$

Inequality (2.8) shows that u is also a *local* minimizer of the functional (1.1); that is, u satisfies (1.4) for all $\varphi \in W^{1,p}(\Omega, \mathbb{R}^m)$ with $\operatorname{supp}\varphi \Subset \Omega$.

Thus, the following theorem holds true.

THEOREM 2.2. *Let Ω be a bounded open subset of $\mathbb{R}^n$, $n \geq 2$. Let $f : \Omega \times \mathbb{R}^{nm} \to \mathbb{R}$ be a C^1 function with respect to z, and $f(x,0)$ is measurable, such that*

$$c_1|z|^p - c_2 \leq f(x,z) \leq c_3|z|^q + c_4,$$

where $0 < c_1 \leq c_3$, $0 \leq c_2, c_4$. Assume that $\frac{\partial f}{\partial z_i^\alpha}(x,z)$ satisfies (2.2), (2.3), and (2.4), with p and q as in (2.5) and (2.7).

Let b be a $L^{p/(p-1)}(\Omega, \mathbb{R}^m)$ function. Then, for all $\tilde{u} \in W^{1,p\frac{q-1}{p-1}}(\Omega, \mathbb{R}^m)$, there exists a local minimizer

$$u \in \big(\tilde{u} + W_0^{1,p}(\Omega, \mathbb{R}^m)\big) \cap W_{\mathrm{loc}}^{1,s}(\Omega, \mathbb{R}^m)$$

of the functional

$$F(u, \Omega) = \int_\Omega \Big[f\big(x, Du(x)\big) + \sum_{\alpha=1}^{m} b^\alpha(x) u^\alpha(x) \Big]\,dx,$$

for all $q \leq s < p\frac{n}{n-\gamma}$. Moreover, for every $\Omega' \Subset \Omega$, it holds that

$$\inf_{v \in u + W_0^{1,p}(\Omega')} F = \inf_{v \in u + W_0^{1,q}(\Omega')} F.$$

REMARK 2.3. Let us assume that $b = 0$ and $f = f(z)$ in (1.1); in [26], it is shown that *every* local minimizer $u \in W_{\mathrm{loc}}^{1,p}(\Omega, \mathbb{R}^m)$ enjoys higher integrability:

$$u \in W_{\mathrm{loc}}^{1,q}(\Omega, \mathbb{R}^m) \cap W_{\mathrm{loc}}^{1,\frac{np}{n-2}}(\Omega, \mathbb{R}^m),$$

provided

$$2 \le p < q < p + 2\min\left\{1, \frac{p}{n}\right\}.$$

When $p \le n$, the previous restriction becomes

$$2 \le p < q < p\frac{n+2}{n}. \tag{2.9}$$

On the other hand, the best result in the case $f(x, z)$ is obtained in our assumptions when $\gamma = 1$ and (2.5) becomes

$$2 \le p < q < p\frac{n+1}{n}.$$

We remark that the gap between $\frac{n+1}{n}$ and $\frac{n+2}{n}$ is due to the basic difference between the non-autonomous case $f(x, z)$ and the autonomous one $f(z)$. Indeed, as far as p, q, n, γ satisfy $p < n < n + \gamma < q$, in [27], an example of $f(x, z)$ is given for which a global minimizer is not in $W^{1,q}_{\text{loc}}(\Omega, \mathbb{R}^m)$. Note that such an example is the so-called double phase functional

$$f(x, z) = |z|^p + a(x)|z|^q.$$

For the double phase functional, we refer to Colombo–Mingione [12] and Colombo–Mingione–Baroni [2].

For a more general structure of the energy function, we recall Cupini–Giannetti–Giova–Passarelli di Napoli [13], Esposito, the second author, and Vincenzo Petricca [25], Chlebicka–Borowski–Miasojedow [7], Bulíček–Gwiazda–Skrzeczkowski [8], Hästö–Ok [28], Balci–Diening–Surnachev [1], Eleuteri–Marcellini–Mascolo [23], De Filippis–Mingione [20], Koch [30], and De Filippis–Leonetti [22].

The gap between $\frac{n+1}{n}$ and $\frac{n+2}{n}$ shows up when dealing with the autonomous case $A_i^\alpha(z)$, $f(z)$ and comparing weak solutions of (3.1) with the minimizers of (1.1): for weak solutions, we need $q < p\frac{n+1}{n}$, and for minimizers, we need $q < p\frac{n+2}{n}$; see, for instance, the introduction of [35] and [3, Theorems 1.2, 1.3, and 1.17]. In the scalar case, we have $q < p\frac{n+2}{n}$ for weak solutions too, provided an additional restriction on A_i is assumed; see the next section for details.

Let us remark that the recent work of Shäffner [39] shows that the higher integrability $W^{1,\frac{np}{n-2}}$ for minimizers holds true under the restriction

$$2 \le p < q < p\frac{n+1}{n-1}.$$

Note that $\frac{n+2}{n} < \frac{n+1}{n-1}$. Then, Shäffner's result improves on bound (2.9). See also [4,5,29].

For details and references on problems with p, q growth, we quote the classical starting results in [32, 33], the well-known article by Mingione [37], and the recent surveys [34, 35] and Mingione–Rădulescu [38]; see also [15, 17, 18, 24, 36].

3. Scalar case

Consider the Dirichlet problem

$$\begin{cases} \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(A^i\left(x, Du(x)\right)\right) = b(x) & \text{in } \Omega, \\ u(x) = \tilde{u}(x) & \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

In the scalar case $m = 1$, to solve (3.1), we refer to the existence and regularity results in [33, Theorem 4.1] and [16, Theorem 2.1] that we merge into the following.

Theorem 3.1. *Let Ω be a bounded open subset of $\mathbb{R}^n$, $n \geq 2$. Let $A^i : \Omega \times \mathbb{R}^n \to \mathbb{R}$, $i = 1, \dots, n$, be locally Lipschitz continuous functions in $\Omega \times \mathbb{R}^n$ such that there exist $\mu, M > 0$: for a.e. $x \in \Omega$ and $\forall z, \tilde{z} \in \mathbb{R}^n$,*

$$\mu\left(1 + |z|^2\right)^{(p-2)/2} |\tilde{z}|^2 \leq \sum_{i,j=1}^{n} A^i_{z_j}(x, z) \tilde{z}_i \tilde{z}_j, \tag{3.2}$$

$$\left|A^i_{z_j}(x, z)\right| \leq M\left(1 + |z|^2\right)^{(q-2)/2}, \tag{3.3}$$

$$\left|A^i_{z_j}(x, z) - A^j_{z_i}(x, z)\right| \leq M\left(1 + |z|^2\right)^{(p+q-4)/4}, \tag{3.4}$$

$$\left|A^i_{x_s}(x, z)\right| \leq M\left(1 + |z|^2\right)^{(p+q-2)/4}, \quad s = 1, \dots, n, \tag{3.5}$$

$$\left|A^i(x, 0)\right| \leq M, \quad \forall x \in \Omega, \tag{3.6}$$

with p and q such that

$$\begin{cases} p \leq q \leq p + 1, \ q < p\frac{n-1}{n-p} & \text{if } 1 < p < 2, \\ p \leq q \leq p + 1, \ q < p\frac{n-1}{n-p} & \text{if } n > 4, \ \frac{3n}{n+2} < p \leq \frac{n}{2}, \\ 2 \leq p \leq q < p\frac{n+2}{n} & \text{otherwise}. \end{cases} \tag{3.7}$$

Assume $b \in L^{p/(p-1)}(\Omega, \mathbb{R}) \cap L^{\infty}_{\text{loc}}(\Omega, \mathbb{R})$.
Then, for all $\tilde{u} \in W^{1, p\frac{q-1}{p-1}}(\Omega, \mathbb{R})$, there exists a weak solution

$$u \in \left(\tilde{u} + W^{1,p}_0(\Omega, \mathbb{R})\right) \cap W^{1,\infty}_{\text{loc}}(\Omega, \mathbb{R}) \cap W^{2,2}_{\text{loc}}(\Omega, \mathbb{R})$$

of the Dirichlet problem (3.1).

A function $u \in W^{1,q}_{\mathrm{loc}}(\Omega, \mathbb{R})$ is a weak solution to the equation when

$$\int_\Omega \sum_{i=1}^n A^i\big(x, Du(x)\big) D_i\varphi(x)\, dx + \int_\Omega b(x)\varphi(x)\, dx = 0, \tag{3.8}$$

for all $\varphi \in W^{1,q}(\Omega, \mathbb{R})$ with $\operatorname{supp}\varphi \Subset \Omega$.

We show that the weak solution u is also a local minimizer of the functional

$$\tilde{F}(u, \Omega) = \int_\Omega \big[f\big(x, Du(x)\big) + b(x)u(x)\big]\, dx. \tag{3.9}$$

For this purpose, we observe that (3.3)–(3.7) imply that there exists $\widetilde{M} \in (0, +\infty)$ such that

$$\big|A^i\big(x, Du(x)\big)\big| \le \widetilde{M}\big(1 + |Du(x)|^{q-1}\big).$$

Since $|Du|^{q-1} \in L^\infty_{\mathrm{loc}}$, we get $A(x, Du(x)) \in L^\infty_{\mathrm{loc}}$. Therefore, if $\varphi \in W^{1,p}(\Omega, \mathbb{R})$ with $\operatorname{supp}\varphi \Subset \Omega$, we have that $A(x, Du(x)) \in L^{\frac{p}{p-1}}(\operatorname{supp}\varphi)$. Hence, we can repeat the same argument as above and obtain (3.8) for all $\varphi \in W^{1,p}(\Omega, \mathbb{R})$ with $\operatorname{supp}\varphi \Subset \Omega$.

Now, we consider the functional (3.9), where f satisfies (1.2) and $f(x, z)$ is C^2 with respect to z. We assume that $b \in L^{p/(p-1)}(\Omega, \mathbb{R}) \cap L^\infty_{\mathrm{loc}}(\Omega, \mathbb{R})$. Moreover,

$$A^i(x, z) = \frac{\partial f}{\partial z_i}(x, z)$$

is locally Lipschitz continuous in $\Omega \times \mathbb{R}^n$ and satisfies (3.2)–(3.6), with p, q, n as in (3.7). We observe that (3.2) implies the convexity of $z \mapsto f(x, z)$.

For a fixed boundary value $\tilde{u} \in W^{1,p\frac{q-1}{p-1}}(\Omega, \mathbb{R}^m)$, by Theorem 3.1, there exists $u \in (\tilde{u} + W^{1,p}_0(\Omega, \mathbb{R})) \cap \in W^{1,\infty}_{\mathrm{loc}}(\Omega, \mathbb{R}) \cap W^{2,2}_{\mathrm{loc}}(\Omega, \mathbb{R})$ verifying (3.8). By (3.8), we have now for the scalar case

$$\int_{\operatorname{supp}\varphi} f\big(x, Du(x) + D\varphi(x)\big)\, dx \ge \int_{\operatorname{supp}\varphi} \big[f\big(x, Du(x)\big) - b(x)\varphi(x)\big]\, dx,$$

for all $\varphi \in W^{1,p}(\Omega, \mathbb{R})$ with $\operatorname{supp}\varphi \Subset \Omega$. Then, we have just obtained that

$$u \in \big(\tilde{u} + W^{1,p}_0(\Omega, \mathbb{R})\big) \cap W^{1,\infty}_{\mathrm{loc}}(\Omega, \mathbb{R}) \cap W^{2,2}_{\mathrm{loc}}(\Omega, \mathbb{R})$$

is a *local* minimizer of the functional (3.9).

Thus, we have proved the following corollary.

THEOREM 3.2. *Let Ω be a bounded open subset of $\mathbb{R}^n$, $n \ge 2$. Let $f : \Omega \times \mathbb{R}^n \to \mathbb{R}$; $f(x, 0)$ is measurable, such that*

$$c_1|z|^p - c_2 \le f(x, z) \le c_3|z|^q + c_4,$$

where $0 < c_1 \leq c_3$, $0 \leq c_2, c_4$. *Moreover,* $f(x,z)$ *is* C^2 *with respect to* z. *Assume that* $A^i = \frac{\partial f}{\partial z^i}$ *is locally Lipschitz continuous in* $\Omega \times \mathbb{R}^n$ *and satisfies* (3.2)–(3.6)*, with* p, q, n *as in* (3.7) *of Theorem* 3.1.

Let b *be a* $L^{p/(p-1)}(\Omega,\mathbb{R}) \cap L^\infty_{\mathrm{loc}}(\Omega,\mathbb{R})$ *function. Then, for all* $\tilde{u} \in W^{1,p\frac{q-1}{p-1}}(\Omega,\mathbb{R})$*, there exists a local minimizer*

$$u \in \big(\tilde{u} + W^{1,p}_0(\Omega,\mathbb{R})\big) \cap W^{1,\infty}_{\mathrm{loc}}(\Omega,\mathbb{R}) \cap W^{2,2}_{\mathrm{loc}}(\Omega,\mathbb{R})$$

of the functional

$$\tilde{F}(u,\Omega) = \int_\Omega \big[f\big(x, Du(x)\big) + b(x)u(x)\big]\,dx,$$

and for every $\Omega' \Subset \Omega$*, it holds that*

$$\inf_{v \in u + W^{1,p}_0(\Omega')} F = \inf_{v \in u + W^{1,\infty}_0(\Omega')} F.$$

REMARK 3.3. The relation between the minimizer and weak solution of the Euler equation has been studied by Carozza–Kristensen–Passarelli di Napoli in [10,11], for the vectorial case $m \geq 1$, when $f = f(z)$; it has also been studied by Bonfanti–Cellina–Mazzola in [6], for the scalar case $m = 1$, when $f = f(x,u,z)$ and u is locally bounded.

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