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Algebraic Geometry. – *Unirationality of varieties described by families of projective hypersurfaces*, by Ciro Ciliberto and Duccio Sacchi, communicated on 10 February 2023.

Abstract. – Let $\mathscr{X} \to W$ be a flat family of generically irreducible hypersurfaces of degree $d \geqslant 2$ in $\mathbb{P}^n$ with singular locus of dimension t, with W unirational of dimension r. We prove that if n is large enough with respect to d, r and t, then $\mathscr{X}$ is unirational. This extends results by J. Harris, B. Mazur and R. Pandharipande in [Duke Math. J. 95 (1998), 125–160] and A. Predonzan in [Rend. Sem. Mat. Univ. Padova 31 (1961), 281–293].

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1. Introduction

A classical theorem by U. Morin says that if X is a general hypersurface of degree d in $\mathbb{P}^n$ and n is large enough with respect to d, then X is unirational (see [7] and also [1]). This result has been extended by A. Predonzan in [8] to any hypersurface X of degree d in $\mathbb{P}^n$, even singular in dimension t: X turns out to be unirational provided n is large enough with respect to d and t (see Theorem 5.4 for a precise statement). This theorem has been rediscovered by J. Harris, B. Mazur and R. Pandharipande in [5] although they give a lower bound for n that is worse than Predonzan's.

The purpose of this paper is to prove an extension of Predonzan's result, namely, Theorem 5.5, that asserts that if $\mathscr{X} \to W$ is a flat family of hypersurfaces of degree $d \geqslant 2$ in $\mathbb{P}^n$, whose general member is irreducible and singular in dimension t, and W is irreducible, unirational of dimension r, if n is large enough with respect to d, r, t, then $\mathscr{X}$ is unirational. The case $d = 2$ is already included in [3]. This theorem, for instance, implies, under suitable numerical conditions, the unirationality of hypersurfaces in Segre products of projective spaces.

As for the proof, Predonzan shows in [8] that if $X \subset \mathbb{P}^n$ is an irreducible hypersurface of degree $d \geqslant 2$, defined over a field $\mathbb{K}$ of characteristic zero, containing a k-plane Λ along which X is smooth, and if k is large enough with respect to the degree d, then X is unirational over the extension of $\mathbb{K}$ with the Plücker coordinates of Λ (see Theorem 5.3 for a precise statement). The key step in our proof is to show that if $\mathscr{X} \to W$ is a flat family of generically irreducible hypersurfaces of degree $d \geqslant 2$

in $\mathbb{P}^n$, with W irreducible of dimension r, and n is large enough with respect to d, r, k, then there is a rationally determined k-plane over the generic hypersurface of the family (see Remark 4.6 for the meaning of being *rationally determined*). This is the so-called Section Lemma (see Lemma 4.5 below). Theorem 5.5 follows by this result and the aforementioned Predonzan's theorem, in view of a unirationality criterion by L. Roth (see Proposition 5.2). The proof of Section Lemma is inspired by a beautiful and elegant idea of F. Conforto in [3], and it uses a birational description of the Fano scheme of k-planes in a projective hypersurface, as in Section 3. This in turn requires some preliminaries about Grassmannians stated in Section 2, which essentially appear in a paper by J. G. Semple [12], and which we expose here for the reader's convenience.

This paper extends some of the results by the second author in his Ph.D. thesis [11].

In this paper, we work over an algebraically closed field $\mathbb{K}$ of characteristic zero.

2. Some preliminaries on Grassmannians

In this section, we expose some preliminaries on Grassmann varieties, following [12].

2.1. Let $\mathbb{G}(k,n)$ be the Grassmann variety of k-planes in $\mathbb{P}^n = \mathbb{P}(V)$, where V is a $\mathbb{K}$-vector space of dimension $n+1$. One has $\mathbb{G}(k,n) \cong \mathbb{G}(n-k-1,n)$; hence, without loss of generality, we may and will assume $2k < n$.

The variety $\mathbb{G}(k,n)$ is naturally embedded in $\mathbb{P}^{N(k,n)}$, with $N(k,n) = \binom{n+1}{k+1} - 1$, via the *Plücker embedding*. Explicitely, in coordinates, we have the following. Fix a basis $B = \{e_1, \dots, e_{n+1}\}$ of V. Then, we can associate with any k-plane Λ of $\mathbb{P}^n = \mathbb{P}(V)$ a $(k+1) \times (n+1)$ matrix

$$M_\Lambda = \begin{bmatrix} v_{1,1} & \cdots & v_{1,n+1} \\ \vdots & & \vdots \\ v_{k+1,1} & \cdots & v_{k+1,n+1} \end{bmatrix}$$

whose rows are the coordinate vectors with respect to the basis B of $k+1$ vectors corresponding to independent points of Λ. Two matrices M and M' represent the same k-plane if and only if there exists $A \in \mathrm{GL}(k+1, \mathbb{K})$ such that $M = AM'$.

The homogeneous coordinates of the point in $\mathbb{P}^{N(k,n)}$ corresponding to Λ are given by the minors of order $k+1$ of M_Λ. They depend only on Λ and not on the matrix M_Λ. These are the *Plücker coordinates* of Λ, and we denote them by z_I where $I = (i_1, \dots, i_{k+1})$ is a multi-index with $1 \leqslant i_1 < i_2 < \cdots < i_{k+1} \leqslant n+1$ denoting the order of the columns of M_Λ which determine the corresponding minor. The Plücker coordinates are lexicographically ordered.

If we consider the subset U of points of $\mathbb{P}^{N(k,n)}$ where the first coordinate $z_{1,\dots,k+1}$ is different from zero, each point $\Lambda \in U \cap \mathbb{G}(k,n)$ represents a k-plane such that there is an associated matrix M_Λ to Λ that can be uniquely written in the form

$$\begin{bmatrix} 1 & 0 & \cdots & 0 & v_{1,k+2} & \cdots & v_{1,n+1} \\ 0 & 1 & \cdots & 0 & v_{2,k+2} & \cdots & v_{2,n+1} \\ \vdots & & \ddots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 & v_{k+1,k+2} & \cdots & v_{k+1,n+1} \end{bmatrix}.$$

From this description, it follows that $U \cap \mathbb{G}(k,n)$ is isomorphic to $\mathbb{A}^{(k+1)(n-k)}$, where the coordinates are the (lexicographically ordered) $v_{i,j}$'s, with $1 \leqslant i \leqslant k+1$ and $k+2 \leqslant j \leqslant n+1$. We will soon give a geometric interpretation of this isomorphism (see Proposition 2.8 below).

Remark 2.1. We can describe geometrically the open subset $U \cap \mathbb{G}(k,n)$: it is the set of k-planes of $\mathbb{P}^n$ that do not intersect the $(n-k-1)$-plane spanned by the points corresponding to $e_{k+2}, \dots, e_{n+1}$. Similarly, for any choice of a totally decomposable element of $\wedge^{n-k} V$ (i.e., a vector which can be expressed as $v_1 \wedge \cdots \wedge v_{n-k}$), we can construct a birational map between $\mathbb{G}(k,n)$ and $\mathbb{P}^{(k+1)(n-k)}$.

2.2. Now, we set $M(k,n) = (k+1)(n-k)$ and consider $\mathbb{P}^{M(k,n)}$ with homogeneous coordinates given by y and $x_{i,j}$ for $i = 1, \dots, k+1$ and $j = k+2, \dots, n+1$: in this setting, the affine space with coordinates $x_{i,j}$ for $i = 1, \dots, k+1$ and $j = k+2, \dots, n+1$ is the complement of the hyperplane H with equation $y = 0$. We define the rational map

$$\psi_{k,n} : \mathbb{P}^{M(k,n)} \dashrightarrow \mathbb{P}^{N(k,n)} \tag{1}$$

sending the point with coordinates $[y, x_{1,k+2}, \dots, x_{k+1,n+1}]$ to the point whose coordinates are the minors of order $k+1$ of the matrix

$$\begin{bmatrix} y & 0 & \cdots & 0 & x_{1,k+2} & \cdots & x_{1,n+1} \\ 0 & y & \cdots & 0 & x_{2,k+2} & \cdots & x_{2,n+1} \\ \vdots & & \ddots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & y & x_{k+1,k+2} & \cdots & x_{k+1,n+1} \end{bmatrix}. \tag{2}$$

If we consider the open subset $U' = \{y \neq 0\} \subset \mathbb{P}^{M(k,n)}$, $\psi_{k,n}|_{U'}$ is the inverse isomorphism of the one described above between $U \cap \mathbb{G}(k,n)$ and $\mathbb{A}^{M(k,n)}$. Therefore, the image of $\psi_{k,n}$ is $\mathbb{G}(k,n)$.

We will describe the linear system $\mathfrak{d}_{k,n}$ of hypersurfaces associated with the map $\psi_{k,n}$. It corresponds to the vector space $W \subset H^0(\mathbb{P}^M, \mathscr{O}_{\mathbb{P}^M}(k+1))$ spanned by y^{k+1} and by the forms

$$y^{k+1-r} D^r_{i_1,\dots,i_r;j_1,\dots,j_r} \tag{3}$$

for $r = 1, \dots, k+1$, with $1 \leqslant i_1 < \cdots < i_r \leqslant k+1$ and $k+2 \leqslant j_1 < \cdots < j_r \leqslant n+1$, where $D^r_{i_1,\dots,i_r;j_1,\dots,j_r}$ denotes the minor of order r of the matrix

$$\begin{bmatrix} x_{1,k+2} & \cdots & x_{1,n+1} \\ x_{2,k+2} & \cdots & x_{2,n+1} \\ \vdots & & \vdots \\ x_{k+1,k+2} & \cdots & x_{k+1,n+1} \end{bmatrix} \tag{4}$$

determined by the rows of place $i_1, \dots, i_r$ and by the columns of place $j_1, \dots, j_r$.

For a fixed $r = 1, \dots, k+1$, we define m_r as the number of the minors of type D^r. Thus, $m_r = \binom{k+1}{r}\binom{n-k}{r}$.

Note that the subvariety of H defined by the 2×2 minors of the matrix (4) is a Segre variety $\mathrm{Seg}(k, n-k-1) \cong \mathbb{P}^k \times \mathbb{P}^{n-k-1}$ (see [4, p. 98]).

We want to geometrically characterize the linear system $\mathfrak{d}_{k,n}$. Before doing that, we need the following.

Lemma 2.2. *Let $r \geqslant 1$. There is no hypersurface of degree $r+2$ in $\mathbb{P}^r$ with multiplicity at least $r+1$ at $r+1$ independent assigned points of $\mathbb{P}^r$.*

Proof. Consider the linear system of $\mathbb{P}^r$ of hypersurfaces of degree r with multiplicity at least $r-1$ at the $r+1$ independent assigned points. This is well known to be a homaloidal linear system, determining a birational map $\omega : \mathbb{P}^r \dashrightarrow \mathbb{P}^r$, such that the counterimages of the lines of the target $\mathbb{P}^r$ are the rational normal curves in the domain $\mathbb{P}^r$ passing through the $r+1$ independent assigned points. The intersection of a hypersurface of degree $r+2$ in $\mathbb{P}^r$ with multiplicity at least $r+1$ at the $r+1$ independent assigned points with the rational normal curve off the $r+1$ independent assigned points is -1. Thus, such a hypersurface is empty and the assertion follows. ∎

Next, we can give the desired geometric description of the linear system $\mathfrak{d}_{k,n}$.

Proposition 2.3. *The linear system $\mathfrak{d}_{k,n}$ consists of the hypersurfaces of degree $k+1$ in $\mathbb{P}^{M(k,n)}$ passing with multiplicity at least k through the Segre variety $\mathrm{Seg}(k, n-k-1)$ contained in the hyperplane H with equation $y = 0$ and defined by the 2×2 minors of the matrix (4).*

PROOF. The linear system $\mathfrak{d}_{k,n}$ has a base locus scheme B_1. By looking at the basis of W in (3), B_1 is defined by the equations

$$y = 0, \quad D^{k+1}_{1,\dots,k+1;j_1,\dots,j_{k+1}}, \qquad \text{for all } k+2 \leqslant j_1 < \cdots < j_r \leqslant n+1.$$

Then, B_1 is the $(k-1)$-th secant variety of $\mathrm{Seg}(k, n-k-1)$ defined by the 2×2 minors of the matrix (4) inside H (see [4, p. 99]). Moreover, $\mathfrak{d}_{k,n}$ is the whole linear system of hypersurfaces of degree $k+1$ of $\mathbb{P}^{M(k,n)}$ containing B_1.

For each $r = 2, \dots, k$, we can also consider the subscheme B_r of B_1 consisting of those points of B_1 where all hypersurfaces of $\mathfrak{d}_{k,n}$ have multiplicity at least r. Looking again at the basis of W in (3), it is immediate that B_r is the $(k-r)$-secant variety of $\mathrm{Seg}(k, n-k-1)$, for $r = 2, \dots, k$ (see again [4, p. 99]). Note that B_1 itself has points of multiplicity at least r along B_r, for all $r = 2, \dots, k$. In particular, each hypersurface in the linear system $\mathfrak{d}_{k,n}$ passes with multiplicity k through the Segre variety $\mathrm{Seg}(k, n-k-1)$ in H, which is B_k.

Conversely, let F be a hypersurface of degree $k+1$ in $\mathbb{P}^{M(k,n)}$ passing with multiplicity at least k through the Segre variety $\mathrm{Seg}(k, n-k-1)$ lying in the hyperplane H and defined by the 2×2 minors of the matrix (4). Then, we claim that F contains the $(k-1)$-th secant variety of $\mathrm{Seg}(k, n-k-1)$; hence, F belongs to $\mathfrak{d}_{k,n}$. Indeed, this is clear for $k = 1$, so we may assume $k \geqslant 2$, in which case the claim follows right away by Lemma 2.2. ∎

2.3. Next, we need a description of the *osculating spaces* to the Grassmann varieties (for the concept of osculating spaces, see [10, p. 141]). First, we need some lemmata.

LEMMA 2.4. *Let* $\mathrm{Seg}(1, k)$ *be a Segre variety in* $\mathbb{P}^n$, *with* $n \geqslant 2k+1$. *Let* $\phi : \mathbb{P}^1 \to \mathbb{G}(k, n)$ *be the morphism which sends a point* p *to the* k*-plane* $\{p\} \times \mathbb{P}^k \subset \mathrm{Seg}(1, k)$ *in* $\mathbb{P}^n$. *Then, the image of* ϕ *is a rational normal curve of degree* $k+1$ *inside* $\mathbb{G}(k, n)$.

PROOF. We can assume $n = 2k+1$. If $[x_0, x_1]$ are homogeneous coordinates of $\mathbb{P}^1$ and z_{ij}, $i = 0, 1$ and $j = 0, \dots, k$, are the homogenous coordinates of $\mathbb{P}^{2k+1}$, we can assume that ϕ is the map which sends $[\alpha_0, \alpha_1]$ to the k-plane whose equations in $\mathbb{P}^{2k+1}$ are

$$\alpha_1 z_{0j} - \alpha_0 z_{1j} = 0 \quad \text{for } j = 0, \dots, k.$$

In particular, the image of a point $[x_0, x_1]$ under ϕ is the point of $\mathbb{G}(k, 2k+1)$ whose coordinates are the minors of maximal order of the matrix

$$\begin{bmatrix} x_0 & 0 & \cdots & 0 & x_1 & 0 & \cdots & 0 \\ 0 & x_0 & \cdots & 0 & 0 & x_1 & \cdots & 0 \\ \vdots & & \ddots & & & & \ddots & \vdots \\ 0 & 0 & \cdots & x_0 & 0 & 0 & \cdots & x_1 \end{bmatrix}.$$

There are only $k+2$ non-vanishing Plücker coordinates of this k-plane and they have as entries the monomials of degree $k+1$ in x_0 and x_1. The assertion follows. ∎

We can generalize the above result.

LEMMA 2.5. *Let k, r, n be positive integers with $k > r$. Let* $\mathrm{Seg}(1,r)$ *be a Segre variety in $\mathbb{P}^n$, with $n \geqslant k+r+1$, and let Π be a $(k-r-1)$-plane, which does not intersect the $(2r+1)$-plane spanned by* $\mathrm{Seg}(1,r)$. *Let $\phi : \mathbb{P}^1 \to \mathbb{G}(k,n)$ be the morphism which sends a point p to the k-plane spanned by Π and by the r-plane in* $\mathrm{Seg}(1,r)$ *given by $\{p\} \times \mathbb{P}^r$. Then, its image is a rational normal curve of degree $r+1$ inside $\mathbb{G}(k,n)$.*

PROOF. We can assume $n = k+r+1$. Let $\mathrm{Seg}(1,r)$ be a Segre variety inside the $(2r+1)$-plane L given by the vanishing of the last $k-r$ homogeneous coordinates of $\mathbb{P}^{k+r+1}$, and let Π be the $k-r-1$ plane given by the vanishing of the first $2r+2$ coordinates. Then, we can associate with the point $[x_0, x_1]$ of $\mathbb{P}^1$ the matrix whose rows span the join of Π and $[x_0, x_1] \times \mathbb{P}^r$. This is the $(k+1) \times (k+r+2)$ matrix

$$\left[\begin{array}{c|c} A_{x_0,x_1} & 0_1 \\ \hline 0_2 & I_{k-r} \end{array}\right],$$

where A_{x_0,x_1} is the $(r+1) \times (2r+2)$ matrix associated with the r-plane $[x_0, x_1] \times \mathbb{P}^r$ in L, 0_1 is the $(r+1) \times (k-r)$ zero matrix, 0_2 is the $(k-r) \times (2r+2)$ zero matrix and I_{k-r} is the $(k-r)$ identity matrix. As in the proof of Lemma 2.4, we see that the only non-vanishing Plücker coordinates of the point of the Grassmaniann associated with this matrix are given by the monomials of order $r+1$ in x_0 and x_1. The assertion follows. ∎

LEMMA 2.6. *Let L_1, L_2 be two distinct k-planes in $\mathbb{P}^n$ intersecting in a $(k-r)$-plane M, with $1 \leqslant r \leqslant k+1$. Then, there is a Segre variety* $\mathrm{Seg}(1, r-1)$ *in $\mathbb{P}^n$ such that there are two distinct points $p_1, p_2 \in \mathbb{P}^1$ such that $L_i \cap \mathrm{Seg}(1, r-1) = \{p_i\} \times \mathbb{P}^{r-1}$, for $i = 1, 2$.*

PROOF. Projecting from M to a $\mathbb{P}^{n-k+r-1}$, the images of L_1, L_2 are two disjoint $(r-1)$-planes L'_1 and L'_2. If we fix an isomorphism $\tau : L'_1 \to L'_2$, the variety defined as the union of the lines joining $p \in L'_1$ to $\tau(p) \in L'_2$ is the desired Segre variety. ∎

The following proposition describes the osculating spaces of Grassmannians.

PROPOSITION 2.7. *Let Λ_0 be a point of $\mathbb{G}(k,n)$ and let $1 \leqslant r \leqslant k$. Then, the r-osculating space $T^{(r)}_{\mathbb{G}(k,n),\Lambda_0}$ to $\mathbb{G}(k,n)$ at Λ_0 is the linear space spanned by the Schubert variety*

$$W_{r,\Lambda_0} = \left\{\Lambda \in \mathbb{G}(k,n) \mid \dim(\Lambda \cap \Lambda_0) \geqslant k-r\right\}$$

and one has

$$\dim\big(T^{(r)}_{\mathbb{G}(k,n),\Lambda_0}\big)=\sum_{i=1}^{r}\binom{k+1}{i}\binom{n-k}{i}. \tag{5}$$

Proof. First of all, we claim that $W_{r,\Lambda_0}\subseteq T^{(r)}_{\mathbb{G}(k,n),\Lambda_0}$. To prove this, note that by Lemmata 2.5 and 2.6, for any k-plane Λ intersecting Λ_0 in a linear space of dimension at least $k-r$, we can construct a rational normal curve of degree r in $\mathbb{G}(k,n)$ passing through Λ_0 and Λ. Such a curve must be contained in $T^{(r)}_{\mathbb{G}(k,n),\Lambda_0}$, and this proves the claim.

To prove that $T^{(r)}_{\mathbb{G}(k,n),\Lambda_0}=\langle W_{r,\Lambda_0}\rangle$, we will compute the dimensions of both $T^{(r)}_{\mathbb{G}(k,n),\Lambda_0}$ and $\langle W_{r,\Lambda_0}\rangle$ and we will prove they are equal.

First, let us prove (5). Without loss of generality, we may assume that Λ_0 is spanned by the points corresponding to the vectors $e_1,\dots,e_{k+1}$ of the basis B of V, so that Λ_0 is the point where only the first Plücker coordinate is different from zero. Consider the local parametrization of $\mathbb{G}(k,n)$ around Λ_0 given by the restriction of the map $\psi_{k,n}$ as in (1) to $\mathbb{A}^{M(k,n)}=\mathbb{P}^{M(k,n)}\setminus H$, so that $\psi_{k,n}$ maps the origin of $\mathbb{A}^{M(k,n)}$ to Λ_0. The r-osculating space $T^{(r)}_{\mathbb{G}(k,n),\Lambda_0}$ is spanned by the points that are derivatives up to order r of the parametrization at the origin.

Each coordinate function of $\psi_{k,n}$ is given by a minor D^s as above (in the affine coordinates $x_{i,j}$, for $i=1,\dots,k+1$ and $j=k+2,\dots,n+1$, of $\mathbb{A}^{M(k,n)}$). The derivatives up to order r of the minors D^s with $s\geqslant r+1$ vanish at $0\in\mathbb{A}^{M(k,n)}$. Hence, $T^{(r)}_{\mathbb{G}(k,n),\Lambda_0}$ has dimension at most $\sum_{i=1}^r m_i$, where we recall that $m_i=\binom{k+1}{i}\binom{n-k}{i}$ is the number of the D^i's.

Moreover, for each minor D^s with $s\leqslant r$, there exists a derivative of order s of the parametrization at the origin such that all of its coordinates, except the one corresponding to D^s, vanish. This implies (5).

Next, we compute the dimension of $\langle W_{r,\Lambda_0}\rangle$ and prove that it equals the right-hand side of (5). Let Λ be an element of W_{r,Λ_0}. It is spanned by $k+1$ points, and we may assume the first $k-r+1$ of them lie on Λ_0. Then, the Plücker coordinates of Λ are given by the maximal minors of a matrix $M_\Lambda=[v_{i,j}]_{i=1,\dots,k+1;\ j=1,\dots,n+1}$ where $v_{i,j}=0$ if $i\in\{1,\dots,k-r+1\}$ and $j\in\{k+2,\dots,n+1\}$. Moreover, varying Λ in W_{r,Λ_0}, we may consider the non-zero $v_{i,j}$ as variables.

The vanishing maximal minors of a matrix of type M_Λ are those involving at most $r+1$ of the last $n-k$ columns. Hence, their number is

$$c=\sum_{i=r+1}^{k+1}\binom{n-k}{i}\binom{k+1}{i}=\sum_{i=r+1}^{k+1}m_i$$

and therefore

$$\dim\big(\langle W_{r,\Lambda_0}\rangle\big)=N(k,n)-c.$$

On the other hand, we have

$$N(k,n) = \sum_{i=1}^{k+1} m_i;$$

hence

$$\dim\big(\langle W_{r,\Lambda_0}\rangle\big) = \sum_{i=1}^{r} m_i = \dim\big(T^{(r)}_{\mathbb{G}(k,n),\Lambda_0}\big),$$

as desired. ■

2.4. Next, we give the announced geometric description of the isomorphism of $U \cap \mathbb{G}(k,n)$ with $\mathbb{A}^{M(k,n)}$.

PROPOSITION 2.8. *Let Π be an element of $\mathbb{G}(n-k-1,n)$ and W_Π the Schubert variety*

$$\big\{\Lambda \in \mathbb{G}(k,n) \mid \dim(\Lambda \cap \Pi) \geqslant 1\big\}.$$

Then, the projection $\varphi : \mathbb{G}(k,n) \dashrightarrow \mathbb{P}^{M(k,n)}$ from the linear space spanned by W_Π is the inverse map of a $\psi_{k,n} : \mathbb{P}^{M(k,n)} \dashrightarrow \mathbb{G}(k,n)$ as in (1).

PROOF. We use the notation of Lemma 2.3. First of all, we observe that the linear system $\mathfrak{d}_{k,n}$ contains the linear system of hyperplanes of $\mathbb{P}^{M(k,n)}$ as a subsystem: this is $kH + |\pi|$, where π is any hyperplane. Via the map $\psi_{k,n}$, the hypersurfaces of $\mathfrak{d}_{k,n}$ are sent to hyperplane sections of $\mathbb{G}(k,n)$. Thus, the inverse of $\psi_{k,n}$ is a projection whose center is the intersection of all hyperplanes of $\mathbb{P}^{N(k,n)}$ whose intersection with $\mathbb{G}(k,n)$ contains $\psi_{k,n}(H)$ with multiplicity at least k.

The image of H under $\psi_{k,n}$ is the Grassmannian $\mathbb{G}_0 = \mathbb{G}(k, n-k-1)$ of all subspaces of dimension k contained in a fixed subspace Π of $\mathbb{P}^n$ dimension $n-k-1$. Indeed, if we set $y = 0$ in (2), we obtain the Plücker embedding associated with a $(k+1) \times (n-k)$ matrix.

A hyperplane H' in $\mathbb{P}^{N(k,n)}$ contains $\mathbb{G}_0$ with multiplicity at least k if and only if H' contains $T^{(k-1)}_{\mathbb{G}(k,n),P}$ for any $P \in \mathbb{G}_0$ and the center of the projection is the intersection of these hyperplanes. Then, from Proposition 2.7, the center of projection is the linear span of W_Π. This proves the assertion. ■

REMARK 2.9. With a dimension count similar to the one at the end of Proposition 2.7, one checks that the linear space spanned by W_Π has dimension $N(k,n) - m_1 - 1 = N(k,n) - (k+1)(n-k) - 1 = N(k,n) - M(k,n) - 1$. This fits with the result of Proposition 2.8.

REMARK 2.10. From the above considerations it follows that the birational map $\psi_{n,k}$ induces an isomorphism between $\mathbb{P}^{M(k,n)}$ minus a hyperplane H and $\mathbb{G}(k,n)$ minus a hyperplane section $\mathfrak{S}'$, precisely the hyperplane section corresponding to the hypersurface $(k+1)H$ in $\mathfrak{d}_{k,n}$. Looking at the proof of Proposition 2.8, we see that $\mathfrak{S}'$ contains $\mathbb{G}_0$ with multiplicity $k+1$; hence, it contains $T^{(k)}_{\mathbb{G}(k,n),P}$ for any $P \in \mathbb{G}_0$. From Proposition 2.7, one deduces that $\mathfrak{S}'$ coincides with the set of all $\Lambda \in \mathbb{G}(k,n)$ that have non-empty intersection with the $(n-k-1)$-plane Π. We will call the hyperplane H' cutting out such a $\mathfrak{S}'$ on $\mathbb{G}(k,n)$ a *k-osculating hyperplane* to $\mathbb{G}(k,n)$.

LEMMA 2.11. *Let $\check{\mathbb{P}}^{N(k,n)}$ be the dual space of $\mathbb{P}^{N(k,n)}$. Then, the k-osculating hyperplanes to $\mathbb{G}(k,n)$ are parametrized by a $\mathbb{G}(n-k-1,n)$ in $\check{\mathbb{P}}^{N(k,n)}$. In particular, since $\mathbb{G}(n-k-1,n)$ is non-degenerate in $\check{\mathbb{P}}^{N(k,n)}$, there is no point of $\mathbb{P}^{N(k,n)}$ contained in all k-osculating hyperplanes.*

PROOF. We have $\check{\mathbb{P}}^{N(k,n)} = \mathbb{P}(\wedge^{k+1}\check{V}) = \mathbb{P}(\wedge^{n-k}V)$.

Let Π be an $(n-k-1)$-plane spanned by $n-k$ points corresponding to the vectors $v_1,\dots,v_{n-k}$ of V. A k-plane Λ, spanned by $k+1$ points corresponding to the vectors $w_1,\dots,w_{k+1}$ of V, intersects Π if and only if the square matrix of order $n+1$ whose rows are $v_1,\dots,v_{n-k},w_1,\dots,w_{k+1}$ has zero determinant. The set of these k-planes is the section of $\mathbb{G}(k,n)$ with the k-osculating hyperplane of $\mathbb{P}^{N(k,n)}$ of equation

$$\sum_{1\le i_1<\cdots<i_{n-k}\le n+1} S_{i_1,\dots,i_{n-k}}\, p_{i_1,\dots,i_{n-k}}\, x_{\overline{i_1,\dots,i_{n-k}}} = 0,$$

where the $p_{i_1,\dots,i_{n-k}}$'s are the Plücker coordinates of Π in $\mathbb{G}(n-k-1,n)$, the $x_{\overline{i_1,\dots,i_{n-k}}}$'s are the homogeneous coordinates of $\mathbb{P}^{N(k,n)}$, where we denote by $\overline{i_1,\dots,i_{n-k}}$ the $(k+1)$-tuple of indices obtained by deleting $\{i_1,\dots,i_{n-k}\}$ from $(1,\dots,n+1)$, and $S_{i_1,\dots,i_{n-k}}$ is the sign of the permutation $(i_1,\dots,i_{n-k},\overline{i_1,\dots,i_{n-k}})$.

So the coordinates of this hyperplane in $\check{\mathbb{P}}^{N(k,n)}$ are $[S_{i_1,\dots,i_{n-k}}\, p_{i_1,\dots,i_{n-k}}]_{(i_1,\dots,i_{n-k})}$. The assertion follows. ■

COROLLARY 2.12. *Let X be an irreducible subvariety of $\mathbb{G}(k,n)$. Then, for a general projection $\varphi : \mathbb{G}(k,n) \dashrightarrow \mathbb{P}^{M(k,n)}$ as in Proposition 2.8, X is not contained in the indeterminacy locus of φ and the restriction of φ to X is a birational map of X to its image.*

PROOF. Given a projection $\varphi : \mathbb{G}(k,n) \dashrightarrow \mathbb{P}^{M(k,n)}$ as in Proposition 2.8, its indeterminacy locus and the subvariety contracted by the projection are contained in a k-osculating hyperplane section of the Grassmannian. By Lemma 2.11, these hyperplanes vary in a Grassmannian $\mathbb{G}(n-k-1,n)$ in $\check{\mathbb{P}}^{N(k,n)}$, and there is no point of $\mathbb{P}^{N(k,n)}$ contained in all these hyperplanes. Hence, given the subvariety X in $\mathbb{G}(k,n)$, there is certainly a k-osculating hyperplane not containing it. The corresponding projection enjoys the required property. ■

3. Fano schemes

Let $X \subset \mathbb{P}^n$ be an irreducible projective variety. Given any positive integer k, we will denote by $F_k(X)$ the Hilbert scheme of k-planes of $\mathbb{P}^n$ contained in X. This is also called the *k-Fano scheme* of X. We will not be interested in the scheme structure on $F_k(X)$, but rather on its support. In particular, we will be interested in $F_k(X)$ when X is an irreducible hypersurface of degree $d \geqslant 2$ in $\mathbb{P}^n$.

This short section is devoted to prove the following.

Proposition 3.1. *Let X be an irreducible hypersurface of degree $d \geqslant 2$ in $\mathbb{P}^n$. Let $\varphi : \mathbb{G}(k,n) \dashrightarrow \mathbb{P}^{M(k,n)}$ be a general projection map as in Proposition 2.8. Then, $\varphi|_{F_k(X)}$ is a birational map on each component of $F_k(X)$ and $\varphi(F_k(X))$ is defined by the vanishing of $\binom{d+k}{k}$ polynomials of degree d.*

Proof. The first assertion follows directly from Corollary 2.12.

Let us fix homogeneous coordinates $[x_0,\dots,x_n]$ in $\mathbb{P}^n$ and let $f=0$ be the equation of X in this system, with

$$f(x_0,\dots,x_n) = \sum_{d_0+\dots+d_n=d} \alpha_{d_0\dots d_n} x_0^{d_0}\dots x_n^{d_n}.$$

We assume, without loss of generality, that the projection is an isomorphism on the open set U of the Grassmannian where the first Plücker coordinate is different from zero. For every $\Lambda \in U$, we can give a parametrization $\phi_\Lambda : \mathbb{P}^k \to \Lambda \subseteq \mathbb{P}^n$ of Λ as

$$[s_0,\dots,s_k] \mapsto [s_0,\dots,s_k]\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & a_{1,k+2} & a_{1,k+3} & \cdots & a_{1,n+1} \\ 0 & 1 & 0 & \cdots & 0 & a_{2,k+2} & a_{2,k+3} & \cdots & a_{2,n+1} \\ \vdots & & & & & & & & \\ 0 & 0 & 0 & \cdots & 1 & a_{k+1,k+2} & a_{k+1,k+3} & \cdots & a_{k+1,n+1} \end{bmatrix}$$

with $a_{i,j}$, for $1 \leqslant i \leqslant k+1$, $k+2 \leqslant j \leqslant n+1$, depending on Λ.

Then, $f(\phi_\Lambda([s_0,\dots,s_k]))$ is a form of degree d in $s_0,\dots,s_k$ with coefficient polynomials in the $a_{i,j}$'s and in the $\alpha_{d_0\dots d_n}$'s. Imposing that Λ sits in $F_k(X)$ is equivalent to impose that $f(\phi_\Lambda([s_0,\dots,s_k]))$ is identically zero as a form in $s_0,\dots,s_k$. This translates in imposing that the $\binom{d+k}{k}$ coefficients of $f(\phi_\Lambda([s_0,\dots,s_k]))$ all vanish, and these are linear in the $\alpha_{d_0,\dots,d_n}$'s and of degree d in the $a_{i,j}$'s. The assertion follows. ∎

4. Families of hypersurfaces and the section lemma

In this section, we introduce the definition of a *family of hypersurfaces* and we prove a crucial result, *Section Lemma* 4.5, in whose proof we use an idea of Conforto [3], which extends previous work by Comessatti [2].

4.1. We start with some definitions. We will denote by $\mathcal{L}_{n,d}$ the linear system of all hypersurfaces of degree d in $\mathbb{P}^n$, and by $p : \mathcal{H}_{n,d} \to \mathcal{L}_{n,d}$ the universal family, so that $\mathcal{H}_{n,d} \subset \mathcal{L}_{n,d} \times \mathbb{P}^n$ and p is the projection to the first factor.

Definition 4.1. Let W be an irreducible variety. We call a *family of hypersurfaces (of degree d and dimension $n-1$) parametrized by W* any morphism $f : \mathcal{X} \to W$, such that there exists a morphism $g : W \to \mathcal{L}_{n,d}$ so that the diagram

$$\begin{tikzcd} \mathcal{X} \arrow[r] \arrow[d, "f"'] & \mathcal{H}_{n,d} \arrow[d, "p"] \\ W \arrow[r, "g"] & \mathcal{L}_{n,d} \end{tikzcd}$$

is cartesian. In particular, $f : \mathcal{X} \to W$ is flat. For any point $w \in W$, we will denote by $X_w \subset \mathbb{P}^n$ the corresponding hypersurface, that is, the fiber of $f : \mathcal{X} \to W$ over w.

Definition 4.2. Given two families of hypersurfaces $\mathcal{X} \to W$ and $\mathcal{Y} \to T$ as in Definition 4.1, we say that $\mathcal{X}$ is *birationally equivalent* to $\mathcal{Y}$ if there exist two birational maps $f : \mathcal{X} \dashrightarrow \mathcal{Y}$ and $g : W \dashrightarrow T$ such that the diagram

$$\begin{tikzcd} \mathcal{X} \arrow[r, dashed, "f"] \arrow[d] & \mathcal{Y} \arrow[d] \\ W \arrow[r, dashed, "g"] & T \end{tikzcd}$$

commutes.

We will be interested in families of hypersurfaces up to birational equivalence. The following lemma gives us a sort of canonical way of representing a family of hypersurfaces up to birational equivalence.

Lemma 4.3. *Let $\mathcal{X} \to W$ be a family of hypersurfaces of degree d in $\mathbb{P}^n$ with* $\dim(W) = r$. *Then, there is a birationally equivalent family $\mathcal{X}' \to W'$ such that W' is a dense open subset of a hypersurface in $\mathbb{P}^{r+1}$ which is birational to W and $\mathcal{X}' \subset W' \times \mathbb{P}^n$ has an equation of the form*

$$\sum_{i_1,\dots,i_d \in \{0,\dots,n\}} a_{i_1\dots i_d}(u_0,\dots,u_{r+1}) \prod_{j=1}^{d} x_{i_j} = 0, \tag{6}$$

where $a_{i_1\dots i_d} \in H^0(W', \mathcal{O}_{W'}(\mu))$ for some $\mu \in \mathbb{N}$, for all $i_1,\dots,i_d \in \{0,\dots,n\}$.

Proof. To give the family, $\mathcal{X} \to W$ is equivalent to give the corresponding morphism $g : W \to \mathcal{L}_{n,d}$. Let $\mathfrak{W} \subset \mathbb{P}^{r+1}$ be a hypersurface with a birational map $h : \mathfrak{W} \dashrightarrow W$.

Then, $g' = g \circ h : \mathfrak{W} \dashrightarrow \mathcal{L}_{n,d}$ is a rational map, and there is a dense open subset W' of $\mathfrak{W}$ where g' is defined. Then, we have a morphism $g' : W' \to \mathcal{L}_{n,d}$, and accordingly we have a family $\mathscr{X}' \to W'$ that is birationally equivalent to $\mathscr{X} \to W$. On the other hand, giving $g' : W' \to \mathcal{L}_{n,d}$ is equivalent to give a suitable $\binom{n+d}{n}$-tuple of elements

$$a_{i_1 \dots i_d} \in H^0\big(W', \mathscr{O}_{W'}(\mu)\big), \quad \text{for all } i_1, \dots, i_d \in \{0, \dots, n\}$$

and some positive integer μ, so that $\mathscr{X}' \subset W' \times \mathbb{P}^n$ has equation (6). ■

4.2. Next, we want to prove the announced *Section Lemma*.

Let $\mathscr{X} \to W$ be a family of hypersurfaces of degree $d \geqslant 2$ in $\mathbb{P}^n$. We denote by $F_k(\mathscr{X}) \to W$ the *relative Fano scheme* of k-planes in $\mathbb{P}^n$ contained in the fibers of $\mathscr{X} \to W$. For any point $w \in W$, the fiber of $F_k(\mathscr{X}) \to W$ over w is $F_k(X_w)$.

We recall the following result (see [6]).

THEOREM 4.4. *Let k, n, d be positive integers with $d \geqslant 2$ and*

$$n \geqslant \begin{cases} 2k+1, & \text{if } d = 2 \text{ and } k \geqslant 2 \\ \frac{1}{k+1}\binom{k+d}{d}, & \text{otherwise.} \end{cases} \tag{7}$$

Then, all hypersurfaces of degree d in $\mathbb{P}^n$ contain a k-plane.

Next, we consider $\mathscr{X} \to W$ a family of hypersurfaces of degree $d \geqslant 2$ in $\mathbb{P}^n$, with $\dim(W) = r$. We will assume that

$$n > k + \frac{1}{k+1}\left[\binom{d+k}{k} d^r - 1\right]. \tag{8}$$

Then, clearly (7) holds, and hence, by Theorem 4.4, the morphism $F_k(\mathscr{X}) \to W$ is surjective. By generic flatness, there is a dense open subset of W over which $F_k(\mathscr{X}) \to W$ is flat.

We are ready to prove the *Section Lemma*.

LEMMA 4.5 (The Section Lemma). *Let $\mathscr{X} \to W$ be a family of hypersurfaces of degree $d \geqslant 2$ in $\mathbb{P}^n$, with* $\dim(W) = r$ *so that* (8) *holds. Then, there is a dense open subset U of W such that over U there is a section of $F_k(\mathscr{X}) \to W$.*

PROOF. Since the problem is birational in nature, by Lemma 4.3, we may assume that W is a dense open subset of a hypersurface of degree $\bar{m}$ in $\mathbb{P}^{r+1}$ with equation

$$\phi(u_0, \dots, u_{r+1}) = 0.$$

The domain $F_k(\mathscr{X})$ of the Fano family $F_k(\mathscr{X}) \to W$, which up to shrinking W we may assume to be flat, is contained in $W \times \mathbb{G}(k,n)$. Consider a general birational projection $\varphi : \mathbb{G}(k,n) \dashrightarrow \mathbb{P}^{M(k,n)}$ as in Proposition 2.8 that determines a birational map

$$\Phi : W \times \mathbb{G}(k,n) \dashrightarrow W \times \mathbb{P}^{M(k,n)}.$$

By applying Corollary 2.12 and up to shrinking W, we may suppose that for all $w \in W$, the restriction of Φ to any irreducible component of $\{w\} \times F_k(X_w)$ is birational onto its image so that Φ restricts to a birational map of $F_k(\mathscr{X})$ to its image, that we denote by $\mathbb{F}_k(\mathscr{X})$, contained in $W \times \mathbb{P}^{M(k,n)}$. By Proposition 3.1, we may assume that $\mathbb{F}_k(\mathscr{X})$ is defined by the vanishing of $\binom{d+k}{k}$ equations in $W \times \mathbb{P}^{M(k,n)}$ of the form

$$\sum_{i_1,\dots,i_d \in \{0,\dots,n\}} b^{\ell}_{i_1\dots i_d}(u_0,\dots,u_{r+1}) \prod_{j=1}^{d} y_{i_j} = 0 \tag{9}$$

for $\ell = 1,\dots,\binom{d+k}{k}$, and $b^{\ell}_{i_1\dots i_d}(u_0,\dots,u_{r+1}) \in H^0(W, \mathscr{O}_W(\mu))$ for a suitable positive integer μ, where the y_i's denote the homogeneous coordinates of $\mathbb{P}^{M(k,n)}$.

To prove the lemma, it clearly suffices to find a rational section $p : W \dashrightarrow \mathbb{F}_k(\mathscr{X})$ of $\mathbb{F}_k(\mathscr{X}) \to W$. Such a rational section is determined by a suitable $(M(k,n)+1)$-tuple of rational functions on W. We may assume that each such rational function is expressed by a homogeneous polynomial in the variables $u_0,\dots,u_{r+1}$ of a fixed degree m modulo $\phi(u_0,\dots,u_{r+1})$.

Supposing $m > \bar{m}$, we can choose M independent elements in $H^0(W, \mathscr{O}_W(m))$, where

$$M = \binom{m+r+1}{r+1} - \binom{m-\bar{m}+r+1}{r+1}.$$

These can be identified with M forms $\Psi_1,\dots,\Psi_M$ of degree m, modulo $\phi(u_0,\dots,u_{r+1})$.

We want to construct a section p by writing its homogeneous coordinates as linear combinations of the Ψ's as above, that is, by writing them as

$$p_i = \sum_{j=1}^{M} \lambda_{i,j}\Psi_j \quad \text{for } i = 0,\dots,M(k,n)$$

where we take the $\lambda_{i,j}$'s as indeterminates. The number of the λ's is

$$\begin{aligned}&\big[(k+1)(n-k)+1\big]M\\ &\quad = \big[(k+1)(n-k)+1\big]\left[\binom{m+r+1}{r+1} - \binom{m-\bar{m}+r+1}{r+1}\right].\end{aligned}$$

We need to find the values of these λ's so that p is a section. For this, we have to replace the y_i's in each of the equations (9) with the $p_i(u_0,\dots,u_{r+1})$'s and we have to impose that the results identically vanish on W, i.e., they must be forms in $\mathbb{K}[u_0, \dots, u_{r+1}]$ that are divisible by $\phi(u_0, \dots, u_{r+1})$.

We make the substitution and for each $\ell = 1, \dots, \binom{d+k}{k}$ we have expressions of the sort

$$\sum_{i_1,\dots,i_d \in \{0,\dots,n\}} b^{\ell}_{i_1\dots i_d}(u_0, \dots, u_{r+1}) \prod_{j=1}^{d} p_{i_j} = \sum_{l_1+\dots+l_{r+1}=dm+\mu} F^{\ell}_{l_0\dots l_{r+1}}(\lambda_{i,j}) u_0^{l_0} \cdots u_{r+1}^{l_{r+1}}$$

where the homogeneous polynomials that we have after the substitution are of degree $dm + \mu$ with respect to $u_0, \dots, u_{r+1}$ and the coefficients $F^{\ell}_{l_0\dots l_{r+1}}$ are polynomials in the λ's.

Thus, for all $\ell = 1, \dots, \binom{d+k}{k}$, we have to impose that

$$\sum_{l_1+\dots+l_{r+1}=dm+\mu} F^{\ell}_{l_0\dots l_{r+1}}(\lambda_{i,j}) u_0^{l_0} \dots u_{r+1}^{l_{r+1}} = \phi(u_0, \dots, u_{r+1}) \Big(\sum_{i_1+\dots+i_{r+1}=dm-\overline{m}+\mu} \alpha^{\ell}_{i_0\cdots i_{r+1}} u_0^{i_0} \cdots u_{r+1}^{i_{r+1}} \Big) \tag{10}$$

where the $\alpha^{\ell}_{i_0\cdots i_{r+1}}$'s are again indeterminates. Their number is

$$\binom{d+k}{k}\binom{dm-\overline{m}+\mu+r+1}{r+1}.$$

Now, to prove the thesis, we need to show that, under condition (8), there exists an *admissible solution* of the system of non-homogeneous equations obtained by equating the coefficients of the monomials of degree $dm + \mu$ in (10) for each $\ell = 1, \dots, \binom{d+k}{k}$. A solution of this system is called *admissible* if it gives rise to a section. Clearly, a solution is admissible if and only if not all the λ's are equal to 0.

In the system, there are

$$\binom{d+k}{k}\binom{dm+\mu+r+1}{r+1}$$

equations in the α's and λ's. The total amount of these variables is

$$\big[(k+1)(n-k)+1\big]\left[\binom{m+r+1}{r+1} - \binom{m-\overline{m}+r+1}{r+1}\right] + \binom{d+k}{k}\binom{dm-\overline{m}+\mu+r+1}{r+1}.$$

We claim that if the number of variables is greater than the number of equations, that is, if the following inequality holds:

$$\text{(11)}\quad \big[(k+1)(n-k)+1\big]\left[\binom{m+r+1}{r+1}-\binom{m-\overline{m}+r+1}{r+1}\right] + \binom{d+k}{k}\binom{dm-\overline{m}+\mu+r+1}{r+1} > \binom{d+k}{k}\binom{dm+\mu+r+1}{r+1},$$

our system has admissible solutions and we do have sections as required.

In general, given a system of non-homogeneous equations, it is not true that if it is *underdeterminate* (i.e., the number of equations is lower than the number of the variables), then the set of solutions is non-empty. However, we do know that, in the associated affine space with coordinates λ's and α's, the origin, where all λ's and all α's vanish, is a solution of the system although it does not give rise to an admissible solution. In any event, this implies that the set of solutions has a component $\mathfrak{S}$ of positive dimension which contains the origin. Moreover, $\mathfrak{S}$ cannot be contained in the subspace defined by the vanishing of all the λ's. Indeed, if all the λ's are equal to 0, from (10), it follows that also the α's are 0. This proves that if (11) holds, there are admissible solutions and therefore there are sections as desired.

Finally, we want to see under which conditions, for m large enough, (11) holds. This can be written as

$$\big[(k+1)(n-k)+1\big]\left[\binom{m+r+1}{r+1}-\binom{m-\overline{m}+r+1}{r+1}\right] + \binom{d+k}{k}\left[\binom{dm-\overline{m}+\mu+r+1}{r+1}-\binom{dm+\mu+r+1}{r+1}\right] > 0.$$

The term on the left is a polynomial in m: the condition in order that it is positive for $m \gg 0$ is that the leading coefficient is positive. The coefficient of the monomial m^{r+1} of maximal degree is equal to zero, so we have to look at the coefficient of m^r. This equals

$$\frac{\big[(k+1)(n-k)+1\big]}{(r+1)!}(r+1)\overline{m} + \frac{\binom{d+k}{k}d^r}{(r+1)!}\big[-(r+1)\overline{m}\big].$$

After dividing for the positive term $\frac{\overline{m}}{r!}$, we obtain

$$(k+1)(n-k)+1-\binom{d+k}{k}d^r$$

and being this positive is equivalent to (8). ∎

REMARK 4.6. Note that the result of the Section Lemma is equivalent to say that if (8) holds, and if w is the generic point of W, defined over the field of rational functions $\mathbb{K}(W)$, then one can find a k-plane Λ in the generic hypersurface X_w of the family, also defined over $\mathbb{K}(W)$. In this case, one says that Λ is *rationally determined* on X_w.

5. UNIRATIONALITY OF FAMILIES OF HYPERSURFACES

In this section, we use the previous results to give a criterion for the unirationality of families of hypersurfaces. We need some preliminaries.

5.1. We recall the following.

DEFINITION 5.1. Let $X \subset \mathbb{P}^n$ be an algebraic variety defined over $\mathbb{K}$ and Λ a k-plane contained in X. One says that X is Λ-*rational* (resp. Λ-*unirational*) if X is $\mathbb{K}(\Lambda)$-rational (resp. $\mathbb{K}(\Lambda)$-unirational), where $\mathbb{K}(\Lambda)$ is the extension of $\mathbb{K}$ obtained by adding to $\mathbb{K}$ the Plücker coordinates of Λ.

Let $\mathscr{X} \to W$ be a flat family of subvarieties of $\mathbb{P}^n$ with W being an irreducible variety. If $w \in W$, we denote, as usual, by $X_w \subset \mathbb{P}^n$ the fiber of $\mathscr{X} \to W$ over w. We assume that there is a dense open subset U of W such that for all $w \in U$, X_w is irreducible. Thus, up to shrinking W, we may assume that this happens for all $w \in W$. Let $F_k(\mathscr{X}) \to W$ be the *relative Fano scheme* of k-planes of $\mathscr{X} \to W$. For all $w \in W$, the fiber of $F_k(\mathscr{X}) \to W$ is $F_k(X_w)$.

The following criterion is due to Roth (see [9]).

PROPOSITION 5.2 (Roth's Criterion). *Let $\mathscr{X} \to W$ be a flat family of varieties with W being an irreducible, unirational variety. Suppose that $F_k(\mathscr{X}) \to W$ is dominant, so that, up to shrinking W, we may assume it is flat. Suppose that there is a section $s : W \to F_k(\mathscr{X})$ of $F_k(\mathscr{X}) \to W$ such that there is a dense open subset U of W such that for all $w \in U$, the variety X_w is $s(w)$-unirational. Then, $\mathscr{X}$ is unirational.*

In addition, if W is rational and for all $w \in U$, the variety X_w is $s(w)$-rational, then $\mathscr{X}$ is rational.

PROOF. We may assume that $U = W$. Let $\phi : \mathbb{P}^r \dashrightarrow W$ be the dominant map which assures the unirationality of W and by $\psi_w : \mathbb{P}^{r'}_{\mathbb{K}(s(w))} \dashrightarrow X_w$ the dominant map which assures the unirationality of X_w, for $w \in W$.

Then, we can construct the map

$$\mathbb{P}^r \times \mathbb{P}^{r'} \dashrightarrow \mathscr{X}$$

such that the pair (t, t') is sent to $\psi_{\phi(t)}(t')$. This is a rational dominant map, and it is defined over $\mathbb{K}$.

It follows furthermore that if ϕ and ψ_w are generically finite of degree a and b, respectively, then this map is generically finite of degree $a \cdot b$. The second assertion follows. ■

5.2. In the paper [8], A. Predonzan proved the following.

THEOREM 5.3. *Let $X \subset \mathbb{P}^n$ be an irreducible hypersurface of degree $d \geqslant 2$ defined over $\mathbb{K}$. Suppose that X contains a k-plane Λ with*

$$k \geqslant k(d)$$

where $k(d)$ is inductively defined as follows:

$$k(d) = \binom{k(d-1)+d-1}{d-1}, \quad k(2) = 0.$$

Suppose that X is smooth along Λ. Then, X is Λ-unirational.

As a consequence of this result, Predonzan also proved in [8] the following.

THEOREM 5.4. *Let $X \subset \mathbb{P}^n$ be an irreducible hypersurface of degree $d \geqslant 2$ defined over $\mathbb{K}$, with a singular locus of dimension t. If*

$$n \geqslant \frac{1}{k(d)+1}\binom{k(d)+d}{d} + k(d) + t + 1,$$

then X is unirational over an extension of $\mathbb{K}$.

This result has been rediscovered in [5], although with a worse lower bound for n. Our aim is to prove the following extension of Theorem 5.4.

THEOREM 5.5. *Let $\mathscr{X} \to W$ be a family of hypersurfaces of degree $d \geqslant 2$ in $\mathbb{P}^n$, with W being irreducible, unirational of dimension r. Assume that if $w \in W$ is the generic point, then X_w is irreducible with a singular locus of dimension t. If*

$$n > k(d) + \frac{1}{k(d)+1}\left[\binom{d+k(d)}{k(d)}d^r - 1\right] + t + 1, \tag{12}$$

then $\mathscr{X}$ is unirational.

PROOF. By the hypotheses, up to shrinking W, we may assume that for all $w \in W$, the hypersurface $X_w \subset \mathbb{P}^n$ is irreducible with singular locus of dimension t. Again up to shrinking W, we may assume that there is an $(n-t-1)$-plane P in $\mathbb{P}^n$ such that for all $w \in W$, the intersection of X_w with P is smooth. In this way, we get a new family

$\mathscr{X}' \to W$ of hypersurfaces of degree d in $\mathbb{P}^{n-t-1}$ such that for all $w \in W$, X'_w is the intersection of X_w with P.

Taking into account (12), by Section Lemma 4.5, up to shrinking W, we may assume there is a section s of $F_{k(d)}(\mathscr{X}') \to W$. Note that for all $w \in W$, X'_w is smooth, and therefore, X_w is smooth along $s(w)$. Then, by Theorem 5.3, for all $w \in W$, X_w is $s(w)$-unirational. Thus, by applying Roth's Criterion 5.2, the assertion follows. ■

We notice that if $d = 2$, then Theorem 5.5 is basically the main result of [3].

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