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Partial Differential Equations. – *Regularity results of solutions of quasilinear systems having singularities in the coefficients*, by FLAVIA GIANNETTI and GIOCONDA MOSCARIELLO, communicated on 21 April 2023.

ABSTRACT. – We consider non-degenerate elliptic systems of the type

$$-\operatorname{div} A(x, Du) = g(x) \quad \text{in } \Omega \subset \mathbb{R}^n,$$

where $u : \Omega \to \mathbb{R}^N$, $g \in L^2(\Omega, \mathbb{R}^N)$ and $x \to A(x, \xi)$ has derivatives in the Marcinkiewicz class $L^{n,\infty}(\Omega)$ with sufficiently small distance to $L^\infty(\Omega)$. We prove that every weak solution $u \in W^{1,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ of the system is such that the nonlinear expression of its gradient $V_\mu(Du) := (\mu^2 + |Du|^2)^{\frac{p-2}{4}} Du$ is weakly differentiable with $D(V_\mu(Du)) \in L^2_{\mathrm{loc}}(\Omega)$. Then, we deduce higher differentiability properties for u itself and some higher integrability results for its gradient.

KEYWORDS. – Elliptic systems, difference quotient, Marcinkiewicz classes, regularity of second order.

2020 MATHEMATICS SUBJECT CLASSIFICATION. – Primary 35B65; Secondary 35R05, 35J47, 49N99.

1. INTRODUCTION

We consider nonlinear elliptic systems of the type

$$-\operatorname{div} A(x, Du) = g(x) \quad \text{in } \Omega, \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$, $n > 2$, and $u : \Omega \to \mathbb{R}^N$, $N \geq 1$. We suppose that $g \in L^2(\Omega, \mathbb{R}^N)$, while the vector field $A : \Omega \times \mathbb{R}^{Nn} \to \mathbb{R}^{Nn}$ is assumed to be a Carathéodory function satisfying, for a.e. $x, y \in \Omega$ and all $\xi, \eta \in \mathbb{R}^{Nn}$, the following conditions:

$$\langle A(x,\eta) - A(x,\zeta), \eta - \zeta \rangle \geq \big(\mu^2 + |\eta|^2 + |\zeta|^2\big)^{\frac{p-2}{2}} |\eta - \zeta|^2, \tag{1.2}$$

$$\big|A(x,\eta) - A(x,\zeta)\big| \leq \alpha |\eta - \zeta| \big(\mu^2 + |\eta|^2 + |\zeta|^2\big)^{\frac{p-2}{2}}, \tag{1.3}$$

$$\big|A(x,\eta) - A(y,\eta)\big| \leq |x - y| \big[k(x) + k(y)\big] \big(\mu^2 + |\eta|^2\big)^{\frac{p-1}{2}}, \tag{1.4}$$

$$A(x, 0) = 0 \tag{1.5}$$

for $\mu \in (0, 1]$, $\alpha > 0$ and some exponent $2 \leq p < n$.

Regarding the function $k : \Omega \to [0, +\infty)$, we assume $k \in L^{n,\infty}(\Omega)$ such that $\operatorname{dist}_{L^{n,\infty}}(k, L^\infty)$ is sufficiently small.

Note that, by virtue of a characterization of the Sobolev functions due to Hajlasz [17], the function k above plays the role of the derivative $D_x A$. Therefore, the condition (1.4) describes the continuity of $A(x,\xi)$ with respect to the x-variable. Obviously, this is a weak form of continuity since the function k may blow up at some points.

The model case of the systems we have in mind is given by the non-degenerate system

$$-\operatorname{div}\left(A(x)\left(1+|Du|^2\right)^{\frac{p-2}{2}} Du\right) = g(x) \quad \text{in } \Omega,$$

where the function $A(x) \in L^\infty(\Omega, \mathbb{R}^{Nn})$ is weakly differentiable with derivatives belonging to the Marcinkiewicz class $L^{n,\infty}$ and having sufficiently small distance to L^∞.

In the linear case, the study of the second-order regularity of solutions to equations with discontinuous coefficients goes back to Miranda (see [20, 21]). More recently, in particular in connection with the regularity of minimizers of functionals of the Calculus of Variations, the regularity theory for solutions to problems of the type (1.1) has been extensively studied. For an almost complete treatment, see [19] and the references therein.

The aim of this paper is to study the second-order regularity for weak solutions $u \in W^{1,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ to

$$-\operatorname{div} A(x, Du) = g(x) \quad \text{in } \Omega. \tag{1.6}$$

As for p-Laplacian systems with coefficients differentiable in the spatial variable, where the higher differentiability of solutions holds in the sense that the nonlinear expressions $V(Du) = V_\mu(Du) := (\mu^2 + |Du|^2)^{\frac{p-2}{4}} Du$ of their gradients are weakly differentiable, our result will be stated in terms of $V(Du)$. More precisely, we shall prove the following.

THEOREM 1.1. *Let $u \in W^{1,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ be a weak solution of (1.6), under the assumptions (1.2)–(1.5) with $k \in L^{n,\infty}(\Omega)$. Further assume $g \in L^2(\Omega, \mathbb{R}^N)$. There exists a positive constant $\sigma = \sigma(n,p)$ such that if*

$$\operatorname{dist}_{L^{n,\infty}}(k, L^\infty) < \sigma, \tag{1.7}$$

then $D(V(Du)) \in L^2_{\mathrm{loc}}(\Omega)$. Moreover, the following estimate holds:

$$\begin{aligned} \int_{B_R} \left|D\big(V(Du)\big)\right|^2 dx \\ \le C\left(1+\frac{1}{R^2}\right)\int_{B_{2R}} (\mu^2+|Du|^2)^{\frac{p}{2}}\,dx + c\int_{B_{2R}} |g|^2\,dx \end{aligned} \tag{1.8}$$

for every ball $B_{2R} \Subset \Omega$ *and constants* $C = C(\mathrm{dist}_{L^{n,\infty}}(k, L^{\infty}), p, \alpha, n, N)$ *and* $c = c(p, \mu)$.

Obviously, for $p = 2$, Theorem 1.1 provides directly the $W^{2,2}_{\mathrm{loc}}$ regularity of solutions to (1.6). Anyway, inequality (1.8), thanks to the properties of $V(Du)$ (see Lemma 2.4 below), yields the $W^{2,2}_{\mathrm{loc}}$ regularity also for $2 < p < n$.

Embedding theorem for Nikolskii spaces gives the existence of a fractional derivative of Du as shown in Corollary 4.1 and a higher integrability result is achieved whenever $g \in L^r(\Omega, \mathbb{R}^N)$, for some $r > 2$ (see Corollary 4.2).

Let us discuss condition (1.7) on the distance of the function $k(x)$ to L^{∞}. First of all, we notice that it is clearly satisfied if the derivatives of $A(x, \xi)$ with respect to x belong to any subspace of $L^{n,\infty}$ in which L^{∞} is dense and then, in particular, if they belong to $L^{n,q}$ with $1 < q < \infty$, since their distances to L^{∞} are null. On the contrary, we underline that L^{∞} is not dense in $L^{p,\infty}$ for any $p > 1$.

Secondly, we note that condition (1.7) does not imply the smallness of the norm of $k(x)$ in $L^{n,\infty}$ (as shown in the example below) but rather it measures how the function $k(x)$ is far from being a regular function. It follows that assuming (1.7) is more general than considering a condition on the norm and allows us to present different settings of our result in a unified way.

We also observe that the norm in $L^{n,\infty}$ is not absolutely continuous; namely, a function can have large norm even if restricted to a set with small measure.

It is worth pointing out that the assumption on the coefficients of the type (1.7) has been firstly introduced in [9] to study the solvability and the regularity of some linear elliptic equations with lower-order terms. Moreover, a condition like (1.7) also turned out crucial in [10] to study the $W^{2,2}$ solvability of the Dirichlet problem for some linear nonvariational elliptic equations as well as in [7, 8] to treat the solvability of some noncoercive nonlinear operators. See also the recent paper [22].

Now, let us spend some words on the strategy of the proof. We first obtain an a priori estimate for regular solutions by using the classic difference quotient method (for details, see, for example, [1, 14, 15]). Neverthless, the fact that the coefficients could have discontinuous derivatives causes, as pointed out above, some difficulties. More precisely, our arguments result to bound integrals of the type

$$\int_{B_{\lambda r}} k(x)^2 \big(\mu^2 + \left|Du(x)\right|^2\big)^{\frac{p}{2}}\, dx$$

and the estimates we get involve some integrals that have to be reabsorbed to the left-hand side. It is on this occasion that the hypothesis that the function $k \in L^{n,\infty}$ has sufficiently small distance to L^{∞} comes into play.

Once the a priori estimate has been obtained, the proof of Theorem 1.1 consists in the construction of regularized problems whose regularized solutions u_{ε} verify the a priori estimate. The conclusion follows by proving that we can pass to the limit in such estimates.

Notice that second-order estimates are established for solutions to homogeneous parabolic systems with discontinuous coefficients in [11, 12]. We also refer to [6] for the p-Laplace system with the right-hand side in L^2.

We emphasize that here we consider nonlinear elliptic systems with right-hand side in L^2 and, at the same time, with discontinuous coefficients.

We exhibit the following example (see also [10]).

EXAMPLE 1.2. Consider the following system in the cube $Q = (0,1]^n$:

$$-\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} A_{ij} \frac{\partial}{\partial x_j} u^\nu = g^\nu(x) \quad \nu = 1,\dots,N$$

with $g = (g^1,\dots,g^N) \in L^2(\Omega, \mathbb{R}^N)$ and

$$A_{ij}^\nu(x) = \delta_{ij}^\nu + \frac{A x_i x_j}{|x|^2} + \varphi^\nu \quad i,j = 1,\dots,n,$$

where $A > 0$, $\varphi \in C^1(\bar{Q})$, $\varphi \geq 0$ and where $\delta_{i,j}$ is the Kronecker delta.

Observe that the matrix of the coefficients verifies all our assumptions for $p = 2$. In particular, the derivatives of A_{ij}^ν belong to $L^{n,\infty}(Q)$. We note that such derivatives do not belong to the Lebesgue space $L^n(Q)$ (for more details, we refer to [25]). An elementary calculation shows that in this case, $\mathrm{dist}_{L^{n,\infty}}(k, L^\infty) = A\omega_n^{1/n}$, where ω_n stands for the Lebesgue measure of the unit ball in $\mathbb{R}^n$. The condition (1.7) is verified provided the constant A is sufficiently small.

In the homogeneous case, i.e. for $g = 0$, our argument works also for degenerate operators. Indeed it holds the following theorem.

THEOREM 1.3. *Let $u \in W^{1,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ be a weak solution of the system*

$$-\operatorname{div} A(x, Du) = 0 \quad \text{in } \Omega \tag{1.9}$$

under the assumptions (1.2)–(1.5) *with $k \in L^{n,\infty}(\Omega)$, $\mu \in [0,1]$. There exists a positive constant $\sigma = \sigma(n,p)$ such that if*

$$\mathrm{dist}_{L^{n,\infty}}(k, L^\infty) < \sigma,$$

then $D(V(Du)) \in L^2_{\mathrm{loc}}(\Omega)$. Moreover, the following estimate holds:

$$\int_{B_R} \left|D\big(V(Du)\big)\right|^2 dx \leq C\left(1 + \frac{1}{R^2}\right) \int_{B_{2R}} (\mu + |Du|^2)^{\frac{p}{2}}\, dx \tag{1.10}$$

for every ball $B_{2R} \Subset \Omega$ and for a constant $C = C(\mathrm{dist}_{L^{n,\infty}}(k, L^\infty), p, \alpha, n, N)$.

Note that in the degenerate case $\mu = 0$, the previous result gives

$$|Du|^{\frac{p-2}{2}}|D^2u| \in L^2_{\mathrm{loc}}(\Omega)$$

with the estimate

$$\int_{B_R} |Du|^{p-2}|D^2u|^2\,dx \le C\left(1+\frac{1}{R^2}\right)\int_{B_{2R}} |Du|^p\,dx.$$

2. Preliminaries

This section is devoted to notation and preliminary results useful for our aims. We start specifying the meaning of the solution to system (1.6).

Definition 2.1. A function $u \in W^{1,p}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ is a local weak solution of (1.6) if

$$\int_\Omega \langle A(x,Du), D\varphi\rangle\,dx = \int_\Omega \langle g(x),\varphi\rangle\,dx$$

for every $\varphi \in C^\infty_0(\Omega,\mathbb{R}^N)$.

For $R > 0$ and $x_0 \in \mathbb{R}^n$, we define

$$B_R(x_0) = \{x \in \mathbb{R}^n : |x - x_0| < R\},$$

but in the case no ambiguity arises, we shall use the short notation B_R. We shall denote by c a general constant that may vary on different occasions, even within the same line of estimates. Relevant dependencies on parameters and special constants will be suitably emphasized using parentheses.

In order to obtain the a priori estimate, we shall use the difference quotient method. Therefore, in the following, we introduce the finite difference operator $\tau_{h,i}$ and recall some basic properties.

Given a vector valued function $F : \mathbb{R}^n \to \mathbb{R}^N$, we use the notation

$$\tau_{h,i}F(x) := F(x + he_i) - F(x),$$

where $h \in \mathbb{R}$, $i \in \{1,\dots,n\}$ and e_i is the unit vector in the x_i direction.

The difference quotient is defined for $h \in \mathbb{R}\setminus\{0\}$ as

$$\Delta_{h,i}F(x) = \frac{\tau_{h,i}F(x)}{h}$$

and the next two lemmas, whose proof can be found, for example, in [15], hold.

LEMMA 2.2. *If* $0 < \rho < R$, $|h| < R - \rho$, $1 < p < +\infty$, $i \in \{1, \dots, n\}$ *and* $F, D_i F \in L^p(B_R)$, *then*

$$\int_{B_\rho} \left|\tau_{h,i} F(x)\right|^p dx \le |h|^p \int_{B_R} \left|D_i F(x)\right|^p dx.$$

Moreover,

$$\int_{B_\rho} \left|F(x + he_i)\right|^p dx \le c(n, p) \int_{B_R} \left|F(x)\right|^p dx.$$

LEMMA 2.3. *If at least one of the functions* F *or* G *has support contained in the set*

$$\left\{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > |h|\right\},$$

then

$$\int_\Omega F \tau_{h,i} G \, dx = -\int_\Omega G \tau_{-h,i} F \, dx.$$

Moreover,

$$\tau_{h,i}(FG)(x) = F(x + he_i)\tau_{h,i} G(x) + G(x)\tau_{h,i} F(x).$$

To shorten the notation, we shall use in the sequel τ_h instead of $\tau_{h,i}$.

To handle with the nonlinear expression of the gradient

$$V(Du) = \left(\mu^2 + |Du|^2\right)^{\frac{p-2}{4}} Du,$$

the inequalities contained in the following lemma will be fundamental.

LEMMA 2.4 ([14, Lemma 2.2]). *For any* $p \ge 2$, *we have*

$$\begin{aligned} &c^{-1}\left(\mu^2 + |\eta|^2 + |\zeta|^2\right)^{\frac{p-2}{2}} |\eta - \zeta|^2 \\ &\qquad \le \left|V(\eta) - V(\zeta)\right|^2 \le c\left(\mu^2 + |\eta|^2 + |\zeta|^2\right)^{\frac{p-2}{2}} |\eta - \zeta|^2 \end{aligned}$$

for any $\eta, \zeta \in \mathbb{R}^{Nn}$ *and a constant* $c = c(p) > 0$.

We also recall the well-known iteration lemma which is essential in the hole filling method.

LEMMA 2.5 ([13, Lemma V.3.1]). *For* $R_0 < R_1$, *consider a bounded function*

$$f : [R_0, R_1] \to [0, \infty)$$

with

$$f(s) \le \vartheta f(t) + \frac{A}{(s - t)^\alpha} + B \quad \textit{for all } R_0 < s < t < R_1$$

where A, B and α denote non-negative constants and $\vartheta \in (0,1)$. Then, we have

$$f(R_0) \le c(\alpha,\vartheta)\Big(\frac{A}{(R_1-R_0)^\alpha} + B\Big).$$

The rest of the present section is dedicated to some definitions and results concerning the functional spaces we shall use.

For $1 < p, q < +\infty$, the *Lorentz space* $L^{p,q}(\Omega)$ consists of all measurable functions f defined on Ω such that

$$\|f\|^q_{p,q} = p\int_0^{+\infty} |\Omega_t|^{\frac{q}{p}} t^{q-1} dt < +\infty$$

with $\Omega_t = \{x \in \Omega : |f(x)| > t\}$, for $t \ge 0$, and $|\Omega_t|$ its Lebesgue measure. For $p = q$, the space $L^{p,q}(\Omega)$ coincides with the Lebesgue space $L^p(\Omega)$.

The class $L^{p,\infty}(\Omega)$, also known as the Marcinkiewicz class weak-$L^p(\Omega)$, consists of all functions f such that

$$|f|^p_{p,\infty} = \sup_{t>0} t^p |\Omega_t| < +\infty.$$

It is a Banach space equipped with the norm

$$\|f\|_{p,\infty} = \sup_{E\subset\Omega} |E|^{\frac{1}{p}-1} \int_E |f|\,dx. \tag{2.1}$$

Since it holds that

$$\frac{(p-1)^p}{p^{p+1}} \|f\|^p_{p,\infty} \le |f|^p_{p,\infty} \le \|f\|^p_{p,\infty}$$

(see [4, Lemma A.2]), we shall use the notation $L^{p,\infty}$ or weak-L^p, with the norm (2.1), indifferently.

Observe that for f belonging to weak-$L^p(\mathbb{R}^n)$ and $g \in L^1(\mathbb{R}^n)$, the convolution $f * g$ belongs to weak-$L^p(\mathbb{R}^n)$ and

$$\|f * g\|_{L^{p,\infty}} \le \|f\|_{L^{p,\infty}} \|g\|_{L^1} \tag{2.2}$$

(see [4, 26]). The distance of a given $f \in L^{p,\infty}$ to L^∞ is defined as

$$\mathrm{dist}_{L^{p,\infty}}(f, L^\infty) = \inf_{g\in L^\infty} \|f - g\|_{p,\infty}.$$

For an exhaustive discussion on the distance to L^∞, we refer to [5]. We remark that assuming that $\mathrm{dist}_{L^{p,\infty}}(f, L^\infty)$ is small does not give any smallness control on the norm in $L^{p,\infty}$ (see [9]).

The next Sobolev Embedding theorem in Lorentz spaces will be useful for us (see [3, 16, 23]).

THEOREM 2.6. *Let us assume $1 < p < n$, $q \geq 1$. Then, every function $u \in W_0^{1,1}(\Omega, \mathbb{R}^N)$ verifying $|Du| \in L^{p,q}$ actually belongs to $L^{p^*,q}$, where $p^* = \frac{np}{n-p}$ and*

$$\|u\|_{p^*,q} \leq S_p \|Du\|_{p,q}, \tag{2.3}$$

where $S_p = \omega_n^{-1/n} \frac{p}{n-p}$.

Hölder's inequality in Lorentz spaces states the following (see [24]).

THEOREM 2.7. *If $0 < p_1, p_2, p < \infty$ and $0 < q_1, q_2, p \leq \infty$ obey $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ and $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$, then*

$$\|fg\|_{L^{p,q}} \leq c\|f\|_{L^{p_1,q_1}} \|g\|_{L^{p_2,q_2}}$$

whenever the right-hand side norms are finite.

We conclude recalling that the fractional Sobolev space $W^{\beta,p}(\Omega, \mathbb{R}^N)$, $\beta \in (0, 1)$, $1 \leq p < \infty$, is made up of measurable functions f such that

$$\|f\|_{W^{\beta,p}} := \|f\|_{L^p} + \left(\int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x - y|^{n+p\beta}} dx\, dy \right)^p$$

is finite.

3. THE A PRIORI ESTIMATE

THEOREM 3.1. *Under the assumptions of Theorem 1.1, assume that $u \in W^{1,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ is a weak solution of (1.6) such that $|D(V(Du))|^2 \in L^1_{\mathrm{loc}}(\Omega)$. Then, there exists a positive constant $\sigma = \sigma(n, p)$ such that if*

$$\mathrm{dist}_{L^{n,\infty}}(k, L^\infty) < \sigma, \tag{3.1}$$

the following estimate holds:

$$\begin{aligned} &\int_{B_R} \left|D\big(V(Du)\big)\right|^2 dx \\ &\qquad \leq C\left(1 + \frac{1}{R^2}\right) \int_{B_{2R}} (\mu^2 + |Du|^2)^{\frac{p}{2}}\, dx + C \int_{B_{2R}} |g|^2\, dx \end{aligned} \tag{3.2}$$

for every ball $B_{2R} \Subset \Omega$ and for a constant $C = C(\mathrm{dist}_{L^{N,\infty}}(k, L^\infty), p, \alpha, n, N, \mu)$.

PROOF. Fix radii $R < s < t < 2R$, consider a cut-off function $\xi \in C_0^\infty(B_t)$, $0 \leq \xi \leq 1$, $\xi \equiv 1$ on B_s, $|\nabla \xi| \leq \frac{c}{t-s}$ and set $\psi = \xi^2 \tau_h u$.

Since u is a weak solution of (1.6), we can choose $\varphi = \tau_{-h}\psi$ as a test function and have by virtue of the properties of the finite difference operator

$$\int_{B_t} \big\langle \tau_h A(x, Du), D\psi \big\rangle\, dx = -\int_{B_t} \langle g, \tau_{-h}\psi\rangle\, dx;$$

that is,

$$\int_{B_t} \big\langle \tau_h A(x, Du), D(\xi^2 \tau_h u)\big\rangle\, dx = -\int_{B_t} \big\langle g, \tau_{-h}(\xi^2\tau_h u)\big\rangle\, dx.$$

It follows that

$$\begin{aligned} &\int_{B_t} \xi^2\big\langle \tau_h A(x, Du), \tau_h Du\big\rangle\, dx + 2\int_{B_t} \xi\big\langle \tau_h A(x, Du), \nabla\xi \otimes \tau_h u\big\rangle\, dx \\ &\quad = -\int_{B_t} \big\langle g, \tau_{-h}(\xi^2\tau_h u)\big\rangle\, dx \end{aligned} \tag{3.3}$$

and observing that

$$\begin{aligned} \tau_h A(x, Du) &= \big[A\big(x + he_i, Du(x + he_i)\big) - A\big(x + he_i, Du(x)\big)\big] \\ &\quad + \big[A\big(x + he_i, Du(x)\big) - A\big(x, Du(x)\big)\big] =: \mathcal{A}_h + \mathcal{A}'_h, \end{aligned}$$

equality (3.3) can be rewritten as

$$\begin{aligned} &\int_{B_t} \xi^2\langle \mathcal{A}_h, \tau_h Du\rangle\, dx \\ &\quad = -\int_{B_t} \xi^2\langle \mathcal{A}'_h, \tau_h Du\rangle\, dx - 2\int_{B_t} \xi\langle \mathcal{A}_h, \nabla\xi \otimes \tau_h u\rangle\, dx \\ &\qquad - 2\int_{B_t} \xi\langle \mathcal{A}'_h, \nabla\xi \otimes \tau_h u\rangle\, dx - \int_{B_t} \big\langle g, \tau_{-h}(\xi^2\tau_h u)\big\rangle\, dx \\ &\quad =: I_1 + I_2 + I_3 + I_4. \end{aligned}$$

By assumption (1.2), we immediately obtain for the left-hand side that

$$\begin{aligned} &\int_{B_t} \xi^2\langle \mathcal{A}_h, \tau_h Du\rangle\, dx \\ &\quad \geq \int_{B_t} \xi^2\big(\mu^2 + \big|Du(x)\big|^2 + \big|Du(x + he_i)\big|^2\big)^{\frac{p-2}{2}} |\tau_h Du|^2\, dx \end{aligned}$$

and hence

$$\begin{aligned} &\int_{B_t} \xi^2\big(\mu^2 + \big|Du(x)\big|^2 + \big|Du(x + he_i)\big|^2\big)^{\frac{p-2}{2}} \frac{|\tau_h Du|^2}{|h|^2}\, dx \\ &\quad \leq \frac{1}{|h|^2}\big(|I_1| + |I_2| + |I_3| + |I_4|\big). \end{aligned} \tag{3.4}$$

In order to estimate $|I_j|$, $j = 1, \dots, 4$, we introduce the notation

$$\mathcal{K}(h) := k(x + he_i) + k(x), \quad \mathcal{D}(h) = \left(\mu^2 + \left|Du(x)\right|^2 + \left|Du(x + he_i)\right|^2\right)^{\frac{1}{2}}.$$

By assumption (1.4), we immediately have

$$\begin{aligned} |I_1| &\le \int_{B_t} \xi^2 |\mathcal{A}'_h| \, |\tau_h Du| \, dx \\ &\le \int_{B_t} \xi^2 |h| \mathcal{K}(h) \left(\mu^2 + |Du|^2\right)^{\frac{p-1}{2}} |\tau_h Du| \, dx. \end{aligned} \tag{3.5}$$

By the definition of the distance $\mathrm{dist}_{L^{n,\infty}}(k, L^\infty)$, for every $\sigma > 0$, there exists $k_0 \in L^\infty(\Omega)$ such that $\|k - k_0\|_{L^{n,\infty}(\Omega)} < \sigma$. Then, define

$$\mathcal{K}_0(h) := k_0(x + he_i) + k_0(x).$$

The use of Young's inequality with a constant $\nu \in (0, 1)$ that will be chosen later yields

$$\begin{aligned} |I_1| &\le \int_{B_t} \xi^2 |h| \big(\mathcal{K}(h) - \mathcal{K}_0(h)\big) \left(\mu^2 + \left|Du(x)\right|^2\right)^{\frac{p-1}{2}} |\tau_h Du| \, dx \\ &\quad + \int_{B_t} \xi^2 |h| \mathcal{K}_0(h) \left(\mu^2 + \left|Du(x)\right|^2\right)^{\frac{p-1}{2}} |\tau_h Du| \, dx \\ &\le \frac{|h|^2}{2\nu} \int_{B_t} \big(\mathcal{K}(h) - \mathcal{K}_0(h)\big)^2 \left(\xi \left(\mu^2 + |Du|^2\right)^{\frac{p}{4}}\right)^2 dx \\ &\quad + \frac{\nu}{2} \int_{B_t} \xi^2 \left(\mu^2 + |Du|^2\right)^{\frac{p-2}{2}} |\tau_h Du|^2 \, dx \\ &\quad + \frac{|h|^2}{2\nu} \|\mathcal{K}_0(h)\|_\infty^2 \int_{B_t} \left(\xi \left(\mu^2 + |Du|^2\right)^{\frac{p}{4}}\right)^2 dx \\ &\quad + \frac{\nu}{2} \int_{B_t} \xi^2 \left(\mu^2 + |Du|^2\right)^{\frac{p-2}{2}} |\tau_h Du|^2 \, dx \\ &\le \frac{|h|^2}{2\nu} \int_{B_t} 2\big(k(x) - k_0(x)\big)^2 \left(\xi \left(\mu^2 + |Du|^2\right)^{\frac{p}{4}}\right)^2 dx \\ &\quad + \frac{|h|^2}{2\nu} \int_{B_t} 2\big(k(x + he_i) - k_0(x + he_i)\big)^2 \left(\xi \left(\mu^2 + |Du|^2\right)^{\frac{p}{4}}\right)^2 dx \\ &\quad + \nu \int_{B_t} \xi^2 \left(\mu^2 + |Du|^2\right)^{\frac{p-2}{2}} |\tau_h Du|^2 \, dx \\ &\quad + \frac{|h|^2}{2\nu} \|\mathcal{K}_0(h)\|_\infty^2 \int_{B_t} \left(\xi \left(\mu^2 + |Du|^2\right)^{\frac{p}{4}}\right)^2 dx. \end{aligned}$$

Now, note that, thanks to Lemma 2.4, the assumption $V(Du) \in W^{1,2}_{\mathrm{loc}}(\Omega)$ guarantees that

$$\left(\mu^2 + |Du|^2\right)^{\frac{p}{2}} \in W^{1,2}_{\mathrm{loc}}(\Omega)$$

and hence, by Sobolev embedding at (2.3), that

$$\xi\left(\mu^2 + |Du|^2\right)^{\frac{p}{2}} \in L^{\frac{n}{n-2},1}.$$

Consequently, we can estimate the first and the second integrals in the right-hand side of the previous inequality as follows:

$$\begin{aligned}
|I_1| &\le 2\frac{|h|^2}{\nu}\left\|(k-k_0)^2\right\|_{\frac{n}{2},\infty}\left\|\xi\left(\mu^2+|Du|^2\right)^{\frac{p}{4}}\right\|^2_{\frac{2n}{n-2},2} \\
&\quad + \nu\int_{B_t} \xi^2\left(\mu^2+|Du|^2\right)^{\frac{p-2}{2}}|\tau_h Du|^2 \\
&\quad + \frac{|h|^2}{2\nu}\|\mathcal{K}_0(h)\|^2_\infty \int_{B_t}\left(\xi\left(\mu^2+|Du|^2\right)^{\frac{p}{4}}\right)^2 dx \\
&\le 2\sigma^2\frac{|h|^2}{\nu}S_2^2\left\|D\left[\xi\left(\mu^2+|Du|^2\right)^{\frac{p}{4}}\right]\right\|_2^2 + \nu\int_{B_t}\xi^2\mathcal{D}(h)^{p-2}|\tau_h Du|^2\,dx \\
&\quad + |h|^2 c\left(\nu,\|k_0\|_\infty\right)\int_{B_t}\xi^2\left(\mu^2+|Du|^2\right)^{\frac{p}{2}}dx.
\end{aligned}$$

Now, let us estimate $|I_2|$. Observe that assumption (1.3) yields

$$\begin{aligned}
|\mathcal{A}_h| &\le \alpha\left(\mu^2 + \left|Du(x)\right|^2 + \left|Du(x+he_i)\right|^2\right)^{\frac{p-2}{2}}|\tau_h Du| \\
&= \alpha\mathcal{D}(h)^{p-2}|\tau_h Du|,
\end{aligned}$$

and hence, by the aid of Young's and Hölder's inequalities, we obtain

$$\begin{aligned}
|I_2| &\le 2\alpha\int_{B_t}\xi\mathcal{D}(h)^{p-2}|\tau_h Du|\cdot|\nabla\xi|\cdot|\tau_h u|\,dx \\
&\le \nu\int_{B_t}\xi^2\mathcal{D}(h)^{p-2}|\tau_h Du|^2\,dx \\
&\qquad + c(\alpha,\nu)\|\nabla\xi\|^2_\infty\int_{B_t}\mathcal{D}(h)^{p-2}|\tau_h u|^2\,dx.
\end{aligned}$$

For I_3, we proceed as follows.
The assumption (1.4) and the properties of ξ yield

$$\begin{aligned}
|I_3| &\le 2\int_{B_t}\xi|\mathcal{A}'_h|\,|\nabla\xi|\,|\tau_h u|\,dx \\
&\le c\|\nabla\xi\|_\infty\int_{B_t}\xi|h|\mathcal{K}(h)\left(\mu^2+|Du|^2\right)^{\frac{p-1}{2}}|\tau_h u|\,dx.
\end{aligned}$$

Reasoning as we have done for the integral in the right-hand side of (3.5), we have

$$
\begin{aligned}
|I_3| &\le c\|\nabla\xi\|_\infty \int_{B_t} \xi|h|\big(\mathcal{K}(h)-\mathcal{K}_0(h)\big)\big(\mu^2+|Du|^2\big)^{\frac{p-1}{2}}|\tau_h u|\,dx \\
&\quad + c\|\nabla\xi\|_\infty \int_{B_t} \xi|h|\mathcal{K}_0(h)\big(\mu^2+\big|Du(x)\big|^2\big)^{\frac{p-1}{2}}|\tau_h u|\,dx \\
&\le \frac{1}{2\nu}\int_{B_t} \xi^2|h|^2\big(\mathcal{K}(h)-\mathcal{K}_0(h)\big)^2\big(\mu^2+|Du|^2\big)^{\frac{p}{2}}\,dx \\
&\quad + \frac{1}{2\nu}|h|^2\big\|\mathcal{K}_0(h)\big\|_\infty^2 \int_{B_t} \xi^2\big(\mu^2+|Du|^2\big)^{\frac{p}{2}}\,dx \\
&\quad + c(\nu)\|\nabla\xi\|_\infty^2 \int_{B_t} \big(\mu^2+|Du|^2\big)^{\frac{p-2}{2}}|\tau_h u|^2\,dx \\
&\le 2\sigma^2\frac{|h|^2}{\nu}S_2^2\big\|D\big[\xi\big(\mu^2+|Du|^2\big)^{\frac{p}{4}}\big]\big\|_2^2 \\
&\quad + c(\nu)\|\nabla\xi\|_\infty^2 \int_{B_t} \big(\mu^2+|Du|^2\big)^{\frac{p-2}{2}}|\tau_h u|^2\,dx \\
&\quad + c\big(\nu,\|k_0\|_\infty\big)|h|^2 \int_{B_t} \xi^2\big(\mu^2+|Du|^2\big)^{\frac{p}{2}}\,dx.
\end{aligned}
$$

Finally, since we obviously have

$$
I_4 = -\int_{B_t} \big\langle g, \tau_{-h}(\xi^2\tau_h u)\big\rangle\,dx = -h\int_{B_t} \left\langle g, \frac{\tau_{-h}(\xi^2\tau_h u)}{h}\right\rangle dx,
$$

by using Young's inequality and Lemma 2.2, we get

$$
\begin{aligned}
|I_4| &\le c(\nu,\mu)|h|^2\int_{B_t}|g|^2\,dx + \nu\mu^{p-2}\int_{B_t}\left|\frac{\tau_{-h}(\xi^2\tau_h u)}{h}\right|^2 dx \\
&\le c(\nu,\mu)|h|^2\int_{B_t}|g|^2\,dx + \nu\mu^{p-2}\int_{B_t}\big|D(\xi^2\tau_h u)\big|^2\,dx \\
&\le c(\nu,\mu)|h|^2\int_{B_t}|g|^2\,dx + 4\nu\mu^{p-2}\int_{B_t}\xi^2|\nabla\xi|^2|\tau_h u|^2\,dx \\
&\quad + \nu\mu^{p-2}\int_{B_t}\xi^4\big|D(\tau_h u)\big|^2\,dx.
\end{aligned}
$$

Combining (3.4) with the estimates of $|I_j|$, $j=1,\dots,4$, above, we obtain

$$
\begin{aligned}
\int_{B_t}\xi^2\mathcal{D}(h)^{p-2}\frac{|\tau_h Du|^2}{|h|^2}\,dx &\le \frac{4\sigma^2}{\nu}S_2^2\big\|D\big[\xi\big(\mu^2+|Du|^2\big)^{\frac{p}{4}}\big]\big\|_2^2 \\
&\quad + 2\nu\int_{B_t}\xi^2\mathcal{D}(h)^{p-2}\frac{|\tau_h Du|^2}{|h|^2}\,dx + c\big(\nu,\|k_0\|_\infty\big)\int_{B_t}\xi^2\big(\mu^2+|Du|^2\big)^{\frac{p}{2}}\,dx
\end{aligned}
$$

$$
\begin{aligned}
&+ c(\alpha,\nu)\|\nabla\xi\|_\infty^2 \int_{B_t} \mathcal{D}(h)^{p-2}\frac{|\tau_h u|^2}{|h|^2}\,dx \\
&+ c(\nu,\mu)\int_{B_t} |g|^2\,dx + 4\nu\mu^{p-2}\int_{B_t} \xi|\nabla\xi|^2\left|\frac{\tau_h u}{h}\right|^2 dx \\
&+ \nu\int_{B_t} \xi^2\mathcal{D}(h)^{p-2}\frac{\left|D(\tau_h u)\right|^2}{|h|^2}\,dx
\end{aligned}
$$

and therefore, choosing $\nu = \frac{1}{4}$,

$$
\begin{aligned}
&\frac{1}{4}\int_{B_s} \mathcal{D}(h)^{p-2}\frac{|\tau_h Du|^2}{|h|^2}\,dx \\
&\quad\le 16\sigma^2 S_2^2\left\|D\left[\xi\left(\mu^2+|Du|^2\right)^{\frac{p}{4}}\right]\right\|_2^2 \\
&\qquad + c\int_{B_t} \xi^2\left(\mu^2+|Du|^2\right)^{\frac{p}{2}}dx + c\|\nabla\xi\|_\infty^2\int_{B_t}\mathcal{D}(h)^{p-2}\frac{|\tau_h u|^2}{|h|^2}\,dx \\
&\qquad + c\int_{B_t} |g|^2\,dx + c\int_{B_t}\xi|\nabla\xi|^2\left|\frac{\tau_h u}{h}\right|^2 dx.
\end{aligned}
$$

By Lemma 2.2, we are legitimate to pass to the limit for $h \to 0$ having

$$
\begin{aligned}
\int_{B_s} \left|D\left(V(Du)\right)\right|^2 dx \le 64p^2\sigma^2 S_2^2 &\int_{B_t}\left|D\left(V(Du)\right)\right|^2 dx \\
&+ \frac{c}{(t-s)^2}\int_{B_t}\left(\mu^2+|Du|^2\right)^{\frac{p}{2}}dx \\
&+ c\int_{B_t}\left(\mu^2+|Du|^2\right)^{\frac{p}{2}}dx + c\int_{B_t}\left|g(x)\right|^2 dx
\end{aligned}
$$

where we also used the properties of the function ξ. For $\sigma < \frac{1}{8pS_2}$, we can apply Lemma 2.5 having

$$
\begin{aligned}
&\int_{B_R} \left|D\left(V(Du)\right)\right|^2 dx \\
&\quad\le C\left(1+\frac{1}{R^2}\right)\int_{B_{2R}}\left(\mu^2+|Du|^2\right)^{\frac{p}{2}}dx + C\int_{B_{2R}}|g|^2\,dx
\end{aligned}
$$

that is the conclusion. ∎

REMARK 3.2. Note that even if we do not provide the precise value of the constant σ in (3.1), a bound on it is given at the end of the proof of Theorem 3.1.

REMARK 3.3. We point out that the dependence of the constant C appearing in (3.2) on the $\mathrm{dist}_{L^{N,\infty}}(k, L^\infty)$ occurs only through the norm of k_0 in L^∞.

4. The main results

In this section, we shall prove Theorems 1.1 and 1.3 as well as higher integrability results for the gradient of the solutions we deduce from them. In particular, the proofs of the main theorems will consist in constructing approximating regular problems on a ball $B_{2R} \Subset \Omega$ whose solutions u_ε, agreeing to the solution u to (1.6) (respectively to (1.9)) on ∂B_{2R}, verify estimate (1.8) (respectively (1.10)). The conclusion will follow showing that the estimates will be preserved passing to the limit.

Proof of Theorem 1.1. Let us show that there exists a sequence of approximating problems for which we are allowed to use the a priori estimate (3.2).

Set $A(x,\eta) = 0$ for any $x \in \mathbb{R}^n \setminus \Omega$ and then consider a standard mollifier $\rho : \mathbb{R}^n \to \mathbb{R}$ with compact support in $B_1 \subset \mathbb{R}^n$. For $0 < \varepsilon < \min\{R, 1\}$, consider

$$A_\varepsilon(x,\eta) := \int_{B_1} A(x+\varepsilon y,\eta)\rho(y)\,dy, \quad k_\varepsilon(x) := \int_{B_1} k(x+\varepsilon y)\rho(y)\,dy,$$

for any $x \in \overline{B}_{2R-\varepsilon}$ and $\eta \in \mathbb{R}^{Nn}$. Using assumptions (1.2)–(1.5), one can easily check that

$$\langle A_\varepsilon(x,\eta) - A_\varepsilon(x,\zeta), \eta-\zeta\rangle \geq \left(\mu^2 + |\eta|^2 + |\zeta|^2\right)^{\frac{p-2}{2}} |\xi - \eta|^2, \tag{4.1}$$

$$\left|A_\varepsilon(x,\eta) - A_\varepsilon(x,\zeta)\right| \leq \alpha|\eta-\zeta|\left(\mu^2 + |\eta|^2 + |\zeta|^2\right)^{\frac{p-2}{2}}, \tag{4.2}$$

$$\left|A_\varepsilon(x,\eta) - A_\varepsilon(y,\eta)\right| \leq |x-y|\left[k_\varepsilon(x) + k_\varepsilon(y)\right]\left(\mu^2 + |\eta|^2\right)^{\frac{p-1}{2}}, \tag{4.3}$$

$$A_\varepsilon(x,0) = 0 \tag{4.4}$$

for all $x \in \Omega$ and all $\eta, \zeta \in \mathbb{R}^{Nn}$.

Consider the unique solution $u_\varepsilon \in W^{1,p}(\Omega,\mathbb{R}^N)$ of the problem

$$\begin{cases} -\operatorname{div} A_\varepsilon(x, Du_\varepsilon) = g(x) & \text{in } B_{2R}, \\ u_\varepsilon = u & \text{on } \partial B_{2R}, \end{cases} \tag{4.5}$$

and observe that, reasoning as we have done in the proof of Theorem 3.1, it can be proven that $D(V(|Du_\varepsilon|))^2 \in L^1_{\mathrm{loc}}(B_{2R})$. Moreover, let $\sigma > 0$ be the constant such that (3.1) holds, and notice that (2.2) guarantees that

$$\operatorname{dist}_{L^{n,\infty}}(k_\varepsilon, L^\infty) < \sigma.$$

Indeed, let $k_0 \in L^\infty$ be such that $\|k - k_0\|_{n,\infty} < \sigma$, and then observe that

$$\begin{aligned} \|k_\varepsilon - k_0\|_{n,\infty} &\leq \left\|k_\varepsilon - (k_0)_\varepsilon\right\|_{n,\infty} + \left\|(k_0)_\varepsilon - k_0\right\|_{n,\infty} \\ &\leq \|k - k_0\|_{n,\infty} + \left\|(k_0)_\varepsilon - k_0\right\|_{n,\infty} \end{aligned} \tag{4.6}$$

and that the second term in the right-hand side is negligible for ε sufficiently small.

Hence, we are legitimate to apply the a priori estimate in Theorem 3.1 to u_ε; that is,

$$\begin{aligned}\int_{B_R} \left|D\big(V(Du_\varepsilon)\big)\right|^2 dx \\ \le C\left(1+\frac{1}{R^2}\right)\int_{B_{2R}} (\mu^2+|Du_\varepsilon|^2)^{\frac{p}{2}}\,dx + c\int_{B_{2R}} |g|^2\,dx,\end{aligned} \tag{4.7}$$

for constants $C = C(\mathrm{dist}_{L^{n,\infty}}(k, L^\infty), p, \alpha, n, N)$ and $c = c(p,\mu)$. Notice that Remark 3.3 and inequality (4.6), make evident that the dependence of C on $\mathrm{dist}_{L^{n,\infty}}(k_\varepsilon, L^\infty)$ is actually uniform with respect to ε and hence can be expressed as a dependence on $\mathrm{dist}_{L^{n,\infty}}(k, L^\infty)$. In order to prove that u satisfies the same inequality, let us use $\varphi = u - u_\varepsilon$ as a test function in (1.1) and (4.5) getting

$$\int_{B_{2R}} \langle A(x,Du) - A_\varepsilon(x,Du_\varepsilon), Du - Du_\varepsilon\rangle\,dx = 0$$

and then

$$\begin{aligned}\int_{B_{2R}} &\langle A_\varepsilon(x,Du) - A_\varepsilon(x,Du_\varepsilon), Du - Du_\varepsilon\rangle\,dx \\ &= \int_{B_{2R}} \langle A_\varepsilon(x,Du) - A(x,Du), Du - Du_\varepsilon\rangle\,dx.\end{aligned}$$

It follows that

$$\begin{aligned}&\int_{B_{2R}} |Du - Du_\varepsilon|^p\,dx \\ &\le c\int_{B_{2R}} (\mu^2 + |Du|^2 + |Du_\varepsilon|^2)^{\frac{p-2}{2}} |Du - Du_\varepsilon|,dx \\ &\le c\int_{B_{2R}} \langle A_\varepsilon(x,Du) - A(x,Du), Du - Du_\varepsilon\rangle\,dx \\ &\le c\left(\int_{B_{2R}} \left|A_\varepsilon(x,Du) - A(x,Du)\right|^{p'} dx\right)^{\frac{1}{p'}} \left(\int_{B_{2R}} |Du - Du_\varepsilon|^p\,dx\right)^{\frac{1}{p}}\end{aligned}$$

so that

$$\left[\left(\int_{B_{2R}} |Du - Du_\varepsilon|^p\,dx\right)^{\frac{1}{p'}}\right]^{p'} \le \int_{B_{2R}} \left|A_\varepsilon(x,Du) - A(x,Du)\right|^{p'} dx.$$

Assumptions (1.3) and (1.5), together with conditions (4.2) and (4.4) and the convergence $A_\varepsilon(x,Du) \to A(x,Du)$ a.e., imply that we can use the dominated convergence theorem to obtain that $Du_\varepsilon \to Du$ strongly in L^p. At this point, estimates (4.7) yield $\|D(V(Du_\varepsilon))\|_{L^2(B_R)} \le C$, so that we deduce that, up to a subsequence, $D(V(Du_\varepsilon))$

is weakly converging to $D(V(Du))$ in $L^2(B_R)$. Therefore, we can pass to the limit for $\varepsilon \to 0$ in the estimate (4.7) having the validity of the desired inequality for the function u. ■

For $2 < p < n$, the following corollaries of fractional higher integrability easily derive from Theorem 1.1.

COROLLARY 4.1. *Let $u \in W^{1,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ be a weak solution of (1.6), under the assumptions (1.2)–(1.5) with $k \in L^{n,\infty}(\Omega)$. Further assume $g \in L^2(\Omega, \mathbb{R}^N)$. There exists a positive constant $\sigma = \sigma(n, p)$ such that if*

$$\operatorname{dist}_{L^{n,\infty}}(k, L^\infty) < \sigma,$$

then $Du \in W^{\beta,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ for every $\beta \in (0, \frac{2}{p})$.

PROOF. Since we can estimate for every $i \in \{1, \dots, n\}$

$$\begin{aligned} |\tau_{h,i} Du|^p &\le c(n,p)\big(\mu^2 + \big|Du(x)\big|^2 + \big|Du(x+he_i)\big|^2\big)^{\frac{p-2}{2}} |\tau_{h,i} Du|^2 \\ &\le c(n,p)\big|\tau_{h,i} V(Du)\big|^2, \end{aligned} \tag{4.8}$$

summing up on $i \in \{1, \dots, n\}$ and taking into account either Lemma 2.2 or the estimate given by Theorem 1.1, we get for $\rho \in (0, R)$ and h sufficiently small

$$\begin{aligned} &\int_{B_\rho} \sum_{i=1}^n |\tau_{h,i} Du|^p \, dx \\ &\quad\le c(n,p)|h|^2 \int_{B_R} \big|D\big(V(Du)\big)\big|^2 \, dx \\ &\quad\le C \cdot \big(|h|^{\frac{2}{p}}\big)^p \left[\left(1 + \frac{1}{R^2}\right) \int_{B_{2R}} (\mu^2 + |Du|^2)^{\frac{p}{2}} \, dx + c \int_{B_{2R}} |g|^2 \, dx\right]. \end{aligned}$$

It follows that Du belongs to the Nikolskii space $\mathcal{H}^{\frac{2}{p},p}$ and hence the conclusion by embedding (see [2, Section 7.73] and also [18]). ■

In the next corollary, we show that assuming a higher integrability of the function g improves the integrability of the fractional derivatives.

COROLLARY 4.2. *Let $u \in W^{1,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ be a weak solution of (1.6), under the assumptions (1.2)–(1.5) with $k \in L^{n,\infty}(\Omega)$. Further assume $g \in L^r(\Omega, \mathbb{R}^N)$, for some $r > 2$. There exists a positive constant $\sigma = \sigma(n, p)$ such that if*

$$\operatorname{dist}_{L^{n,\infty}}(k, L^\infty) < \sigma,$$

then $Du \in W^{\beta,q}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ for some $q > p$ and for every $\beta \in (0, \frac{2}{p})$.

Proof. Without loss of generality, we assume $0 < R < 1$. The estimate given by Theorem 1.1 and the use of Lemma 2.4 yield

$$\begin{aligned} &\fint_{B_R} |DV(Du)|^2 dx \\ &\quad \le c\Big(1+\frac{1}{R^2}\Big)\Big(\fint_{B_{2R}} \big|V(Du)-\big(V(Du)\big)_{B_{2R}}\big|^2 dx + \fint_{B_{2R}} |g|^2 dx\Big). \end{aligned}$$

Hence, applying Sobolev–Poincaré inequality, we have the following reverse Hölder's inequality:

$$\fint_{B_R} |DV(Du)|^2 dx \le c\Big[\Big(\fint_{B_{2R}} \big(|DV(Du)|^2\big)^{\frac{n}{n+2}} dx\Big)^{\frac{n+2}{n}} + \fint_{B_{2R}} |g|^2 dx\Big]$$

getting the existence of an exponent $s > 2$ such that $|DV(Du)| \in L^s_{\mathrm{loc}}$ and

$$\fint_{B_{R/2}} |DV(Du)|^s dx \le c\Big[\Big(\fint_{B_{R/2}} |DV(Du)|^2 dx\Big)^{\frac{s}{2}} + \fint_{B_{2R}} |g|^s dx\Big].$$

Then, using the pointwise inequality in (4.8), we easily obtain that

$$\frac{\|\tau_h Du\|_{\frac{ps}{2}}}{|h|^s} \le c\,\|DV(Du)\|_s^{\frac{2}{p}}$$

which allows us to conclude that Du belongs to the Nikolskii space $\mathcal{H}^{\frac{2}{p},\frac{ps}{2}}$ and hence, setting $q := \frac{ps}{2}$, by embedding $Du \in W^{\beta,q}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$ for every $\beta \in (0,\frac{2}{p})$. ∎

Proof of Theorem 1.3. By a careful analysis of the proofs of Theorems 3.1 and 1.1, it is evident that the degenerate case, that is, for $\mu = 0$, causes further difficulties only when dealing with the integral involving the datum g. More specifically, in the estimate of $|I_4|$, as well as in the corresponding estimate in the approximation part, an integral which can blow up appears. Then, the proof can proceed according to the ones of previous theorems since, for $g \equiv 0$, the term I_4 is not present. ∎

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Flavia Giannetti
Dipartimento di Matematica e Applicazioni "R. Caccioppoli", Università di Napoli "Federico II"
Via Cintia, 80126 Napoli, Italy;
giannett@unina.it

Gioconda Moscariello
Dipartimento di Matematica e Applicazioni "R. Caccioppoli", Università di Napoli "Federico II"
Via Cintia, 80126 Napoli, Italy;
gmoscari@unina.it