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Partial Differential Equations. – *Existence of two non-zero weak solutions for a $p(\cdot)$-biharmonic problem with Navier boundary conditions*, by Gabriele Bonanno, Antonia Chinnì and Vicenţiu D. Rădulescu, communicated on 23 June 2023.

Abstract. – In this paper, the existence of non-trivial weak solutions for some problems with Navier boundary conditions driven by the $p(\cdot)$-biharmonic operator is investigated. The proofs combine variational methods with topological arguments.

Keywords. – $p(\cdot)$-biharmonic-type operators, Navier boundary value problem, variational methods.

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1. Introduction

The paper is devoted to study of a class of elliptic problems driven by $p(\cdot)$-biharmonic operator. In particular, we deal with the existence and multiplicity of solutions for the problem

$$(P_\lambda) \qquad \begin{cases} \Delta(|\Delta u|^{p(x)-2}\Delta u) = \lambda f(x, u(x)) \text{ in } \Omega, \\ u = \Delta u = 0 \text{ in } \partial\Omega, \end{cases}$$

where Ω is an open bounded subset of $\mathbb{R}^N$ $(N \geq 1)$ with smooth boundary $\partial\Omega$, $p \in C(\overline{\Omega})$ with

$$\max\left\{1, \frac{N}{2}\right\} < p^- : \min_{x\in\overline{\Omega}} p(x) \leq p^+ : \max_{x\in\overline{\Omega}} p(x) < +\infty, \tag{1.1}$$

$\Delta^2_{p(x)} u := \Delta(|\Delta u|^{p(x)-2}\Delta u)$ is the operator which is often called $p(\cdot)$-biharmonic, $f \in C^0(\overline{\Omega} \times \mathbb{R})$, and λ is a positive parameter.

Due to the simultaneous involvement of the variable exponent $p(\cdot)$ and the biharmonic operator, problems as (P_λ) are of interest to several fields of application of the study of elliptic problems.

The presence of variable exponent allows to frame the problem within the modeling of various physical phenomena such as flows of electrorheological fluids or fluids with temperature-dependent viscosity and nonlinear viscoelasticity; even filtration processes through a porous media and image processing give rise to equations with nonstandard growth conditions, that is, equations with variable exponents of nonlinearities (see [11, 14, 36] for more details).

On the other hand, the presence of biharmonic operator allows the problem to be framed in the study of fourth-order differential equations that arise from the study of beam deflection problems on nonlinear elastic foundation, first dealt by Nečas and Kratochvíl in [31].

In the literature, there are several papers in which existence and multiplicity of solutions related to problems involving the $p(\cdot)$-biharmonic operator has been investigated. Below we list some of the most recent publications in which these issues have been addressed:

- nonlocal elliptic problem involving $p(\cdot)$-biharmonic operator with Navier boundary conditions (see for instance [1, 12, 13, 19, 21, 24, 28, 39]);
- $(p(\cdot), q(\cdot))$-biharmonic systems (see for instance [4]);
- elliptic problems involving $p(\cdot)$-biharmonic operator with different boundary conditions (see for instance [2,3,10,13,15,16,18,20,22,23,25,27,29,32,37,38,40,41,43]).

Many of the results are obtained through variational methods by applying mountain pass theorem, Krasnosel'skii genus theory and critical point theorems established by Bonanno–Marano [9] and Ricceri [35] (see also [5, 6]).

In this paper, we prove the existence of at least two non-zero weak solutions for problem (P_λ) assuming that the nonlinear term f verifies (AR)-condition and its antiderivative has a suitable growth (see Theorem 3.1). This result will be extended to the more general problem

$$(P_{\lambda,\mu}) \qquad \begin{cases} \Delta\big(|\Delta u|^{p(x)-2}\Delta u\big) = \lambda f\big(x, u(x)\big) + \mu g\big(x, u(x)\big) \text{ in } \Omega, \\ u = \Delta u = 0 \text{ in } \partial\Omega \end{cases}$$

(see Theorem 3.2) and, by way of application, we present a consequence of obtained results (see Theorem 3.3) with an example. It is opportune to precise that the results presented are a generalization of those ones obtained in [7] when exponent p is assumed constant.

The abstract result we will use is contained in [8] and concerns the existence of at least two non-trivial critical points for an appropriate functional.

2. Preliminaries

In order to introduce the space in which solutions of problem (P_λ) are defined, it is necessary to recall some definitions concerning the variable exponent spaces. We refer to the monograph by Rădulescu and Repovš [34] (see also [30]) for more details.

With $p \in C(\overline{\Omega})$ such that

$$1 < p^- =: \min_{x\in\overline{\Omega}} p(x) \le p^+ =: \max_{x\in\overline{\Omega}} p(x) < +\infty, \tag{2.1}$$

the variable exponent Lebesgue space $L^{p(x)}(\Omega)$ is defined as

$$L^{p(x)}(\Omega) := \left\{u : \Omega \to \mathbb{R} : \text{measurable and } \rho_{p(x)}(u) := \int_\Omega |u(x)|^{p(x)}\, dx < +\infty\right\}$$

and

$$\|u\|_{L^{p(x)}(\Omega)} := \inf\left\{\delta > 0 : \int_\Omega \left|\frac{u(x)}{\delta}\right|^{p(x)} dx \le 1\right\} \tag{2.2}$$

defines a norm on it. The function $\rho_{p(x)}$ is called "modular" and it is in close relation with the norm (2.2) as pointed out by Fan and Zhao in [17, Theorem 1.3].

Proposition 2.1. *Let $u \in L^{p(x)}(\Omega)$; then*

(1) $\|u\|_{L^{p(x)}(\Omega)} < 1\ (= 1;\ > 1) \iff \rho_{p(x)}(u) < 1\ (= 1;\ > 1)$;

(2) *if* $\|u\|_{L^{p(x)}(\Omega)} > 1$, *then* $\|u\|^{p^-}_{L^{p(x)}(\Omega)} \le \rho_{p(x)}(u) \le \|u\|^{p^+}_{L^{p(x)}(\Omega)}$;

(3) *if* $\|u\|_{L^{p(x)}(\Omega)} < 1$, *then* $\|u\|^{p^+}_{L^{p(x)}(\Omega)} \le \rho_{p(x)}(u) \le \|u\|^{p^-}_{L^{p(x)}(\Omega)}$.

For $m \in \mathbb{N}$, we introduce the variable exponent Sobolev space $W^{m,p(x)}(\Omega)$ defined as

$$W^{m,p(x)}(\Omega) := \left\{u \in L^{p(x)}(\Omega) : D^\alpha u \in L^{p(x)}(\Omega),\ \forall |\alpha| \le m\right\}$$

and relative norm

$$\|u\|_{m,p(x)} := \sum_{|\alpha|\le m} \|D^\alpha u\|_{L^{p(x)}(\Omega)}$$

with $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_N)$ multi-index of $\mathbb{R}^N$,

$$|\alpha| = \sum_{N}^{i=1} \alpha_i \quad \text{and} \quad D^\alpha = D^{\alpha_1}_{x_1} D^{\alpha_2}_{x_2} \cdots D^{\alpha_N}_{x_N}.$$

Condition $p^- > 1$ in (2.1) ensures that $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$ are separable and reflexive Banach spaces for each $m \in \mathbb{N}$ (see for instance [17]).

We put $X := W^{2,p(x)}(\Omega) \cap W^{1,p(x)}_0(\Omega)$, where $W^{1,p(x)}_0(\Omega)$ is the closure of $C^\infty_0(\Omega)$ in $W^{1,p(x)}(\Omega)$. As proved by Zang and Fu in [42], a norm on X equivalent to the standard one $\|\cdot\|_{2,p(x)}$ is the following:

$$\|u\| := \|\Delta u\|_{L^{p(x)}(\Omega)}$$

for each $u \in X$

By standard results on variable exponent Sobolev spaces (see for example [26, Theorem 3.1]) we know that the embedding

$$X \hookrightarrow W^{2,p^-}(\Omega) \cap W_0^{1,p^-}(\Omega)$$

is continuous. Moreover, by extension of Rellich–Kondrachov theorem to spaces $W^{m,p}(\Omega)$, condition $p^- > \frac{N}{2}$ in (1.1) ensures that $W^{2,p^-}(\Omega)$ is compactly embedded in $C^0(\overline{\Omega})$ and so the embedding $X \hookrightarrow C^0(\overline{\Omega})$ is compact. In particular, there exists $k > 0$ such that

$$\|u\|_\infty \leq k\|u\|$$

for each $u \in X$.

In the sequel, for $\alpha > 0$ and $q \in C(\overline{\Omega})$ with $q^- > 1$, we put

$$[\alpha]^q := \max\{\alpha^{q^-}, \alpha^{q^+}\},$$
$$[\alpha]_q := \min\{\alpha^{q^-}, \alpha^{q^+}\}.$$

It is easy to verify that

(i) $[\alpha]^{\frac{1}{q}} = \max\{\alpha^{\frac{1}{q^-}}, \alpha^{\frac{1}{q^+}}\}$,

(ii) $[\alpha]_{\frac{1}{q}} = \min\{\alpha^{\frac{1}{q^-}}, \alpha^{\frac{1}{q^+}}\}$,

(iii) $[\alpha]_{\frac{1}{q}} = a \iff \alpha = [a]^q$, $[\alpha]^{\frac{1}{q}} = a \iff \alpha = [a]_q$,

(iv) $[\alpha]_q[\beta]_q \leq [\alpha\beta]_q \leq [\alpha\beta]^q \leq [\alpha]^q[\beta]^q$.

Following what was done in several papers, we denote by D and x_0, respectively, the radius and the center of the greatest ball contained in Ω; i.e.

$$D := \sup_{x\in\Omega} \sup\left\{r > 0 : B(x,r) \subseteq \Omega\right\}$$

and $B(x_0, D) \subseteq \Omega$.

Put $h(t) := t^2(t-D)^2$ for each $t \in \mathbb{R}$, and fixed $\delta > 0$; we denote by v_δ the function

$$v_\delta(x) = \begin{cases} 0 & x \in \Omega \setminus B(x^0, D), \\ \frac{\delta}{D^4} 16h\big(|x - x^0|\big) & x \in B(x^0, D) \setminus B\big(x^0, \frac{D}{2}\big), \\ \delta & x \in B\big(x^0, \frac{D}{2}\big). \end{cases}$$

Clearly, $v_\delta \in X$ for each $\delta > 0$ and Figure 1 shows the trend of the function v_δ for $N = 2$.

The following proposition provides the estimate of $\rho_{p(x)}(v_\delta)$ which will play an important role in what we will say.

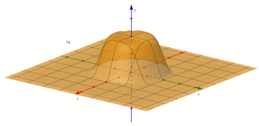

FIGURE 1. Example of v_δ for $N = 2$.

PROPOSITION 2.2. *For each $\delta > 0$, it results that*

$$\rho_{p(x)}(v_\delta) \leq [\delta]^p l_D,$$

where

$$l_D := \left[\frac{32}{D^2}\frac{(N+5)^2}{8(N+2)}\right]^p m\left(D^N - \left(\frac{D}{2}\right)^N\right)$$

and m denotes the measure of unit ball of $\mathbb{R}^N$.

PROOF. By standard arguments, for each $x \in \Omega$ and $i \in \{1, 2, \ldots, N\}$, it results that

$$\frac{\partial v_\delta}{\partial x_i}(x) = \begin{cases} 0 & x \in \Omega \setminus B(x^0, D), \\ \frac{\delta}{D^4} 16 h'\left(|x - x^0|\right) \frac{x_i - x_i^0}{|x - x^0|} & x \in B(x^0, D) \setminus B\left(x^0, \frac{D}{2}\right), \\ 0 & x \in B\left(x^0, \frac{D}{2}\right) \end{cases}$$

and

$$\frac{\partial^2 v_\delta}{\partial x_i^2}(x) = \begin{cases} 0 & x \in \Omega \setminus B(x^0, D), \\ \frac{\delta}{D^4} 16\Big[h''\left(|x - x^0|\right)\frac{(x_i - x_i^0)^2}{|x - x^0|^2} + h'\left(|x - x^0|\right)\frac{|x - x^0|^2 - (x_i - x_i^0)^2}{|x - x^0|^3}\Big], & \\ & x \in B(x^0, D) \setminus B\left(x^0, \frac{D}{2}\right), \\ 0 & x \in B\left(x^0, \frac{D}{2}\right). \end{cases}$$

Therefore, one has

$$\Delta v_\delta(x) = \begin{cases} 0 & x \in \Omega \setminus B(x^0, D), \\ \delta \frac{32}{D^4}\left[2(N+2)|x - x^0|^2 - 3D(N+1)|x - x^0| + ND^2\right], & \\ & x \in B(x^0, D) \setminus B\left(x^0, \frac{D}{2}\right), \\ 0 & x \in B\left(x^0, \frac{D}{2}\right). \end{cases}$$

In order to estimate $\rho_{p(x)}(v_\delta)$, we consider the function

$$K(t) := 2(N+2)t^2 - 3D(N+1)t + ND^2$$

and we observe that

$$-\frac{D^2}{2} = K\left(\frac{D}{2}\right) < 0 < K(D) = D^2.$$

Moreover, arguing as in [7], we obtain that

$$\begin{aligned}
\rho(\Delta v_\delta) &= \int_\Omega \left|\Delta v_\delta(x)\right|^{p(x)} dx \\
&= \int_{B(x^0,D)\setminus B(x^0,\frac{D}{2})} \left(\frac{32\delta}{D^4}\left|K\left(|x-x^0|\right)\right|\right)^{p(x)} dx \\
&\le \int_{B(x^0,D)\setminus B(x^0,\frac{D}{2})} \left(\frac{32\delta}{D^4}\frac{(N+5)^2}{8(N+2)}\right)^{p(x)} dx \\
&\le \left[\frac{32\delta}{D^4}\frac{(N+5)^2}{8(N+2)}\right]^p m\left(D^N - \left(\frac{D}{2}\right)^N\right) \le [\delta]^p l_D.
\end{aligned}$$

■

Now, we introduce the functionals $\Phi, \Psi : X \to \mathbb{R}$ defined as follows:

$$\begin{aligned}
\Phi(u) &:= \int_\Omega \frac{1}{p(x)}\left|\Delta u(x)\right|^{p(x)} dx, \\
\Psi(u) &:= \int_\Omega F\left(x, u(x)\right) dx
\end{aligned}$$

for each $u \in X$, where $F(x,t) := \int_0^t f(x,\xi)\, d\xi$ for each $(x,t) \in \Omega \times \mathbb{R}$. Standard arguments ensure that Φ and Ψ are in $C^1(X)$ with

$$\begin{aligned}
\left\langle \Phi'(u), v\right\rangle &= \int_\Omega \left|\Delta u(x)\right|^{p(x)-2} \Delta u(x) \Delta v(x)\, dx, \\
\left\langle \Psi'(u), v\right\rangle &= \int_\Omega f\left(x, u(x)\right) v(x)\, dx
\end{aligned}$$

for each $u, v \in X$. These relations highlight the variational meaning of problem (P_λ) in the sense that for each $\lambda > 0$, the critical points of the functional $I_\lambda := \Phi - \lambda\Psi$ are its weak solutions.

The main tool that will allow us to obtain weak solutions of (P_λ) is the following result of Bonanno and D'Aguì (see [8]) in which existence of at least two non-zero critical points for functionals type I_λ is guaranteed.

THEOREM 2.1. *Let X be a real Banach space and let $\Phi, \Psi : X \to \mathbb{R}$ be two continuously Gateaux differentiable functionals such that* $\inf_{x\in X} \Phi(x) = \Phi(0) = \Psi(0) = 0$*. Assume that there exist $r > 0$ and $\bar{x} \in X$, with $0 < \Phi(\bar{x}) < r$, such that*

(a_1) $\frac{\sup_{\Phi(x)\le r} \Psi(x)}{r} < \frac{\Psi(\bar{x})}{\Phi(\bar{x})}$,

(a_2) *for each* $\lambda \in \Lambda_r := \left]\frac{\Phi(\bar{x})}{\Psi(\bar{x})}, \frac{r}{\sup_{\Phi(x)\leq r}\Psi(x)}\right[$, *the functional* $I_\lambda : \Phi - \lambda\Psi$ *satisfies (PS)-condition and it is unbounded from below.*

Then, for each $\lambda \in \Lambda_r$, *the functional* I_λ *admits at least two non-zero critical points* $u_{\lambda,1}, u_{\lambda,2}$ *such that* $I_\lambda(u_{\lambda,1}) < 0 < I_\lambda(u_{\lambda,2})$.

3. Existence of two weak non-zero solutions

A first result on problem (P_λ) concerns the existence of at lest two non-zero weak solutions. In the sequel, with $\alpha > 0$ and $H \in C^0(\Omega \times \mathbb{R})$, we put

$$H^\alpha := \int_\Omega \max_{|\xi|\leq\alpha} H(x,\xi)\,dx$$

and we observe that $H^\alpha \geq 0$ for each $\alpha > 0$.

Theorem 3.1. *Assume that*

(f_1) *there exist* $\delta, \gamma \in \mathbb{R}$, *with* $0 < \delta < \gamma$, *such that*

$$\frac{F^\gamma}{\left[\frac{\gamma}{k(p^+)^{\frac{1}{p^-}}}\right]_p} < \frac{p^-}{l_D}\frac{\int_{B(x^0,\frac{D}{2})} F(x,\delta)\,dx}{[\delta]^p},$$

(f_2) $F(x,t) \geq 0$ *for every* $x \in \Omega$ *and for all* $t \in [0,\delta]$,

(f_3) *there exist* $m > p^+$, $s > 0$ *such that*

$$0 < mF(x,t) \leq tf(x,t)$$

for each $x \in \Omega$ *and* $|t| \geq s$.

Then, put

$$\Lambda_{\gamma,\delta} := \left]\frac{[\delta]^p l_D}{p^- \int_{B(x^0,\frac{D}{2})} F(x,\delta)\,dx}, \frac{\left[\frac{\gamma}{k(p^+)^{\frac{1}{p^-}}}\right]_p}{F^\gamma}\right[,$$

for each $\lambda \in \Lambda_{\gamma,\delta}$ *the problem* (P_λ) *admits at least two non-zero weak solutions.*

Proof. Fixing γ, δ as in (f_1) and $\lambda \in \Lambda_{\gamma,\delta}$, we apply Theorem 2.1 to the functional

$$I_\lambda : \Phi - \lambda\Psi$$

by choosing

$$r = \left[\frac{\gamma}{k(p^+)^{\frac{1}{p^-}}}\right]_p. \tag{3.1}$$

First, we observe that condition (f_3) ensures (PS)-condition and unboundedness from below for functional I_λ for each $\lambda > 0$. To reach this condition, it is enough to use arguments similar to those contained in [33] taking into account that the functional Φ is related to norm defined on X.

From (2) and (3) of Proposition 2.1, it results that

$$[\|u\|]_p = [\|\Delta u\|_{p(x)}]_p \leq \rho_{p(x)}(\Delta u) \leq [\|\Delta u\|_{p(x)}]^p = [\|u\|]^p$$

and so

$$\frac{1}{p^+}[\|u\|]_p \leq \Phi(u) \leq \frac{1}{p^-}[\|u\|]^p$$

for each $u \in X$. In particular, if $\Phi(u) \leq r$, then one has $[\|u\|]_p \leq p^+ r$ that, thanks to (3.1) and (iv), is equivalent to

$$\|u\| \leq [p^+ r]^{\frac{1}{p}}.$$

The continuous embedding $X \hookrightarrow C^0(\overline{\Omega})$ leads to

$$\|u\|_\infty \leq k\|u\| \leq k[p^+ r]^{\frac{1}{p}} \leq k[p^+]^{\frac{1}{p}}[r]^{\frac{1}{p}} = k(p^+)^{\frac{1}{p^-}}[r]^{\frac{1}{p}} = \gamma$$

and so

$$\Psi(u) = \int_\Omega F\big(x, u(x)\big)\,dx \leq \int_\Omega \max_{|\xi|\leq\gamma} F(x,\xi)\,dx = F^\gamma.$$

Therefore, it turns out that

$$\frac{1}{r}\sup_{\Phi(u)\leq r}\Psi(u) \leq \frac{1}{r}F^\gamma. \tag{3.2}$$

Moreover, as proven in Proposition 2.2, if we consider v_δ, it results that

$$\Phi(v_\delta) \leq \frac{1}{p^-}\rho_{p(x)}(v_\delta) \leq \frac{1}{p^-}[\delta]^p l_D$$

while, taking into account that $v_\delta(x) \in [0,\delta]$ for each $x \in \Omega$, condition (f_2) ensures that

$$\Psi(v_\delta) = \int_\Omega F\big(x, v_\delta(x)\big)\,dx \geq \int_{B(x^0,\frac{D}{2})} F(x,\delta)\,dx.$$

In conclusion, one has

$$\frac{\Psi(v_\delta)}{\Phi(v_\delta)} \geq \frac{p^-}{l_D}\frac{\int_{B(x^0,\frac{D}{2})} F(x,\delta)\,dx}{[\delta]^p}. \tag{3.3}$$

Conditions (3.2), (3.3), and (f_1) ensure that

$$\frac{1}{r}\sup_{\Phi(u)\leq r}\Psi(u) < \frac{\Psi(v_\delta)}{\Phi(v_\delta)}$$

and so condition (a_1) requested in Theorem 2.1 is verified. Finally, we verify that

$$\Phi(v_\delta) < r = \left[\frac{\gamma}{k(p^+)^{\frac{1}{p^-}}}\right]_p.$$

If $\Phi(v_\delta) \geq r$, then we obtain

$$\frac{1}{p^-}[\delta]^p l_D \geq \Phi(v_\delta) \geq r.$$

Taking into account that $\gamma > \delta$, one has

$$\max_{|\xi|\leq\gamma} F(x,\xi) \geq F(x,\delta)$$

and so $F^\gamma \geq \int_{B(x_0,D)} F(x,\delta)\,dx$. This leads to

$$\frac{F^\gamma}{\left[\frac{\gamma}{k(p^+)^{\frac{1}{p^-}}}\right]_p} \geq \frac{p^-}{l_D}\frac{\int_{B(x^0,\frac{D}{2})} F(x,\delta)\,dx}{[\delta]^p},$$

which is in contradiction with condition (f_1). Since $\lambda \in \Lambda_{\gamma,\delta} \subseteq \left]\frac{\Phi(v_\delta)}{\Psi(v_\delta)}, \frac{r}{\sup_{\Phi(u)\leq r}\Psi(u)}\right[$, Theorem 2.1 ensures that functional I_λ admits at least two non-zero critical points that, as observed before, are non-trivial weak solutions of problem (P_λ). ■

Remark 3.1. When $F^\gamma = 0$, it results that $\frac{\sup_{\Phi(x)\leq r}\Psi(x)}{r} = 0$ and so

$$\Lambda_{\gamma,\delta} := \left]\frac{[\delta]^p l_D}{p^- \int_{B(x^0,\frac{D}{2})} F(x,\delta)\,dx}, +\infty\right[.$$

In this case, condition (f_3) implies that $s \geq \gamma$, while condition (f_2) leads to $F(x,\xi) = 0$ for each $\xi \in [0,\delta]$ for a.e. $x \in \Omega$.

Remark 3.2. If $f(x,0) = 0$, then in Theorem 3.1 condition (f_3) can be replaced by the weaker condition

$(\tilde{f_3})$ there exist $m > p^+, s > 0$ such that

$$0 < mF(x,t) \leq tf(x,t)$$

for each $x \in \Omega$ and $t \geq s$

in order to obtain the existence of at least two non-zero and non-negative weak solutions for problem (P_λ).

Now we present an existence result for the perturbed problem $(P_{\lambda,\mu})$.

THEOREM 3.2. *Assume that $f \in C^0(\Omega \times \mathbb{R})$ verifies conditions (f_1), (f_2), and (f_3) of Theorem 3.1.*

Then, for each $\lambda \in \Lambda_{\gamma,\delta}$ and $g \in C^0(\Omega \times \mathbb{R})$ verifying that

(g_2) *$G(x,t) \geq 0$ for every $x \in \Omega$ and for all $t \in [0,\delta]$,*

(g_3) *$|g(x,t)| \leq a_1|t|^\alpha + a_2$ for each $(x,t) \in \Omega \times \mathbb{R}$ and for some a_1, $a_2 > 0$ and $0 < \alpha < p^+ - 1$,*

there exists $\eta_{\lambda,g} > 0$ with

$$\eta_{\lambda,g} = \frac{\lambda}{G^\gamma}\left[\frac{\gamma}{k(p^+)^{\frac{1}{p^-}}}\right]_p \left(\frac{p^-}{l_D}\frac{\int_{B(x^0,\frac{D}{2})} F(x,\delta)\,dx}{[\delta]^p} - \frac{F^\gamma}{\left[\frac{\gamma}{k(p^+)^{\frac{1}{p^-}}}\right]_p}\right) \tag{3.4}$$

such that for all $\mu \in]0, \eta_{\lambda,g}[$ the problem $(P_{\lambda,\mu})$ admits at least two non-zero weak solutions.

PROOF. Fixing $\lambda \in \Lambda_{\gamma,\delta}$, g verifying (g_2) and (g_3) and $\mu \in]0, \eta_{\lambda,g}[$, we apply Theorem 2.1 by choosing

$$r = \left[\frac{\gamma}{k(p^+)^{\frac{1}{p^-}}}\right]_p$$

and taking into account that the energy functional related to problem $(P_{\lambda,\mu})$ is

$$I_{\lambda,\mu} : \Phi - \lambda\Psi_{\lambda,\mu}$$

with

$$\Psi_{\lambda,\mu}(u) = \int_\Omega \left(F\big(x,u(x)\big) + \frac{\mu}{\lambda}G\big(x,u(x)\big)\right)dx$$

for each $u \in X$. Conditions (f_3) and (g_3) ensure that $I_{\lambda,\mu}$ satisfies (PS)-condition and it is unbounded from below.

Arguing as in Theorem 3.1, thanks to (g_2) one has

$$\Psi_{\lambda,\mu}(v_\delta) = \int_\Omega \left(F\big(x,v_\delta(x)\big) + \frac{\mu}{\lambda}G\big(x,v_\delta(x)\big)\right)dx \geq \int_{B(x^0,\frac{D}{2})} F(x,\delta)\,dx$$

and this ensures

$$\frac{\Psi_{\lambda,\mu}(v_\delta)}{\Phi(v_\delta)} \geq \frac{p^-}{l_D}\frac{\int_{B(x^0,\frac{D}{2})} F(x,\delta)\,dx}{[\delta]^p}. \tag{3.5}$$

Moreover, if $\Phi(u) \leq r$, then one has $\|u\|_\infty \leq \gamma$ (see Proof of Theorem 3.1) and so

$$\Psi_{\lambda,\mu}(u) = \int_\Omega \left(F\big(x,u(x)\big) + \frac{\mu}{\lambda}G\big(x,u(x)\big)\right)dx \leq F^\gamma + \frac{\mu}{\lambda}G^\gamma$$

from which

$$\frac{1}{r}\sup_{\Phi_{\lambda,\mu}(u)\le r}\Psi_{\lambda,\mu}(u)\le\frac{1}{r}\Big(F^\gamma+\frac{\mu}{\lambda}G^\gamma\Big). \tag{3.6}$$

Because of the condition (3.4), it results that

$$\frac{1}{r}\Big(F^\gamma+\frac{\mu}{\lambda}G^\gamma\Big)<\frac{p^-}{l_D}\frac{\int_{B(x^0,\frac{D}{2})}F(x,\delta)\,dx}{[\delta]^p},$$

and so, by (3.5) and (3.6),

$$\frac{1}{r}\sup_{\Phi(u)\le r}\Psi_{\lambda,\mu}(u)<\frac{\Psi(v_\delta)}{\Phi_{\lambda,\mu}(v_\delta)}$$

which is the assumption (a_1) requested in Theorem 2.1. ■

REMARK 3.3. The values of $\Lambda_{\gamma,\delta}$ and $\eta_{\lambda,g}$ in the various particular cases are shown below:

- $F^\gamma G^\gamma>0$

$$\Lambda_{\gamma,\delta}:=\left]\frac{[\delta]^p l_D}{p^-\int_{B(x^0,\frac{D}{2})}F(x,\delta)\,dx},\frac{\left[\frac{\gamma}{k(p^+)^{\frac{1}{p^-}}}\right]_p}{F^\gamma}\right[,$$

$$\eta_{\lambda,g}=\frac{\lambda}{G^\gamma}\left[\frac{\gamma}{k(p^+)^{\frac{1}{p^-}}}\right]_p\left(\frac{p^-}{l_D}\frac{\int_{B(x^0,\frac{D}{2})}F(x,\delta)\,dx}{[\delta]^p}-\frac{F^\gamma}{\left[\frac{\gamma}{k(p^+)^{\frac{1}{p^-}}}\right]_p}\right);$$

- $F^\gamma>0,\ G^\gamma=0$

$$\Lambda_{\gamma,\delta}:=\left]\frac{[\delta]^p l_D}{p^-\int_{B(x^0,\frac{D}{2})}F(x,\delta)\,dx},\frac{\left[\frac{\gamma}{k(p^+)^{\frac{1}{p^-}}}\right]_p}{F^\gamma}\right[,\quad \eta_{\lambda,g}=+\infty;$$

- $F^\gamma=0,\ G^\gamma>0$

$$\Lambda_{\gamma,\delta}:=\left]\frac{[\delta]^p l_D}{p^-\int_{B(x^0,\frac{D}{2})}F(x,\delta)\,dx},+\infty\right[,$$

$$\eta_{\lambda,g}=\frac{\lambda}{G^\gamma}\left[\frac{\gamma}{k(p^+)^{\frac{1}{p^-}}}\right]_p\left(\frac{p^-}{l_D}\frac{\int_{B(x^0,\frac{D}{2})}F(x,\delta)\,dx}{[\delta]^p}-\frac{F^\gamma}{\left[\frac{\gamma}{k(p^+)^{\frac{1}{p^-}}}\right]_p}\right);$$

- $F^\gamma=G^\gamma=0$

$$\Lambda_{\gamma,\delta}:=\left]\frac{[\delta]^p l_D}{p^-\int_{B(x^0,\frac{D}{2})}F(x,\delta)\,dx},+\infty\right[,\quad \eta_{\lambda,g}=+\infty.$$

A more applicable version of result presented in Theorem 3.2 is the following.

THEOREM 3.3. *Assume that* $f \in C^0(\Omega \times \mathbb{R})$ *verifies condition* (f_3) *of Theorem* 3.1. *Moreover, we suppose that the following assumptions are verified:*

$(\tilde{f}_1)$ $\limsup_{t\to 0^+} \frac{\inf_{x\in\Omega} F(x,t)}{t^{p^-}} = +\infty$,

$(\tilde{f}_2)$ $F(x,t) \geq 0$ *for every* $x \in \Omega$ *and for all* $t \in [0, k(p^+)^{\frac{1}{p^-}}]$.

Then, put $\overline{\gamma} := k(p^+)^{\frac{1}{p^-}}$ *and*

$$\lambda^* := \begin{cases} \frac{1}{F^{\overline{\gamma}}} & F^{\overline{\gamma}} > 0, \\ +\infty & F^{\overline{\gamma}} = 0, \end{cases}$$

for each $\lambda \in]0, \lambda^*[$, *for each* $g \in C^0(\Omega \times \mathbb{R})$ *verifying* (g_3) *of Theorem* 3.2 *and*

$(\tilde{g}_2)$ $G(x,t) \geq 0$ *for every* $x \in \Omega$ *and for all* $t \in [0, k(p^+)^{\frac{1}{p^-}}]$

and for each $\mu \in]0, \frac{1}{G^{\overline{\gamma}}}(1 - \lambda F^{\overline{\gamma}})[$, *the problem* $(P_{\lambda,\mu})$ *admits at least two non-zero weak solutions.*

PROOF. Fix $\lambda \in]0, \lambda^*[$, g, and μ as requested in the thesis. By $(\tilde{f}_1)$ there exists $\overline{\delta} < \min\{1, \overline{\gamma}\}$ such that

$$\frac{p^- m\left(\frac{D}{2}\right)^N \inf_{x\in\Omega} F(x,t)}{\overline{\delta}^{p^-} l_D} > \frac{1}{\lambda}. \tag{3.7}$$

We apply Theorem 3.2 by choosing $\delta = \overline{\delta}$ and $\gamma = \overline{\gamma}$ and by taking into account that $[\overline{\delta}]^p = \overline{\delta}^{p^-}$. Condition (3.7) ensures that

$$\frac{p^-}{l_D} \frac{\int_{B(x^0,\frac{D}{2})} F(x,\overline{\delta})\,dx}{[\overline{\delta}]^p} \geq \frac{p^- m\left(\frac{D}{2}\right)^N \inf_{x\in\Omega} F(x,t)}{\overline{\delta}^{p^-} l_D} > \frac{1}{\lambda} > F^{\overline{\gamma}}$$

and so condition (f_1) is verified. Moreover, because of the choice of $\overline{\gamma}$ conditions $(\tilde{f}_2)$ and $(\tilde{g}_2)$ imply, respectively, (f_2) and (g_2). Since it results that

$$\left]0, \frac{1}{G^{\overline{\gamma}}}(1 - \lambda F^{\overline{\gamma}})\right[\subseteq]0, \eta_{\lambda,g}[,$$

Theorem 3.2 ensures the existence of at lest two non-zero solutions for problem $(P_{\lambda,\mu})$. ∎

REMARK 3.4. If $f(x,0) = g(x,0) = 0$, then in Theorems 3.2 and 3.3 condition (f_3) can be replaced by the weaker condition:

$(\tilde{f}_3)$ there exist $m > p^+$, $s > 0$ such that

$$0 < mF(x,t) \leq tf(x,t)$$

for each $x \in \Omega$ and $t \geq s$

in order to obtain the existence of at least two non-zero and non-negative weak solutions for problem ($P_{\lambda,\mu}$).

Finally, we present an example of application of the previous result.

EXAMPLE 3.1. Let $s, q, h \in]0, +\infty[$ such that $s \neq q$ and

$$0 < \min\{s, q\} + 1 < p^- \leq p^+ < \max\{s, q\} + 1.$$

Then, for each $\lambda \in]0, \frac{1}{|\Omega|(\frac{\gamma^{s+1}}{s+1} + \frac{\gamma^{q+1}}{q+1})}[$, $0 < h < p^+ - 1$ and

$$\mu \in \left]0, \frac{h+1}{|\Omega|\gamma^{h+1}}\left(1 - \lambda|\Omega|\left(\frac{\gamma^{s+1}}{s+1} + \frac{\gamma^{q+1}}{q+1}\right)\right)\right[$$

problem

$$\begin{cases} \Delta\left(|\Delta u|^{p(x)-2}\Delta u\right) = \lambda\left(|t|^s + |t|^q\right) + \mu|t|^h \text{ in } \Omega, \\ u = \Delta u = 0 \text{ in } \partial\Omega \end{cases}$$

admits at least two non-zero and non-negative weak solutions.

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