Rend. Lincei Mat. Appl. 34 (2023), 465–490
DOI 10.4171/RLM/1015

Complex Variable Functions. – *Holomorphic functions on the disk with infinitely many zeros*, by SEBASTIANO BOSCARDIN, communicated on 10 February 2023.

ABSTRACT. – Suppose that $\phi(z) = \sum_{n=1}^{\infty} d_n z^n$ is a series with radius of convergence greater than 1 and suppose that d_n are real numbers such that $\phi(1) \neq 0$. We prove that for any integer a greater than 1, the complex variable function

$$f(z) = \sum_{n=0}^{\infty} \phi(z^{a^n}) = \phi(z) + \phi(z^a) + \phi(z^{a^2}) + \phi(z^{a^3}) + \cdots$$

has infinitely many zeros on the unit disk. It even takes every complex value in every disk centered in any point of the boundary.

KEYWORDS. – Zeros of holomorphic functions, Fredholm's series, great Picard theorem, Jensen's formula.

2020 MATHEMATICS SUBJECT CLASSIFICATION. – Primary 30B10; Secondary 30B30.

1. INTRODUCTION

Following the analogy provided by the great Picard theorem, we say that a holomorphic function on the disk $D = \{z \in \mathbb{C} \mid |z| < 1\}$ has the *Picard property* if it takes every complex value infinitely many times at every neighborhood of any point of the boundary of D. The domain of holomorphy of such functions is D since the zeros accumulate at every point of the boundary (and the function is not always zero). We are interested in finding power series with this property.

A power series given by $\sum_{n=1}^{\infty} c_n z^{k_n}$ with $c_n \in \mathbb{C}$, $k_n \in \mathbb{N}^+$ is called *lacunary* with *Hadamard gaps* if there exists $q > 1$ such that $k_{n+1}/k_n \geq q$. W. H. J. Fuchs proved in 1967 [1] that a lacunary series convergent on the disk D, such that $\limsup |c_n| > 0$, has infinitely many zeros. A more general result of T. Murai in 1981 [6] affirms that a lacunary and unbounded series convergent on D has the Picard property.

Another class of lacunary series with the Picard property was provided by F. Nazarov in an unpublished work. There exist some sets $\Lambda \subset \mathbb{N}$ such that every series $\sum_{k\in\Lambda} c_k z^k$ with $\sum_{k\in\Lambda} |c_k|^2 = \infty$ has the Picard property. For example, a set of the form $\Lambda = \{a^n \mid n \in \mathbb{N}\}$ with $a \geq 2$ an integer falls in this category and corresponds to a lacunary series.

The so-called Fredholm series

$$f(z) = \sum_{n=0}^{\infty} z^{2^n} = z + z^2 + z^4 + z^8 + \cdots$$

satisfies the functional equation

$$f(z) = z + f(z^2).$$

It is a very simple example of a lacunary series with Hadamard gaps. It is interesting both from an arithmetic and an analytic point of view. For example, its values at the points $\frac{1}{n}$ are linked to the *iterated paperfolding* [8]. K. Mahler proved the transcendence of the values taken at the algebraic points different from zero (the Fredholm series is part of a vast class of functions satisfying certain functional equations for which the same result holds) [2] and he studied in detail the behavior at the boundary [4]. However, results about algebraic dependence or independence of the zeros are not known and their position is little understood. In the last section of this paper, we give an estimation of the number of zeros of modulus less than or equal to r, for $0 < r < 1$.

U. Zannier [10] proved, independently from the previous results, that the Fredholm series has the Picard property. This falls as a special case of Murai's or Nazarov's theorem, but Zannier's proof proceeds in a different way and contains further results. The object of this paper is to generalize Zannier's method to obtain the following without lacunarity hypothesis.

THEOREM 1.1. *Let $\phi(z) = \sum_{n=1}^{\infty} d_n z^n$ (with $\phi(0) = 0$) be a series convergent on a disk of radius greater than 1, where d_n are real, such that $\phi(1) \neq 0$. Let $a \geq 2$ be an integer and let $f : D \to \mathbb{C}$ be a holomorphic function such that $f(z) = \phi(z) + f(z^a)$ for every $z \in D$. Then f has the Picard property.*

That means that the function

$$f(z) = \sum_{n=0}^{\infty} \phi(z^{a^n}) = \phi(z) + \phi(z^a) + \phi(z^{a^2}) + \phi(z^{a^3}) + \cdots \tag{1}$$

has the Picard property. It should be noted that the function defined in (1) is actually holomorphic and it is the only one (up to an additive constant) which satisfies the functional equation

$$f(z) = \phi(z) + f(z^a).$$

We can write $f(z) = \sum_{n=1}^{\infty} b_n z^n$ with the coefficients given by

$$b_n = \sum_{j=0}^{\infty} d_{n/a^j},$$

where we agree that $d_k = 0$ if k is not an integer. Also, with the same convention

$$d_n = b_n - b_{n/a}.$$

The Fredholm series is obtained with $\phi(z) = z$ and $a = 2$, hence Theorem 1.1 proves that it has the Picard property.

The series (1) is not necessarily lacunary, therefore its Picard property is not a consequence of Murai's or Nazarov's theorem. As an example of non-lacunary series, one can consider the f arising for $a = 2$ and $\phi(z) = e^z - 1$:

$$f(z) = (e^z - 1) + (e^{z^2} - 1) + (e^{z^4} - 1) + \cdots.$$

Actually, we note that for most of the choices of ϕ, f is not lacunary. Another significant example suggested by the referee is the following. Fix a parameter $w \in \mathbb{C} \setminus \{0\}$ and put $a = 2$ and

$$\phi(z) = \frac{z}{w - z^2} = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{w^{n+1}}.$$

Then $f(z) = \sum_{n=0}^{\infty} \phi(z^{2^n}) = \sum_{n=1}^{\infty} b_n z^n$ with

$$b_n = \frac{1}{w^{m+1}} \quad \text{if } n = 2^k(2m+1) \text{ for } m, k \in \mathbb{N}.$$

The radius of convergence of ϕ in zero is $\sqrt{|w|}$. If w is real and $|w| > 1$, then Theorem 1.1 tells us that f has the Picard property. If instead $w = 1$, we get $f(z) = \frac{z}{1-z}$ [5, Exercise 3.3] contrasting with the conclusion of Theorem 1.1. Functions of this sort come up when studying simple linear recurrences. Consider for example the following recurrence depending on a parameter $\alpha \in \mathbb{C}$:

$$X_{n+2} = \alpha X_{n+1} - X_n \quad n \geq 0. \tag{2}$$

Fix $w \in \mathbb{C} \setminus \{0\}$ and put initial values X_0 and X_1 such that the explicit expression for X_n is

$$X_n = w\xi_2^n - \xi_1^n,$$

where ξ_1 and ξ_2 are the two roots of the polynomial

$$z^2 - \alpha z + 1$$

and $|\xi_1| < |\xi_2|$ (suppose that their moduli are different). Since $\xi_1 \xi_2 = 1$, we observe that $\phi(\xi_1^{2^n}) = (X_{2^n})^{-1}$ for every $n \geq 0$. Hence

$$f(\xi_1) = \sum_{n=0}^{\infty} \phi(\xi_1^{2^n}) = \sum_{n=0}^{\infty} (X_{2^n})^{-1}.$$

Theorem 1.1 applies when w is real and $|w| > 1$. For such a fixed w, we then know that $f(\xi) = 0$ for infinitely many $\xi \in D$. Put $\alpha = \xi + 1/\xi$. Then for these infinitely many α for the recurrence (2) (with initial values $X_0 = w - 1$ and $X_1 = w/\xi - \xi$), we get

$$\sum_{n=0}^{\infty} (X_{2^n})^{-1} = 0.$$

This function f also possesses nice arithmetic properties that we briefly discuss in the next section.

We return to the main theorem. The hypothesis $\phi(1) \neq 0$ in Theorem 1.1 is natural because we have the following lemma.

Lemma 1.2. *If $\phi(1) \neq 0$ and $f(z) = \phi(z) + f(z^a)$ for some integer $a \geq 2$, then the domain of holomorphy of f is exactly D.*

Conversely, without the hypothesis $\phi(1) \neq 0$, Theorem 1.1 does not hold. We can construct a counterexample. Take any f holomorphic on a disk with radius greater than 1 with $f(0) = 0$ (e.g. $f(z) = z$) and put $\phi(z) = f(z) - f(z^a)$ in order to satisfy the functional equation. Formula (1) holds. However, f does not have the Picard property because its domain of holomorphy is bigger than D. Note that with this construction we get $\phi(1) = f(1) - f(1^a) = 0$.

We now prove Lemma 1.2.

Proof. Assume that f can be extended to a holomorphic function on an open set which contains a point z_0 on the boundary of D. It is not restrictive to assume that there exists an integer k such that $z_0^{a^k} = 1$. Then the limit $\lim_{r\to 1^-} f(r)$ exists because, using repeatedly the functional equation, we get

$$f(rz_0) = \phi(rz_0) + \phi\big((rz_0)^a\big) + \phi\big((rz_0)^{a^2}\big) + \cdots + \phi(r^{a^k}) + f(r^{a^k}),$$

$$\lim_{r\to 1^-} f(r^{a^k}) = \lim_{r\to 1^-} \big[f(rz_0) - \phi(rz_0) - \phi\big((rz_0)^a\big) - \cdots - \phi(r^{a^k})\big].$$

The limit on the right-hand side exists finite since ϕ has radius of convergence greater than 1 and f can be extended at z_0, thus

$$0 = \lim_{r\to 1^-} f(r) - \lim_{r\to 1^-} f(r^a) = \lim_{r\to 1^-} \big[f(r) - f(r^a)\big] = \lim_{r\to 1^-} \phi(r) = \phi(1),$$

while we assumed that $\phi(1) \neq 0$. ∎

In order to prove Theorem 1.1, we recall the standard notation $e(z) = \exp(2\pi i z)$, $D(r) = \{z \in \mathbb{C} \mid |z| < r\}$, $D = D(1)$, and $H = \{z \in \mathbb{C} \mid \Im z > 0\}$.

We will first state an intermediate proposition, the proof of which follows a generalization of the method proposed by Zannier in [10] for the special case $a = 2$ and $\phi = \mathrm{Id}$ (the Fredholm series). From there, Theorem 1.1 will be deduced easily in many cases (e.g. for the Fredholm series). In general, we will need some lemmas about real-valued analytic functions. I thank Professor Zannier for helping me in this last step.

2. A REMARK ABOUT TRANSCENDENCE

One of the main motivations for studying solutions of a functional equation comes from the example of the Fredholm series. Mahler proved that it takes transcendental values on algebraic z satisfying $0 < |z| < 1$. His method also applies to the previously mentioned

$$f(z) = \sum_{n=0}^{\infty} \phi(z^{2^n}) \quad \text{with } \phi(z) = \frac{z}{w - z^2}.$$

If $w \in \mathbb{C} \setminus \{0, 1\}$ is algebraic, it can be verified that $f(z)$ and z are algebraically independent over $\mathbb{C}$ and that the same transcendence property holds for f. Conversely, f is rational for $w = 1$.

Mahler had already noticed in [3] that from its method the transcendence of a series involving the Fibonacci numbers F_n follows. He used $\phi(z) = (\frac{z}{1-z^2})^k$ for an integer $k \geq 2$. The corresponding function

$$f(z) = \sum_{n=0}^{\infty} \phi(z^{2^n})$$

takes transcendental values on algebraic z satisfying $0 < |z| < 1$. By evaluating in $z = \frac{1-\sqrt{5}}{2}$, he found that the following number is transcendental (for $k \geq 2$):

$$\sum_{n=0}^{\infty} (F_{2^n})^{-k},$$

while for $k = 1$ it is algebraic and equal to $\frac{7-\sqrt{5}}{2}$ [5, Exercise 3.7].

Before going on with the proof of Theorem 1.1, we note a simple transcendence property of the Fredholm series following from Mahler's theorem. As a consequence we get that the Fredholm series does not have any zeros on some lines through the origin, the union of which is dense in the disk D.

THEOREM 2.1. *Let f be the Fredholm series. If $z = te^{2\pi i\theta} = te(\theta)$ with $t \in \mathbb{R}$, $0 < t < 1$, $\theta \in \mathbb{R}$, $\theta = \frac{k}{2^n}$ for some integers k and n such that z is not real, then $f(z)$ is a transcendental number.*

PROOF. Suppose (by contradiction) that $f(z)$ is algebraic. Mahler's theorem ensures that z is transcendental. Put $\alpha = z(\bar{z})^{-1}$. Since $\theta = \frac{k}{2^n}$ with $k, n \in \mathbb{N}$, α is a 2^n-th root of the unity, so it is algebraic. We have

$$f(z) = \sum_{j=0}^{n-1} z^{2^n} + f(z^{2^n}),$$

$$f(\bar{z}) = f(\alpha z) = \sum_{j=0}^{n-1} (\alpha z)^{2^n} + f\big((\alpha z)^{2^n}\big).$$

Subtracting these two equations and recalling that $\alpha^{2^n} = 1$, we find that

$$f(z) - f(\bar{z}) = \sum_{j=0}^{n-1} (1 - \alpha^{2^n}) z^{2^n}.$$

Note that $\alpha \neq 1$ because we have chosen $z \notin \mathbb{R}$, therefore this equation provides a polynomial with algebraic coefficients of degree at least 1 in the variable z ($f(\bar{z}) = \overline{f(z)}$ is algebraic). Such a polynomial has z as root, contradicting the fact that z is transcendental. ■

3. Structure of the proof of Theorem 1.1

For convenience, we change the variable we use by composing with the function e. The domain of our functions changes from the disk D to the half plane H. For $w \in H$, put

$$F(w) = f\big(e(w)\big),$$
$$P(w) = \phi\big(e(w)\big).$$

Doing so, the functional equation defining f becomes more manageable.

$$F(w) = P(w) + F(aw).$$

When f is the Fredholm series, the proof is actually contained in [10]. It articulates as follows.

(1) We find an approximate formula for $F = f \circ e$ evaluated at some points near the boundary of H. It will equal a sum of exponentials and a remainder. The remainder is F evaluated at some point which we keep away from the boundary.

(2) Part of the sum will vanish using Ramanujan sums.

(3) What we are left with is an expression for F evaluated at some points near the boundary. Inside suitably small disks it almost equals, with arbitrary precision, a function S (with a rescaled domain) plus a constant number. S only depends on f.

(4) Thanks to Rouchè's theorem, we find a zero of F if S plus that constant is zero.

(5) We note that the image of S is invariant under translation by some constants c_m and $\mathbb{Z}$.

(6) We prove that the group generated by $\Re(c_m)$ and $\mathbb{Z}$ is dense in $\mathbb{R}$, hence we conclude that S is surjective.

For the general case, we just change the sum of exponentials (evaluations of the function e) with a sum of evaluations of $P = \phi \circ e$. We also use a different kind of Ramanujan sums which do not vanish but get arbitrarily small.

4. An intermediate proposition

Proposition 4.1. *Let $\phi(z) = \sum_{n=1}^{\infty} d_n z^n$ be a series where $d_n \in \mathbb{R}$ are such that $\sum_{n=1}^{\infty} n|d_n|$ converges and $\phi(1) \neq 0$. Let $a \geq 2$ be an integer and $f : D \to \mathbb{C}$ a holomorphic function such that $f(z) = \phi(z) + f(z^a)$ for every $z \in D$. If the subgroup of $\mathbb{R}$ generated by $\phi(1)$ and by the real part of the constants*

$$c_m = \sum_{l=1}^{\infty} \left[\phi\left(e\left(\frac{m}{a^l} \right) \right) - \phi(1) \right]$$

for m varying in $\mathbb{Z}$ is dense in $\mathbb{R}$, and if there exists at least one c_m with non-zero imaginary part, then f has the Picard property.

Note that both ϕ and ϕ' converge on $\bar{D}$ because $\sum_{n=1}^{\infty} n|d_n|$ converges. Then the series that defines c_m converges because $\phi \circ e$ is Lipschitz.

The proof of this proposition follows steps 1 through 5 of the aforementioned plan. This proposition brings us close to Theorem 1.1. In many cases, for example with the Fredholm series, proving the density of the group generated by $\Re(c_m)$ and $\phi(1)$ (step 6 generalized) is quite easy. In general, even verifying that they do not all vanish is a crucial difficulty. We will address this problem later.

4.1. About certain sums of Ramanujan type

This section is done in order to accomplish step 2 of the plan. Given the integer a and the function $P = \phi \circ e$, we would like to find an integer q such that the sum $\sum_{m=0}^{\varphi(q)-1} P(\frac{a^m}{q})$ is small.

Lemma 4.2. *Put $q = p^k$. If p is an odd prime and $a \equiv 1 \bmod p$ (or $p = 2$ and $a \equiv 1 \bmod 4$) but $a \not\equiv 1 \bmod q$, then*

$$\sum_{m=0}^{\varphi(q)-1} e\left(\frac{a^m}{q} \right) = 0.$$

Proof. Let $t = v_p(a-1) < k$ be the p-adic valuation of $a-1$. It is enough to show that

$$\{a^m | 0 \le m \le \varphi(q)-1\} = \{x | x \equiv 1 \bmod p^t\}$$

modulo q. Clearly, $a \equiv 1 \bmod p^t$ implies that $a^m \equiv 1 \bmod p^t$. On the other hand, both sets have cardinality p^{k-t}. This holds for the left-hand side set since the multiplicative order of a modulo q is p^{k-t}. It follows easily by the *lifting-the-exponent lemma*: $v_p(a^m-1) = v_p(a-1) + v_p(m)$. ■

An immediate generalization that will be needed is the following. Put $t = v_p(a-1)$ the p-adic valuation of $a-1$. Under the same hypothesis, if n is an integer not divisible by p^{k-t}, then also

$$\sum_{m=0}^{\varphi(q)-1} e\left(\frac{a^m n}{q}\right) = 0.$$

Let now s be an integer coprime with p. If $n < p^{k-t}$, we can write

$$\sum_{m=0}^{\varphi(q)-1} e\left(\frac{a^m s}{q}\right)^n = \sum_{m=0}^{\varphi(q)-1} e\left(\frac{a^m sn}{q}\right) = 0$$

and thus if $\varphi(z) = \sum_{n=1}^{\infty} d_n z^n$,

$$\left|\sum_{m=0}^{\varphi(q)-1} P\left(\frac{a^m s}{q}\right)\right| \le \varphi(q) \sum_{n=q/a}^{\infty} |d_n| \le a \sum_{n=q/a}^{\infty} n|d_n|. \tag{3}$$

If $a \ne 2, 3$ is an assigned integer, choose p an odd prime which divides $a-1$ (or $p=2$ if 4 divides $a-1$). Then the inequality (3) holds with $q = p^k$ for every integer k big enough so that $a \not\equiv 1 \bmod q$. Instead, if $a = 2$ or $a = 3$, a prime p with the required property cannot be found and we must follow another path (exactly as Zannier proposed in [10]).

In the case $a = 2$ (putting $p = 3$) and in the case $a = 3$ (putting $p = 5$), a is a primitive root modulo p^k for every k. Put $q = p^k$ for $k \ge 2$. If n is an integer not divisible by p^{k-2} and if s is coprime with p, we obtain

$$\sum_{m=0}^{\varphi(q)-1} e\left(\frac{a^m sn}{q}\right) = 0$$

and thus if $\varphi(z) = \sum_{n=1}^{\infty} d_n z^n$, then similarly to before

$$\left|\sum_{m=0}^{\varphi(q)-1} P\left(\frac{a^m s}{q}\right)\right| \le \varphi(q) \sum_{n=q/p^2}^{\infty} |d_n| \le p^2 \sum_{n=q/p^2}^{\infty} n|d_n|. \tag{4}$$

4.2. An approximate formula

Given a, choose p such that the inequality (3) or (4) holds with $q = p^k$ for k big enough. Choose r a power of p such that $r \equiv 1 \bmod a$. For every integer $t \geq 1$, define $k = a^t$, $q = r^k$ and choose any $s \equiv 1 \bmod ra^t$. Note that q is coprime with s and that $q > a$. Put $\theta_0 = \frac{s}{q}$ and $n_0 = \varphi(q)$. By induction we observe that $q \equiv 1 \bmod a^t$. For every $l \leq t$ and for every $n \geq l$, multiplying the congruence $\theta_0 \equiv (1-q)\theta_0 \bmod \mathbb{Z}$ by a^{n-l} we obtain

$$a^{n-l}\theta_0 \equiv a^n \frac{1-q}{a^l}\theta_0 \mod \mathbb{Z}.$$

Since $a^{n_0} \equiv 1 \bmod q$ and $q \equiv s \equiv 1 \bmod a^t$, we find that, for $0 \leq l \leq t \leq n_0$,

$$a^{n_0-l}\theta_0 \equiv a^{n_0}\frac{1-q}{a^l}\theta_0 \equiv \frac{1-q}{a^l}\theta_0 = \frac{\theta_0 - s}{a^l} \equiv \frac{\theta_0 - 1}{a^l} \mod \mathbb{Z} \tag{5}$$

(the second congruence is the only one not obvious).

We can write the following equalities:

$$F\left(\frac{w}{a^{n_0}} + \theta_0\right) = \sum_{l=0}^{n_0-1} P\left(\frac{w}{a^{n_0-l}} + a^l\theta_0\right) + F(w + a^{n_0}\theta_0) \tag{6}$$

$$= \sum_{l=0}^{n_0-1} P\left(\frac{w}{a^{n_0-l}} + a^l\theta_0\right) + F(w + \theta_0) \tag{7}$$

$$= \sum_{l=0}^{n_0-1} \left[P\left(\frac{w}{a^{n_0-l}} + a^l\theta_0\right) - P(a^l\theta_0)\right] + F(w+\theta_0) + \Delta$$

$$= \sum_{l=1}^{t} \left[P\left(\frac{w}{a^l} + \frac{\theta_0 - 1}{a^l}\right) - P\left(\frac{\theta_0 - 1}{a^l}\right)\right]$$

$$+ \sum_{l=t+1}^{n_0} \left[P\left(\frac{w}{a^l} + a^{n_0-l}\theta_0\right) - P(a^{n_0-l}\theta_0)\right] + F(w+\theta_0) + \Delta, \tag{8}$$

where $\Delta = \sum_{l=0}^{n_0-1} P(a^l\theta_0)$ satisfies by (3) or (4) (depending on a) $|\Delta| \leq a\sum_{n=q/a}^{\infty} n|d_n|$ or $|\Delta| \leq p^2 \sum_{n=q/p^2}^{\infty} n|d_n|$ when t is big enough. In (6), we use recursively the functional equation $F(z) = P(z) + F(az)$. Formula (7) is valid because $a^{n_0}\theta_0 \equiv \theta_0 \bmod \mathbb{Z}$ and F is 1-periodic. Finally, in (8) we change the variable l with $n_0 - l$ and we use (5).

Note that for fixed a,

$$\lim_{t\to\infty} |\Delta| = \lim_{q\to\infty} |\Delta| = 0$$

since $\sum_{n=1}^{\infty} n|d_n|$ converges by hypothesis.

4.3. Two auxiliary functions

In this section, we accomplish steps 3 and 4 of the plan. Similarly to what was done in [10], we define for $w \in H$ two new functions:

$$G(w) = \sum_{l=1}^{\infty} \left[P\left(\frac{w-1}{a^l}\right) - P\left(-\frac{1}{a^l}\right) \right],$$
$$S(w) = F(w) + G(w).$$

Fix α real. For every t, we look for an integer $n_0 = n_0(t)$ and a real number $\theta_0 = \theta_0(t)$ which, when t tends to infinity, tend respectively to infinity and to α, such that the number

$$F\left(\frac{w}{a^{n_0}} + \theta_0\right) - S(w+\alpha) + G(\alpha)$$

tends to 0 when t tends to infinity, uniformly for w varying in a compact. Through the use of Rouchè's theorem, this fact will lead to the following proposition.

Proposition 4.3. *Let $\alpha \in \mathbb{R}$ and let U be an open disk centered in $e(\alpha)$. For all $\zeta \in H$, the function f takes on $U \cap D$ the value $S(\zeta) - G(\alpha)$ infinitely many times.*

Choose t and s such that

$$|\theta_0 - \alpha| \leq \frac{ra^t}{q} = \frac{k}{r^{k-1}} < a^{-t},$$

where $k = a^t$, $q = r^k$, $s \equiv 1 \bmod ra^t$, and $\theta_0 = \frac{s}{q}$ are defined as usual. The first inequality can always be satisfied by choosing the right s, the second one holds for k big enough. Put $n_0 = \varphi(q)$.

Fix $K \subset H$ a compact set. Fix α too, and let t vary. F is locally Lipschitz (that is on compact sets) over H since it is of class C^1, while P is Lipschitz over the whole $\overline{H}$ because $\sum_{n=1}^{\infty} n|d_n|$ converges, hence ϕ and ϕ' converge on $\overline{D}$. Therefore, there exist some constants C_1, C_2, C_3 (dependent on K but not on t) such that, for all $w \in K$,

$$\sum_{l=t+1}^{n_0} \left| P\left(\frac{w}{a^l} + a^{n_0-l}\theta_0\right) - P(a^{n_0-l}\theta_0) \right| \leq C_1 \sum_{l=t+1}^{n_0} \left| \frac{w}{a^l} \right| \leq C_1|w|a^{-t} \leq C_2 a^{-t},$$

$$\sum_{l=1}^{t} \left| P\left(\frac{w+\alpha-1}{a^l}\right) - P\left(\frac{w+\theta_0-1}{a^l}\right) \right| \leq C_1 \sum_{l=1}^{t} \left| \frac{\alpha-\theta_0}{a^l} \right| \leq C_1|\alpha-\theta_0| < C_1 a^{-t},$$

$$\sum_{l=1}^{t} \left| P\left(\frac{\theta_0-1}{a^l}\right) - P\left(\frac{\alpha-1}{a^l}\right) \right| \leq C_1 \sum_{l=1}^{t} \left| \frac{\alpha-\theta_0}{a^l} \right| \leq C_1|\alpha-\theta_0| < C_1 a^{-t},$$

$$\sum_{l=t+1}^{\infty}\left|P\left(\frac{w+\alpha-1}{a^l}\right)-P\left(\frac{\alpha-1}{a^l}\right)\right|\leq C_1\sum_{l=t+1}^{n_0}\left|\frac{w}{a^l}\right|\leq C_1|w|a^{-t}\leq C_2a^{-t},$$
$$|F(w+\theta_0)-F(w+\alpha)|\leq C_3|\theta_0-\alpha|<C_3a^{-t}.$$

Also note that

$$\begin{aligned}G(w+\alpha)-G(\alpha)-\sum_{l=1}^{t}&\left[P\left(\frac{w}{a^l}+\frac{\theta_0-1}{a^l}\right)-P\left(\frac{\theta_0-1}{a^l}\right)\right]\\ &=\sum_{l=1}^{t}\left[P\left(\frac{w+\alpha-1}{a^l}\right)-P\left(\frac{w+\theta_0-1}{a^l}\right)\right]\\ &\quad+\sum_{l=1}^{t}\left[P\left(\frac{\theta_0-1}{a^l}\right)-P\left(\frac{\alpha-1}{a^l}\right)\right]\\ &\quad+\sum_{l=t+1}^{\infty}\left[P\left(\frac{w+\alpha-1}{a^l}\right)-P\left(\frac{\alpha-1}{a^l}\right)\right].\end{aligned}$$

We can now use formula (8) and the previous bounds to estimate the quantity we are interested in (remember that $S=F+G$):

$$\begin{aligned}&\left|F\left(\frac{w}{a^{n_0}}+\theta_0\right)-S(w+\alpha)+G(\alpha)\right|\\ &=\left|\sum_{l=1}^{t}\left[P\left(\frac{w}{a^l}+\frac{\theta_0-1}{a^l}\right)-P\left(\frac{\theta_0-1}{a^l}\right)\right]\right.\\ &\quad+\sum_{l=t+1}^{n_0}\left[P\left(\frac{w}{a^l}+a^{n_0-l}\theta_0\right)-P\left(a^{n_0-l}\theta_0\right)\right]\\ &\quad\left.+F(w+\theta_0)-F(w+\alpha)-G(w+\alpha)+G(\alpha)+\Delta\right|\\ &\leq\left|\sum_{l=t+1}^{n_0}\left[P\left(\frac{w}{a^l}+a^{n_0-l}\theta_0\right)-P(a^{n_0-l}\theta_0)\right]\right|+\left|F(w+\theta_0)-F(w+\alpha)\right|\\ &\quad+\left|-G(w+\alpha)+G(\alpha)+\sum_{l=1}^{t}\left[P\left(\frac{w}{a^l}+\frac{\theta_0-1}{a^l}\right)-P\left(\frac{\theta_0-1}{a^l}\right)\right]\right|+|\Delta|\\ &<\frac{2C_1+2C_2+C_3}{a^t}+|\Delta|=\frac{C_4}{a^t}+|\Delta|\end{aligned}\tag{9}$$

for a constant C_4. $\Delta=\sum_{l=0}^{n_0-1}P(a^l\theta_0)$ satisfies by (3) or (4) (depending on a) $|\Delta|\leq a\sum_{n=q/a}^{\infty}n|d_n|$ or $|\Delta|\leq p^2\sum_{n=q/p^2}^{\infty}n|d_n|$ when t is big enough. Hence $|\Delta|$ tends to zero when t tends to infinity.

Put $v = S(\zeta)$ and define

$$F_t(w) = F\left(\frac{w}{a^{n_0}} + \theta_0\right),$$

where n_0 and θ_0 depend on t as usual. In order to complete the proof of Proposition 4.3, consider a disk C centered in $\zeta - \alpha$ such that $\overline{C} \subseteq H$ and such that $S(w+\alpha) \neq v$ for every w on the boundary of C. It exists since S is not constant, otherwise $\lim_{x\to 0^+} F(ix) = S - \lim_{x\to 0^+} G(ix) = S - G(0) = S$ while the left side term diverges (see the functional equation). When t is big enough, we have

$$\left|\left(F_t(w) - v + G(\alpha)\right) - \left(S(w+\alpha) - v\right)\right| < \frac{C_4}{a^t} + |\Delta| < \left|S(w+\alpha) - v\right|$$

for every w on the boundary of C since the central term goes to zero when t tends to infinity, while the right one has a positive minimum (for w varying on the boundary of C) which does not depend on t. The functions $F_t(w) - v + G(\alpha)$ and $S(w+\alpha) - v$ are holomorphic on the whole H, hence the inequality allows us to apply Rouchè's theorem to obtain that these two functions have the same number of zeros (counted with multiplicity) inside C. The second one has a zero at $\zeta - \alpha$, thus there exists in C a zero ζ_t of the first function. This means that

$$F\left(\frac{\zeta_t}{a^{n_0}} + \theta_0\right) = F_t(\zeta_t) = S(\zeta) - G(\alpha).$$

Letting t rise to infinity, the quantity $\frac{\zeta_t}{a^t} + \theta_0$ tends to α because $\zeta_t \in C$ is contained in a fixed compact and $|\theta_0 - \alpha| < a^{-t}$. By the continuity of e, the numbers $e(\frac{\zeta_t}{a^t} + \theta_0)$ tend to $e(\alpha)$. Therefore, we have found a sequence of complex numbers that tends to $e(\alpha)$ at which f takes the value $S(\zeta) - G(\alpha)$. Proposition 4.3 is proven.

4.4. The image of S

From Proposition 4.3, the intermediate proposition follows if we ensure S to be surjective. That is exactly guaranteed by the hypothesis on the density of the group, since we will show that the image of S is invariant under translation by c_m, $\phi(1)$, and $-\phi(1)$.

Again, we proceed as in [10]. It is more convenient to study the image of

$$S_1(w) := S(w+1) = F(w) + \sum_{l=1}^{\infty}\left[P\left(\frac{w}{a^l}\right) - P\left(-\frac{1}{a^l}\right)\right].$$

Remember that the constants c_m are defined for every integer m as

$$c_m = \sum_{l=1}^{\infty}\left[P\left(\frac{m}{a^l}\right) - P(0)\right].$$

Let us see where the c_m appear. For every m and s integers and $w \in H$,

$$\begin{aligned}
&S_1(w+ma^s) - S_1(w)\\
&\quad= \sum_{l=0}^{\infty}\left[P\left(\frac{w+ma^s}{a^l}\right) - P\left(\frac{w}{a^l}\right)\right]\\
&\quad= \sum_{l=s+1}^{\infty}\left[P\left(\frac{w+ma^s}{a^l}\right) - P\left(\frac{w}{a^l}\right)\right]\\
&\quad= \sum_{l=1}^{\infty}\left[P\left(\frac{w+ma^s}{a^{l+s}}\right) - P\left(\frac{m}{a^l}\right) - P\left(\frac{w}{a^{l+s}}\right) + P(0)\right] + c_m\\
&\quad= \Delta_{m,s}(w) + c_m.
\end{aligned}$$

We should now verify that $\Delta_{m,s}(w)$ is very small. Fix a compact set K. By the Lipschitz property of P, there exist constants C_1 and C_2 (independent from m and s) such that for all $w \in K$ we have

$$\begin{aligned}
\left|\Delta_{m,s}(w)\right| &\le \sum_{l=1}^{\infty}\left|P\left(\frac{wa^{-s}+m}{a^l}\right) - P\left(\frac{m}{a^l}\right)\right| + \sum_{l=1}^{\infty}\left|P\left(\frac{w}{a^{l+s}}\right) - P(0)\right|\\
&\le 2C_1\sum_{l=1}^{\infty}\left|\frac{w}{a^{l+s}}\right| \le 2C_1\frac{|w|}{a^s} \le \frac{C_2}{a^s}.
\end{aligned}$$

If $v = S_1(w_0)$ is a point on the image of S_1, consider the holomorphic functions (in the variable $w \in H$) $S_1(w) - v$ and $S_1(w+ma^s) - c_m - v$. Like before, through Rouchè's theorem we obtain that there exists w such that $S_1(w+ma^s) = S_1(w_0) + c_m$. Then $S_1(w_0) + c_m \in S_1(H)$. Moreover, we easily note that

$$\begin{aligned}
S_1(aw) &= F(aw) + \sum_{l=1}^{\infty}\left[P\left(\frac{aw}{a^l}\right) - P\left(-\frac{1}{a^l}\right)\right]\\
&= F(aw) + \sum_{l=1}^{\infty}\left[P\left(\frac{aw}{a^l}\right) - P(0)\right] - \sum_{l=1}^{\infty}\left[P\left(-\frac{1}{a^l}\right) - P(0)\right]\\
&= F(w) - P(w) + \sum_{l=2}^{\infty}\left[P\left(\frac{aw}{a^l}\right) - P(0)\right]\\
&\quad + P(w) - P(0) - \sum_{l=1}^{\infty}\left[P\left(-\frac{1}{a^l}\right) - P(0)\right]\\
&= F(w) + \sum_{l=1}^{\infty}\left[P\left(\frac{w}{a^l}\right) - P\left(-\frac{1}{a^l}\right)\right] - P(0) = S_1(w) - P(0).
\end{aligned}$$

Hence $S_1(w) - P(0) \in S_1(H)$ and also $S_1(aw) + P(0) \in S_1(H)$. We have proven that $S_1(H)$ is invariant under translation by c_m, $\phi(1)$, and $-\phi(1)$. Thus it is invariant also under translation by any element of the semigroup generated by $\mathbb{Z}P(0) = \mathbb{Z}\phi(1)$ and $\Re c_m = c_m + c_{-m}$ (we do not know if it is invariant under translation by $-\Re c_m$). This semigroup is dense in $\mathbb{R}$ since, by hypothesis, the group is dense too (the closure of the group and the closure of the semigroup coincide because among the generators there are both $\phi(1)$ and $-\phi(1) \neq 0$).

Thus if $s \in S_1(H)$, then $s + r \in S_1(H)$ for all $r \in \mathbb{R}$ thanks to the open mapping theorem, since $s + a \in S_1(H)$ for all a in the dense group. Moreover, for some m, $\Im(c_m) = -\Im(c_{-m}) \neq 0$ (by hypothesis), therefore $S_1(H)$ contains numbers with arbitrarily big and arbitrarily small imaginary parts. The image of S_1 is connected given that H is connected and S_1 is continuous. Then $S_1(H)$ contains numbers with any imaginary part. Along with the previous observation, we conclude that $S_1(H) = \mathbb{C}$.

This completes the proof of the intermediate proposition (Proposition 4.1).

5. Some lemmas about real-valued analytic functions

In order to prove Theorem 1.1, we have to verify the density of the group generated by $\Re(c_m)$ and $\phi(1)$, and the existence of a non-real c_m. To do so, we need some lemmas.

In this paper, a function $f : \mathbb{R} \to \mathbb{R}$ of class C^∞ is called *analytic* if for all $x_0 \in \mathbb{R}$ there exists $\varepsilon > 0$ such that the Taylor series

$$\sum_{n=0}^{\infty} \frac{f^n(x_0)}{n!}(x - x_0)^n$$

converges for all x such that $|x - x_0| < \varepsilon$ and the limit of the series is $f(x)$.

Lemma 5.1. *Let $h : \mathbb{R} \to \mathbb{R}$ be an analytic and periodic function with period 1 with $h(0) = 0$. Let $a \geq 2$ be an integer. Then the series $\sum_{n=1}^{\infty} h(\frac{x}{a^n})$ converges for all x real. Moreover, the function $H : \mathbb{R} \to \mathbb{R}$*

$$H(x) = \sum_{n=1}^{\infty} h\left(\frac{x}{a^n}\right) \tag{10}$$

which satisfies $H(x) = h(\frac{x}{a}) + H(\frac{x}{a})$ is analytic.

Proof. There is $\varepsilon > 0$ such that, if $|x| < \varepsilon$, then

$$\big|h(x)\big| \leq \big(\big|h'(0)\big| + 1\big)|x|$$

since $h(0) = 0$. Fix $M > 0$. Let n_0 be an integer such that $a^{n_0} > \frac{M}{\varepsilon}$, then for all x

such that $|x| < M$,

$$\left|h\left(\frac{x}{a^n}\right)\right| \le \sup|h| =: M_n \qquad \text{for } 1 \le n \le n_0 - 1,$$

$$\left|h\left(\frac{x}{a^n}\right)\right| \le \left(\left|h'(0)\right| + 1\right)\frac{M}{a^n} =: M_n \quad \text{for } n \ge n_0.$$

Then $\sum_{n=1}^{\infty} M_n < \infty$ and so by the Weierstrass criterion the series (10) converges absolutely and uniformly on the compact sets. Then the limit is analytic because so are the addends. ■

LEMMA 5.2. *Let $h : \mathbb{R} \to \mathbb{R}$ be an analytic and periodic function with period 1 with $h(0) = 0$. Let $a \ge 2$ be an integer. Define the analytic function (by Lemma 5.1) $H : \mathbb{R} \to \mathbb{R}$*

$$H(x) = \sum_{n=1}^{\infty} h\left(\frac{x}{a^n}\right).$$

If there is k_0 such that for every integer $k > k_0$, $H(1 + a^k) = H(1 + a^{k+1})$, then H is periodic with period 1.

PROOF. First of all, we prove by induction on s that, for every $s \ge 1$,

$$H^{(s)}(1) = \frac{1}{a^s - 1} h^{(s)}(0). \tag{11}$$

Fix s. Assume that for every $1 \le t < s$ we know $(a^t - 1)H^{(t)}(1) = h^{(t)}(0)$, and we want to prove (11). We achieve this with some calculations starting from the hypothesis

$$0 = H(1 + a^k) - H(1 + a^{k+1}).$$

For every $k > k_0$, we can write

$$\begin{aligned}
0 &= H(1 + a^k) - H(1 + a^{k+1}) \\
&= \sum_{n=k+1}^{\infty} \left[h\left(\frac{1 + a^k}{a^n}\right) - h\left(\frac{1 + a^{k+1}}{a^n}\right)\right] \\
&= \sum_{n=k+1}^{\infty} \left[h\left(\frac{1 + a^k}{a^n}\right) - h\left(\frac{1 + a^{k+1}}{a^{n+1}}\right)\right] - h\left(\frac{1 + a^{k+1}}{a^{k+1}}\right) \\
&= \sum_{n=1}^{\infty} \left[h\left(\frac{1/a^k + 1}{a^n}\right) - h\left(\frac{1/a^{k+1} + 1}{a^n}\right)\right] - h\left(\frac{1}{a^{k+1}}\right) \\
&= \sum_{n=1}^{\infty} \left[h\left(\frac{1/a^k + 1}{a^n}\right) - h\left(\frac{1/a^{k+1} + 1}{a^n}\right)\right] + h(0) - h\left(\frac{1}{a^{k+1}}\right).
\end{aligned}$$

Now we approximate these differences with a Taylor series of order s and we introduce the Lagrange remainder

$$= \sum_{n=1}^{\infty}\sum_{m=1}^{s-1}\frac{1}{m!}h^{(m)}\left(\frac{1/a^{k+1}+1}{a^n}\right)\left(\frac{a-1}{a^{k+1+n}}\right)^m - \sum_{m=1}^{s-1}\frac{1}{m!}h^{(m)}(0)\left(\frac{1}{a^{k+1}}\right)^m$$
$$+ \sum_{n=1}^{\infty}\frac{1}{s!}h^{(s)}(\zeta_n)\left(\frac{a-1}{a^{k+1+n}}\right)^s - \frac{1}{s!}h^{(s)}(\zeta_0)\left(\frac{1}{a^{k+1}}\right)^s$$

for some $\frac{1/a^k+1}{a^n} \leq \zeta_n \leq \frac{1/a^{k+1}+1}{a^n}$ and $0 \leq \zeta_0 \leq \frac{1}{a^{k+1}}$. We continue from the last expression using the inductive hypothesis:

$$\begin{aligned}
(12) \qquad &= \sum_{n=1}^{\infty}\sum_{m=1}^{s-1}\frac{1}{m!}h^{(m)}\left(\frac{1/a^{k+1}+1}{a^n}\right)\left(\frac{a-1}{a^{k+1+n}}\right)^m \\
&\quad - \sum_{n=1}^{\infty}\sum_{m=1}^{s-1}\frac{1}{m!}(a^m-1)h^{(m)}\left(\frac{1}{a^n}\right)\left(\frac{1}{a^{k+1+n}}\right)^m \\
&\quad + \sum_{n=1}^{\infty}\frac{1}{s!}h^{(s)}(\zeta_n)\left(\frac{a-1}{a^{k+1+n}}\right)^s - \frac{1}{s!}h^{(s)}(\zeta_0)\left(\frac{1}{a^{k+1}}\right)^s \\
&= \sum_{n=1}^{\infty}\sum_{m=1}^{s-1}\frac{1}{m!}\left(\frac{1}{a^{k+1+n}}\right)^m\Bigg[(a-1)^m h^{(m)}\left(\frac{1/a^{k+1}+1}{a^n}\right) \\
&\qquad\qquad - (a^m-1)h^{(m)}\left(\frac{1}{a^n}\right)\Bigg] \\
&\quad + \sum_{n=1}^{\infty}\frac{1}{s!}h^{(s)}(\zeta_n)\left(\frac{a-1}{a^{k+1+n}}\right)^s - \frac{1}{s!}h^{(s)}(\zeta_0)\left(\frac{1}{a^{k+1}}\right)^s.
\end{aligned}$$

By induction on m_0, it may be observed that, for every $m_0 \geq 1$, we have

$$\begin{aligned}
&\sum_{n=1}^{\infty}\sum_{m=1}^{s-1}\frac{1}{m!}\left(\frac{1}{a^{k+1+n}}\right)^m\left[(a-1)^m h^{(m)}\left(\frac{1/a^{k+1}+1}{a^n}\right) - (a^m-1)h^{(m)}\left(\frac{1}{a^n}\right)\right] \\
&\quad = \sum_{n=1}^{\infty}\sum_{m=m_0}^{s-1}\frac{1}{m!}\left(\frac{1}{a^{k+1+n}}\right)^m\Bigg[(a-1)^m h^{(m)}\left(\frac{1/a^{k+1}+1}{a^n}\right) \\
&\qquad\qquad - \left(a^m - 1 - \sum_{j=1}^{m_0-1}(a-1)^j\binom{m}{j}\right)h^{(m)}\left(\frac{1}{a^n}\right)\Bigg] \\
&\quad + \sum_{n=1}^{\infty}\sum_{j=1}^{m_0-1}\frac{1}{s!}\binom{s}{j}(a-1)^j h^{(s)}(\zeta_{n,j})\left(\frac{1}{a^{k+1+n}}\right)^s
\end{aligned}$$

for certain $\frac{1}{a^n} \le \zeta_{n,j} \le \frac{1/a^{k+1}+1}{a^n}$. Putting $m_0 = s-1$ and substituting the last expression in (12), we get

$$0 = \sum_{n=1}^{\infty} \frac{1}{(s-1)!}\left(\frac{1}{a^{k+1+n}}\right)^{s-1}\left[(a-1)^{s-1}h^{(s-1)}\left(\frac{1/a^{k+1}+1}{a^n}\right) - (a-1)^{s-1}h^{(s-1)}\left(\frac{1}{a^n}\right)\right]$$
$$+\sum_{n=1}^{\infty}\frac{1}{s!}h^{(s)}(\zeta_n)\left(\frac{a-1}{a^{k+1+n}}\right)^s - \frac{1}{s!}h^{(s)}(\zeta_0)\left(\frac{1}{a^{k+1}}\right)^s$$
$$+\sum_{n=1}^{\infty}\sum_{j=1}^{s-2}\frac{1}{s!}\binom{s}{j}(a-1)^j h^{(s)}(\zeta_{n,j})\left(\frac{1}{a^{k+1+n}}\right)^s.$$

Here again we use the Lagrange remainder to simplify the first line of the previous equation. We are left with a linear combination of $h^{(s)}$ valued at some points:

$$0 = \sum_{n=1}^{\infty} \frac{1}{(s-1)!}\left(\frac{1}{a^{k+1+n}}\right)^{s}(a-1)^{s-1}h^{(s)}(\zeta'_n)$$
$$+\sum_{n=1}^{\infty}\frac{1}{s!}h^{(s)}(\zeta_n)\left(\frac{a-1}{a^{k+1+n}}\right)^s - \frac{1}{s!}h^{(s)}(\zeta_0)\left(\frac{1}{a^{k+1}}\right)^s$$
$$+\sum_{n=1}^{\infty}\sum_{j=1}^{s-2}\frac{1}{s!}\binom{s}{j}(a-1)^j h^{(s)}(\zeta_{n,j})\left(\frac{1}{a^{k+1+n}}\right)^s$$

for certain $\frac{1}{a^n} \le \zeta'_n \le \frac{1/a^{k+1}+1}{a^n}$. Multiply this last expression by $s!a^{(k+1)s}$ and take the limit for k which tends to infinity. Remembering that $\zeta_{n,j}$, ζ_n, and ζ'_n tend to $\frac{1}{a^n}$ $(n > 0)$ and that ζ_0 tends to 0, we finally find that

$$0 = \sum_{n=1}^{\infty} s\left(\frac{1}{a^n}\right)^s (a-1)^{s-1}h^{(s)}\left(\frac{1}{a^n}\right)$$
$$+\sum_{n=1}^{\infty} h^{(s)}\left(\frac{1}{a^n}\right)\left(\frac{a-1}{a^n}\right)^s - h^{(s)}(0)$$
$$+\sum_{n=1}^{\infty}\sum_{j=1}^{s-2}\binom{s}{j}(a-1)^j h^{(s)}\left(\frac{1}{a^n}\right)\left(\frac{1}{a^n}\right)^s$$
$$= \sum_{n=1}^{\infty}\left(\frac{1}{a^n}\right)^s (a^s-1)h^{(s)}\left(\frac{1}{a^n}\right) - h^{(s)}(0).$$

We get (11) as wanted:

$$(a^s-1)H^{(s)}(1)=\sum_{n=1}^{\infty}\left(\frac{1}{a^n}\right)^s(a^s-1)h^s\left(\frac{1}{a^n}\right)=h^{(s)}(0).$$

Therefore, we find the equality of the derivatives of order $s\geq 1$ at 0 and 1:

$$H^{(s)}(1)=\frac{1}{a^s-1}h^{(s)}(0)=\sum_{n=1}^{\infty}\frac{1}{a^{ns}}h^{(s)}(0)=H^{(s)}(0).$$

From this it follows, by the analyticity of H, that $H(x+1)=H(x)+c$ for some constant c. Our hypothesis was $H(1+a^k)=H(1+a^{k+1})$ for some k (and $a\geq 2$ is an integer), then $c=0$, and hence H is periodic. ∎

Lemma 5.3. *Let $h:\mathbb{R}\to\mathbb{R}$ be an analytic and periodic function with period 1 with $h(0)=0$. Let $a\geq 2$ be an integer. Define (as above) the function $H:\mathbb{R}\to\mathbb{R}$*

$$H(x)=\sum_{n=1}^{\infty}h\left(\frac{x}{a^n}\right).$$

If H is 1-periodic, then h has zero integral in a period.

Proof. Note that H is well defined and continuous (Lemma 5.1 does not use the analyticity for this step). Since a is an integer and H is 1-periodic,

$$\frac{1}{a}\int_0^a H(x)dx=\int_0^1 H(x)dx.$$

It suffices to observe that $h(x)=H(ax)-H(x)$, hence changing variable we see

$$\begin{aligned}\int_0^1 h(x)dx&=\int_0^1 H(ax)dx-\int_0^1 H(x)dx\\&=\frac{1}{a}\int_0^a H(x)dx-\int_0^1 H(x)dx\\&=\int_0^1 H(x)dx-\int_0^1 H(x)dx=0.\end{aligned}$$ ∎

6. Proof of Theorem 1.1

Since $\phi(z)=\sum_{n=1}^{\infty}d_nz^n$ has radius of convergence greater than 1, $\sum_{n=1}^{\infty}n|d_n|$ converges. By the previous intermediate proposition (Proposition 4.1), it suffices to prove that the group generated by $\phi(1)$ and by the real parts of

$$c_m = \sum_{l=1}^{\infty}\left[\phi\left(e\left(\frac{m}{a^l}\right)\right) - \phi(1)\right]$$

is dense in $\mathbb{R}$, and that there exists some c_m not real. We need Lemmas 5.2 and 5.3.

The hypothesis that ϕ has radius of convergence greater than 1 will help us, but in some particular cases may not be needed. For example, the function

$$f(z) = \sum_{m,n\geq 0} \frac{z^{2^n 3^m}}{4^m}$$

obtained by choosing $a = 2$ and $\phi(z) = \sum_{m=0}^{\infty} \frac{z^{3^m}}{4^m}$ has the Picard property too, as one can easily verify using Proposition 4.1.

6.1. *Density of the group generated by* $\phi(1)$ *and* $\Re c_m$

Put $\psi(z) = \phi(z) - \phi(1)$ and we must verify that the group generated by $\phi(1)$ and by the quantities

$$\sigma_m = \Re c_m = \sum_{l=1}^{\infty} \Re\psi\left(e\left(\frac{m}{a^l}\right)\right)$$

is dense in $\mathbb{R}$. If (by contradiction) that did not happen, given that $\phi(1) \neq 0$, the real numbers $\sigma_m/\phi(1)$ for varying m would form a set of rationals with bounded denominator. This will not be possible since for every $\varepsilon > 0$ we will find m_1 and m_2 such that $|\sigma_{m_1} - \sigma_{m_2}| < \varepsilon$ but $|\sigma_{m_1} - \sigma_{m_2}| \neq 0$. We choose $m_1 = 1 + a^k$ and $m_2 = 1 + a^{k+1}$ for k big enough.

First of all, it is easy to verify that $\sigma_{1+a^k} - \sigma_{1+a^{k+1}}$ tends to 0 when k tends to infinity. Now we have to prove that it cannot happen that $\sigma_{1+a^k} - \sigma_{1+a^{k+1}} = 0$ definitely in k. This would be a very particular case. As we can believe, it does not arise. Put for x real $h(x) = \Re\psi(e(x))$. It is an analytic function because it is the real part of a holomorphic function. We obtain

$$H(m) := \sum_{n=1}^{\infty} h\left(\frac{m}{a^n}\right) = \sigma_m.$$

If $\sigma_{1+a^k} - \sigma_{1+a^{k+1}} = 0$ definitely in k, we would have $H(1 + a^k) = H(1 + a^{k+1})$ for all $k > k_0$ for some k_0. Lemma 5.2 affirms that H is periodic. Apply Lemma 5.3 to H to find that h has zero integral on a period; that is the real part of ψ has mean value 0 on the boundary of the disk. This is not possible since the mean value on the boundary (the real part is a harmonic function) must coincide with the value at the center $\Re\psi(0) = \Re(\phi(0) - \phi(1)) = -\phi(1)$ which is not zero by hypothesis. We have proved that the group is dense.

6.2. *A non-real c_m*

To apply the intermediate proposition (Proposition 4.1), it remains to prove that there exists a c_m with non-zero imaginary part. In many particular cases, it is easy to verify this request. In general, we need some work. I thank Professor Zannier for the valuable contribution he gave me for completing the proof. Put as before $\psi(z) = \phi(z) - \phi(1)$ and consider the function

$$u(z) = \psi\big(e(z)\big) - \psi\big(e(-z)\big)$$

which is holomorphic on the stripe $-\varepsilon < \Im z < \varepsilon$ where $\varepsilon = \log(r) > 0$ if ψ is holomorphic on the disk of radius $r > 1$. Note that if z is real $u(z) = 2\Im\psi(e(z))$. Put

$$U(z) = \sum_{n=1}^{\infty} u\left(\frac{z}{a^n}\right)$$

which is holomorphic on the same stripe $-\varepsilon < \Im z < \varepsilon$ since $u(0) = 0$. Then

$$U(m) = \sum_{n=1}^{\infty} u\left(\frac{m}{a^n}\right) = 2\Im \sum_{n=1}^{\infty} \psi\left(e\left(\frac{m}{a^n}\right)\right) = 2\Im c_m \quad \text{for every integer } m.$$

If (by contradiction) $\Im c_m = 0$ for all integers m, then in particular $U(1 + a^k) = 2\Im c_{1+a^k} = 0$ for all integers k. Since u restricted to the real line is a real analytic function, Lemma 5.2 implies that U restricted to the real line is 1-periodic, thus it is 1-periodic on the whole stripe. Then there is a function $V : D(r) \setminus \overline{D}(1/r) \to \mathbb{C}$ such that

$$U(z) = V(q) \quad \text{for } q = e(z).$$

Let $V^+(q)$ be the holomorphic function on $D(r)$ obtained by summing the non-negative terms of the expansion of V in Laurent series. It holds that

$$V(q^a) - V(q) = U(az) - U(z) = u(z) = \psi(q) - \psi(q^{-1}),$$

therefore, since ψ is a holomorphic function and the expansion in Laurent series at a point is unique (also $a > 0$),

$$V^+(q^a) - V^+(q) = \psi(q) - \psi(0).$$

Inserting $q = 1$, we find that $0 = \psi(1) - \psi(0)$; that is $\psi(1) = \psi(0)$, so $\phi(1) = \phi(0)$ but $\phi(0) = 0$ and $\phi(1) \neq 0$ by hypothesis.

The proof of Theorem 1.1 is complete.

We do not know if Theorem 1.1 also holds for $d_n \in \mathbb{C}$ and not only when we suppose them real. Proposition 4.3 still holds. Moreover, the image of S is invariant under translation by c_m, $\phi(1)$, and $-\phi(1)$ but we are not able to prove S to be surjective with this information.

7. Estimate of the number of zeros

We discuss some quantitative estimations about the number of zeros contained in $D(r)$ of some function with the Picard property. We give an upper bound for the Fredholm series and a lower bound for every function that emerges from Theorem 1.1. We then improve this lower bound for the Fredholm series following Fuchs' approach.

All lower bounds will be of the form: there exists a constant $\mathcal{C}$ and some $r_0 < 1$ such that for every $r > r_0$ the number of zeros $N(r)$ contained in $D(r)$ is greater than $\mathcal{C} \cdot h(r)$ (for some function h with $\lim_{r\to 1^-} h(r) = +\infty$), that is $h(r) = O(N(r))$ for $r \to 1^-$.

7.1. Upper bound

Let g be a holomorphic function on $D(R)$. For $r < R$, let $a_1, \dots, a_{n(r)}$ be the zeros of g inside $\overline{D}(r)$ and let $n(r)$ be their number. Denote by $M(r)$ the maximum of the modulus of g on the boundary of $D(r)$. Suppose that $g(0) \neq 0$. For every k such that $1 < k < \frac{R}{r}$, we obtain by Jensen's formula

$$\begin{aligned} \log M(kr) &\geq \int_0^1 \log \left| g\big(kre(\theta)\big)\right| d\theta \\ &= \log\left|g(0)\right| + \sum_{l=1}^{n(kr)} \log\left(\frac{kr}{|a_l|}\right) \\ &\geq \log\left|g(0)\right| + \sum_{l=1}^{n(r)} \log\left(\frac{kr}{|a_l|}\right) \\ &\geq \log\left|g(0)\right| + n(r)\log k. \end{aligned} \tag{13}$$

We apply this inequality to the function $g(z) = f(z)/z$, where f is the Fredholm series. g is holomorphic on D and $g(0) = 1$. For every $1 < k < \frac{1}{r}$, from (13) we get ($n(r)$ is the number of zeros of g inside $\overline{D}(r)$, that is the ones of f inside $\overline{D}(r)$ different from 0)

$$n(r) \leq \frac{\log M(kr)}{\log k} < -\frac{\log(1-kr)}{\log k}$$

since $M(r) < \sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$. The idea behind this kind of Jensen's bound was taken from [7, Section 15.20].

If $r = e^{x\log x}$ for some $x > 0$, choose $k = \frac{1-x}{r}$ ($k > 1$ for x small enough, and $k < \frac{1}{r}$) and we find that

$$n(r) \leq -\frac{\log x}{\log\big((1-x)/r\big)}. \tag{14}$$

With simple algebraic steps, from (14) we obtain that, asymptotically, the number of zeros $n(r)$ of the Fredholm series contained in $\bar{D}(r)\setminus\{0\}$ (which goes to infinity) increases at most like $\frac{\log(-\log r)}{\log r}$. That is, for every $\mathcal{C}>1$, for r big enough

$$n(r)\leq\mathcal{C}\frac{\log(-\log r)}{\log r}$$

or equivalently for $d>0$ small enough

$$n(1-d)\leq-\mathcal{C}\frac{\log d}{d}.$$

7.2. *Lower bound*

Let f be any function which arises from Theorem 1.1. We retrace the proof of the infinity of the zeros to get a quantitative estimate.

Fix $\alpha\in\mathbb{R}$ and $\zeta\in H$. Let C be a disk centered in $\zeta-\alpha$ such that $\bar{C}\subseteq H$ and such that $S(w+\alpha)\neq S(\zeta)$ for all w on the boundary of C. We proved that for every $t\in\mathbb{N}$ big enough there exists $\zeta_t\in C$ such that

$$f\left(e\left(\frac{\zeta_t}{a^{n_0}}+\theta_0\right)\right)=S(\zeta)-G(\alpha),$$

where θ_0 (dependent on t) is a real number satisfying $|\theta_0-\alpha|<a^{-t}$ and $n_0=\varphi(b^{a^t})$ (here φ is the Euler function) for some constants a and b dependent on the function f.

Remark 7.1. As noted in [10, Remark 4.1], f almost shows a "fractal" behavior. The values taken in the circles $C_t=\frac{C}{a^{n_0}}+\theta_0$ for varying $t\in\mathbb{N}$ are almost the same but at rescaled points (see expression (9)).

Fix $0<r<1$. Put $C_0=\min\Im C$, we have

$$\left|e\left(\frac{\zeta_t}{a^{n_0}}+\theta_0\right)\right|=\exp\left(-2\pi\Im\left(\frac{\zeta_t}{a^{n_0}}+\theta_0\right)\right)=\exp\left(-2\pi\Im\frac{\zeta_t}{a^{n_0}}\right)$$
$$\leq\exp(-2\pi C_0a^{-n_0}).$$

Thus the number of $t\in\mathbb{N}$ such that $|e(\frac{\zeta_t}{a^{n_0}}+\theta_0)|<r$ is at least

$$\left\lfloor\log_a\left(\log_b\left(-\log_a\left(-\frac{\log r}{2\pi C_0}\right)\right)\right)\right\rfloor\geq\mathcal{C}\log\left(\log\left(-\log(-\log r)\right)\right)$$

for a constant $\mathcal{C}>0$ and for r close enough to 1. One can choose C small enough such that the sets $C_t=\frac{C}{a^{n_0}}+\theta_0$ are disjoint.

Given $\alpha \in \mathbb{R}$, we choose ζ satisfying $S(\zeta) - G(\alpha) = 0$. Pick any sector

$$S = \{z \in \mathbb{C} \mid 2\pi\alpha_0 < \arg z < 2\pi\alpha_1\} \quad \text{with } \alpha_0 < \alpha < \alpha_1.$$

Definitively in t, ζ_t belongs to S. Let $n_S(r)$ be the number of zeros of f of modulus smaller than r contained in S. Then (we may need to decrease $\mathscr{C}$ since the argument works only definitively) for every r big enough, we have

$$n_S(r) \geq \mathscr{C} \log\big(\log\big(-\log(-\log r)\big)\big) \tag{15}$$

or equivalently for $d > 0$ small enough

$$n_S(1-d) \geq \mathscr{C} \log\big(\log(-\log d)\big).$$

REMARK 7.2. Here we have obtained the same estimation for every sector regardless of the angle α. We believe that the zeros are equally distributed in every direction. Recall that Nazarov proved that some lacunary series $f(z) = \sum_{k\in\Lambda} c_k z^k$ have the Picard property when $\sum_{k\in\Lambda} |c_k|^2 = \infty$. The zeros of these f are equidistributed in a precise sense. For $0 < r < 1$, define the discrete measure

$$\nu_r = \sum_{\zeta : f(\zeta)=0,\, |\zeta|<r} \log \frac{r}{|\zeta|} \delta_\zeta$$

and put $\tau(r)^2 = \sum_{k\in\Lambda} |c_k|^2 r^{2k}$, then $\frac{1}{\log \tau(r)} \nu_r$ weakly converges to the Lebesgue measure on the boundary of D when r tends to 1.

Moreover, we observe that this convergence of measures implies that for every $\mathscr{C} > 0$ and for every r big enough $n(r) \geq \mathscr{C} \log \tau(r)$.

7.3. Another estimate with Fuchs' method

In the particular case of f the Fredholm series, we find a lower bound for $n(r)$, the number of zeros of f contained in $D(r)$ different from 0. We will find better estimates but not as general as the previous ones because they are only valid for the Fredholm series.

For every constant $C > 0$, exactly one among the following scenarios happens.

(1) There exists a sequence of $(r_k)_{k\in\mathbb{N}}$ of real numbers smaller than 1 which tend to 1 such that $|f(z)| \geq C$ for every z such that $|z| = r_k$.

(2) There exists $r_0 < 1$ such that for all $r > r_0$ there is a z of modulus r such that $|f(z)| < C$.

If we had $f(z) = \sum_{k=1}^{\infty} c_k z^{n_k}$ unbounded with $\lim_{k\to\infty} |c_k| = 0$ (but here $|c_k| = 1$), we would affirm that Scenario 2 happens and not 1 ([9] or [6, Lemma 5]).

As noted in [10, Remark 4.2], we can approximate f with $s_n(\theta) = \sum_{m=0}^{n-1} e(2^m\theta)$. Mahler precisely proved that [4, Theorem 1]

$$|f(re(\theta)) - s_n(\theta)| \leq C_0 = 3, \tag{16}$$

where $n = \lfloor -\log_2(-\log r)\rfloor$.

REMARK 7.3. For $C > 8$, Scenario 2 happens. In fact, choose $\theta = \frac{1}{9}$ in (16). Since $s_n(1/9) = 0$ whenever 6 divides n, we get $|f(re(1/9))| \leq 5 + 3 = 8$.

We obtain two different bounds.

Scenario 1. In [10, Remark 4.2], the following is proved.

LEMMA 7.4. *If $\mathcal{A}_n = \{\theta \in [0,1] \mid |s_n(\theta)| > \frac{\sqrt{n}}{2}\}$, then the Lebesgue measure of $|\mathcal{A}_n|$ is greater than a constant ($\frac{9}{32}$) which does not depend on n.*

Using this result and (16), we can write (for $n = \lfloor -\log_2(-\log r)\rfloor$)

$$\int_0^1 \log^+ |f(re(\theta))| d\theta \geq \int_{\mathcal{A}_n} \log^+ (|s_n(\theta)| - C_0) d\theta \geq \frac{9}{32} \log^+ \left(\frac{\sqrt{n}}{2} - C_0\right).$$

For $r = r_k$, we have supposed that $|f(re(\theta))| > C$. Then if $C < 1$ (we change $\log^+$ to log), then

$$\begin{aligned}\int_0^1 \log |f(re(\theta))| d\theta &\geq \frac{9}{32} \log^+ \left(\frac{\sqrt{n}}{2} - C_0\right) + \log(C)\\ &\geq C_1 \log n - C_2\end{aligned}$$

for n big enough and for some constants $C_1 > 0$ and $C_2 > 0$. We use Jensen's formula. Given that there is $\alpha > 0$ such that $|a_l| > \alpha$ for all a_l zeros of $g(z) = f(z)/z$, we have (note that $g > f$)

$$n(r) \log\left(\frac{1}{\alpha}\right) \geq \sum_{l=1}^{n(r)} \log\left(\frac{r}{|a_l|}\right) = \int_0^1 \log |g(re(\theta))| d\theta \geq C_1 \log n - C_2$$

whence

$$n(r) \geq \mathcal{C} \log(n) \geq \mathcal{C} \log\big(-\log(-\log r)\big)$$

valid for r big enough and for some positive constants which, for simplicity, we just call $\mathcal{C}$. Equivalently for $d > 0$ small enough,

$$n(1-d) \geq \mathcal{C} \log(-\log d).$$

We have removed one log from the estimate (15).

Scenario 2. Fuchs proved [1] that there exist a constant $C > 0$ and $r_0 < 1$ such that if $r > r_0$ and $|f(z)| < C$ with $|z| = r(s_0 s_1)^{1/2}$, then there exists a zero of f of modulus between rs_0 and rs_1, where s_0, s_1 are functions of r. In the special case of the Fredholm series, for some constant p

$$s_0(r) = \exp\Big(-\frac{p}{2^{\lceil -\log_r 2\rceil+1}}\Big),$$
$$s_1(r) = \exp\Big(-\frac{p}{2^{\lceil -\log_r 2\rceil+2}}(1+\log 2)\Big).$$

For every $r < 1$, consider the annulus $S_r = \{z \in \mathbb{C} \big| rs_0 < |z| < rs_1\}$. In this case, we are supposing that for every $r > r_0$ there exists z of modulus r such that $|f(z)| < C$. In particular, there is one such z of modulus $r(s_0 s_1)^{1/2}$ for all r big enough since $r(s_0 s_1)^{1/2}$ tends to 1 when r tends to 1. Then, thanks to Fuchs' proof, there exists a zero in S_r.

We make some approximations in the limit of r tending to 1. We find a constant $C_0 > 0$ such that

$$\begin{aligned}\log(rs_1) &= \log(rs_0) + \frac{p}{2^{\lceil -\log_r 2\rceil+2}}(1-\log 2)\\ &\le \log(rs_0) + C_0 2^{1/(C_0 \log r)}\\ &\le \log(rs_0) + C_0 2^{1/(C_0 \log(rs_0))}\end{aligned}$$

for r close enough to 1 (we used $\log_r 2 = \frac{\log 2}{\log r}$).

For every $\alpha > 0$, consider the annulus

$$S'_\alpha = \big\{z \in \mathbb{C} \mid \alpha > -\log|z| > \alpha - C_0 2^{-1/(C_0\alpha)}\big\}.$$

For $\alpha = -\log(rs_0)$, we have $S_r \subseteq S'_\alpha$. Then there exists a zero of f in S'_α for every α small enough.

To obtain a lower bound to $n(r)$, we compute how many disjoint annuli can stay inside $D(r)$. Given that for α small enough $C_0 2^{-1/(C_0\alpha)} < \alpha/2$, the sets $S'_{2^{-n}}$ are disjoint for $n \ge n_0$ big enough. Moreover, if $C_0 2^{-1/(C_0\alpha)} < \alpha/2$ and if $\alpha > -2\log r$, we have $S'_\alpha \subseteq D(r)$. Therefore, when $n_0 \le n < -\log_2(-2\log r)$ the sets $S'_{2^{-n}}$ are disjoint and contained in $D(r)$. For $n \ge n_1$ big enough, they all contain a zero of f. We conclude that, for r close enough to 1,

$$n(r) \ge -\log_2(-2\log r) - C_1 \ge -\mathscr{C}\log(-\log r),$$

where $C_1 = \max\{n_0, n_1\}$ and $\mathscr{C}$ is an appropriate positive constant as usual. Equivalently for $d > 0$ small enough,

$$n(1-d) \ge -\mathscr{C}\log d.$$

We have removed two log functions from the estimate (15).

ACKNOWLEDGMENTS. – I express my thanks to Professor Umberto Zannier for the precious suggestions, and for the punctuality and scrupulousness he used when supervising my work. In particular, I recall the crucial contribution he gave me for completing the proof of the theorem when, near the end, I was not able to conclude. I thank Professor Mikhail Sodin for indicating some useful references unknown to us. An interesting example of a class of non-lacunary series to which we can apply the main theorem was kindly suggested by the referee.

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Received 19 October 2022,
and in revised form 1 February 2023

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