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Calculus of Variations, Differential Geometry. – *Cylindrical singular minimal surfaces*, by Ulrich Dierkes and Rafael López, communicated on 10 February 2023.

Dedicated to the memory of Stefan Hildebrandt.

Abstract. – We construct a foliation of the upper half space in $\mathbb{R}^3$ consisting of minimizing cylindrical α-singular minimal surfaces when $\alpha < 0$. Furthermore, we discuss stability results for the α-catenaries when $\alpha > 0$ and relate these to the geodesics of a conformal metric.

Keywords. – Singular minimal surfaces, cylindrical surface, $\alpha + 1$-catenary, foliation.

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1. Introduction

Let α be a fixed real number and let $\vec{a}$ be a unit vector in the Euclidean space $\mathbb{R}^3$. An oriented connected surface Σ in the half space $\mathbb{R}^3_+ = \{p \in \mathbb{R}^3 : \langle p, \vec{a}\rangle > 0\}$ is called an *α-singular minimal surface* with respect to the vector $\vec{a}$ if its mean curvature H satisfies

$$H(p) = \frac{\alpha}{2}\frac{\langle N(p), \vec{a}\rangle}{\langle p, \vec{a}\rangle}, \quad p \in \Sigma, \tag{1}$$

where N is the Gauß map of Σ. Here, $\langle\cdot,\cdot\rangle$ denotes the usual scalar product of the Euclidean space. The case $\alpha = 0$ corresponds to a minimal surface; if $\alpha = 1$, Σ is the two-dimensional analogue of the catenary [6]; if $\alpha = -2$, Σ is a minimal surface in the upper half space model of hyperbolic space $\mathbb{H}^3$. In nonparametric form, and choosing $\vec{a} = (0, 0, 1)$, equation (1) is equivalent to

$$\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = \frac{\alpha}{u\sqrt{1+|Du|^2}}, \tag{2}$$

where Σ is the graph of $z = u(x, y)$, $(x, y) \in \Omega \subset \mathbb{R}^2$, and (x, y, z) are canonical coordinates of $\mathbb{R}^3$. This equation is non-uniformly elliptic with a strong singularity at $u = 0$. Every α-singular minimal surface is also a minimal surface in a Riemannian manifold. Indeed, consider in $\mathbb{R}^3_+$ the conformal metric $\tilde{g} = z^\alpha\langle\cdot,\cdot\rangle$. Then, Σ viewed as an isometric immersion $\Sigma \hookrightarrow (\mathbb{R}^3_+, \tilde{g})$ is minimal if and only if Σ is an α-singular minimal surface. For further geometrically relevant interpretation of (1) or (2), see the papers [8, 11].

Suppose that Σ is a cylindrical surface, that is, a surface invariant by a uniparametric group of translations along the direction $\vec{v} \in \mathbb{R}^3$. Let us parametrize Σ as $X(s,t) = \gamma(s) + t \cdot \vec{v}$, $s \in I \subset \mathbb{R}$, $t \in \mathbb{R}$, where γ is a planar curve whose trace is contained in a plane orthogonal to $\vec{v}$. It was proved in [11] that if Σ is an α-singular minimal surface, then Σ is a plane parallel to $\vec{a}$ or $\vec{v}$ is orthogonal to $\vec{a}$. Discarding the first case when Σ is a plane parallel to $\vec{a}$, and after a rigid motion of $\mathbb{R}^3$, we will suppose that $\vec{a} = (0,0,1)$ and $\vec{v} = (0,1,0)$. Then, the generating curve γ is contained in the xz-plane and we have $\gamma(s) = (x(s), 0, z(s))$. If κ is the curvature of γ, then equation (1) is equivalent to

$$\kappa(s) = \alpha \frac{\langle \mathbf{n}(s), \vec{a}\rangle}{\langle \gamma(s), \vec{a}\rangle}. \tag{3}$$

A curve γ which satisfies (3) will be called *α-catenary* (the usual catenary, if $\alpha = 1$), and the corresponding surface Σ will be named *α-singular minimal catenary*. The generating curves of α-singular minimal catenaries are important in two different scenarios. First, they also are the profiles of rotational α-singular minimal surfaces. Indeed, suppose that Σ is an α-singular minimal surface that, in addition, is a surface of revolution about an axis L. Let us observe that there is no *a priori* relation between the axis L and the vector $\vec{a}$. However, equation (1) imposes a relation between L and $\vec{a}$ as follows.

THEOREM 1.1 ([11]). *Let Σ be an α-singular minimal surface which is a surface of revolution about L. Then, either L and $\vec{a}$ are parallel or L is orthogonal to $\vec{a}$ and L is included in the coordinate plane $z = 0$. In the latter case, the generating curve of Σ is an $\alpha + 1$-catenary.*

In particular, we have two families of rotational singular minimal surfaces. One of them corresponds to the case that L is parallel to $\vec{a}$: see [11] for a full description of these surfaces, also in [8]. On the other hand, the profile curves of the second family of surfaces are just solutions of (3) for a different constant α. As a particular case, if we take $\alpha = 0$ in Theorem 1.1, then the usual catenary when rotated about a horizontal axis generates a minimal surface of rotational type, which is, obviously, the catenoid.

The purpose of this paper is as follows:

(a) To give a qualitative geometric description of the generating α-catenaries.

(b) To investigate the stability and/or minimizing properties of the corresponding α-singular minimal catenaries.

(c) To compute the geodesics of the conformal metric $\tilde{g}$ and relate these with α-catenaries.

The organization of this paper is the following. In Section 2, we present the geometric description of α-catenaries. Here, we distinguish between the cases $\alpha < 0$ and $\alpha > 0$ (Theorems 2.2 and 2.4). Furthermore, we investigate when the α-singular minimal catenaries are the minimizers of the α-area functional in a class of functions with bounded variation, cp. Theorem 2.3 and Theorem 2.4. The proofs of Theorems 2.2 and 2.3 are carried out in Section 3, whereas Theorem 2.4 will be proved in Section 4. In Section 5, we study the stability of α-catenaries. By Theorem 2.2, we know that if $\alpha < 0$, these curves are minimizers; in particular, they are stable. However, if $\alpha > 0$, conjugate points occur and hence solutions are in general not minimizers (i.e. if the arc under investigation contains a pair of conjugate points). For the cases $\alpha = 1$ and $\alpha = 1/2$, where we know explicit solutions of (3), we give sharp estimates for the range of stability of these α-catenaries (Propositions 5.2 and 5.3). While in Section 2, we have proved that the upper halfplane $\mathbb{R}^2_+$ can be foliated by a uniparametric family of α-catenaries for each α, in Section 6, we show that $\mathbb{R}^2_+$ can also be foliated by all α-catenaries once we have fixed the same initial conditions by varying $\alpha \in (-\infty, \infty)$ (Theorem 6.1). Moreover, all these α-catenaries are monotonically ordered in $\mathbb{R}^2_+$ according to the value of α. Finally, in Section 7, we give a new interpretation of α-catenaries, proving that α-catenaries coincide with the geodesics of the manifold $(\mathbb{R}^3_+, z^{2\alpha}\langle\cdot,\cdot\rangle)$; see Theorem 7.5.

2. Geometric description of α-catenaries

Suppose that $\gamma(s) = (x(s), z(s))$ is parametrized by arc-length and let

$$\gamma'(s) = \big(\cos\theta(s), \sin\theta(s)\big)$$

for some smooth function $\theta = \theta(s)$. Then, γ is an α-catenary if and only if the functions $\{x(s), z(s), \theta(s)\}$ satisfy

$$\begin{aligned} x'(s) &= \cos\theta(s), \\ z'(s) &= \sin\theta(s), \\ \theta'(s) &= \alpha\frac{\cos\theta(s)}{z(s)}. \end{aligned} \tag{4}$$

Proposition 2.1. *The solutions of (4) are graphs over the x-axis or vertical straight-lines.*

Proof. Suppose that γ is not a graph over the x-axis. Then, there is $s_0 \in I$ such that γ is vertical at $s = s_0$; i.e. $x'(s_0) = \cos\theta(s_0) = 0$. Define the functions $\{\bar{x}(s), \bar{z}(s), \overline{\theta}(s)\}$ by

$$\bar{x}(s) = x(s_0),$$

$$\bar{z}(s) = s - s_0 + z(s_0),$$
$$\overline{\theta}(s) = \frac{\pi}{2}.$$

It is immediate that the triple functions $\{\bar{x}(s), \bar{z}(s), \overline{\theta}(s)\}$ satisfy the same system (4) with the same initial conditions at $s = s_0$ as the functions $\{x(s), z(s), \theta(s)\}$. Uniqueness proves they agree and thus γ is a vertical straight-line. ∎

Thanks to this result, we can assume, without loss of generality, that the tangent angle θ is included in the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$. Proposition 2.1 also asserts that we can write $\gamma(s) = (s, f(s))$ for a positive function $f = f(s)$, $s \in I$, and by (3), the function f satisfies

$$\frac{f''(s)}{1 + f'(s)^2} = \frac{\alpha}{f(s)}. \tag{5}$$

In particular, f is convex (resp. concave) if $\alpha > 0$ (resp. $\alpha < 0$). Let us observe that f is a constant function if and only if $\alpha = 0$. If $f' \not\equiv 0$, multiplying in (5) by f', we obtain

$$f' = \pm\sqrt{c^2 f^{2\alpha} - 1}, \quad c \neq 0. \tag{6}$$

This implies, together with (5),

$$f''(s) = \alpha c^2 f(s)^{2\alpha - 1}. \tag{7}$$

This equation is known in the literature as Emden–Fowler type equation [12]. Some explicitly known solutions are semicircles of the type $f(x) = \sqrt{R^2 - (x-a)^2}$ for $\alpha = -1$, and the catenaries $f(x) = a\cosh(\frac{x-b}{a})$, when $\alpha = 1$.

Introducing the polar angle φ by

$$\tan\varphi = \frac{z}{x},$$

we find by (4)

$$\frac{d\varphi}{d\theta} = \frac{\sin\varphi \cdot \sin(\theta - \varphi)}{\alpha\cos\theta},$$

which leads to the two-dimensional system

$$\begin{aligned} \Phi(\varphi, \theta) &:= \frac{d\varphi}{dt} = \sin\varphi \cdot \sin(\theta - \varphi), \\ \Psi(\varphi, \theta) &:= \frac{d\theta}{dt} = \alpha\cos\theta. \end{aligned} \tag{8}$$

Later in Sections 3 and 4, we will carefully analyze system (8) in the phase space (φ, θ).

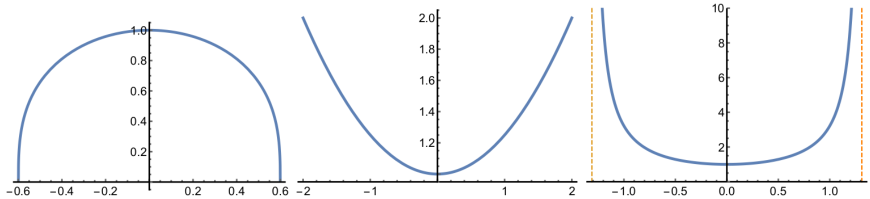

FIGURE 1. α-catenaries. Case $\alpha = -2$ (left); Case $\alpha = 0.5$ (middle); Case $\alpha = 2$ (right), where the curve is asymptotic to the vertical lines of equation $x = \pm R$, $R \approx 1.311$.

In the following, we tacitly assume that $f'(0) = 0$, $f(0) > 0$ for a solution f of (5) or, equivalently, $\theta(0) = z'(0) = x(0) = 0$ for any solution $(x(s), z(s), \theta(s))$ of (4).

THEOREM 2.2. *Let $\alpha < 0$. Any solution $z = f(x)$ of (5) is defined on a maximal interval $(-R, R)$, where $R < \infty$ depends on f and satisfies the following:*

(a) *f is strictly concave and symmetric about the z-axis.*

(b) *We have $\lim_{x\to\pm R} f(x) = 0$ and $\lim_{x\to\pm R} f'(x) = \mp\infty$.*

(c) *Every solution $z = f(x)$, $x \in (-R, R)$, of (5) is embedded in a field of extremals for the variational integral*

$$I_\alpha(u) = \int u(x)^\alpha \sqrt{1 + u'(x)^2}\, dx$$

and hence minimizes $I_\alpha(-)$ with respect to arbitrary variations in the upper halfplane $\mathbb{R} \times \mathbb{R}^+$ or $(-R, R) \times \mathbb{R}^+$, respectively. See Figure 1, left.

In fact, much more is true. To this end, consider the α-singular minimal catenaries Σ given by

$$\mathcal{X}(s, t) = \big(x(s), 0, z(s)\big) + t \cdot (0, 1, 0),$$

which are generated by any solution $z = f(x)$ of (5); that is,

$$\big(x(s), 0, z(s)\big) = \big(s, 0, f(s)\big) \quad \text{for } s \in (-R, R).$$

Clearly, Σ is the graph of the function $z = v(x, y) := f(x)$ defined for (x, y) in the infinite strip

$$\mathcal{I}_R = (-R, R) \times \mathbb{R}$$

and $z = v(x, y)$ is a solution of the nonparametric singular minimal surface equation (2) on $\mathcal{I}_R$. We show that these cylinders determine a foliation of the upper half space $\mathbb{R}^3_+$ such that every single cylinder minimizes the corresponding energy-functional in a very general sense.

THEOREM 2.3. *Let $\alpha < 0$ be fixed. There exists a foliation of $\mathbb{R}^3_+ := \mathbb{R}^2 \times \mathbb{R}^+$, $\mathbb{R}^+ := \{t > 0\}$, determined by concave functions $z = v_\lambda(x,y) : \mathcal{I}_\lambda \to \mathbb{R}^+$, $\lambda > 0$ arbitrary, and $v_\lambda(0,0) = \lambda v_1(0,0)$, $Dv_\lambda(0,0) = 0$, such that each $v_\lambda \in C^\omega(\mathcal{I}_\lambda) \cap C^{0,\frac{1}{1+|\alpha|}}(\overline{\mathcal{I}_\lambda})$ solves the Dirichlet problem*

$$\begin{cases} \operatorname{div}\left(\dfrac{Dv_\lambda}{\sqrt{1+|Dv_\lambda|^2}}\right) = \dfrac{\alpha}{v_\lambda\sqrt{1+|Dv_\lambda|^2}} & \text{in } \mathcal{I}_\lambda, \\ v_\lambda = 0 & \text{on } \partial\mathcal{I}_\lambda. \end{cases}$$

In fact, the graph of every v_λ is a cylinder; i.e. $\frac{\partial v_\lambda(x,y)}{\partial y} = 0$ for all $(x,y) \in \mathcal{I}_\lambda$ and $|Dv_\lambda(x,y)| \to \infty$ as $(x,y) \to \partial\mathcal{I}_\lambda$. Finally, the subgraphs

$$\mathcal{V}_\lambda := \big\{(x,y,z) \in \mathcal{I}_\lambda \times \mathbb{R}^+ : 0 < z \le v_\lambda(x,y)\big\},$$

$\lambda > 0$, have boundaries of least α-area in $\mathbb{R}^3_+$. In particular, the function $v_\lambda = v_\lambda(x,y)$ minimizes locally the α-area

$$\mathcal{E}_\alpha(u) := \int u^\alpha \sqrt{1+|Du|^2}$$

in $\mathrm{BV}_{\mathrm{loc}}(\mathcal{I}_\lambda)$. (For definitions of "$\mathrm{BV}_{\mathrm{loc}}$", "$\alpha$-area", etc., see Remark (2) after Theorem 2.4.)

THEOREM 2.4. *Let $\alpha > 0$. Every solution $z = f(x)$ of (5) is defined on a maximal interval $(-R, R)$, $R > 0$ depending on f and fulfills the following:*

(i) *f is symmetric about the z-axis and strictly convex with minimum at $x = 0$.*

(ii) *For $\alpha > 1$, we have $R < \infty$, while for $\alpha \in (0,1]$, also $R = \infty$. In both cases, $\lim_{s\to\pm R} f(s) = \infty$. In particular, if $\alpha > 1$, the graph of f is asymptotic to two vertical lines.*

(iii) *Let $f_1(x)$ denote the solution of (5) such that $f_1(0) = 1$. Then, every solution of (5) (with $f'(0) = 0$) is of the form $f_\lambda(x) := \lambda^{-1} f_1(\lambda x)$, $x \in (-\frac{R}{\lambda}, \frac{R}{\lambda})$. On the domain of definition, there exists a unique pair of conjugate values $\pm x_\lambda \in (-\frac{R}{\lambda}, \frac{R}{\lambda})$, $x_\lambda > 0$, so that $P_\lambda := (-x_\lambda, f_\lambda(-x_\lambda))$ and $P^*_\lambda := (x_\lambda, f_\lambda(x_\lambda))$ are conjugate points on the curve $\mathcal{C}_\lambda := \{(x, f_\lambda(x)) : x \in (-\frac{R}{\lambda}, \frac{R}{\lambda})\}$, considered as extremal of the variational integral*

$$I_\alpha(u) := \int u(x)^\alpha \sqrt{1+u'(x)^2}\,dx.$$

Consequently, $u = f_\lambda(\cdot)$ is not a weak (local) minimizer of I_α on any interval containing $(-x_\lambda, x_\lambda)$, while f_λ furnishes a weak minimum for I_α on every interval $[a,b] \subset (-x_\lambda, x_\lambda)$.

See Figure 1, middle and right.

REMARKS. (1) By adding more redundant variables, we obtain a result analogous to Theorem 2.3 in the upper half space $\mathbb{R}^n \times \mathbb{R}^+$, $\mathbb{R}^+ = \{t > 0\}$, for arbitrary dimension $n > 2$.

(2) For arbitrary open $\Omega \subset \mathbb{R}^n$ and locally integrable function $f \in L^1_{\mathrm{loc}}(\Omega)$, we write $f \in \mathrm{BV}_{\mathrm{loc}}(\Omega)$ if for each open set $U \Subset \Omega$, we have

$$\sup\left\{\int_U f \operatorname{div}\varphi dx;\ \varphi \in C^1_c(U,\mathbb{R}^n),\ |\varphi(x)| \le 1\right\} < \infty.$$

Hence, $f \in \mathrm{BV}_{\mathrm{loc}}(\Omega)$ if the distributional derivative Df is a Radon measure.

Let $A \subset \mathbb{R}^n \times \mathbb{R}^+$ be an open set, $\alpha < 0$, and $f \in \mathrm{BV}_{\mathrm{loc}}(A)$. Then, we have the following.

DEFINITION 2.5. f has "*least α-weighed gradient*" or simply "*least α-gradient*" in A if

$$\int_K x_{n+1}^\alpha |Df| \le \int_K x_{n+1}^\alpha |D(f+g)|$$

for every function $g \in \mathrm{BV}_{\mathrm{loc}}(A)$ with compact support $K \subset A$. Moreover, a set $C \subset A$ has a boundary of "*least α-area*" with respect to A if the characteristic function $\varphi_C \in \mathrm{BV}_{\mathrm{loc}}(A)$ and has least α-gradient in A.

For more pertinent comments and references to the literature, we refer to [3,5,7].

3. CASE $\alpha < 0$: PROOF OF THEOREMS 2.2 AND 2.3

The proof invokes the classical ideas of Jacobi and Weierstraß, in particular, the concept of a "*field of curves or extremals*", nowadays more commonly denoted as "*foliation*" or "*calibration*", and we refer to the monographs of Bolza [2], Carathéodory [4], and Giaquinta and Hildebrandt [9] for a complete description of the classical theory of the calculus of variations. In particular, a variant of a method due to Bombieri, De Giorgi, and Giusti, which was introduced in their celebrated paper on minimal cones [3], will be used here, similarly as in [5,7].

We prove both theorems simultaneously, and the idea is the following (cp. [5,7]). First, we will see that any function f which has least α-gradient according to the definition above has sub-level sets of least α-area (Lemma 3.3 below). Furthermore, any function f, which belongs to the Sobolev-class $H^1_{1,\mathrm{loc}}(A)$ and solves the equation

$$\int_A x_{n+1}^\alpha |Df|^{-1} Df D\varphi \, dx = 0, \quad \forall \varphi \in C_c^\infty(A), \tag{9}$$

where $Df = (D_1 f, \dots, D_{n+1} f)$ stands for the weak gradient, also has least α-gradient in A (see Lemma 3.1). Finally, a function $f(x) = F(u,v)$ of the two variables

$$u = u(x_1,x_2,x_3) = |x_1|, \quad v = v(x_1,x_2,x_3) = x_3$$

is constructed, which solves (9) on the domain $A = \mathbb{R}^2 \times \mathbb{R}^+$, and hence has least α-gradient in $\mathbb{R}^2 \times \mathbb{R}^+$. In particular, the level sets of f are given by the cylindrical surfaces or the curves which are described in Theorem 2.3 or Theorem 2.2, respectively.

For the convenience of the reader, we recall the following lemma (cp. [3, 5, 7]).

LEMMA 3.1. *Let $A \subset \mathbb{R}^n \times \mathbb{R}^+$ be an open set and let $\mathcal{N} \subset \mathbb{R}^{n+1}$ be closed with $\mathcal{H}_n(\mathcal{N}) = 0$. Suppose $f \in H^1_{1,\mathrm{loc}}(A)$ satisfies*

$$\mathcal{H}_{n+1}\big(\{x \in A : |Df(x)| = 0\}\big) = 0$$

and

$$\int_A x_{n+1}^{\alpha} |Df|^{-1} Df \cdot D\varphi \, dx = 0$$

for every $\varphi \in C_c^\infty(A \setminus \mathcal{N})$. Then, f has least α-gradient in A. ■

LEMMA 3.2. *Suppose $A \subset \mathbb{R}^n \times \mathbb{R}^+$ is an open set and $\mathcal{S} \subset A$ is closed with $\mathcal{H}_n(\mathcal{S} \cap K) < \infty$ for every compact set $K \subset A$. Furthermore, assume that $f \in C^2(A \setminus \mathcal{S})$ fulfills $|Df| \neq 0$ for all $x \in A \setminus \mathcal{S}$ and $|Df|^{-1} \cdot Df$ extends continuously to all of A. Finally, if the equation*

$$\sum_{i=1}^{n+1} D_i\{x_{n+1}^{\alpha} |Df|^{-1} D_i f\} = 0$$

holds classically in $A \setminus \mathcal{S}$. Then, also

$$\int_A x_{n+1}^{\alpha} |Df|^{-1} Df \cdot D\varphi \, dx = 0$$

for every $\varphi \in C_c^\infty(A)$. ■

LEMMA 3.3. *Let $A \subset \mathbb{R}^n \times \mathbb{R}^+$ and $f \in \mathrm{BV}_{\mathrm{loc}}(A)$ be a function of least α-gradient in A. Then, every (nonempty) level set*

$$\mathcal{E}_\lambda := \{x \in A : f(x) \le \lambda\}$$

has a boundary of least α-area in A.

PROOF. For a proof of Lemmas 3.1, 3.2, and 3.3, see [3, 5, 7]. ■

Since we are looking for a field of cylindrical surfaces in the upper half space $\mathbb{R}^2 \times \mathbb{R}^+$, it is natural to introduce the variables

$$\begin{cases} u = u(x, y, z) = +\sqrt{x^2} = |x|, \\ v = v(x, y, z) = z, \end{cases} \tag{10}$$

where $(x_1, x_2, x_3) \equiv (x, y, z) \in \mathbb{R}^2 \times \mathbb{R}^+$, and consider the function

$$f(x, y, z) = F\big(u(x, y, z), v(x, y, z)\big),$$

where $F : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ has to be determined. Suppose $\widetilde{\Omega} \subset \mathbb{R}^+ \times \mathbb{R}^+$ is open and corresponds to $\Omega \subset \mathbb{R}^2 \times \mathbb{R}^+$ under the transformation (10). Then, a function $f = f(x, y, z) = f(x_1, x_2, x_3) \in C^2(\Omega)$, $|\nabla f| \neq 0$, satisfies the Euler equation (9) or

$$\sum_{i=1}^{3} D_i \left(x_3^\alpha \frac{D_i f}{|Df|} \right) = 0 \quad \text{in } \Omega,$$

if and only if $F = F(u, v)$ fulfills

$$\frac{\partial}{\partial u}\left(\frac{v^\alpha F_u}{|\nabla F|}\right) + \frac{\partial}{\partial v}\left(\frac{v^\alpha F_v}{|\nabla F|}\right) = 0$$

or

$$F_v^2 F_{uu} - 2F_u F_v F_{uv} + F_u^2 F_{vv} + \frac{\alpha}{v} F_v |\nabla F|^2 = 0 \quad \text{in } \widetilde{\Omega}, \tag{11}$$

where we have put $F_u = \frac{\partial F}{\partial u}$, $F_v = \frac{\partial F}{\partial v}$, $|\nabla F|^2 = F_u^2 + F_v^2$, etc.

The function $F = F(u, v)$ will in turn be constructed from its level sets $\{F = \text{const}\}$. Along a regular level curve $(u(t), v(t)) \in \mathbb{R}^+ \times \mathbb{R}^+$, we have $dF = F_u\, du + F_v\, dv = 0$, and by (11), also

$$u''v' - v''u' + \alpha(u'^2 + v'^2)\frac{u'}{v} = 0, \tag{12}$$

where $u' = \frac{du}{dt}$ and $v' = \frac{dv}{dt}$, etc. Introducing "polar" (resp. "tangent") angular coordinates φ and θ, respectively, by

$$\begin{cases} \varphi := \operatorname{arctg}\left(\dfrac{v}{u}\right), \\ \theta := \operatorname{arctg}\left(\dfrac{v'}{u'}\right), \end{cases}$$

which are both invariant under homotheties $(u, v) \mapsto \lambda(u, v)$, for $\lambda > 0$, we find the relations

$$\varphi' = \frac{v'u - u'v}{u^2 + v^2} \quad \text{and} \quad \theta' = \frac{v''u' - u''v'}{(u')^2 + (v')^2}.$$

On the other hand, we infer from

$$u = \sqrt{u^2 + v^2}\cos\varphi, \qquad v = \sqrt{u^2 + v^2}\sin\varphi,$$
$$u' = \sqrt{(u')^2 + (v')^2}\cos\theta, \quad v' = \sqrt{(u')^2 + (v')^2}\sin\theta$$

that also

$$\varphi' = \left\{\frac{(u')^2 + (v')^2}{u^2 + v^2}\right\}^{\frac{1}{2}} \sin(\theta - \varphi),$$

while from (12), we have

$$\theta' = \alpha \frac{u'}{v} = \frac{\alpha\sqrt{(u')^2 + (v')^2}\cos\theta}{\sqrt{u^2 + v^2}\sin\varphi}.$$

The last relations yield

$$\frac{\theta'}{\varphi'} = \frac{\alpha\cos\theta}{\sin\varphi\sin(\theta - \varphi)},$$

which is equivalent to the two-dimensional system

$$\begin{aligned} \frac{d\varphi}{dt} &= \sin\varphi\sin(\theta - \varphi), \\ \frac{d\theta}{dt} &= \alpha\cos\theta, \end{aligned} \tag{13}$$

which coincides with the system (8).

In the following, it is necessary to analyze (13) according to Poincaré–Bendixson theory in the phase space (φ, θ). See Figure 2. As will be clear later, it is sufficient to concentrate on the rectangle $\overline{R} = [0, \pi] \times [-\frac{\pi}{2}, \frac{\pi}{2}]$, or even $\overline{R}_1 = [0, \frac{\pi}{2}] \times [-\frac{\pi}{2}, 0]$. First, observe that there are no singular points (φ, θ) of (13) in the open rectangle

$$R = (0, \pi) \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right),$$

and hence also no limit cycles of (13) exist within R. The only singular points in $\overline{R}$ are contained in the boundary ∂R, namely, the six points

$$(\varphi, \theta) = \left(0, \pm\frac{\pi}{2}\right), \left(\frac{\pi}{2}, \pm\frac{\pi}{2}\right), \left(\pi, \pm\frac{\pi}{2}\right).$$

These equilibria can be classified as follows (assuming $\alpha < 0$):

(1) The points $(\varphi, \theta) = (0, \frac{\pi}{2})$ or $(\pi, \frac{\pi}{2})$ are unstable nodes with principal or exceptional directions $(\pm 1, 0)$ or $(0, \pm 1)$ (depending on the value of $\alpha < 0$). The linearized system has matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & -\alpha \end{pmatrix}.$$

(2) The points $(\varphi, \theta) = (0, -\frac{\pi}{2})$ or $(\pi, -\frac{\pi}{2})$ are stable nodes with principal or exceptional direction $(\pm 1, 0)$ or $(0, \pm 1)$. The linearized system has matrix

$$\begin{pmatrix} -1 & 0 \\ 0 & \alpha \end{pmatrix}.$$

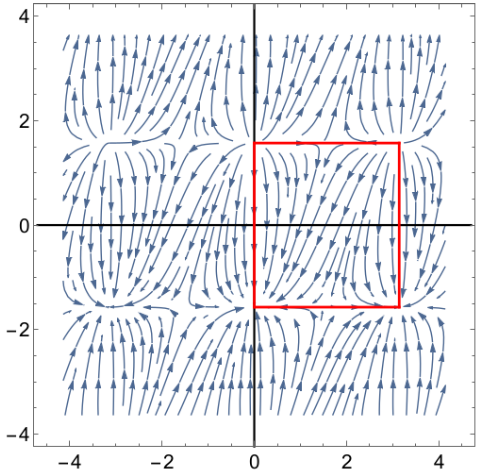

FIGURE 2. The (φ, θ)-phase space for $\alpha = -2$ including the rectangle R.

(3) The singular points $(\varphi, \theta) = (\frac{\pi}{2}, \pm\frac{\pi}{2})$ are saddle points with linearized system matrix

$$\begin{cases} \begin{pmatrix} -1 & 1 \\ 0 & -\alpha \end{pmatrix}, & (\varphi, \theta) = \left(\frac{\pi}{2}, \frac{\pi}{2}\right), \\ \begin{pmatrix} 1 & -1 \\ 0 & \alpha \end{pmatrix}, & \text{otherwise.} \end{cases}$$

The stable manifold has direction $\pm(1, 0)$ or $\pm(1, 1-\alpha)$ for $(\varphi, \theta) = (\frac{\pi}{2}, \frac{\pi}{2})$ or $(\frac{\pi}{2}, -\frac{\pi}{2})$, respectively, while the unstable manifolds have direction $\pm(1, 1-\alpha)$ or $\pm(1, 0)$, respectively.

LEMMA 3.4. *Let $\alpha < 0$. There exists a trajectory $\sigma(t) = (\varphi(t), \theta(t))$, $t \in \mathbb{R}$, which solves system (13) and has the following properties:*

(i) $\sigma(+\infty) = (0, -\frac{\pi}{2})$, $\sigma(-\infty) = (\pi, \frac{\pi}{2})$, *and* $\sigma(0) = (\frac{\pi}{2}, 0)$.

(ii) $\sigma(t) \in R$ *for all finite* $t \in (-\infty, \infty)$*; in fact,* $\sigma(t) \in (0, \frac{\pi}{2}) \times (-\frac{\pi}{2}, 0)$ *for* $t > 0$ *and* $\sigma(t) \in (\frac{\pi}{2}, \pi) \times (0, \frac{\pi}{2})$ *for every* $t < 0$.

(iii) $\frac{d\varphi}{dt} < 0$ *and* $\frac{d\theta}{dt} < 0$ *for all* $t \in \mathbb{R}$.

In particular, the integral curve σ has a nonparametric representation $\theta = \Theta(\varphi)$ for some differentiable function

$$\Theta : (0, \pi) \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

with $\Theta(0) = -\frac{\pi}{2}$, $\Theta(\frac{\pi}{2}) = 0$, $\Theta(\pi) = \frac{\pi}{2}$, *and* $\frac{d\Theta}{d\varphi} > 0$.

PROOF. Let l_1, l_2, l_3, and l_4 denote the sides of the (φ, θ)-rectangle R; that is,

$$l_1 = [0, \pi) \times \left\{ -\frac{\pi}{2} \right\}, \quad l_2 = \{\pi\} \times \left[-\frac{\pi}{2}, \frac{\pi}{2} \right),$$
$$l_3 = (0, \pi] \times \left\{ \frac{\pi}{2} \right\}, \quad l_4 = \{0\} \times \left(-\frac{\pi}{2}, \frac{\pi}{2} \right].$$

The vectorfield $(\Phi(\varphi, \theta), \Psi(\varphi, \theta)) = (\sin\varphi \sin(\theta - \varphi), \alpha \cos\theta)$ determined by (13) satisfies on the sides l_1, l_2, l_3, and l_4, respectively,

$$\Phi_{|l_1} = \Phi\left(\varphi, -\frac{\pi}{2}\right) = \sin\varphi \sin\left(-\frac{\pi}{2} - \varphi\right) = -\sin\varphi\cos\varphi = \begin{cases} <0 & \text{if } \varphi \in \left(0, \frac{\pi}{2}\right) \\ >0 & \text{if } \varphi \in \left(\frac{\pi}{2}, \pi\right), \end{cases}$$
$$\Psi_{|l_1} = \Psi\left(\varphi, -\frac{\pi}{2}\right) = \alpha\cos\left(-\frac{\pi}{2}\right) = 0,$$
$$\Phi_{|l_2} = \Phi(\pi, \theta) = 0, \quad \Psi_{l_2} = \Psi(\pi, \theta) = \alpha\cos\theta < 0 \quad \text{for } \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right),$$
$$\Phi_{|l_3} = \Phi\left(\varphi, \frac{\pi}{2}\right) = \sin\varphi\sin\left(\frac{\pi}{2} - \varphi\right) = \sin\varphi\cos\varphi = \begin{cases} >0, & \text{if } \varphi \in \left(0, \frac{\pi}{2}\right), \\ <0, & \text{if } \varphi \in \left(\frac{\pi}{2}, \pi\right), \end{cases}$$
$$\Psi_{|l_3} = \Psi\left(\varphi, \frac{\pi}{2}\right) = \alpha\cos\frac{\pi}{2} = 0,$$
$$\Phi_{|l_4} = \Phi(0, \theta) = 0, \quad \Psi_{|l_4} = \Psi(0, \theta) = \alpha\cos\theta < 0, \quad \text{if } \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

This calculation shows that $(\Phi(\varphi, \theta), \Psi(\varphi, \theta))$ at non-singular boundary points of the rectangle $\partial R = l_1 \cup l_2 \cup l_3 \cup l_4$ is directed into the closure $\overline{R} = [0, \pi] \times [-\frac{\pi}{2}, \frac{\pi}{2}]$ of R; hence, any trajectory of (13) which starts in $\overline{R}$ remains trapped in $\overline{R}$. Note also that any trajectory which starts at a singular point on the boundary ∂R and is directed into the interior R has to end at another singular point on ∂R.

Now it is convenient to consider the subrectangles $R_1 = (0, \frac{\pi}{2}) \times (-\frac{\pi}{2}, 0)$ and $R_2 = (\frac{\pi}{2}, \pi) \times (0, \frac{\pi}{2})$ of the quadrilateral R, with boundaries $\partial R_1 = L_1 \cup L_2 \cup L_3 \cup L_4$ and $\partial R_2 = s_1 \cup s_2 \cup s_3 \cup s_4$, where

$$L_1 = \left[0, \frac{\pi}{2}\right) \times \left\{-\frac{\pi}{2}\right\}, \quad L_2 = \left\{\frac{\pi}{2}\right\} \times \left[-\frac{\pi}{2}, 0\right),$$
$$L_3 = \left(0, \frac{\pi}{2}\right] \times \{0\}, \quad L_4 = \{0\} \times \left(-\frac{\pi}{2}, 0\right],$$

$$s_1 = \left[\frac{\pi}{2}, \pi\right) \times \{0\}, \qquad s_2 = \{\pi\} \times \left[0, \frac{\pi}{2}\right),$$
$$s_3 = \left(\frac{\pi}{2}, \pi\right] \times \left\{\frac{\pi}{2}\right\}, \qquad s_4 = \left\{\frac{\pi}{2}\right\} \times \left(0, \frac{\pi}{2}\right].$$

An inspection of the vectorfield $(\Phi(\varphi, \theta), \Psi(\varphi, \theta))$ on the respective boundary segments yields

$$\begin{aligned}
\Phi_{|L_1} &= \Phi\left(\varphi, -\frac{\pi}{2}\right) = -\sin\varphi\cos\varphi < 0, \quad \text{if } \varphi \in \left(0, \frac{\pi}{2}\right),\\
\Psi_{|L_1} &= \Psi\left(\varphi, -\frac{\pi}{2}\right) = \alpha\cos\left(-\frac{\pi}{2}\right) = 0,\\
\Phi_{|L_2} &= \Phi\left(\frac{\pi}{2}, \theta\right) = -\cos\theta < 0, \quad \text{if } \theta \in \left(-\frac{\pi}{2}, 0\right),\\
\Psi_{|L_2} &= \Psi\left(\frac{\pi}{2}, \theta\right) = \alpha\cos\theta < 0, \quad \text{if } \theta \in \left(-\frac{\pi}{2}, 0\right),\\
\Phi_{|L_3} &= \Phi(\varphi, 0) = -\sin^2\varphi < 0,\\
\Psi_{|L_3} &= \Psi(\varphi, 0) = \alpha < 0,\\
\Phi_{|L_4} &= \Phi(0, \theta) = 0,\\
\Psi_{|L_4} &= \Psi(0, \theta) = \alpha\cos\theta < 0, \quad \text{if } \theta \in \left(-\frac{\pi}{2}, 0\right).
\end{aligned}$$

Furthermore, we have the following system on the boundary ∂R_2:

$$\begin{aligned}
\Phi_{|s_1} &= \Phi(\varphi, 0) = -\sin^2\varphi < 0,\\
\Psi_{|s_1} &= \Psi(\varphi, 0) = \alpha < 0,\\
\Phi_{|s_2} &= \Phi(\pi, \theta) = 0,\\
\Psi_{|s_2} &= \Psi(\pi, \theta) = \alpha\cos\theta < 0,\\
\Phi_{|s_3} &= \Phi\left(\varphi, \frac{\pi}{2}\right) = \sin\varphi\cos\varphi < 0, \quad \text{if } \varphi \in \left(\frac{\pi}{2}, \pi\right),\\
\Psi_{|s_3} &= \Psi\left(\varphi, \frac{\pi}{2}\right) = 0,\\
\Phi_{|s_4} &= \Phi\left(\frac{\pi}{2}, \theta\right) = -\cos\theta < 0, \quad \text{for } \theta \in \left(0, \frac{\pi}{2}\right),\\
\Psi_{|s_4} &= \Psi\left(\frac{\pi}{2}, \theta\right) = \alpha\cos\theta < 0, \quad \text{if } \theta \in \left[0, \frac{\pi}{2}\right).
\end{aligned}$$

Concluding, we have shown that the vectorfield $(\Phi(\varphi, \theta), \Psi(\varphi, \theta))$ points

(1) strictly into the interior of R_1 along the boundary segments L_2 and L_3;

(2) into the closure $\overline{R}_1 = [0, \frac{\pi}{2}] \times [-\frac{\pi}{2}, 0]$ of R_1 along the boundary lines L_1 and L_4, whereas (Φ, Ψ) is directed to:

(3) the exterior of R_2 along the lines s_1 and s_4;

(4) the closure $\overline{R}_2 = [\frac{\pi}{2}, \pi] \times [0, \frac{\pi}{2}]$ along the boundary segments s_2 and s_3.

Now consider the trajectory $\sigma(t) = (\varphi(t), \theta(t))$, $t \geq 0$, of the system (13) with $\sigma(0) = (\frac{\pi}{2}, 0)$. Since $\frac{\Psi(\frac{\pi}{2},0)}{\Phi(\frac{\pi}{2},0)} = -\alpha > 0$, σ enters the rectangle R_1 and therefore has to end at one of the singular boundary points on ∂R_1, i.e. at $(0, \frac{\pi}{2})$ or $(\frac{\pi}{2}, -\frac{\pi}{2})$. Recall that by the classification above, $(0, -\frac{\pi}{2})$ is a stable node, while $(\frac{\pi}{2}, -\frac{\pi}{2})$ is a saddle point whose stable manifold has direction $\pm(1, 1-\alpha)$. Hence, $\sigma(t)$ must end at the node $(0, -\frac{\pi}{2})$; i.e. $\lim_{t\to\infty} \sigma(t) = (0, -\frac{\pi}{2})$. By a similar argument, we find that also $\lim_{t\to-\infty} \sigma(t) = (\pi, \frac{\pi}{2})$ and (i) of the lemma follows. We have $\sigma(t) \in R_1 \cup R_2 \subset R$ for all finite $t \neq 0$, which implies (ii).

Finally, we observe that $\Psi(\varphi, \theta) = \alpha \cos\theta < 0$ for all $(\varphi, \theta) \in R$, whereas $\Phi(\varphi, \theta) = \sin\varphi \sin(\theta - \varphi) < 0$ for all $(\varphi, \theta) \in R_1 \cup R_2$, since $\sin(\theta - \varphi) < 0$ for $(\varphi, \theta) \in R_1 \cup R_2$. Because of $\sigma(t) \in R_1 \cup R_2 \cup \{(\frac{\pi}{2}), 0\}$ for all finite t, also (iii) of Lemma 3.4 follows.

Since $\varphi'(t) < 0$ for all $t \in (-\infty, \infty)$, we can write $t = T(\varphi)$ for $\varphi \in (0, \pi)$ to denote the inverse function, and hence $\theta = \theta(t) = \theta(T(\varphi)) =: \Theta(\varphi)$ where $\Theta : (0, \pi) \to (-\frac{\pi}{2}, \frac{\pi}{2})$ is differentiable with $\frac{d\Theta}{d\varphi} > 0$ and $\Theta(0) = -\frac{\pi}{2}$ and $\Theta(\pi) = \frac{\pi}{2}$. Lemma 3.4 is completely proved. ∎

Now we are in a position to prove Theorems 2.2 and 2.3. Observe that by virtue of the transformation

$$\varphi = \operatorname{arctg}\left(\frac{v}{u}\right) \quad \text{and} \quad \theta = \operatorname{arctg}\left(\frac{v'}{u'}\right),$$

the solution curve $\sigma(t) = (\varphi(t), \theta(t))$, $t > 0$, from Lemma 3.4 induces an analytic curve Γ given by $(u(t), v(t))$, $t > 0$, in the (u, v)-plane which, by virtue of Lemma 3.4, (i)–(iii), lies in the first quadrant

$$Q = \{(u, v) : u \geq 0,\ v \geq 0\}.$$

In fact, since φ, θ are invariant under homotheties, we obtain a "field" in the sense of Weierstraß of extremal curves Γ_λ, $\lambda > 0$, which covers Q simply (quarter circles with center at the origin, if $\alpha = -1$).

Let Γ_1 denote the curve issuing vertically from the point $(1, 0)$. Again by Lemma 3.4, we can introduce the angle φ as a parameter on Γ_1 and hence obtain the representation $u = u_1(\varphi)$, $v = v_1(\varphi)$, $\varphi \in [0, \frac{\pi}{2}]$, for Γ_1 ($u_1(\varphi) = \cos\varphi$, $v_1(\varphi) = \sin\varphi$, if $\alpha = -1$). Alternatively, since $\theta \in (-\frac{\pi}{2}, 0)$ for $\varphi \in (0, \frac{\pi}{2})$, the curve Γ_1 can be written as a graph $v = v_1(u)$, $0 \leq u \leq 1$, for some function $v_1 \in C^1((0, 1)) \cap C^0([0, 1])$, with the properties

$$v_1(0) > 0, \quad \lim_{u\to 0+} v_1'(0) = 0, \quad \lim_{u\to 1-} v_1'(u) = -\infty.$$

Furthermore, since $\frac{d\theta}{dt} < 0$, the representation $v_1(\cdot)$ must be a concave function.

Now, any function $F = F(u, v)$ with $|\nabla F| \neq 0$ which is defined on the open quadrant $\mathring{Q} = \{(u, v) : u, v > 0\}$ and has the curves Γ_λ, $\lambda > 0$, as level curves must satisfy equation (11). The simplest choice would be the homogeneous function

$$F(u, v) := (u^2 + v^2) h\left(\operatorname{arctg} \frac{v}{u}\right), \quad (u, v) \in \mathring{Q},$$

which will be normalized by the requirement

$$F\big(u_1(\varphi), v_1(\varphi)\big) = 1, \quad \varphi \in \left(0, \frac{\pi}{2}\right);$$

that is,

$$h(\varphi) = \big(u_1^2(\varphi) + v_1^2(\varphi)\big)^{-1}, \quad \varphi \in \left(0, \frac{\pi}{2}\right),$$

and $\lim_{\varphi\to 0^+} h(\varphi) = 1$, $\lim_{\varphi\to\frac{\pi}{2}^-} h(\varphi) = v_1^{-2}(\frac{\pi}{2}) \neq 0$. Then, $F = F(u, v)$ extends continuously onto the closed quadrant Q, and, by construction, it is an analytic solution of (11) in the interior $\mathring{Q}$. We have $F(\Gamma_\lambda) = \lambda^2$ and, furthermore, $F_u = 2uh - vh'$ and $F_v = 2vh + uh'$. Hence,

$$|\nabla F| = (u^2 + v^2)^{\frac{1}{2}} (4h^2 + h'^2)^{\frac{1}{2}} \neq 0 \quad \text{on } \mathring{Q}.$$

We claim that the quotient $\frac{\nabla F}{|\nabla F|}$ extends continuously to the domain $\{(u, v) \in \mathbb{R}^2 : u \geq 0,\ v > 0\}$. Hence, it is sufficient to show that the limits

$$\lim_{\varphi\to\frac{\pi}{2}^-} \frac{h}{(4h^2 + h'^2)^{\frac{1}{2}}}, \quad \lim_{\varphi\to\frac{\pi}{2}^-} \frac{h'}{(4h^2 + h'^2)^{\frac{1}{2}}}$$

exist. To see this, we compute h', obtaining

$$h'(\varphi) = -2(u_1^2 + v_1^2)^{-2}(u_1 u_1' + v_1 v_1') = -2h \frac{u_1 u_1' + v_1 v_1'}{u_1^2 + v_1^2}.$$

From $\varphi = \operatorname{arctg} \frac{v_1}{u_1}$, we have $u_1^2 + v_1^2 = v_1' u_1 - v_1 u_1'$, whence

$$h'(\varphi) = -2h \frac{1 + \frac{v_1}{u_1}\frac{v_1'}{u_1'}}{\frac{v_1'}{u_1'} - \frac{v_1}{u_1}} = -2h \cdot \operatorname{cotg}(\varphi - \theta),$$

where $\theta = \operatorname{arctg} \frac{v_1'}{u_1'}$. Recall that $\theta \to 0$ as $\varphi \to \frac{\pi}{2}$; therefore, $h'(\varphi) \to 0$ as $\varphi \to \frac{\pi}{2}^-$

and also $F_u \to 0$ as $u \to 0^+$, $v > 0$, and $F_v \to 2vh(\frac{\pi}{2}) = \frac{2v}{v_1^2(\frac{\pi}{2})} \neq 0$ as $u \to 0^+$, $v > 0$.

Concluding, we have shown that the function $\frac{\nabla F}{|\nabla F|}$ extends continuously to the set $\{(u,v) : u \geq 0,\ v > 0\}$. We are now in a position to apply the removability result, Lemma 3.2, to the function

$$f(x,y,z) := F(u,v), \quad u := +\sqrt{x^2} = |x|,\ v = z > 0,$$

where $n = 2$, $A = \mathbb{R}^2 \times \mathbb{R}^+$, and $\mathcal{S}$ denotes the closed set

$$\mathcal{S} = \{(x,y,z) \in \mathbb{R}^2 \times \mathbb{R}^+ : u = 0\}.$$

Hence, Lemma 3.1 is applicable with $A = \mathbb{R}^2 \times \mathbb{R}^+$ and $\mathcal{N} = \varnothing$ to conclude that $f = f(x,y,z)$ has least α-gradient in $\mathbb{R}^2 \times \mathbb{R}^+$, and finally Lemma 3.3 shows that the level sets

$$\mathcal{U}_\lambda = \{(x,y,z) \in \mathbb{R}^2 \times \mathbb{R}^+ : f(x,y,z) \leq \lambda\},$$

for $\lambda > 0$, have boundaries of least α-area in $\mathbb{R}^2 \times \mathbb{R}^+$.

Recall that the level curves Γ_λ, $\lambda > 0$, of $F = F(u,v)$ are determined by concave functions $v = v_\lambda(u) = v_\lambda(|x|)$, $\lambda > 0$, $v_\lambda(\lambda) = 0$, which may now be considered as functions $v_\lambda = v_\lambda(x,y)$ defined on the strips $\mathcal{I}_\lambda = (-\lambda,\lambda) \times \mathbb{R}$; i.e., $v_\lambda(x,y) = v_\lambda(|x|)$, such that $v_\lambda \in C^1(\mathcal{I}_\lambda) \cap C^\omega(\mathcal{I}_\lambda \setminus \{0\} \times \mathbb{R}) \cap C^0(\overline{\mathcal{I}_\lambda})$ and

$$\begin{aligned} &v_\lambda(0,0) = \lambda v_1(0,0) > 0, \quad v_{\lambda|\partial\mathcal{I}_\lambda} = 0,\\ &Dv_\lambda(0,0) = 0,\\ &\big|Dv_\lambda(x,y)\big| \to \infty \quad \text{as } (x,y) \to \partial\mathcal{I}_\lambda. \end{aligned}$$

The level sets $\mathcal{U}_\lambda$ may hence be viewed as sub-level sets $\mathcal{V}_\lambda$ of the functions $v_\lambda(x,y)$; i.e.

$$\mathcal{U}_\lambda = \{(x,y,z) \in \mathbb{R}^2 \times \mathbb{R}^+ : (x,y) \in \mathcal{I}_\lambda,\ z \leq v_\lambda(x,y)\}.$$

In particular, the cylinders $v = v_\lambda(x,y)$, $(x,y) \in \mathcal{I}_\lambda$ minimize (locally) the α-area

$$\mathcal{E}_\alpha = \int u^\alpha \sqrt{1 + |Du|^2}$$

defined as a measure in the class of non-negative functions with bounded variation $\mathrm{BV}_{\mathrm{loc}}(\mathcal{I})$. Clearly, $v_\lambda(x,y)$ then satisfies the singular minimal surface equation (2) classically on $\mathcal{I}_\lambda \setminus (\{0\} \times \mathbb{R})$, and, by elliptic regularity theory, we have $v_\lambda \in C^\omega(\mathcal{I}_\lambda)$ and (2) is fulfilled classically on all of $\mathcal{I}_\lambda$.

To conclude with the proof of Theorems 2.2 and 2.3, recall that from the discussion preceding Lemma 3.4, we have for any trajectory of (13) close to the singular point

$(\varphi,\theta)=(0,-\frac{\pi}{2})$, the asymptotic relation $\theta=-\frac{\pi}{2}+\varphi^\tau+o(\varphi^\tau)$, as $\varphi\to 0^+$, with $\tau=|\alpha|$. A discussion similar to the one in [7] leads to

$$\frac{u_1'}{u_1}\approx -\cos\theta\approx -\cos\Big(-\frac{\pi}{2}+\varphi^\tau\Big)\approx -\varphi^\tau+o(\varphi^\tau),\quad \text{as } \varphi\to 0^+,$$

whence

$$u_1(\varphi)\approx 1-\frac{1}{1+\tau}\varphi^{\tau+1}+o(\varphi^{\tau+1}),\quad \text{as } \varphi\to 0^+.$$

However, $v_1=u_1\operatorname{tg}\varphi$ or $v_1(\varphi)=\varphi+o(\varphi)$, $\varphi\to 0^+$, from which we conclude $v_1\in C^{0,\frac{1}{1+\tau}}(\overline{\mathcal{I}_1})$ or $v_\lambda\in C^{0,\frac{1}{1+\tau}}(\overline{\mathcal{I}_\lambda})$, $\lambda>0$, with $\tau=|\alpha|$. Theorems 2.2 and 2.3 are proved.

4. Case $\alpha>0$: Proof of Theorem 2.4

Proof. *Ad* (i), (ii) *of Theorem* 2.4. Here, we refer to [11, Theorem 2], cp. also [8].

Ad (iii). We may follow arguments from the classical theory of calculus of variations; see the monograph of Giaquinta and Hildebrandt [9], in particular, Chapter 5. Recall that according to Lindelöf's construction device (see [9]), two points P and P^* on an extremal $\mathcal{C}$ are conjugate to each other if the tangents to $\mathcal{C}$ at P and P^*, respectively, intersect the x-axis in the same point. This device applies in particular to variational integrals of the kind $I_\alpha(\cdot),\alpha>0$, as is shown in [9].

Furthermore, the non-minimizing properties described in (iii) of Theorem 2.4 follow from general theory, cp. [9], since here the strict Legendre condition holds. It remains to prove the existence of two finite conjugate values $\pm x_\lambda$ for each extremal curve $y=f_\lambda(x)$, $x\in(-\frac{R}{\lambda},\frac{R}{\lambda})$ (and arbitrary but fixed $\alpha>0$). Since the solution curves $y=f_\lambda(x)=\frac{1}{\lambda}f_1(\lambda x)$ correspond to the trajectory $\sigma(t)=(\varphi(t),\theta(t))$ of system (13) which emanates at $(\frac{\pi}{2},0)$, we have to show, by Lindelöf's device, the existence of some finite $t_0>0$ such that $\varphi(t_0)=\theta(t_0)$. To see this, we consider system (13) for positive α in the triangle

$$T=\Big\{(\varphi,\theta):0<\varphi<\frac{\pi}{2},\ 0<\theta<\varphi\Big\}$$

with sides

$$l_1=\Big\{(\varphi,\theta):\theta=0,\ 0<\varphi\le\frac{\pi}{2}\Big\},$$
$$l_2=\Big\{(\varphi,\theta):\varphi=\frac{\pi}{2},\ 0<\theta\le\frac{\pi}{2}\Big\},$$
$$l_3=\Big\{(\varphi,\theta):\varphi=\theta,\ 0\le\varphi<\frac{\pi}{2}\Big\}.$$

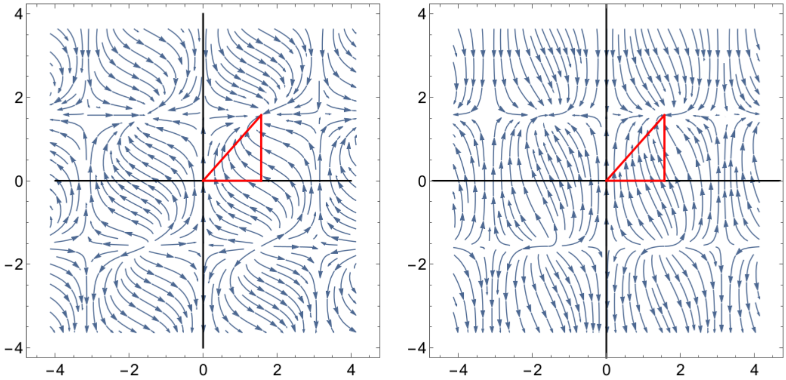

FIGURE 3. The (φ, θ)-phase space when $\alpha = 0.5$ (left) and $\alpha = 2$ (right). We have also indicated the triangle T.

Note that the only singular point of (13) in the closure $\overline{T}$ is the point $(\frac{\pi}{2}, \frac{\pi}{2})$, which is now a stable node, improper only for $\alpha = 1$. See Figure 3. Moreover, the vectorfield (Φ, Ψ) of (13) points strictly into T along the sides l_1 and l_2, while (Φ, Ψ) strictly points outward $\overline{T}$ when restricted to l_3. Hence, the trajectory $\sigma(t) = (\varphi(t), \theta(t))$ which starts into T at $\sigma(0) = (\frac{\pi}{2}, 0)$ with direction $\dot{\sigma}(0) = (-1, \alpha)$ either has to leave T in finite time t_0 across the side l_3 or we have

$$\lim_{t\to\infty} \sigma(t) = \left(\frac{\pi}{2}, \frac{\pi}{2}\right).$$

However, the principal directions in $(\frac{\pi}{2}, \frac{\pi}{2})$ are

$$\pm(1,0), \text{ if } \alpha > 1 \quad \text{and} \quad \pm(1, 1-\alpha), \text{ if } 0 < \alpha < 1,$$

while the exceptional directions are

$$\pm(1, 1-\alpha), \text{ if } \alpha > 1 \quad \text{and} \quad \pm(1,0), \text{ if } 0 < \alpha < 1,$$

and $\pm(1, 0)$ for the improper node $\alpha = 1$. Therefore, also in this case, the trajectory σ has to intersect the segment l_3 in finite time t_0, whence in any case $\varphi(t_0) = \theta(t_0)$ for some finite $t_0 > 0$. ∎

5. STABILITY OF α-CATENARIES

In this section, we consider the stability problem of α-catenaries. We know that the α-energy of a curve $y : [a, b] \to \mathbb{R}$, $y = y(x)$, is

$$I_\alpha(y) = \int_a^b y^\alpha \sqrt{1 + y'^2}\, dx.$$

Suppose now that $y(x)$ is an α-catenary. We need to calculate the second derivative $\delta^2 I_\alpha$ of the energy. Since $I_\alpha(y)$ is of type $\int_a^b F(x, y, y')\,dx$, we find by following Giaquinta–Hildebrandt [9] or Bolza [2, Chapter 2], and putting $P = F_{yy}$, $Q = F_{yy'}$, and $R = F_{y'y'}$:

$$\delta^2 I_\alpha(u; y) = \int_a^b u \cdot \Psi[u]\,dx,$$

where

$$\Psi[u] = (P - Q')u - R'u' - Ru'',$$

and u belongs to the class of admissible functions $\mathcal{A} = \{u \in C^2[a, b] : u(a) = u(b) = 0\}$. The curve $y = y(x)$ is said to be *stable* if $\delta^2 I_\alpha(u; y) \geq 0$ for all $u \in \mathcal{A}$. If it is clear from the context, we write simply $\delta^2 I_\alpha(u)$ instead of $\delta^2 I_\alpha(u; y)$. The second derivative $\delta^2 I_\alpha(u; y)$ of the energy has an associated elliptic operator called the *Jacobi operator* which is defined by

$$J[u] = -\frac{1}{R} \cdot \Psi[u] = u'' + \frac{R'}{R}u' + \frac{Q' - P}{R}u.$$

The operator J has an associated spectral theory so the stability of the α-catenary $y = y(x)$ is equivalent to the positivity of all eigenvalues of J. Since

$$\delta^2 I[u] = -\int_a^b Ru \cdot J[u]\,dx = \int_a^b \big(-Ruu'' - R'uu' + (P - Q')u^2\big)\,dx,$$

we get, upon integrating by parts,

$$\delta^2 I[u] = \int_a^b Ru'^2 + (P - Q')u^2\,dx = \int_a^b R\Big(u'^2 + \frac{P - Q'}{R}u^2\Big)\,dx.$$

The computations of P, Q, and R give

$$P = \alpha(\alpha - 1)y^{\alpha-2}\sqrt{1 + y'^2}, \quad Q = \alpha\frac{y^{\alpha-1}y'}{\sqrt{1 + y'^2}}, \quad R = \frac{y^\alpha}{(1 + y'^2)^{3/2}}.$$

After a dilation from the origin, we will assume that $y'(0) = 0$ and $y(0) = 1$. In particular, this implies $c = 1$ in (6). Using (6), $y^{2\alpha} = 1 + y'^2$ and consequently

$$P = \alpha(\alpha - 1)y^{2\alpha-2}, \quad Q = \frac{\alpha y'}{y}, \quad R = y^{-2\alpha}.$$

Again, (6) and (7) imply

$$\delta^2 I[u] = \int_a^b y^{-2\alpha}(u'^2 - \alpha y^{2\alpha-2}u^2)\,dx. \tag{14}$$

As an immediate consequence, we obtain the stability in case that α is non-positive.

THEOREM 5.1. *If $\alpha \leq 0$, then α-catenaries are stable.*

Recall that in Theorem 2.2, we have proved that these curves are local minimizers; consequently, they are stable.

Now we consider the stability problem for α-catenaries when α is positive. Following the method of Lindelöf, two points x^* and x'^* are conjugate if the tangent lines of $y(x)$ at x^* and x'^* meet at the x-axis; in particular, one of them is positive and the other is negative by the symmetry of $y(x)$ with respect to the y-axis; see [9, Chapter 5]. In the symmetric case of conjugate points x^* and $-x^*$, the tangent lines to $y(x)$ at the points x^* and $-x^*$ must coincide at the origin, or equivalently, this occurs if $\varphi(x^*) = \theta(x^*)$ (see (iii) of Theorem 2.4). In particular, in any open interval containing $[-x^*, x^*]$, the α-catenary is not a minimizer.

In this section, and when the domain of $y(x)$ is the symmetric interval $[-a, a]$, $a > 0$, we give a different approach to the stability problem without analyzing if the surface is a minimizer. We will find admissible functions u to insert in the integral (14). The objective is to prove the existence of $a_0 > 0$ such that $y = y(x)$ is not stable in the interval $[-a, a]$ for $a > a_0$. Thanks to the symmetry of $y(x)$, we have $y(-a) = y(a)$ and thus $u(x) = y(x) - y(-a)$ is an admissible function. Since $u'^2 = y'^2 = y^{2\alpha} - 1$, we have

$$\delta^2 I[u] = \int_{-a}^{a} \left(1 - y^{-2\alpha} - \alpha \frac{\big(y - y(a)\big)^2}{y^2}\right) dx. \tag{15}$$

PROPOSITION 5.2. *Let $\alpha = 1$. There exists $a_0 \approx 1.20305$ such that if $a > a_0$, the catenary $y(x) = \cosh(x)$ is not stable in the interval $[-a, a]$.*

PROOF. The integral (15) when $\alpha = 1$ is

$$\delta^2 I[u] = \int_{-a}^{a} \frac{\sinh(x)^2 - \big(\cosh(x) - \cosh(a)\big)^2}{\cosh(x)^2}\, dx.$$

This integral can be solved by quadratures, obtaining

$$\delta^2 I[u] = -2\tanh(a) - 2\cosh(a)\left(\sinh(a) - 4\tan^{-1}\left(\tanh\left(\frac{a}{2}\right)\right)\right).$$

The graphic of $\delta^2 I[u]$ considered as a function of the variable a is depicted in Figure 4, left. There is a unique value $a_0 \approx 1.20305$ such that $\delta^2 I[u] > 0$ if $a < a_0$ is 0 at a_0, and $\delta^2 I[u] < 0$ if $a > a_0$. This proves the result. ∎

We can compare the value a_0 with the conjugate point x^* for the catenary. This point x^* is the solution of $\cosh(x) = x\sinh(x)$, obtaining $x^* \approx 1.19968$.

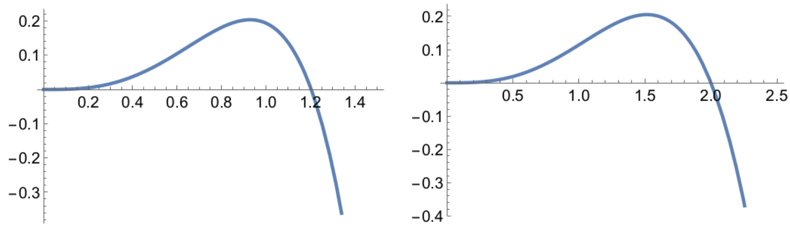

FIGURE 4. The graphics of $\delta^2 I[u]$ in Proposition 5.2 (left) and Proposition 5.3 (right).

The next result is similar for $\frac{1}{2}$-catenaries. The key is again that the integral (15) can be determined by quadratures.

PROPOSITION 5.3. *Let $\alpha = 1/2$. If $a > 2 =: a_0$, then the $\frac{1}{2}$-catenary is not stable in the interval $[-a, a]$.*

PROOF. If $\alpha = 1/2$, then (6) is $y' = \pm\sqrt{y-1}$, whose solution with our initial conditions is $y(x) = x^2/4 + 1$. Now, (15) is

$$\delta^2 I[u] = \int_{-a}^{a} \frac{x^4 + 2(a^2+4)x^2 - a^4}{2(x^2+4)^2}\,dx.$$

A computation leads to

$$\delta^2 I[u] = -\frac{1}{16}(a^2-4)\left((a^2-4)\tan^{-1}\left(\frac{a}{2}\right) + 2a\right).$$

Again, an analysis of this function proves that for $a < 2$, the integral is negative, and if $a > 2$, the integral is positive; see Figure 4, right. ∎

The symmetric conjugate point x^* of $y(x)$ satisfies the equation $y(x^*) = x^* y'(x^*)$; that is, $1 - {x^*}^2/4 = 0$, whose solution is $x^* = 2$. This value coincides with a_0 of the above proposition.

From Propositions 5.2 and 5.3, we can deduce the existence of a value a_0 such that if $a > a_0$, the α-catenary is not stable in the interval $[-a, a]$. The key in the proof of both propositions is that we can integrate by quadratures equation (5) as well as the integral (15). However, numerical computations using MATHEMATICA show that the result may be true in general.

THEOREM 5.4 (numerical). *Let $\alpha > 0$. There exists $a_0 > 0$ such that if $a > a_0$, the α-catenary is not stable in the interval $[-a, a]$.*

In Table 1, we show the results of these computations for a_0 for different values of α and, at the same time, with the numerical computation of the symmetric conjugate point x^*.

α	R_α	a_0	x^*
0.25	∞	3.1521	3.1466
0.5	∞	2.0000	2.0000
0.75	∞	1.4958	1.4945
1	∞	1.2030	1.1997
2	1.311	0.6850	0.6772
3	0.701	0.4820	0.4730
4	0.482	0.3717	0.3636

TABLE 1. Numerical comparison between the values of a_0 where $\delta^2 I[u] \approx 0$ and the conjugate point x^*.

6. FOLIATION BY α-CATENARIES

In this section, (x, y) denotes the coordinates of the halfplane $\mathbb{R}^2_+$. We consider all α-catenaries for all values of the parameter α fixing the same initial conditions in (5). To be precise, after a horizontal translation and a dilation from the origin, we will assume f_α is the solution of (5) with initial conditions

$$f_\alpha(0) = 1, \quad f'_\alpha(0) = 0.$$

Let $\mathcal{J}_\alpha = (-R_\alpha, R_\alpha)$ be the maximal interval, $R_\alpha \in \mathbb{R}^+ \cup \{\infty\}$. The graph of f_α is the set $G_\alpha = \{(x, f_\alpha(x)) : x \in \mathcal{J}_\alpha\}$. We will prove that all graphs G_α fill the halfplane $\mathbb{R}^2_+$ in the sense that for each point of $\mathbb{R}^2_+$, except the points of $L = (\{0\} \times (0, 1)) \cup (\{0\} \times (1, \infty))$, passes a unique graph G_α. We also show that these graphs are ordered according to the parameter α; that is, if $\alpha > \beta$, then $f_\alpha(x) > f_\beta(x)$ for all $x \in \mathcal{J}_\alpha \cap \mathcal{J}_\beta$ and $x \neq 0$.

THEOREM 6.1. *For each $(x, y) \in \mathbb{R}^2_+$, $(x, y) \neq (0, 1)$, there is a unique $\alpha \in \mathbb{R}$ such that $(x, y) \in G_\alpha$. Furthermore, if $\alpha < \beta$, then $f_\alpha(x) < f_\beta(x)$ for all $x \in \mathcal{J}_\alpha \cap \mathcal{J}_\beta$, $x \neq 0$.*

Prior to the proof, it deserves to point out the following observations. The points of $\mathbb{R}^2_+$ which are not included by the theorem correspond with the points of L because all G_α are graphs with a common point at $(0, 1)$. However, from Proposition 2.1, we know that $\{0\} \times \mathbb{R}^+$ is an α-catenary for *all* α. As a consequence, it is true that the upper halfplane $\mathbb{R}^2_+$ is covered for all α catenaries and, up to the expected points of $\{0\} \times \mathbb{R}^+$, for each point only passes a unique graph G_α. On the other hand, and by Theorems 2.2 and 2.4, we know the position of G_α. If $\alpha = 0$, then the 0-catenary is the horizontal line $y = 1$. This curve separates $\mathbb{R}^2_+$ in two halfs corresponding to different signs of α.

Exactly, the graphs G_α for $\alpha > 0$ fill the halfplane $y > 1$, whereas the graphs G_α for $\alpha < 0$ fill the strip $\{0 < y < 1\}$.

The proof is divided in several parts. First, we prove the monotonicity of G_α in terms of α. This is a consequence of the one-dimensional version of the comparison principle of elliptic equations of divergence type for equation (1); see [10, Theorem 10.1]. However, we will give a simple argument of calculus to derive this result of comparison, which is the following.

Suppose $\alpha < \beta$. From (5), we have at $x = 0$, $f_\alpha''(0) < f_\beta''(0)$, so there is an open interval J around $x = 0$ where $f_\alpha(x) < f_\beta(x)$ for all $x \in J$. By contradiction, suppose that x_0 is the first point after $x = 0$ where f_β coincides with f_α which, by the symmetry of G_α and G_β, we can suppose $x_0 > 0$. Since $f_\alpha(x) < f_\beta(x)$ for all $x \in (0, x_0)$ and $f_\alpha(x_0) = f_\beta(x_0)$, then $f_\beta'(x_0) \le f_\alpha'(x_0)$. Thanks to the initial conditions, we know that $c = 1$ in (6). Using (6) again, the inequality $f_\beta'(x_0) \le f_\alpha'(x_0)$ is equivalent to

$$f_\beta(x_0)^{2\beta} \le f_\alpha(x_0)^{2\alpha},$$

which is impossible because $\alpha < \beta$. ∎

The next step is the study of the behavior of the maximal domains $\mathcal{J}_\alpha = (-R_\alpha, R_\alpha)$ in terms of the parameter α. We know that if $\alpha \in [0, 1]$, then $R_\alpha = \infty$; otherwise, R_α is finite. We prove that $R_\alpha = R_\alpha(\alpha)$ is monotonic in $\mathbb{R} \setminus [0, 1]$.

Proposition 6.2. *If $\alpha > 1$ (resp. $\alpha < 0$), then R_α is strictly decreasing in α (resp. strictly increasing). Moreover, in both cases,* $\lim_{\alpha\to 1^+} R_\alpha = \lim_{\alpha\to 0^-} R_\alpha = \infty$ *and* $\lim_{\alpha\to\pm\infty} R_\alpha = 0$.

Proof. Suppose first that $\alpha > 1$. We consider the inverse of the solution f_α. So, for each graph G_α, we write the x-coordinate in terms of y; that is,

$$x_\alpha = x_\alpha(y), \quad f_\alpha(x_\alpha) = y.$$

By the monotonocity proved for f_α with respect to α, once fixed y, the function $\alpha \mapsto x_\alpha(y)$ is decreasing on α. Using (6), we know

$$\frac{dx_\alpha}{dy} = \frac{1}{\sqrt{y^{2\alpha} - 1}}$$

and $x_\alpha(1) = 0$. If $\alpha < \beta$,

$$\begin{aligned} x_\alpha(y) - x_\beta(y) &= \int_1^y \frac{d}{dt}(x_\alpha - x_\beta)\, dt \\ &= \int_1^y \left(\frac{1}{\sqrt{t^{2\alpha} - 1}} - \frac{1}{\sqrt{t^{2\beta} - 1}} \right) dt. \end{aligned}$$

Letting $y \to \infty$, we have

$$R_\alpha - R_\beta = \int_1^\infty \left(\frac{1}{\sqrt{t^{2\alpha} - 1}} - \frac{1}{\sqrt{t^{2\beta} - 1}} \right) dt = c > 0$$

because the integral is convergent and the integrand is positive. This proves that the map $\alpha \mapsto R_\alpha$ is decreasing. We now calculate the limits. First, we have

$$\lim_{\alpha \to +\infty} R_\alpha = \lim_{\alpha \to +\infty} \int_1^\infty \frac{1}{\sqrt{t^{2\alpha} - 1}}\, dt. \tag{16}$$

Defining $g_\alpha(t) = \frac{1}{\sqrt{t^{2\alpha}-1}}$, then $g_\alpha(t) \to 0$ as $\alpha \to \infty$. Since $g_\alpha(t) \leq g_\beta(t)$ for all $\beta \leq \alpha$, $\beta > 1$, the dominated convergence theorem implies that the integrals in (16) converge to 0 as $\alpha \to \infty$. Similarly, if $\alpha \to 1$, we have $g_\alpha(t) \geq \frac{1}{t^\alpha}$, and thus

$$\lim_{\alpha \to 1+} R_\alpha = \lim_{\alpha \to 1+} \int_1^\infty \frac{1}{\sqrt{t^{2\alpha} - 1}}\, dt > \lim_{\alpha \to 1+} \int_1^\infty t^{-\alpha}\, dt = \infty.$$

The arguments are similar when α is negative. In order to compute the limit of R_α as $t \to -\infty$, we have $g_\alpha(t) \to 0$ as $\alpha \to -\infty$, and for $\alpha \leq -2$,

$$\int_0^1 \frac{t^{-\alpha}}{\sqrt{t^{-2\alpha} - 1}}\, dt < \int_0^1 \frac{1}{\sqrt{1 - t^{-4}}}\, dt < \infty.$$

This proves that $R_\alpha \to 0$ as $\alpha \to -\infty$. The other limit to prove is $\lim_{\alpha \to 0^-} R_\alpha = \infty$. For $\alpha < 0$, we have $t^{2\alpha} - 1 \leq (\frac{1}{2})^{2\alpha} - 1$ for all $t \in [\frac{1}{2}, 1)$. Then,

$$R_\alpha = \int_0^1 g_\alpha(t)\, dt \geq \int_{\frac{1}{2}}^1 g_\alpha(t)\, dt \geq \int_{\frac{1}{2}}^1 \frac{dt}{\sqrt{\left(\frac{1}{2}\right)^{2\alpha} - 1}} \to \infty$$

as $\alpha \to 0^-$. ■

We complete the proof of Theorem 6.1. The uniqueness is a consequence of the monotonicity of f_α in the variable α, so we only need to prove the existence.

Let $(x_0, y_0) \in \mathbb{R}^2_+ \setminus L$, $(x, y) \neq (0, 1)$. The proof distinguishes the case that $y_0 > 1$ and $y_0 < 1$. Notice that if $y_0 = 1$, then $(x_0, y_0) \in G_0$.

(1) Case $y_0 > 1$. The value α will be positive, as it is expected. Consider $f_1(x) = \cosh(x)$. If $y_0 = f_1(x_0)$, then $(x_0, y_0) \in G_1$. If $y_0 < f_1(x_0)$, consider the interval $[0, 1]$ of the parameter α and the map

$$g\colon [0, 1] \to \mathbb{R}, \quad g(\alpha) = f_\alpha(x_0).$$

This map is continuous by the theorem of dependence of parameters with $g(0) = 1$ and $g(1) = f_1(x_0)$. By the intermediate value theorem, there is $\alpha_0 \in (0, 1)$ such that $g(\alpha_0) = f_{\alpha_0}(x_0) = y_0$, proving the result.

If $y_0 > f_1(x_0)$, and by Proposition 6.2, let $\alpha_1 > 0$ such that $R_{\alpha_1} < x_0$. Consider the function

$$h\colon [1, \alpha_1] \to \mathbb{R}, \quad h(\alpha) = x_\alpha(y_0).$$

This map is continuous by the theorem of dependence of parameters with $h(1) = x_1(y_0)$ and $h(x_0) < R_{\alpha_1}$. Since $R_{\alpha_1} < x_0 < x_1(y_0)$, the intermediate value theorem asserts that there is $\alpha_0 \in (1, \alpha_1)$ such that $h(\alpha_0) = x_{\alpha_0}(y_0) = x_0$, proving the result.

(2) Case $y_0 < 1$. Now, the value of α will be negative. Again, using Proposition 6.2, let $\alpha_1 < \alpha_2$ such that

$$R_{\alpha_1} < x_0, \quad \text{and} \quad \frac{x_0}{1 - y_0} < R_{\alpha_2}.$$

The number $\frac{x_0}{1-y_0}$ is the x-coordinate of the intersection point between the line $K \subset \mathbb{R}^2_+$ joining the point (0, 1) (of the initial conditions for all α) with (x_0, y_0) and the x-axis. Define

$$h\colon [\alpha_1, \alpha_2] \to \mathbb{R}, \quad h(\alpha) = x_\alpha(y_0).$$

This map is continuous by the theorem of dependence of parameters with $h(\alpha_1) = x_{\alpha_1}(y_0) < R_{\alpha_1}$. On the other hand, since $R_{\alpha_2} > \frac{x_0}{1-y_0}$, together with the property that the graph of f_{α_2} is concave (Theorem 2.2), implies that G_{α_2} cannot meet K, hence f_{α_2} lies above the line K. In particular,

$$h(\alpha_2) = x_{\alpha_2}(y_0) > x_0.$$

Again, the intermediate value theorem proves the result.

This completes the proof of Theorem 6.1. ∎

7. Geodesics of the conformal metric

In this section, let (x_1, x_2, x_3) denote the canonical coordinates of $x \in \mathbb{R}^3$, and $\{e_1, e_2, e_3\}$ is the canonical basis of $\mathbb{R}^3$. In Section 1, we have pointed out that α-singular minimal surfaces are also minimal surfaces in $\mathbb{R}^3_+$ endowed with the conformal metric $\tilde{g} = x_3^\alpha \langle\,,\rangle$. For completeness, we prove this fact because the notation of the proof will be used in subsequent results.

Proposition 7.1. *Let $\Sigma \hookrightarrow \mathbb{R}^3_+$ be an immersion of a surface Σ. Then, Σ is an α-singular minimal surface if and only if Σ is a minimal surface in $(\mathbb{R}^3_+, \tilde{g})$.*

Proof. Since $\tilde{g}$ is conformal to the Euclidean metric, the Levi-Civita connections $\tilde{\nabla}$ and $\overline{\nabla}$ for $\tilde{g}$ and $\langle\,,\rangle$ are related by

$$\tilde{\nabla}_X Y = \overline{\nabla}_X Y + \frac{\alpha}{2x_3}\big(X(x_3)Y + Y(x_3)X - \langle X, Y\rangle \overline{\nabla} x_3\big). \tag{17}$$

See [1, Chapter 1]. We can also compare the mean curvature H and $\widetilde{H}$ of Σ with the induced Euclidean metric and with the conformal metric $\tilde{g}$, respectively. If A and $\widetilde{A}$ are the respective second fundamental forms of $\langle\, ,\rangle$ and $\tilde{g}$, then

$$\widetilde{A}(X,Y) = x_3^{\alpha/2}\Big(A(X,Y) - \frac{\alpha}{2x_3}\langle\overline{\nabla}x_3, N\rangle\langle X,Y\rangle\Big).$$

Hence, the mean curvatures H and $\widetilde{H}$ satisfy

$$\widetilde{H} = x_3^{-\alpha/2}\Big(H - \frac{\alpha}{2x_3}\langle\overline{\nabla}x_3, N\rangle\Big) = x_3^{-\alpha/2}\Big(H - \frac{\alpha\langle N, e_3\rangle}{2x_3}\Big).$$

This proves that $\widetilde{H} = 0$ if and only Σ is α-singular minimal surface. ∎

In this section, we will relate the geodesics of the Riemannian manifold $(\mathbb{R}^3_+, \tilde{g})$ with the α-catenaries, the generating curves of cylindrical α-singular minimal surfaces. As a first step, we investigate the equations that satisfy the geodesics of $(\mathbb{R}^3_+, \tilde{g})$. A parametrized curve $\gamma = \gamma(s)$ by the arc-length with respect to the metric $\tilde{g}$ is a geodesic for $\tilde{g}$ if and only if $\widetilde{\nabla}_{\gamma'}\gamma' = 0$. Thus, from (17), we have

$$\gamma'' = \widetilde{\nabla}_{\gamma'}\gamma' = -\frac{\alpha}{2x_3}\big(2\langle\gamma', e_3\rangle\gamma' - |\gamma'|^2 e_3\big). \tag{18}$$

If we write $\gamma(s) = (x_1(s), x_2(s), x_3(s))$ and since γ is parametrized by arc-length with the metric $\tilde{g}$, we have $1 = x_3(s)^\alpha|\gamma'(s)|^2$, or equivalently,

$$1 = x_3(s)^\alpha(x_1'^2 + x_2'^2 + x_3'^2). \tag{19}$$

Then, (18) leads to the ordinary different system

$$\begin{cases} x_1'' &= -\alpha\dfrac{x_3'}{x_3}x_1', \\ x_2'' &= -\alpha\dfrac{x_3'}{x_3}x_2', \\ x_3'' &= -\dfrac{\alpha}{2x_3}(x_3'^2 - x_1'^2 - x_2'^2). \end{cases} \tag{20}$$

In case that both functions $x_k = x_k(s)$ for $k = 1, 2$ are constant, then by solving the third equation of (20), we deduce that vertical straight-lines are geodesics. Otherwise, if one of the functions x_1 or x_2 is not constant, we can do a first integration of the corresponding equation in (20). By simplicity, we can assume that both functions x_1 and x_2 are not constant. Then, the first two equations of (20) give

$$x_k' = \frac{a_k}{x_3^\alpha}, \quad k = 1, 2, \tag{21}$$

for some nonzero real numbers $a_k \neq 0$. Substituting into (19),

$$1 = (a_1^2 + a_2^2)x_3^{-\alpha} + x_3^\alpha x_3'^2. \tag{22}$$

Let $m = a_1^2 + a_2^2$. Notice that x_3 cannot be a constant function by the third equation of (20). Therefore, the function x_3 is given by

$$x_3' = \pm\frac{\sqrt{x_3^\alpha - m}}{x_3^\alpha}. \tag{23}$$

Then, the third equation of (20) is now

$$x_3'' = -\frac{\alpha}{2x_3}\frac{x_3^\alpha - 2m}{x_3^{2\alpha}}, \tag{24}$$

which is equivalent to (22).

Proposition 7.2. *The geodesics of* $(\mathbb{R}^3_+, \tilde{g})$ *are included in vertical planes.*

Proof. Let γ be a geodesic of $(\mathbb{R}^3_+, \tilde{g})$. The result is true if γ is a vertical straight-line. Otherwise, and from (21), the velocity vector of γ is

$$\gamma'(s) = \frac{1}{x_3^\alpha}(a_1, a_2, 0) + x_3' e_3;$$

hence, $\gamma'(s)$ is contained in the plane spanned by the vectors $(a_1, a_2, 0)$ and e_3 for all $s \in I$, obtaining the result. ■

We discuss explicit solutions of the system (20) for the values $\alpha = -2$ and $\alpha = 2$.

Example 7.3. Let $\alpha = -2$. Here, we obtain the geodesics of the hyperbolic space. We know that vertical lines are geodesics. Other geodesics are obtained by solving (22). For simplicity, we assume that the geodesic γ is included in the x_1x_3-plane. Thus, take $a_2 = 0$ and $a_1 = 0$. Then, the integration of (22) gives $x_3(s) = 1/\cosh(s)$. Now, the fist equation of (21) yields $x_1(s) = \tanh(s)$. In this case, $\gamma(s) = (\tanh(s), 0, 1/\cosh(s))$, which is a parametrization of the hemicircle of radius 1 centered at the origin. Let us notice that this curve corresponds to the solutions of (5) for the value $\alpha = -1$.

Example 7.4. Let $\alpha = 2$. As in the previous example and after discarding the vertical straight-lines, we assume $a_1 = 1$ and $a_2 = 0$. Now, (22) is

$$\int \frac{x_3^2}{\sqrt{x_3^2 - 1}} x_3' = s.$$

The integration gives

$$x_3\sqrt{x_3^2-1}+\log\left(x_3+\sqrt{x_3^2-1}\right)=2s.$$

Letting $x_3(s)=\cosh\phi(s)$ for some function $\phi(s)$, this identity is

$$\phi(s)+\cos\phi(s)\sin\phi(s)=2s.$$

If we take $x_1(s)=\phi(s)$, then it is easy to check that $x_1(s)$ satisfies equation (21) for $a_1=1$. Thus, the parametrization of γ is $\gamma(s)=(\phi(s),\cosh\phi(s))$ proving that γ is a catenary. Let us notice that now γ is a solution of (5) for the value $\alpha=1$.

The above two examples are not exceptional and we will relate the geodesics of $(\mathbb{R}^3_+,\tilde{g})$ with the solutions of (5).

THEOREM 7.5. *The geodesics of* $(\mathbb{R}^3_+, x_3^\alpha\langle\,,\rangle)$ *are* $\alpha/2$*-catenaries and vice versa.*

PROOF. Vertical geodesics are generating curves of α-minimal singular surfaces of cylindrical type for any α, and vice versa.

Let now γ be a geodesic of $(\mathbb{R}^3_+, x_3^\alpha\langle\,,\rangle)$ and suppose that γ is not a vertical straight-line. We compute the (signed) curvature κ_γ of γ as planar curve. After a rotation about e_3, we suppose $\gamma(s)=(x_1(s),0,x_3(s))\equiv(x_1(s),x_3(s))$. Then,

$$\kappa_\gamma(s)=\frac{x_1'x_3''-x_3'x_1''}{\left|\gamma'(s)\right|^3}. \tag{25}$$

From (21), $\gamma'=(a_1x_3^{-\alpha},x_3')$, $a_1\neq 0$, and thus $\gamma''=(-\alpha_1x_3^{-\alpha-1}x_3',x_3'')$. We know that $|\gamma'|^3=x_3^{-3\alpha/2}$. Since now $m=a_1^2$, using (23) and (24), we have

$$\kappa_\gamma(s)=a_1\frac{\alpha}{2x_3(s)^{1+\frac{\alpha}{2}}}.$$

We now compute the curvature of the graph of the solutions of (5). In order to not confuse the parameter α and the variable of these solutions, we denote by $f_\beta=f_\beta(t)$ the solution of (5) where the parameter α is now β. The curvature of the graph of $y=f_\beta(t)$ is

$$\kappa_{f_\beta}(t)=\frac{f_\beta''(t)}{\left(1+f_\beta'(t)^2\right)^{3/2}}.$$

Using the expression of $f_\beta''(t)$ in (5) together with that of $f_\beta'(t)$ in (6), the curvature κ_{f_β} is

$$\kappa_{f_\beta}(t)=\frac{1}{c}\frac{\beta}{f_\beta(t)^{1+\beta}}.$$

Comparing this identity with (25), we deduce that, up to a constant, κ_γ coincides with κ_{f_β} if and only if $\beta = \alpha/2$. Finally, using the classical theorem of the local theory of planar curves, we conclude that γ and the graph of f_β coincide up to a dilation of $\mathbb{R}^3_+$ from the origin and a horizontal translation of $\mathbb{R}^3_+$, which proves the theorem. ∎

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