



Mathematical Analysis. – *Overdetermined problems and relative Cheeger sets in unbounded domains*, by **DANILO GREGORIN AFONSO**, **ALESSANDRO IACOPETTI** and **FILOMENA PACELLA**, communicated on 10 February 2023.

Dedicated to the memory of Professor A. Ambrosetti.

ABSTRACT. – In this paper, we study a partially overdetermined mixed boundary value problem for domains Ω contained in an unbounded set \mathcal{C} . We introduce the notion of the Cheeger set relative to \mathcal{C} and show that if a domain $\Omega \subset \mathcal{C}$ admits a solution of the overdetermined problem, then it coincides with its relative Cheeger set. We also study the related problem of characterizing constant mean curvature surfaces Γ inside \mathcal{C} . In the case when \mathcal{C} is a cylinder, we obtain further results whenever the relative boundary of Ω or the surface Γ is a graph on the base of the cylinder.

KEYWORDS. – Overdetermined elliptic problem, shape optimization in unbounded domains, cartesian graphs, constant mean curvature surfaces.

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1. INTRODUCTION

Given an unbounded Lipschitz domain $\mathcal{C} \subset \mathbb{R}^N$, $N \geq 2$, we consider a bounded domain $\Omega \subset \mathcal{C}$ and denote by Γ_Ω its *relative (to \mathcal{C}) boundary*, i.e., $\Gamma_\Omega = \partial\Omega \cap \mathcal{C}$. Then, we set $\Gamma_{1,\Omega} = \partial\Omega \cap \partial\mathcal{C}$ and assume that $\mathcal{H}_{N-1}(\Gamma_{1,\Omega}) > 0$, where \mathcal{H}_{N-1} denotes the $(N - 1)$ -dimensional Hausdorff measure.

We study the partially overdetermined mixed boundary value problem

$$(1.1) \quad \begin{cases} -\Delta u = 1 & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_\Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_{1,\Omega}, \\ \frac{\partial u}{\partial \nu} = -c < 0 & \text{on } \Gamma_\Omega, \end{cases}$$

where $c > 0$ is a constant and ν denotes the exterior unit normal, which is defined on the regular part of $\partial\Omega$. The question we address is to determine the domains $\Omega \subset \mathcal{C}$ for which (1.1) admits a solution. Obviously, this depends on the given container \mathcal{C} .

A related question is to characterize the surfaces Γ in the set \mathcal{C} with constant mean curvature (CMC, in short). More precisely, we consider CMC smooth, bounded, connected, orientable $(N - 1)$ -dimensional manifolds Γ inside \mathcal{C} whose boundary is contained in $\partial\mathcal{C}$ and intersects $\partial\mathcal{C}$ orthogonally. The characterization of such surfaces also depends on the set \mathcal{C} .

Both questions were addressed in [15] (see also [16]) in the case when \mathcal{C} is a cone. The overdetermined problem for more general equations in cones was considered in [4].

When the cone is convex, the results of [15] completely characterize the domains Ω for which (1.1) admits a solution, as well as the CMC surfaces which intersect $\partial\mathcal{C}$ orthogonally: they are either spherical sectors centered at the vertex of the cone or half-balls lying on a flat portion of $\partial\mathcal{C}$. For nonconvex cones, a similar characterization has been proved, but only for CMC radial graphs [15, 16]. On the other side, nonradiality results have been proved for a class of nonconvex cones [7].

In this paper, we study the aforementioned questions when \mathcal{C} is a general unbounded smooth set. More precise results will be obtained when the unbounded set \mathcal{C} is a cylinder in \mathbb{R}^N spanned by a smooth bounded domain $\omega \subset \mathbb{R}^{N-1}$. We denote it by \mathcal{C}_ω , i.e.,

$$\mathcal{C}_\omega := \omega \times (0, +\infty) = \{x = (x', x_N) \in \mathbb{R}^N : x' \in \omega, x_N \in (0, +\infty)\}.$$

In this case, we obtain a result about CMC surfaces $\Gamma = \Gamma_\varphi$ inside \mathcal{C}_ω which are graphs of smooth functions φ defined on $\bar{\omega}$. We prove that if Γ_φ meets $\partial\mathcal{C}$ orthogonally, then the mean curvature of Γ_φ is necessarily 0 and Γ_φ is the graph of a constant function (see Proposition 2.1).

Then, we consider the Lipschitz domain

$$\Omega_\varphi := \{(x', x_N) \in \mathbb{R}^N : x' \in \omega, x_N < \varphi(x')\} \subset \mathcal{C}_\omega$$

and study the overdetermined problem (1.1) in Ω_φ . In view of Proposition 2.1 and the results in cones, it would be natural to conjecture that the domains Ω_φ for which (1.1) admits a solution are the ones corresponding to a function $\varphi \equiv h$ for some $h > 0$, so that Ω_φ is the bounded cylinder $\omega \times (0, h)$. In particular, Γ_φ would have zero mean curvature. However, the case of the cylinder looks quite different and we show some partial results.

In the case when \mathcal{C} is a general Lipschitz unbounded domain, we introduce the definition of relative (to \mathcal{C}) Cheeger set for a domain $\Omega \subseteq \mathcal{C}$. It generalizes the classical one which goes back to [3] (see also [13, 17]). Then, we show that if \mathcal{C} is convex, any bounded domain $\Omega \subset \mathcal{C}$ for which (1.1) admits a solution coincides with its relative Cheeger set. This is an interesting geometric property of Ω which sheds light on the connections between overdetermined and isoperimetric problems. We also show some properties of relative self-Cheeger sets.

We delay the precise statement of the results and the comments on the proofs to the corresponding sections.

This paper is organized as follows. In Section 2, we study CMC surfaces in cylinders. In Section 3, we introduce the relative Cheeger problem and prove some properties of relative Cheeger sets. We also provide a bound on the first eigenvalue of the mixed boundary value problem for the Laplacian in terms of the Cheeger constant. The overdetermined problem is studied in Section 4.

2. CMC CARTESIAN GRAPHS

Let $\omega \subset \mathbb{R}^{N-1}$ be a smooth bounded domain and let $\mathcal{C}_\omega \subset \mathbb{R}^N$ be the half-cylinder spanned by ω , i.e., $\mathcal{C}_\omega = \omega \times (0, +\infty)$. We denote by $\partial\mathcal{C}_\omega^+$ the lateral part of $\partial\mathcal{C}_\omega$, i.e.,

$$\partial\mathcal{C}_\omega^+ := \partial\mathcal{C}_\omega \cap \{x_N > 0\}.$$

Let $\varphi \in C^2(\bar{\omega})$ be such that $\varphi > 0$ in $\bar{\omega}$ and consider the associated cartesian graph

$$\Gamma_\varphi := \{(x', x_N) \in \mathbb{R}^N : x' \in \omega, x_N = \varphi(x')\}.$$

By construction, we have that $\Gamma_\varphi \subset \mathcal{C}_\omega$ and Γ_φ meets $\partial\mathcal{C}_\omega$ only on $\partial\mathcal{C}_\omega^+$. The main result of this section is the following.

PROPOSITION 2.1. *Assume that Γ_φ is a CMC surface which meets $\partial\mathcal{C}_\omega^+$ orthogonally. Then, Γ_φ is a minimal hypersurface and φ is a constant function.*

PROOF. Since Γ_φ is a smooth cartesian graph, then the mean curvature of Γ_φ (with respect to the exterior unit normal ν_{Γ_φ}) is given by

$$(2.1) \quad H_{\Gamma_\varphi} = -\frac{1}{N-1} \operatorname{div} \left(\frac{\nabla\varphi}{\sqrt{1+|\nabla\varphi|^2}} \right)$$

and, as H_{Γ_φ} is constant, then integrating (2.1) over ω , we deduce that

$$(2.2) \quad (N-1)H_{\Gamma_\varphi}|\omega| = -\int_\omega \operatorname{div} \left(\frac{\nabla\varphi}{\sqrt{1+|\nabla\varphi|^2}} \right) dx'.$$

On the other hand, by the divergence theorem, we have

$$(2.3) \quad \int_\omega \operatorname{div} \left(\frac{\nabla\varphi}{\sqrt{1+|\nabla\varphi|^2}} \right) dx' = \int_{\partial\omega} \frac{\nabla\varphi \cdot \nu_{\partial\omega}}{\sqrt{1+|\nabla\varphi|^2}} d\sigma,$$

where $\nu_{\partial\omega}$ is the unit outward normal to $\partial\omega$. Let us observe that since Γ_φ meets $\partial\mathcal{C}_\omega^+$

orthogonally, then for all $x \in \bar{\Gamma}_\varphi \cap \partial\mathcal{C}_\omega$, it holds

$$(2.4) \quad \nu_{\Gamma_\varphi}(x) \cdot \nu_{\partial\mathcal{C}_\omega^+}(x) = 0.$$

Moreover, as Γ_φ is a cartesian graph, for all $x = (x', x_N) \in \bar{\Gamma}_\varphi$, we have

$$(2.5) \quad \nu_{\Gamma_\varphi}(x) = \nu_{\Gamma_\varphi}(x', \varphi(x')) = \frac{1}{\sqrt{1 + |\nabla\varphi(x')|^2}} \begin{bmatrix} -\nabla\varphi(x') \\ 1 \end{bmatrix}.$$

In particular, since $\partial\mathcal{C}_\omega^+$ is a cylinder spanned by $\partial\omega$, we have

$$\nu_{\partial\mathcal{C}_\omega^+}(x) = (\nu_{\partial\omega}(x'), 0) \quad \text{for all } x = (x', x_N) \in \partial\mathcal{C}_\omega^+.$$

From these considerations, (2.4), and (2.5), we readily obtain that

$$(2.6) \quad \nabla\varphi(x') \cdot \nu_{\partial\omega}(x') = 0 \quad \forall x \in \partial\omega.$$

Hence, combining (2.2), (2.3), and (2.6), we deduce that

$$H_{\Gamma_\varphi} = 0.$$

To conclude, it remains to prove that φ is a constant function. Since $H_{\Gamma_\varphi} = 0$, then from (2.1), integrating by parts and taking into account (2.6), it follows that

$$\int_\omega \frac{\nabla\varphi \cdot \nabla\psi}{\sqrt{1 + |\nabla\varphi|^2}} dx' = \int_{\partial\omega} \psi \frac{\nabla\varphi \cdot \nu_{\partial\omega}}{\sqrt{1 + |\nabla\varphi|^2}} d\sigma = 0 \quad \forall \psi \in C^1(\bar{\omega}).$$

Finally, choosing $\psi = \varphi$, we readily obtain that

$$\int_\omega \frac{|\nabla\varphi|^2}{\sqrt{1 + |\nabla\varphi|^2}} dx' = 0,$$

which implies $|\nabla\varphi|^2 = 0$ on ω . As ω is connected, we get that φ is a constant function. \blacksquare

Finally, we prove the following Minkowski formula for graphs, which is interesting in itself and will be used in Section 4.

PROPOSITION 2.2. *Assume that Γ_φ meets $\partial\mathcal{C}_\omega^+$ orthogonally. Then,*

$$(2.7) \quad \int_{\Gamma_\varphi} H_{\Gamma_\varphi} \langle x_N e_N, \nu \rangle d\sigma = \frac{1}{N-1} \int_\omega \frac{|\nabla\varphi|^2}{\sqrt{1 + |\nabla\varphi|^2}} dx',$$

where $e_N = (0, \dots, 0, 1) \in \mathbb{R}^N$.

PROOF. Since Γ_φ is the cartesian graph associated with φ , we have

$$(2.8) \quad d\sigma = \sqrt{1 + |\nabla\varphi|^2} dx'.$$

Exploiting (2.1) and (2.5) and integrating by parts, we get that

$$(2.9) \quad \begin{aligned} & \int_{\Gamma_\varphi} H_{\Gamma_\varphi} \langle x_N e_N, \nu \rangle d\sigma \\ &= -\frac{1}{N-1} \int_\omega \operatorname{div} \left(\frac{\nabla\varphi}{\sqrt{1 + |\nabla\varphi|^2}} \right) \varphi dx' \\ &= \frac{1}{N-1} \left(\int_\omega \frac{|\nabla\varphi|^2}{\sqrt{1 + |\nabla\varphi|^2}} dx' - \int_{\partial\omega} \varphi \frac{\nabla\varphi \cdot \nu_{\partial\omega}}{\sqrt{1 + |\nabla\varphi|^2}} d\sigma_{\partial\omega} \right). \end{aligned}$$

Finally, since Γ_φ intersects $\partial\mathcal{C}_\omega^+$ orthogonally, then from (2.6), we infer that the last integral in (2.9) is zero, and (2.7) readily follows. \blacksquare

3. RELATIVE CHEEGER SETS

Let $\mathcal{C} \subset \mathbb{R}^N$ be an unbounded Lipschitz domain. Let us recall the definition of the relative perimeter.

DEFINITION 3.1. The *relative (to \mathcal{C}) perimeter* of a set $E \subset \mathcal{C}$ is

$$P_{\mathcal{C}}(E) := |D\chi_E|_{\mathcal{C}}(E) = \sup \left\{ \int_E \operatorname{div} \psi dx : \psi \in C_c^1(\mathcal{C}, \mathbb{R}^N), \|\psi\|_\infty \leq 1 \right\},$$

where χ_E is the characteristic function of E . Of course, a set is said to be of *finite (relative) perimeter* when $P_{\mathcal{C}}(E) < +\infty$.

Note that if $\Gamma_E := \partial E \cap \mathcal{C}$ is Lipschitz, then

$$(3.1) \quad P_{\mathcal{C}}(E) = \mathcal{H}_{N-1}(\Gamma_E) = \int_{\Gamma_E} 1 d\sigma.$$

Let $|E|$ denote the Lebesgue measure of E , which will also be called the volume of E . Let $\Omega \subseteq \mathcal{C}$ and set, as in Section 1, $\Gamma_\Omega := \partial\Omega \cap \mathcal{C}$.

DEFINITION 3.2. The *relative (to \mathcal{C}) Cheeger constant* of $\Omega \subseteq \mathcal{C}$ is

$$h_{\mathcal{C}}(\Omega) := \inf_{E \subseteq \Omega} \frac{P_{\mathcal{C}}(E)}{|E|}.$$

The sets which attain the minimum will be called the *relative (to \mathcal{C}) Cheeger sets* of Ω . If Ω itself is a Cheeger set, then Ω is said to be *self-Cheeger*.

The problem of finding $h_{\mathcal{C}}(\Omega)$ and the associated Cheeger set (or sets) has many interesting motivations and applications. The Cheeger constant $h(\Omega)$ first appeared in a bound for the first eigenvalue of the Laplacian on manifolds [3]; see [9] for the link with the spectral theory for the p -Laplacian. See [11] and the references therein for fracture problems in mechanics of materials; [8] for a landslide problem; [2] for a generalization of the Cheeger problem in image processing. Regarding the Cheeger problem on its own, see [1] for a study via p -torsion functions and [10] for a characterization of Cheeger sets of convex plane domains. The surveys [13, 17] provide a nice overview.

We now show some properties of relative Cheeger sets.

PROPOSITION 3.3 (Existence). *Let $\Omega \subset \mathcal{C}$ be a bounded domain such that Γ_{Ω} is Lipschitz. Then, there exists at least one Cheeger set for Ω .*

PROOF. The proof is a straightforward adaptation of the proof of [17, Proposition 3.1]. The important thing to note is that the test functions φ have compact support in \mathcal{C} , so they do not “see” the boundary of the container. ■

PROPOSITION 3.4. *Let \mathcal{C} be a cone with vertex at the origin, i.e.,*

$$\mathcal{C} = \{tx : x \in D, t \in (0, +\infty)\},$$

where D is a domain on the unit sphere S^{N-1} . Let $\Omega \subset \mathcal{C}$ be a bounded domain such that Γ_{Ω} is Lipschitz. If E is a Cheeger set for Ω , then $\partial E \cap \Gamma_{\Omega} \neq \emptyset$.

PROOF. We adapt the proof of [17, Proposition 3.5]. Note that the fact that the cone is invariant by dilation plays a crucial role in the proof.

For the sake of contradiction, suppose that $d(\partial E, \Gamma_{\Omega}) \geq \delta$ for some $\delta > 0$. Then, we can find some $t > 1$ such that the set

$$tE := \{x \in \mathcal{C} : t^{-1}x \in E\}$$

is still in Ω . By the change of variables formula, we have

$$|tE| = \int_{tE} 1 \, dx = \int_E t^N \, dx = t^N |E|.$$

To compute the perimeter, one proceeds as follows. By the definition of the relative perimeter, the chain rule, and the change of variables formula, we get

$$\begin{aligned} P_{\mathcal{C}}(tE) &= \sup \left\{ \int_{tE} \operatorname{div} \psi(x) \, dx : \psi \in C_c^1(\mathcal{C}, \mathbb{R}^N), \|\psi\|_{\infty} \leq 1 \right\} \\ &= \sup \left\{ \int_E \operatorname{div} (\psi(t^{-1}x)) t^N \, dx : \psi \in C_c^1(\mathcal{C}, \mathbb{R}^N), \|\psi\|_{\infty} \leq 1 \right\} \\ &= t^{N-1} P_{\mathcal{C}}(E). \end{aligned}$$

Observe that the set of functions where we compute the supremum is the same, but when computing the divergence, we must take into account the dilation of the argument.

Now, it remains to observe that

$$\frac{P_{\mathcal{C}}(tE)}{|tE|} = \frac{t^{N-1} P_{\mathcal{C}}(E)}{t^N |E|} < h_{\mathcal{C}}(\Omega),$$

which contradicts the definitions of the (relative) Cheeger constant and Cheeger set. ■

We can prove a similar result in cylinders.

PROPOSITION 3.5. *Let $\omega \subset \mathbb{R}^{N-1}$ and let \mathcal{C}_ω be the cylinder spanned by ω . Assume that $\Omega \subset \mathcal{C}_\omega$ is a bounded domain such that Γ_Ω is a connected surface whose projection into \mathbb{R}^{N-1} is exactly ω . If some relative Cheeger set E_f of Ω is defined by the graph of a function f on ω , i.e., $E_f = \{(x', x_N) \in \mathcal{C}_\omega : 0 < x_N < f(x')\}$, then $\Gamma_{E_f} \cap \Gamma_\Omega \neq \emptyset$.*

PROOF. Note that $\Gamma_{E_f} = \partial E_f \cap \mathcal{C}$ is just the graph Γ_f of the function f . If $\Gamma_f \cap \Gamma_\Omega = \emptyset$, considering the domain $E_{f+\delta}$ defined by the function $f + \delta$, for $\delta > 0$ sufficiently small so that $E_{f+\delta} \subseteq \Omega$, we have

$$\frac{\mathcal{H}_{N-1}(\Gamma_{f+\delta})}{|E_{f+\delta}|} = \frac{\mathcal{H}_{N-1}(\Gamma_f)}{|E_{f+\delta}|} < \frac{\mathcal{H}_{N-1}(\Gamma_f)}{|E_f|},$$

contradicting the definition of a Cheeger set. ■

Note that in the previous proof, we can allow f to be zero at some strict subset of ω .

REMARK 3.6. The Cheeger constant can be obtained by minimization in the class of smooth subdomains of Ω ; see [14, proof of Theorem 1] and the references therein.

When \mathcal{C} is a cylinder and Ω_φ is the domain defined by the graph of a function $\varphi \in C^2(\bar{\omega})$, we have the following result.

THEOREM 3.7. *Let $\omega \subset \mathbb{R}^{N-1}$ be a smooth bounded domain and consider the cylinder C_ω spanned by ω . Let $\varphi \in C^2(\bar{\omega})$ be a positive function and assume that Γ_φ meets $\partial \mathcal{C}_\omega$ orthogonally. If Ω_φ is self-Cheeger, then*

$$(3.2) \quad H_{\Gamma_\varphi} \leq \frac{P_{\mathcal{C}_\omega}(\Omega_\varphi)}{(N-1)|\Omega_\varphi|}.$$

PROOF. For any C^1 function $v \leq 0$ on $\bar{\omega}$, the relative perimeter of the perturbed domain $\Omega_{\varphi+tv}$ is given by (recall (3.1) and (2.8))

$$\begin{aligned} p_v(t) &:= P_{\mathcal{C}}(\Omega_{\varphi+tv}) = \mathcal{H}_{N-1}(\Gamma_{\varphi+tv}) \\ &= \int_{\omega} \sqrt{1 + |\nabla\varphi + t\nabla v|^2} \, dx'. \end{aligned}$$

We also define

$$V_v(t) := |\Omega_{\varphi+tv}| = \int_{\omega} \varphi + tv \, dx'.$$

Here, we are considering t small enough for these quantities to make sense in the container, that is, $\varphi + tv \geq 0$.

If Ω_{φ} is self-Cheeger, then

$$\left. \frac{d}{dt} \left(\frac{p_v(t)}{V_v(t)} \right) \right|_{t=0} = \left. \frac{p'_v(t)V_v(t) - p_v(t)V'_v(t)}{(V_v(t))^2} \right|_{t=0} \geq 0$$

for all negative $v \in C^1(\bar{\omega})$.

Since $(V_v(t))^2$ is always positive, we have that

$$(3.3) \quad p'_v(0)V_v(0) - p_v(0)V'_v(0) \geq 0.$$

Simple computations yield

$$p'_v(t) = \int_{\omega} \frac{\nabla\varphi \cdot \nabla v + t|\nabla v|^2}{\sqrt{1 + |\nabla\varphi + t\nabla v|^2}} \, dx'$$

and

$$V'_v(t) = \int_{\omega} v \, dx'.$$

Then,

$$(3.4) \quad \begin{aligned} & p'_v(0)V_v(0) - p_v(0)V'_v(0) \\ &= \int_{\omega} \frac{\nabla\varphi \cdot \nabla v}{\sqrt{1 + |\nabla\varphi|^2}} \, dx' |\Omega_{\varphi}| - P_{\mathcal{E}_{\omega}}(\Omega_{\varphi}) \int_{\omega} v \, dx'. \end{aligned}$$

Integrating by parts and using the divergence theorem, we obtain

$$(3.5) \quad \begin{aligned} & \int_{\omega} \frac{\nabla\varphi \cdot \nabla v}{\sqrt{1 + |\nabla\varphi|^2}} \, dx' \\ &= \int_{\omega} \operatorname{div} \left(v \frac{\nabla\varphi}{\sqrt{1 + |\nabla\varphi|^2}} \right) \, dx' - \int_{\omega} v \operatorname{div} \left(\frac{\nabla\varphi}{\sqrt{1 + |\nabla\varphi|^2}} \right) \, dx' \\ &= \int_{\partial\omega} v \frac{\nabla\varphi \cdot \nu_{\partial\omega}}{\sqrt{1 + |\nabla\varphi|^2}} \, dx' - \int_{\omega} v \operatorname{div} \left(\frac{\nabla\varphi}{\sqrt{1 + |\nabla\varphi|^2}} \right) \, dx' \\ &= - \int_{\omega} v \operatorname{div} \left(\frac{\nabla\varphi}{\sqrt{1 + |\nabla\varphi|^2}} \right) \, dx \end{aligned}$$

since Γ_φ meets $\partial\mathcal{C}_\omega$ orthogonally (see the proof of Proposition 2.1). Then, substituting (3.5) into (3.4) and taking into account (3.3), we get

$$\int_{\omega} v \left(|\Omega_\varphi| \operatorname{div} \left(\frac{\nabla\varphi}{\sqrt{1+|\nabla\varphi|^2}} \right) + P_{\mathcal{C}_\omega}(\Omega_\varphi) \right) dx' \leq 0$$

for every negative $v \in C^1(\bar{\omega})$. Hence,

$$0 \leq |\Omega_\varphi| \operatorname{div} \left(\frac{\nabla\varphi}{\sqrt{1+|\nabla\varphi|^2}} \right) + P_{\mathcal{C}_\omega}(\Omega_\varphi) = -(N-1)H_{\Gamma_\Omega}|\Omega_\varphi| + P_{\mathcal{C}_\omega}(\Omega_\varphi),$$

from which (3.2) readily follows. \blacksquare

THEOREM 3.8. *Let $\omega \subset \mathbb{R}^{N-1}$ be a smooth bounded domain and consider the cylinder C_ω spanned by ω . Let $\varphi \in C^2(\bar{\omega})$ be a positive function. Assume that $|\nabla\varphi| \geq \delta$ on $\bar{\omega}$, for some $\delta > 0$. Then, Ω_φ cannot be self-Cheeger.*

PROOF. Again, we apply the idea of perturbing φ , but now the aim is to find a specific function $v \leq 0$ in $C^1(\bar{\omega})$ for which

$$\left. \frac{d}{dt} \left(\frac{p_v(t)}{V_v(t)} \right) \right|_{t=0} = \left. \frac{p'_v(t)V_v(t) - p_v(t)V'_v(t)}{(V_v(t))^2} \right|_{t=0} < 0.$$

Indeed, this will imply that

$$\frac{P_{\mathcal{C}}(\Omega_{\varphi+tv})}{|\Omega_{\varphi+tv}|} < \frac{P_{\mathcal{C}}(\Omega)}{|\Omega|}$$

for every $t < \varepsilon$ for some $\varepsilon > 0$.

As $(V_v(t))^2$ is always positive, the problem reduces to finding v such that

$$p'_v(0)V_v(0) - p_v(0)V'_v(0) < 0.$$

Take

$$v(x') = -e^{\alpha\varphi(x')},$$

where $\alpha > 0$ is a constant to be chosen later. Then,

$$\nabla v(x') = -\alpha e^{\alpha\varphi(x')} \nabla\varphi(x')$$

and

$$\begin{aligned} & p'_v(0)V_v(0) - p_v(0)V'_v(0) \\ &= \int_{\omega} -\alpha |\Omega_\varphi| e^{\alpha\varphi} \frac{|\nabla\varphi|^2}{\sqrt{1+|\nabla\varphi|^2}} dx' + \int_{\omega} P_{\mathcal{C}}(\Omega_\varphi) e^{\alpha\varphi} dx' \\ &\leq \int_{\omega} e^{\alpha\varphi} \left(P_{\mathcal{C}}(\Omega_\varphi) - \alpha |\Omega_\varphi| \frac{|\nabla\varphi|^2}{\sqrt{1+|\nabla\varphi|^2}} \right) dx'. \end{aligned}$$

Since $|\nabla\varphi|$ is bounded on $\bar{\omega}$ (because φ is smooth) and $|\nabla\varphi| \geq \delta$ on $\bar{\omega}$, choosing α big enough such that

$$P_{\mathcal{C}}(\Omega_{\varphi}) - \alpha|\Omega_{\varphi}| \frac{\delta^2}{\sqrt{1 + \|\nabla\varphi\|_{\infty}^2}} < 0,$$

we easily conclude. ■

We now go back to the case when \mathcal{C} is a general unbounded Lipschitz domain and conclude this section with an application.

Assume that Γ_{Ω} is smooth and consider the eigenvalue problem with mixed boundary conditions:

$$(3.6) \quad \begin{cases} -\Delta u = \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \Gamma_{\Omega} \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_{1,\Omega}. \end{cases}$$

Let $H_0^1(\Omega \cup \Gamma_{1,\Omega})$ be the closure in $H^1(\Omega)$ of the space $C_c^1(\Omega \cup \Gamma_{1,\Omega})$. It is the natural space to study problem (3.6), for it coincides with the space of functions in $H^1(\Omega)$ whose trace vanishes on Γ_{Ω} .

The spectral theory for this problem is analogous to the one for the Dirichlet-Laplacian (see [5, Section 1.4] for the spectral theory of mixed boundary value problems in a more general setting). In particular, the eigenvalues are all positive and form an increasing divergent sequence, and the first eigenfunction is positive. Moreover, the first eigenvalue is characterized by

$$\lambda_1(\Omega) = \min_{v \in H_0^1(\Omega \cup \Gamma_{1,\Omega}) \setminus \{0\}} \frac{\int_{\Omega} |\nabla v|^2 dx}{\int_{\Omega} v^2 dx}.$$

THEOREM 3.9. *Let $\Omega \subset \mathcal{C}$ be a bounded domain such that Γ_{Ω} is Lipschitz. Then,*

$$\lambda_1(\Omega) \geq \frac{h_{\mathcal{C}}^2(\Omega)}{4}.$$

PROOF. We adapt the proof presented for the Dirichlet- p -Laplacian in the appendix of [12].

Let $w \in C_c^{\infty}(\Omega \cup \Gamma_{1,\Omega})$ and $E_t = \{x \in \Omega : w(x) > t\}$. By the classical coarea formula, by the definition of $h_{\mathcal{C}}(\Omega)$ and Cavalieri's principle, we have

$$\begin{aligned} \int_{\Omega} |\nabla w| dx &= \int_{-\infty}^{\infty} \mathcal{H}_{N-1}(w^{-1}(t)) dt = \int_{-\infty}^{\infty} \frac{\mathcal{H}_{N-1}(w^{-1}(t))}{|E_t|} |E_t| dt \\ &\geq h_{\mathcal{C}}(\Omega) \int_{\Omega} |w| dx. \end{aligned}$$

Hence,

$$(3.7) \quad h_{\mathcal{C}}(\Omega) \leq \frac{\int_{\Omega} |\nabla w| dx}{\int_{\Omega} w dx} \quad \forall w \in C_c^{\infty}(\Omega \cup \Gamma_{1,\Omega}),$$

and by density, (3.7) holds as well in $W_0^{1,1}(\Omega \cup \Gamma_{1,\Omega})$.

Now, let $v \in H_0^1(\Omega \cup \Gamma_{1,\Omega})$. By Hölder's inequality, it follows that

$$v^2 \in W_0^{1,1}(\Omega \cup \Gamma_{1,\Omega}).$$

Indeed,

$$\int_{\Omega} |\nabla(v^2)| dx = 2 \int_{\Omega} |v| |\nabla v| dx \leq 2 \|v\|_2 \|\nabla v\|_2.$$

By (3.7), it follows that

$$h_{\mathcal{C}}(\Omega) \leq \frac{\int_{\Omega} |\nabla(v^2)| dx}{\int_{\Omega} v^2 dx} \leq 2 \frac{\|v\|_2 \|\nabla v\|_2}{\|v\|_2^2} = 2 \frac{\|\nabla v\|_2}{\|v\|_2}.$$

We conclude the proof by taking into account the variational characterization of the first eigenvalue $\lambda_1(\Omega)$. ■

4. THE OVERDETERMINED PROBLEM

Let \mathcal{C} be an unbounded Lipschitz domain and $\Omega \subset \mathcal{C}$ a bounded domain with smooth relative boundary Γ_{Ω} .

For a solution u of (1.1), we define

$$P(x) = |Du(x)|^2 + \frac{2}{N}u(x), \quad x \in \bar{\Omega}.$$

This function (sometimes called P -function) is often used in the study of overdetermined problems [6, 15, 18].

We can give a bound on the curvature of Γ_{Ω} , with the aid of the following.

LEMMA 4.1. *If \mathcal{C} is convex and u is a solution of (1.1), then either $P \equiv c^2$ in $\bar{\Omega}$ or $\frac{\partial P}{\partial \nu} > 0$ on Γ_{Ω} .*

PROOF. Direct computations yield

$$DP = 2D^2uDu + \frac{2}{N}Du$$

and

$$\Delta P = 2\|D^2u\|^2 + 2\langle Du, D(\Delta u) \rangle - \frac{2}{N} = 2\left(\|D^2u\|^2 - \frac{(\Delta u)^2}{N}\right)$$

since $\Delta u = -1$. Here, D^2u denotes the Hessian matrix of u and $\|D^2u\|^2$ is the sum of the squares of the elements of D^2u . By [15, inequality (2.4)], it then follows that

$$\Delta P \geq 0 \quad \text{in } \Omega.$$

Moreover, by the boundary conditions for u , we obtain the following boundary conditions for P :

$$P \equiv c^2 \text{ on } \Gamma_\Omega, \quad \frac{\partial P}{\partial \nu} = 2\langle D^2u Du, \nu \rangle \text{ on } \Gamma_{1,\Omega}.$$

From the convexity assumption on \mathcal{C} , we have that the second fundamental form $h(\cdot, \cdot)$ on $\partial\mathcal{C}$ is positive semidefinite at any regular point. On the other hand, by the Neumann condition $\frac{\partial u}{\partial \nu} = 0$ on $\Gamma_{1,\Omega}$, we deduce that Du is tangent to $\Gamma_{1,\Omega}$. Thus, denoting by N_u the vector field differentiating along the direction Du , i.e.,

$$N_u = \sum_{k=1}^N u_k(x) \frac{\partial}{\partial x_k},$$

we obtain

$$0 = N_u(\langle Du, \nu \rangle) = \sum_{j,k=1}^N u_k u_{kj} \nu_j + h(Du, Du) \geq \langle D^2u Du, \nu \rangle.$$

Hence, the function P satisfies

$$\begin{cases} \Delta P \geq 0 & \text{in } \Omega \\ P = c^2 & \text{on } \Gamma_\Omega \\ \frac{\partial P}{\partial \nu} \leq 0 & \text{on } \Gamma_{1,\Omega}. \end{cases}$$

By the maximum principle for the mixed boundary value problem (see [5, Section 1.2.1] or [15, Corollary 2.3]), we obtain $P \leq c^2$ in Ω . Then, the strong maximum principle applies, so that either $P \equiv c^2$ in Ω or $P < c^2$ in Ω . In this last case, by Hopf's lemma, we get $\frac{\partial P}{\partial \nu} > 0$ on Γ_Ω . ■

Let $H(x)$ denote the mean curvature at a point $x \in \Gamma_\Omega$.

PROPOSITION 4.2. *If \mathcal{C} is convex and there exists a solution for (1.1) in Ω , then either*

$$H(x) < \frac{1}{Nc}$$

or

$$(4.1) \quad H(x) \equiv \frac{1}{Nc}$$

on for every $x \in \Gamma_\Omega$.

PROOF. The proof of [6, Lemma 3.3] can be easily adapted to our setting. Denoting for brevity $u_v = \frac{\partial u}{\partial v}$, exploiting that $-\Delta u = 1$, and arguing as in [6, Lemma 3.3], we have

$$(4.2) \quad u_{vv} - (N - 1)cH(x) = -1 \quad \text{on } \Gamma_\Omega.$$

Consider the two possible cases given by Lemma 4.1. In the inequality case,

$$(4.3) \quad P_v = 2u_v u_{vv} + \frac{2}{N}u_v > 0.$$

Dividing (4.3) by $2u_v$ and combining with (4.2), we obtain

$$H(x) < \frac{1}{Nc}.$$

The case of equality in Lemma 4.1 develops similarly. ■

As anticipated, we show that the solutions of the overdetermined problem are (relatively) self-Cheeger.

THEOREM 4.3. *If \mathcal{C} is convex and there exists a solution u for (1.1) in Ω , then $|Du| \leq c$ in $\bar{\Omega}$, c as in (1.1), and Ω is self-Cheeger.*

PROOF. Recall from the proof of Lemma 4.1 that

$$P(x) = |Du(x)|^2 + \frac{2}{N}u(x) \leq c^2$$

in $\bar{\Omega}$. By the maximum principle, we have that u is positive, and thus we obtain that $|Du| \leq c$ in $\bar{\Omega}$. It then follows that for any smooth subdomain $E \subset \Omega$ (recall Remark 3.6), we have

$$|E| = \int_E -\Delta u \, dx = \int_{\partial E} \frac{\partial u}{\partial v} \, d\sigma \leq c \mathcal{H}_{N-1}(\partial E \cap \mathcal{C})$$

because $\frac{\partial u}{\partial v} = 0$ on $\Gamma_{1,\Omega}$.

Now, integrating (1.1) in Ω , we have

$$(4.4) \quad \begin{aligned} |\Omega| &= - \int_\Omega \Delta u \, dx = - \int_{\partial\Omega} \frac{\partial u}{\partial v} \, d\sigma = \int_{\Gamma_\Omega} c \, d\sigma \\ &= c \mathcal{H}_{N-1}(\Gamma_\Omega). \end{aligned}$$

Hence,

$$\frac{\mathcal{H}_{N-1}(\Gamma_\Omega)}{|\Omega|} = \frac{1}{c} \leq \frac{\mathcal{H}_{N-1}(\partial E \cap \mathcal{C})}{|E|}.$$

The proof is complete. ■

REMARK 4.4. The previous theorem allows us to study the geometrical properties of the bounded domains Ω which admit a solution of the overdetermined problem (1.1), by looking at those of the self-Cheeger sets. In particular, we point out those provided by Theorems 3.7 and 3.8.

We conclude with a result which applies when $\Omega = \Omega_\varphi$ is defined by the graph of a function φ as in Section 3. It gives a bound for the gradient of φ , whenever Ω_φ admits a solution of the overdetermined problem (1.1).

PROPOSITION 4.5. *Let $\omega \subset \mathbb{R}^{N-1}$ be a convex smooth bounded domain and $\varphi \in C^2(\bar{\omega})$ a positive function. If Γ_φ meets $\partial\mathcal{C}_\omega$ orthogonally and Ω_φ admits a solution to the overdetermined problem (1.1), then we have a bound from above for $|\nabla\varphi|$:*

$$(4.5) \quad \frac{\mathcal{H}_{N-1}(\Gamma_\varphi)}{N} < \int_\omega \frac{1}{\sqrt{1+|\nabla\varphi|^2}} dx'.$$

PROOF. From the Minkowski formula (2.7), we have

$$(4.6) \quad \begin{aligned} & \int_{\Gamma_\varphi} H_{\Gamma_\varphi} \langle x_N e_N, \nu \rangle d\sigma \\ &= \frac{1}{N-1} \int_\omega \frac{|\nabla\varphi|^2}{\sqrt{1+|\nabla\varphi|^2}} dx' \\ &= \frac{1}{N-1} \int_\omega \frac{1+|\nabla\varphi|^2}{\sqrt{1+|\nabla\varphi|^2}} dx' - \frac{1}{N-1} \int_\omega \frac{1}{\sqrt{1+|\nabla\varphi|^2}} dx' \\ &= \frac{\mathcal{H}_{N-1}(\Gamma_\varphi)}{N-1} - \frac{1}{N-1} \int_\omega \frac{1}{\sqrt{1+|\nabla\varphi|^2}} dx'. \end{aligned}$$

Now, from the bound on H_{Γ_φ} given by Proposition 4.2 (observe that the case (4.1), i.e., $H_{\Gamma_\varphi} \equiv \frac{1}{Nc}$, cannot happen, in view of Proposition 2.1), we have

$$(4.7) \quad \begin{aligned} \int_{\Gamma_\varphi} H_{\Gamma_\varphi} \langle x_N e_N, \nu \rangle d\sigma &< \int_{\Gamma_\varphi} \frac{\langle x_N e_N, \nu \rangle}{Nc} d\sigma \\ &= \frac{1}{Nc} \int_\omega \varphi dx' \\ &= \frac{|\Omega_\varphi|}{Nc} = \frac{\mathcal{H}_{N-1}(\Gamma_\varphi)}{N}, \end{aligned}$$

by (4.4). Finally, combining (4.6) and (4.7), we get

$$\frac{\mathcal{H}_{N-1}(\Gamma_\varphi)}{N-1} - \frac{1}{N-1} \int_\omega \frac{1}{\sqrt{1+|\nabla\varphi|^2}} dx' < \frac{\mathcal{H}_{N-1}(\Gamma_\varphi)}{N},$$

and (4.5) readily follows. \blacksquare

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