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Calculus of Variations. – *Minimal extension for the α-Manhattan norm*, by Daniel Campbell, Aapo Kauranen and Emanuela Radici, communicated on 10 November 2023.

Abstract. – Let $\partial\mathcal{Q}$ be the boundary of a convex polygon in $\mathbb{R}^2$, $e_\alpha = (\cos\alpha, \sin\alpha)$ and $e_\alpha^\perp = (-\sin\alpha, \cos\alpha)$ a basis of $\mathbb{R}^2$ for some $\alpha \in [0, 2\pi)$ and $\varphi : \partial\mathcal{Q} \to \mathbb{R}^2$ a continuous, finitely piecewise linear injective map. We construct a finitely piecewise affine homeomorphism $v : \mathcal{Q} \to \mathbb{R}^2$ coinciding with φ on $\partial\mathcal{Q}$ such that the following property holds: $|\langle Dv, e_\alpha\rangle|(\mathcal{Q})$ (resp., $\langle Dv, e_\alpha^\perp\rangle|(\mathcal{Q})$) is as close as we want to $\inf|\langle Du, e_\alpha\rangle|(\mathcal{Q})$ (resp., $\inf|\langle Du, e_\alpha^\perp\rangle|(\mathcal{Q})$) where the infimum is meant over the class of all BV homeomorphisms u extending φ inside $\mathcal{Q}$. This result extends that already proven by Pratelli and the third author in [Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl. 29 (2018), no. 3, 511–555] in the shape of the domain.

Keywords. – Homeomorphic extension, BV homeomorphisms, strict approximation in BV.

Mathematics Subject Classification 2020. – 46E35 (primary); 30E10, 58E20 (secondary).

1. Introduction

In this paper, we are interested in the problem of extending injective continuous and piecewise linear boundary values from a convex polygon by piecewise affine homeomorphisms. The motivation for such a study arises in the context of approximation problems found in regularity theory for non-linear elasticity. There is already a plurality of extension results in a variety of contexts, which have been applied to solve various approximation problems. Let us now give an overview of some examples.

In general, we are interested in the approximation of a weakly differentiable homeomorphism, which we would like to approximate by $\mathcal{C}^1$ homeomorphisms or by locally finite piecewise affine homeomorphisms. The approximation of a planar $W^{1,p}$ homeomorphism $1 < p < \infty$ in [9, 10] relies heavily on the injectivity of the harmonic extension of convex boundary values. In [5], the authors were also able to approximate a bi-Lipschitz map and its inverse simultaneously in the (p, p)-bi-Sobolev setting and to do so used the extension result in [6]. In order to solve the $W^{1,1}$ case in [8], the authors had to develop an independent extension result in that paper which was further examined and improved in [1, 15]. The extension result was also utilized in the $(1,1)$-bi-Sobolev setting in [12]. Finally, let us mention that the authors of [14] approximate planar BV homeomorphisms using an extension result they proved in [13].

More than just the approximation of weakly differentiable homeomorphisms by diffeomorphisms, these extension results have been key in examining the behavior of weak and strong limits of homeomorphisms in their respective classes. Such results include a categorization of the closure of Hom $\cap W^{1,p}$, $p \geq 2$, in [11], a categorization of the closure of Hom $\cap W^{1,p}$, $1 < p < 2$, in [7] and partial BV result in [2, 4].

It was demonstrated in [13] that their main extension result can be "rotated" to approximate a BV homeomorphism strictly and similarly in [14] for the area-strict case. Nevertheless, this approach makes the application of the extension result somewhat cumbersome and technical. The main result of the present paper is a piecewise affine homeomorphic extension that improves on that of [13]. More precisely, we consider extensions of piecewise linear boundary values defined on a boundary of convex quadrilaterals (and not only rectangles parallel to the coordinate axes as in [13]) which are optimal in a particular BV sense. We emphasize that the generality of the class of convex quadrilaterals includes the "rotated" version of the extension result of [13]. Also, our Theorem 1.2 is stronger than the extension theorem there (not only because of the shape of $\mathcal{Q}$) in the sense that it immediately implies their extension theorem but the opposite is not true (see Remark 1.3). Nevertheless, this improvement is a case of separating estimates already conducted in [13].

The motivation for our extension theorem is the full categorization result in [3], where we identify a condition which guarantees that a map is a strict or area-strict limit of BV homeomorphisms. In the course of the approximation, we want to work on grids that are not only made up of rectangles, and we prefer to not have to rotate the rectangles. In that sense, we need the current result, which we present below, after we set some necessary notation.

Let $\mathcal{Q} \subset \mathbb{R}^2$ be a convex polygon, let $\alpha \in [0, 2\pi)$ be fixed and call $e_\alpha = (\cos\alpha, \sin\alpha)$ and $e_\alpha^\perp = (-\sin\alpha, \cos\alpha)$.

We define the following numbers (see Figure 1):

$$
\begin{aligned}
a^- &:= \inf\left\{\langle x, e_\alpha^\perp\rangle : x \in \mathcal{Q}\right\} \quad a^+ := \sup\left\{\langle x, e_\alpha^\perp\rangle : x \in \mathcal{Q}\right\},\\
b^- &:= \inf\left\{\langle x, e_\alpha\rangle : x \in \mathcal{Q}\right\}, \quad b^+ := \sup\left\{\langle x, e_\alpha\rangle : x \in \mathcal{Q}\right\}.
\end{aligned} \tag{1.1}
$$

For each $s \in (a^-, a^+)$, we define V_s^1, V_s^2 uniquely by the conditions

$$
V_s^1, V_s^2 \in \partial\mathcal{Q}, \quad \langle V_s^1, e_\alpha^\perp\rangle = \langle V_s^2, e_\alpha^\perp\rangle = s, \quad \langle V_s^1, e_\alpha\rangle < \langle V_s^2, e_\alpha\rangle. \tag{1.2}
$$

Similarly, for every $t \in (b^-, b^+)$, we define H_t^1, H_t^2 uniquely by

$$
H_t^1, H_t^2 \in \partial\mathcal{Q}, \quad \langle H_t^1, e_\alpha\rangle = \langle H_t^2, e_\alpha\rangle = t, \quad \langle H_t^1, e_\alpha^\perp\rangle < \langle H_t^2, e_\alpha^\perp\rangle. \tag{1.3}
$$

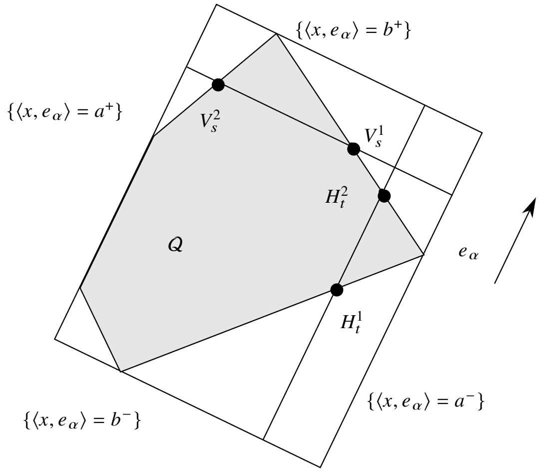

FIGURE 1. Polygon $\mathcal{Q}$ with H_t^1, H_t^2, V_s^1 and V_s^2.

Let $\varphi : \partial\mathcal{Q} \to \mathbb{R}^2$ be continuous, injective and piecewise linear. We denote $\mathcal{P}$ as the bounded component of $\mathbb{R}^2 \setminus \varphi(\partial\mathcal{Q})$. For every pair of points $\mathbf{A}, \mathbf{B} \in \overline{\mathcal{P}}$, we denote by $\rho_{\mathcal{P}}(\mathbf{A}, \mathbf{B})$ the geodesic distance between $\mathbf{A}$ and $\mathbf{B}$ inside $\overline{\mathcal{P}}$. We define the quantity

$$\Psi_\alpha(\varphi) := \int_{a^-}^{a^+} \rho_{\mathcal{P}}\big(\varphi(V_s^1), \varphi(V_s^2)\big) ds + \int_{b^-}^{b^+} \rho_{\mathcal{P}}\big(\varphi(H_t^1), \varphi(H_t^2)\big) dt.$$

Loosely speaking, the quantity $\Psi_\alpha(\varphi)$ accounts for the length of all the geodesics inside $\mathcal{P}$ connecting pairs of points on $\varphi(\partial\mathcal{Q})$ whose preimage in φ is a pair of points in $\partial\mathcal{Q}$ lying on a line parallel to either α or $\alpha^\perp$. Further, for $u \in \mathrm{BV}(\Omega, \mathbb{R}^2)$, we denote the α-Manhattan norm of Du as $\|\cdot\|_\alpha$ which we define as

$$\|Du\|_\alpha(\mathcal{Q}) := \big|\langle Du, e_\alpha\rangle\big|(\mathcal{Q}) + \big|\langle Du, e_\alpha^\perp\rangle\big|(\mathcal{Q}).$$

The main results of the paper are the following.

THEOREM 1.1. *Let $\alpha \in [0, 2\pi)$ be fixed, $\mathcal{Q} \subset \mathbb{R}^2$ a convex polygon and $\varphi : \partial\mathcal{Q} \to \mathbb{R}^2$ a continuous piecewise linear injective map. Then, for every $\varepsilon > 0$, there exists a finitely piecewise affine homeomorphism $v : \mathcal{Q} \to \mathbb{R}^2$ extending φ, such that*

$$\|Dv\|_\alpha(\mathcal{Q}) \le \Psi_\alpha(\varphi) + \varepsilon. \tag{1.4}$$

THEOREM 1.2. *Let $\varepsilon > 0$ and let v be the extension from Theorem 1.1. Then,*

$$\begin{aligned} \big|\langle Dv, e_\alpha\rangle\big|(\mathcal{Q}) &\le \int_{b^-}^{b^+} \rho_{\mathcal{P}}\big(\varphi(H_t^1), \varphi(H_t^2)\big) dt + \varepsilon, \\ \big|\langle Dv, e_\alpha^\perp\rangle\big|(\mathcal{Q}) &\le \int_{a^-}^{a^+} \rho_{\mathcal{P}}\big(\varphi(V_s^1), \varphi(V_s^2)\big) ds + \varepsilon. \end{aligned} \tag{1.5}$$

REMARK 1.3. We observe that Theorem 1.1 is stronger than the result of [13] as it provides the almost optimal extension with respect to any rotated Manhattan norm and not just for the canonical one (where $\alpha = 0$).

Let us also remark that Theorem 1.2 immediately implies Theorem 1.1 but the argument used to construct v is exactly the same. Also, it is immediate that

$$\int_{b^-}^{b^+} \rho_{\mathcal{P}}\big(\varphi(H_t^1), \varphi(H_t^2)\big)dt \\ \leq \inf\big\{\big|\langle Du, e_\alpha\rangle\big|(\overline{\mathcal{Q}}) : u \in \mathrm{Hom} \cap \mathrm{BV}(\overline{\mathcal{Q}}, \mathbb{R}^2),\ u = \varphi \text{ on } \partial\mathcal{Q}\big\}$$

and

$$\int_{a^-}^{a^+} \rho_{\mathcal{P}}\big(\varphi(V_s^1), \varphi(V_s^2)\big)ds \\ \leq \inf\big\{\big|\langle Du, e_\alpha^\perp\rangle\big|(\overline{\mathcal{Q}}) : u \in \mathrm{Hom} \cap \mathrm{BV}(\overline{\mathcal{Q}}, \mathbb{R}^2),\ u = \varphi \text{ on } \partial\mathcal{Q}\big\},$$

and our result in fact shows that there is a sequence of homeomorphisms achieving the infimum and having variation converging to the left-hand side in the sense of (1.5). This fact is actually a direct consequence of the proofs in [13], though it was not explicitly remarked there. The key argument is in Theorem 2.9.

1.1. Sketch of the proof

Before expounding the proof in detail, let us look at an overview of the proof. We start with a convex polygon $\mathcal{Q}$. Up to a rotation of α, we may assume that $\alpha = 0$. Either (the rotated) $\mathcal{Q}$ has horizontal sides, or after removing a tiny triangle called T_1 close to the lowest point of $\mathcal{Q}$ and a triangle called T_2 close to the highest point of $\mathcal{Q}$ we get a convex Δ that has a pair of horizontal sides (see Figure 2). We extend φ on $\partial T_1, \partial T_2$ so that it is continuous injective and piecewise linear. By making the triangles small enough, we guarantee that $\Psi_0(\varphi(\Delta)) + \Psi_0(\varphi(T_1)) + \Psi_0(\varphi(T_2)) \leq \Psi_0(\varphi(\mathcal{Q})) + \varepsilon$. Here, our new φ extends the original φ from $\partial\mathcal{Q}$. This step is Lemma 3.2.

Now, we separate Δ into thin horizontal strips S_i (see Figure 3), defining a continuous injective piecewise linear φ on ∂S_i so that $\sum_{i=1}^{M} \Psi_0(\varphi(\partial S_i)) \leq \Psi_0(\varphi(\partial\Delta)) + \varepsilon$ which extends the original φ from $\partial\Delta \cup \partial T_1 \cup \partial T_2$. This step is Lemma 3.3.

We separate each S_i into a central rectangle and a pair of right-angle triangles at each end. On the rectangular domains R_i, we can use Proposition 2.8 to extend the boundary values and get a piecewise affine homeomorphism w_i on the R_i satisfying an estimate on $|Dw_i|(R_i)$. In Lemma 4.2, we show how we extend the boundary values to get a piecewise affine homeomorphism on the triangles at the ends of the strips; see Figure 6. We do this by further separating them into even thinner rectangles where we can extend and estimate as above. The remaining part of the set is so small that its contribution to the norm is bounded by $2^{-i}\varepsilon$.

The final part of the proof is collating the estimates and summing to estimate that our mapping v satisfies (1.4).

2. Preliminaries

In this section, we recall a list of definitions and known geometrical results which are already available in the literature. Most of them are taken from [13, 14].

Notation 2.1. Throughout the paper, we endeavor to keep to the following norms of notation:

- $\mathcal{Q}$ is a convex polygon,
- $\mathcal{P}$ is a 2-dimensional polygon with boundary $\partial\mathcal{P}$. If $\varphi : \partial\mathcal{Q} \to \mathbb{R}^2$ is injective and piecewise linear continuous, then $\mathcal{P}$ is the polygon corresponding to the bounded component of $\mathbb{R}^2 \setminus \varphi(\partial\mathcal{Q})$ and $\mathcal{P} = \varphi(\partial\mathcal{Q})$,
- $\alpha \in [0, 2\pi)$ is a given angle and the vector $e_\alpha := (\cos\alpha, \sin\alpha)$. Also, we denote $e_\alpha^\perp := (\cos(\alpha + \pi/2), \sin(\alpha + \pi/2))$,
- u and v are planar BV mappings,
- a^-, a^+, b^-, b^+ are the numbers from (1.1), typically $s \in (a^-, a^+)$ and $t \in (b^-, b^+)$ and $\ell = a^+ - a^-$, $h = b^+ - b^-$,
- Δ is a convex polygon with 2 sides parallel to α,
- $T, T_1, T_2, \widetilde{T}, T_i^1, T_i^2$ are all triangles,
- $V_s^1, V_s^2, H_t^1, H_t^2$ are the points satisfying the conditions in (1.2) and (1.3) although we may replace $\mathcal{Q}$ with another convex polygon, for example, Δ or T,
- by $\mathcal{R}_\mathcal{Q} = [a^-, a^+]e_\alpha + [b^-, b^+]e_\alpha^\perp$ we denote the smallest rectangle with sides parallel to e_α and $e_\alpha^\perp$ containing $\mathcal{Q}$,
- $c_1, c_2, d_1, d_2 \in \mathbb{R}$ are ordinates,
- by $\widetilde{C}$ we denote a generic constant whose precise value may vary between estimates,
- the points in the preimage are $A, B, C, D, E, F, G, P, Q$,[1]
- if φ is a piecewise linear injective map defined on the triangle ABC, we call $\mathbf{d} := |\varphi(A) - \varphi(C)|$ the length of the image of the hypotenuse through φ,

(1) We do not need to utilize the notation $B(x, r) = \{y : |y - x| < r\}$ so there is no danger of confusion when using B to denote a point.

- $\beta \in (0, \frac{\pi}{2})$ is the angle at A in the triangle ABC,
- $\eta > 0$ is a small chosen parameter,
- the points in the image are written in bold font, e.g., $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}, \mathbf{X}, \mathbf{Y}, \mathbf{Z}$,
- $\mathcal{P}, \mathcal{P}_0, \mathcal{P}^+$ are polygons in the image, typically the piecewise affine image of a polygon in the preimage, e.g., $\mathcal{P} = \varphi(\partial\mathcal{Q})$,
- φ, ψ are continuous injective piecewise linear maps from one-dimensional "skeletons" (i.e., a finite union of segments) in the preimage,
- given a set $A \subset \mathbb{R}^2$ and a function $\varphi : A \to \mathbb{R}^2$, we denote by $\varphi\rceil_B$ the restriction of φ to a subset $B \subset A$,
- $\gamma_{\mathbf{A},\mathbf{B}}$ is the geodesic curve from $\mathbf{A}$ to $\mathbf{B}$ in $\mathcal{P}$ and $\rho_{\mathcal{P}}(\mathbf{A}, \mathbf{B})$ is the length of that curve.

Definition 2.2 (Geodesics and modified geodesics). Let $\mathcal{P} \subset \mathbb{R}^2$ be a polygon, and let $\mathbf{A}$ and $\mathbf{B}$ be any two distinct points in $\mathcal{P}$. We define $\gamma_{\mathbf{AB}}$ as the unique geodesic (i.e., curve of minimal length) connecting them, lying inside $\mathcal{P}$. Notice that $\gamma_{\mathbf{AB}}$ is a piecewise linear curve, whose vertices are only $\mathbf{A}, \mathbf{B}$ and some vertices of $\partial\mathcal{P}$ whose internal angles have size at least π. Assume now that $\mathbf{A}, \mathbf{B} \in \partial\mathcal{P}$, and let $\mathbf{W}_1, \mathbf{W}_2, \dots, \mathbf{W}_K$ be all the vertices of $\mathcal{P}$ met by $\gamma_{\mathbf{AB}}$, so that

$$\gamma_{\mathbf{AB}} = \mathbf{A}\mathbf{W}_1, \mathbf{W}_2, \dots, \mathbf{W}_K\mathbf{B}.$$

Fix now any $\delta > 0$. For every $1 \leq i \leq K$, let $\widetilde{\mathbf{W}}_i \neq \mathbf{W}_i$ be some arbitrary point in the internal bisector of the angle at $\mathbf{W}_i$ having distance from $\mathbf{W}_i$ smaller than δ. The piecewise linear curve $\widetilde{\gamma}_{\mathbf{AB}} = \mathbf{A}\widetilde{\mathbf{W}}_1, \widetilde{\mathbf{W}}_2, \dots, \widetilde{\mathbf{W}}_K\mathbf{B}$ is then called a δ-*modification* of $\gamma_{\mathbf{AB}}$.

Notice that there exists a constant $\overline{\delta}(\mathcal{P}) > 0$, depending on $\mathcal{P}$ but not on $\mathbf{A}$ and $\mathbf{B}$, such that the interior of $\widetilde{\gamma}_{\mathbf{AB}}$ is contained in the interior of $\mathcal{P}$ if $\delta < \overline{\delta}(\mathcal{P})$, unless the segment $\mathbf{AB}$ is already contained in $\partial\mathcal{P}$, in which case $K = 0$ and $\widetilde{\gamma}_{\mathbf{AB}} = \gamma_{\mathbf{AB}} \subseteq \partial\mathcal{P}$.

Lemma 2.3 ([13, Lemma 2.4]). *Let* $\mathbf{A}, \mathbf{B}, \mathbf{C}$ *and* $\mathbf{D}$ *be four distinct points in a polygon* $\mathcal{P}$. *Then, the intersection* $\gamma_{\mathbf{AB}} \cap \gamma_{\mathbf{CD}}$ *is either empty or closed and connected. Assume now also that* $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D} \in \partial\mathcal{P}$ *and call* $\partial\mathcal{P}_1, \partial\mathcal{P}_2$ *the two components of* $\partial\mathcal{P} \setminus \{\mathbf{C}, \mathbf{D}\}$. *If* $\mathbf{A} \in \partial\mathcal{P}_1$ *and* $\mathbf{B} \in \partial\mathcal{P}_2$, *then* $\gamma_{\mathbf{AB}} \cap \gamma_{\mathbf{CD}} \neq \emptyset$. *If* $\mathbf{A}, \mathbf{B} \in \partial\mathcal{P}_1$ *and* $\gamma_{\mathbf{AB}} \cap \gamma_{\mathbf{CD}} \neq \emptyset$, *then the first and last point of this intersection must either be vertices of* $\mathcal{P}$ *or coincide with one of the points* $\mathbf{A}$ *or* $\mathbf{B}$.

We remark that in the reference the lemma is stated without closedness of the intersection but it follows from the following simple observation. If $\mathbf{A}, \mathbf{B}, \mathbf{C}$ and $\mathbf{D}$ are four distinct points in $\mathcal{P}$ and the intersection $\gamma_{\mathbf{AB}} \cap \gamma_{\mathbf{CD}}$ is not empty, then it is either a point (hence a closed set) or a piecewise linear curve which starts and ends at corners of $\partial\mathcal{P}$ (thus being the finite union of closed connected segments).

LEMMA 2.4 ([13, Lemma 2.5]). *Let $\mathcal{P}$ be a polygon, let $\mathbf{A}, \mathbf{B} \in \partial\mathcal{P}$ be two points such that the segment $\mathbf{AB}$ is not contained in $\partial\mathcal{P}$, then let $\delta < \overline{\delta}(\mathcal{P})$ and let $\widetilde{\gamma}_{\mathbf{AB}}$ be a modified geodesic in the sense of Definition 2.2. Let also $\mathcal{P}_1$ and $\mathcal{P}_2$ be the two polygons in which $\mathcal{P}$ is divided by $\widetilde{\gamma}_{\mathbf{AB}}$, and let $\varepsilon > 0$ be a given constant. If δ is small enough, depending only on ε and $\mathcal{P}$, then the following is true.*

For any two points $\mathbf{C}, \mathbf{D} \in \mathcal{P}_i$ for $i \in \{1, 2\}$, one has

$$\rho_{\mathcal{P}_i}(\mathbf{C}, \mathbf{D}) < \rho_{\mathcal{P}}(\mathbf{C}, \mathbf{D}) + \varepsilon. \tag{2.1}$$

If $\mathbf{C} \in \mathcal{P}_1$, $\mathbf{D} \in \mathcal{P}_2$ and $\mathbf{E} \in \partial\mathcal{P}_1 \cap \partial\mathcal{P}_2$ is any point with distance at most δ from $\gamma_{\mathbf{CD}}$, then

$$\rho_{\mathcal{P}_1}(\mathbf{C}, \mathbf{E}) + \rho_{\mathcal{P}_2}(\mathbf{E}, \mathbf{D}) < \rho_{\mathcal{P}}(\mathbf{C}, \mathbf{D}) + \varepsilon. \tag{2.2}$$

DEFINITION 2.5 (Set of vertices of a geodesic curve). Let $\mathcal{P} \subset \mathbb{R}^2$ be a polygon. For every $\mathbf{A}, \mathbf{B} \in \partial\mathcal{P}$, there is a unique ordered set $\mathcal{X}(\mathbf{A}, \mathbf{B}) = \{\mathbf{X}_1, \ldots, \mathbf{X}_N\}$ such that the geodesic $\gamma_{\mathbf{AB}}$ is exactly the piecewise linear curve $\mathbf{AX}_1, \ldots, \mathbf{X}_N\mathbf{B}$, and the points $\mathbf{X}_j$ are all the vertices of $\mathcal{P}$ met by the geodesic $\gamma_{\mathbf{AB}}$ (except $\mathbf{A}$ and $\mathbf{B}$ themselves, in case they are already vertices). The set $\mathcal{X}(\mathbf{A}, \mathbf{B})$ is called the *set of vertices* of $\gamma_{\mathbf{AB}}$.

DEFINITION 2.6 (δ-linearization of a Jordan curve). Let ψ be a Jordan curve with finite length, and let $\delta > 0$ be much smaller than the diameter of the bounded component of $\mathbb{R}^2 \setminus \psi$. Let $\widehat{\mathbf{A}_1\mathbf{B}_1}, \widehat{\mathbf{A}_2\mathbf{B}_2}, \ldots, \widehat{\mathbf{A}_N\mathbf{B}_N}$ be finitely many essentially disjoint arcs contained in ψ. Let then φ be the closed curve obtained by replacing each arc $\widehat{\mathbf{A}_i\mathbf{B}_i}$ with the segment $\mathbf{A}_i\mathbf{B}_i$. We say that φ is a δ-linearization of ψ if

- φ is injective,
- every arc $\widehat{\mathbf{A}_i\mathbf{B}_i}$ is such that $\mathcal{H}^1(\widehat{\mathbf{A}_i\mathbf{B}_i}) < \delta$,
- $\widehat{\mathbf{A}_i\mathbf{B}_i} \cap \varphi \subset \mathbf{A}_i\mathbf{B}_i$.

The δ-linearization is said *complete* if the union of the arcs $\widehat{\mathbf{A}_i\mathbf{B}_i}$ is the whole curve ψ; hence, φ is piecewise linear.

LEMMA 2.7 ([13, Corollary 4.3]). *Let $\Delta \subset \mathbb{R}^2$ be a convex polygon, and let $\psi : \partial\Delta \to \mathbb{R}^2$ be a parametrized Jordan curve with finite length and let $\varphi : \partial\Delta \to \mathbb{R}^2$ be a δ-linearization of ψ. Then, for every $P, Q \in \partial\Delta$, one has*

$$\rho_{\varphi(\partial\Delta)}\big(\varphi(P), \varphi(Q)\big) \leq \rho_{\psi(\partial\Delta)}\big(\psi(P), \psi(Q)\big) + 2\delta.$$

In particular, for every $\theta \in [0, 2\pi)$, we deduce

$$\Psi_\theta(\varphi) \leq \Psi_\theta(\psi) + 2\delta\mathcal{H}^1(\partial\Delta).$$

We conclude this section recalling two extension results that will be useful in the sequel. The next proposition (Proposition 2.8) is proved in [13, Theorem A], and Corollary 2.10 is a straightforward consequence of Proposition 2.8.

PROPOSITION 2.8 (Minimal extension for standard Manhattan norm). *Let $\mathcal{R} \subset \mathbb{R}^2$ be a rectangle of the form $[a^-, a^+] \times [b^-, b^+]$, and let $\varphi : \partial\mathcal{R} \to \mathbb{R}^2$ be a continuous injective map. Then, for every $\varepsilon > 0$, there exists a piecewise affine homeomorphism $v : \mathcal{R} \to \mathbb{R}^2$ coinciding with φ on $\partial\mathcal{R}$ such that*

$$\|Dv\|_0(\mathcal{R}) \leq \Psi_0(\varphi) + \varepsilon.$$

Moreover, if φ is piecewise linear, then the map v can be chosen finitely piecewise affine.

THEOREM 2.9 (Minimal extension for standard Manhattan norm). *Let $\varepsilon > 0$, and let v be the mapping from Proposition 2.8. Then,*

$$|D_1 v|(\mathcal{Q}) \leq \int_{b^-}^{b^+} \rho_{\mathcal{P}}\big(\varphi(H_t^1), \varphi(H_t^2)\big)dt + \varepsilon,$$
$$|D_2 v|(\mathcal{Q}) \leq \int_{a^-}^{a^+} \rho_{\mathcal{P}}\big(\varphi(V_s^1), \varphi(V_s^2)\big)ds + \varepsilon.$$

PROOF. The finitely piecewise affine homeomorphisms from a rectangle to a polygon in [13] used in the proof of Proposition 2.8 are constructed in [13, Lemma 2.12]. The key estimates we need to extract are the last two unnumbered equations of the proof, found in [13, p. 543]. They say exactly that

$$|D_1 v|(\mathcal{Q}) \leq \int_{b^-}^{b^+} \rho_{\mathcal{P}}\big(\varphi(H_t^1), \varphi(H_t^2)\big)dt + \varepsilon,$$
$$|D_2 v|(\mathcal{Q}) \leq \int_{a^-}^{a^+} \rho_{\mathcal{P}}\big(\varphi(V_s^1), \varphi(V_s^2)\big)ds + \varepsilon. \qquad \blacksquare$$

COROLLARY 2.10 ($W^{1,1}$ extension with non-optimal bound). *There exists $\widetilde{C} > 0$ such that the following holds. Let $\mathcal{R} \subset \mathbb{R}^2$ be a rectangle, let $\partial\mathcal{R}$ be its boundary and let $\varphi : \partial\mathcal{R} \to \mathbb{R}^2$ be a continuous, piecewise linear and injective map. Then, there exists a finitely piecewise affine homeomorphism $v : \mathcal{R} \to \mathbb{R}^2$ extending φ such that*

$$\|Dv\|_{L^1(\mathcal{R})} \leq \widetilde{C}\mathcal{H}^1(\partial\mathcal{R})\mathcal{H}^1\big(\varphi(\partial\mathcal{R})\big).$$

PROOF. The conclusion follows by applying Proposition 2.8 with

$$\varepsilon = \mathcal{H}^1(\partial\mathcal{R})\mathcal{H}^1\big(\varphi(\partial\mathcal{R})\big)$$

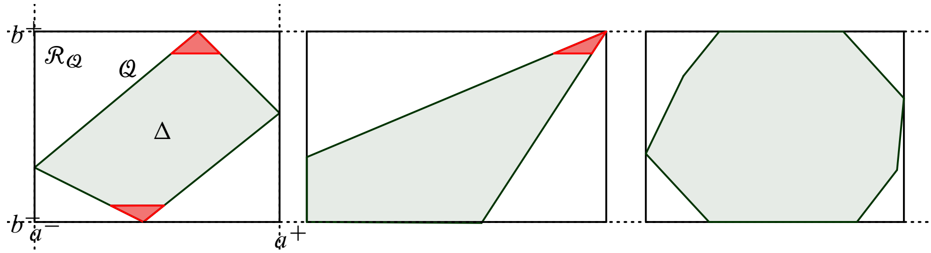

FIGURE 2. The figure shows the set-up in the case that $\alpha = 0$. We have the polygon $\mathcal{Q}$ constituted of the green set Δ and the removed red triangle(s). In the case where the polygon $\mathcal{Q}$ has no sides parallel to α (corresponding to class (iii) from Remark 3.1 pictured on the left), we generate a polygon with two sides, both parallel to α by removing 2 triangles. If there is already one side parallel to some α (as pictured in the middle), then it is enough to remove one triangle and the remaining set has two sides both parallel to α; this corresponds to class (ii) from Remark 3.1. In the scenario, on the right, $\mathcal{Q}$ already has two sides parallel to α (corresponding to class (i) from Remark 3.1), and it is not necessary to remove any triangles.

and by observing that

$$\|Dv\|_{L^1(\mathcal{R})} \leq \widetilde{C}\big(|D_1 v|(\mathcal{R}) + |D_2 v|(\mathcal{R})\big) = \widetilde{C}\|Dv\|_0(\mathcal{R})$$

and

$$\begin{aligned}\Psi_0(\varphi) &= \int_{a^-}^{a^+} \rho_{\varphi(\partial c R)}\big(\varphi(t, b^-), \varphi(t, b^+)\big)dt + \int_{b^-}^{b^+} \rho_{\varphi(\partial c R)}\big(\varphi(a^-, t), \varphi(a^+, t)\big)dt \\ &\leq (a^+ - a^- + b^+ - b^-)\mathcal{H}^1\big(\varphi(\partial\mathcal{R})\big) \\ &\leq \frac{1}{2}\mathcal{H}^1(\partial\mathcal{R})\mathcal{H}^1\big(\varphi(\partial\mathcal{R})\big).\end{aligned}$$ ∎

3. EXTENSION ON A ONE-DIMENSIONAL SKELETON

REMARK 3.1. If $\mathcal{Q}$ coincides with the rectangle $\mathcal{R}_{\mathcal{Q}}$, then the conclusion of Theorem 1.1 follows directly by [13, Theorem A]. So, without loss of generality, we can assume that $\mathcal{Q} \neq \mathcal{R}_{\mathcal{Q}}$. Then, as sketched in Figure 2, there are three cases left to consider:

(i) $\mathcal{Q}$ has two parallel sides parallel to α;

(ii) $\mathcal{Q}$ has exactly one side parallel to α;

(iii) $\mathcal{Q}$ has no side parallel to α.

In this section, we introduce a suitable partition of $\mathcal{Q}$, whose boundary will be referred to as one-dimensional *skeleton*, and we define a continuous, injective and piecewise linear extension of φ on the skeleton. This procedure will be done in two different steps, which eventually correspond to the following technical lemmas.

LEMMA 3.2 (Skeleton-triangles). *Let $\alpha \in [0, 2\pi)$ be fixed, let $\mathcal{Q} \subset \mathbb{R}^2$ be a convex polygon and let $\varphi : \partial\mathcal{Q} \to \mathbb{R}^2$ be a continuous, piecewise linear, injective map.*

Then, for every $\varepsilon > 0$, one of the following holds:

- *the set $\mathcal{Q}$ is of class* (ii) *from Remark 3.1, and there exists a triangle T satisfying the following:*
  (a) *two sides of T are contained in two sides of $\mathcal{Q}$, so T and $\mathcal{Q}$ share the vertex W and the side of T inside $\mathcal{Q}$ is parallel to α (we refer to the point in the intersection of the third side of T and the bisector of the vertex at W as W^*),*
  (b) $\mathcal{H}^1(\partial T) < \varepsilon$,
  (c) $\mathcal{Q} = T \cup \Delta$, *where $\partial\Delta$ is a convex polygon with two sides parallel to α one of which lies in $\partial\mathcal{Q}$ and the other one is the common side of ∂T and $\partial\Delta$,*
  (d) *φ is linear on each side of $\partial T \cap \partial\mathcal{Q}$,*
  (e) *there exists $\overline{\varphi} : \partial T \cup \partial\Delta \to \mathbb{R}^2$ a continuous piecewise linear injective map such that $\overline{\varphi} = \varphi$ on $\partial\mathcal{Q}$, $\overline{\varphi}$ is exactly bi-linear on $\partial T \cap \partial\Delta$ and the singular point is precisely W^*,*
  (f) *the estimates*
  $$\mathcal{H}^1\big(\overline{\varphi}(\partial T)\big) \leq \varepsilon,$$
  $$\Psi_\alpha(\overline{\varphi}_{\rceil\partial T}) + \Psi_\alpha(\overline{\varphi}_{\rceil\partial\Delta}) < \Psi_\alpha(\varphi) + \varepsilon$$
  hold;
- *the set $\mathcal{Q}$ is of class* (iii) *from Remark 3.1, and there exists a pair of disjoint triangles T_1, T_2 satisfying the following:*
  (a) *T_1, T_2 each contain a vertex of $\mathcal{Q}$ which we call W_1 and W_2, resp.,*
  (b) *it holds that*
  $$\mathcal{H}^1(\partial T_1), \mathcal{H}^1(\partial T_2) < \varepsilon\,, \tag{3.1}$$
  (c) $\mathcal{Q} = T_1 \cup T_2 \cup \Delta$, *where $\partial\Delta$ is a convex polygon with two sides parallel to α one of which is the common side of ∂T_1 and $\partial\Delta$ and the other is the common side of ∂T_2 and $\partial\Delta$,*
  (d) *φ is linear on each side of $\partial T_1 \cap \partial\mathcal{Q}$ and $\partial T_2 \cap \partial\mathcal{Q}$,*
  (e) *there exists $\overline{\varphi} : \partial T_1 \cup \partial T_2 \cup \partial\Delta \to \mathbb{R}^2$ a continuous piecewise linear injective map such that $\overline{\varphi} = \varphi$ on $\partial\mathcal{Q}$, $\overline{\varphi}$ is exactly bi-linear on $\partial T_2 \cap \partial\Delta$ and on $\partial T_1 \cap \partial\Delta$ and the singular points are precisely W_1^*, W_2^*,*
  (f) *the estimates*
  $$\mathcal{H}^1\big(\overline{\varphi}(\partial T_1)\big) + \mathcal{H}^1\big(\overline{\varphi}(\partial T_2)\big) \leq \varepsilon \tag{3.2}$$

and

$$\Psi_\alpha(\overline{\varphi}\rceil_{\partial T_1}) + \Psi_\alpha(\overline{\varphi}\rceil_{\partial\Delta}) + \Psi_\alpha(\overline{\varphi}\rceil_{\partial T_2}) < \Psi_\alpha(\varphi) + \varepsilon \tag{3.3}$$

hold.

PROOF. It suffices to prove the claim for the case of $\mathcal{Q}$ being class (ii) since the class (iii) case is just a repetition of the same argument used at a pair of opposing vertices of $\mathcal{Q}$.

Assume that $\mathcal{Q}$ is of class (ii), then there exists a side of $\mathcal{R}_\mathcal{Q}$ (the smallest rectangle containing $\mathcal{Q}$ with sides parallel and perpendicular to $e_\alpha, e_\alpha^\perp$) whose intersection with $\mathcal{Q}$ is exactly the vertex W of $\mathcal{Q}$. Up to a rotation of angle α and a translation of $(-a^-, -b^-)$, we may assume that $\mathcal{R}_\mathcal{Q} = [0,\ell] \times [0,h]$ for $\ell = a^+ - a^-$ and $h = b^+ - b^-$. Further, we may assume that the α rotation of $\mathcal{Q}$ has a horizontal side lying in $[0,\ell] \times \{h\}$ such that $W = (w, 0)$ for some $w \in [0,\ell]$. It suffices to prove our claim for the α-rotated, translated $\mathcal{Q}$ and replacing Ψ_α with Ψ_0.

Since $\mathcal{Q}$ is convex, it holds that for every $t \in (0,h)$, one has that $\partial\mathcal{Q} \cap (\mathbb{R} \times \{t\})$ consists of exactly two points H_t^1 and H_t^2 with $0 \le (H_t^1)_1 < (H_t^2)_1 \le \ell$. Analogously, for every $s \in (0,\ell)$, one has that $\partial\mathcal{Q} \cap (\{s\} \times \mathbb{R})$ consists of exactly two points V_s^1 and V_s^2 with $0 \le (V_s^1)_1 < (V_s^2)_1 \le h$.

Let $\varepsilon > 0$ be arbitrary fixed. We then let $\overline{\delta} = \overline{\delta}(\varphi(\partial\mathcal{Q})) > 0$ be the parameter introduced directly after Definition 2.2, and let $\delta_1 < \overline{\delta}$ be the parameter introduced in Lemma 2.4 for the polygon given by $\varphi(\partial\mathcal{Q})$ and the number $\frac{\varepsilon}{2(h+\ell)}$. Let

$$0 < \eta_1 < \min\left\{\frac{\varepsilon}{24}, \frac{\delta_1}{4}\right\}.$$

For the following, recall the definition of the points $H_t^{1,2}$ in (1.3). Since $\varphi : \mathcal{Q} \to \mathbb{R}^2$ is continuous, injective and piecewise linear, then we can find $0 < t^* \ll h$ such that the following properties hold:

(i) $|H_{t^*}^1 - W| < \eta_1$, $|H_{t^*}^2 - W| < \eta_1$;

(ii) φ is linear on each of the segments $H_{t^*}^1 W$, $H_{t^*}^2 W$;

(iii) $|\varphi(H_{t^*}^1) - \varphi(W)| < \eta_1$ and $|\varphi(H_{t^*}^2) - \varphi(W)| < \eta_1$.

We denote T the triangle $WH_{t^*}^1 H_{t^*}^2$, and $\Delta = \mathcal{Q} \setminus T$. We call $W^* = \frac{1}{2}(H_{t^*}^1 + H_{t^*}^2)$, and then W^* is the intersection of the segment $H_{t^*}^1 H_{t^*}^2$ and the bisector of the angle at W. Then, claim (a) is immediate and claim (b) is immediate from the choice of t^*. By construction, Δ is convex and the third side of T is horizontal (i.e., parallel to α), thus proving (c). By the triangular inequality and by property (i), we have that

$$\mathcal{H}^1(\partial T) < 4\eta_1.$$

We now consider γ^* the geodesic connecting $\varphi(H^1_{t^*})$ and $\varphi(H^1_{t^*})$ inside the polygon identified by $\varphi(\partial\mathcal{Q})$. There are two possibilities: either γ^* is a segment or γ^* is a bilinear path lying inside $\varphi(\partial\mathcal{Q})$ passing through $\varphi(W)$. In both cases, we let $\mathbf{X}$ be on the internal bisector of the corner $\varphi(W)$ such that $|\mathbf{X} - \varphi(W)| < 2\eta_1$, and we call $\tilde{\gamma}^*$ the path $\varphi(H^1_{t^*})\mathbf{X}\varphi(H^2_{t^*})$. Notice that, since $2\eta_1 < \delta_1$, $\tilde{\gamma}^*$ is a δ_1-modification of γ^* in the sense of Definition 2.2. Moreover, $\tilde{\gamma}^*$ lies in the interior of $\varphi(\partial\mathcal{Q})$ and $\mathcal{H}^1(\tilde{\gamma}^*) \leq 4\eta_1$.

We now construct the extension $\overline{\varphi} : \partial T \cup \partial\Omega \to \mathbb{R}^2$ that is continuous, piecewise linear and injective. We let $\overline{\varphi} = \varphi$ on $\partial\mathcal{Q}$, then we have claim (d) by the choice of t^* and we only need to define $\overline{\varphi}$ on the segment $H^1_{t^*}H^2_{t^*}$.

We set $\overline{\varphi}(W^*) := \mathbf{X}$, and then we define $\overline{\varphi} : H^1_{t^*}H^2_{t^*} \to \mathbb{R}^2$ as the map that is linear on $H^1_{t^*}W^*$ and $W^*H^2_{t^*}$, such that

$$\overline{\varphi}(H^1_{t^*}H^2_{t^*}) = \tilde{\gamma}^*.$$

Then, we have claim (e). Thanks to property (iii) from the choice of t^* and the bound on $\mathcal{H}^1(\tilde{\gamma}^*)$, we can compute

$$\mathcal{H}^1\big(\overline{\varphi}(\partial T)\big) \leq 6\eta_1.$$

For claim (f), we now estimate the quantity $\Psi_0(\overline{\varphi}_{\rceil\partial T}) + \Psi_0(\overline{\varphi}_{\rceil\partial\Delta})$. Thanks to our choice of δ_1 and (2.1) of Lemma 2.4, we can observe

$$\begin{aligned}
&\rho_{\overline{\varphi}(\partial T)}\big(\overline{\varphi}(H^1_t), \overline{\varphi}(H^2_t)\big)\\
&\quad \leq \rho_{\varphi(\partial\mathcal{Q})}\big(\varphi(H^1_t), \varphi(H^2_t)\big) + \frac{\varepsilon}{2(h+\ell)} \qquad \text{for all } t \in (0, t^*),\\
&\rho_{\overline{\varphi}(\partial\Delta)}\big(\overline{\varphi}(H^1_t), \overline{\varphi}(H^2_t)\big)\\
&\quad \leq \rho_{\varphi(\partial\mathcal{Q})}\big(\varphi(H^1_t), \varphi(H^2_t)\big) + \frac{\varepsilon}{2(h+\ell)} \qquad \text{for all } t \in (t^*, h),\\
&\rho_{\overline{\varphi}(\partial\Delta)}\big(\overline{\varphi}(V^1_s), \overline{\varphi}(V^2_s)\big)\\
&\quad \leq \rho_{\varphi(\partial\mathcal{Q})}\big(\varphi(V^1_s), \varphi(V^2_s)\big) + \frac{\varepsilon}{2(h+\ell)} \qquad \text{for all } s \in \big(0, (H^1_{t^*})_1\big) \cup \big((H^2_{t^*})_1, \ell\big).
\end{aligned}$$

On the other hand, for every $s \in ((H^1_{t^*})_1, (H^2_{t^*})_1)$, we call $V^3_s = (s, t^*)$ and we notice that the geodesic $\gamma_{\varphi(V^1_s)\varphi(V^2_s)}$ must intersect $\tilde{\gamma}^*$.

Since, by construction, $\overline{\varphi}(V^3_s) \in \tilde{\gamma}^*$, then the maximal distance between $\overline{\varphi}(V^3_s)$ and $\tilde{\gamma}^* \cap \gamma_{\varphi(V^1_s)\varphi(V^2_s)}$ is bounded by $\mathcal{H}^1(\tilde{\gamma}^*) \leq 4\eta_1 < \delta_1$. Then, from (2.2) of Lemma 2.4, we get

$$\begin{aligned}
\rho_{\overline{\varphi}(\partial T)}\big(\overline{\varphi}(V^1_s), \overline{\varphi}(V^3_s)\big) &+ \rho_{\overline{\varphi}(\partial\Delta)}\big(\overline{\varphi}(V^3_s), \overline{\varphi}(V^2_s)\big)\\
&< \rho_{\varphi(\partial\mathcal{Q})}\big(\varphi(V^1_s), \varphi(V^2_s)\big) + \frac{\varepsilon}{2(h+\ell)}.
\end{aligned}$$

Therefore, we can compute

$$
\begin{aligned}
(3.4)\quad & \Psi_0(\overline{\varphi}\rceil_{\partial T}) + \Psi_0(\overline{\varphi}\rceil_{\partial\Delta}) \\
&= \int_0^{t^*} \rho_{\overline{\varphi}(\partial T)}\big(\overline{\varphi}(H_t^1), \overline{\varphi}(H_t^2)\big)dt + \int_{(H_{t^*}^1)_1}^{(H_{t^*}^2)_1} \rho_{\overline{\varphi}(\partial T)}\big(\overline{\varphi}(V_s^1), \overline{\varphi}(V_s^3)\big)ds \\
&\quad + \int_{t^*}^{h} \rho_{\overline{\varphi}(\partial\Delta)}\big(\overline{\varphi}(H_t^1), \overline{\varphi}(H_t^2)\big)dt + \int_{(H_{t^*}^1)_1}^{(H_{t^*}^2)_1} \rho_{\overline{\varphi}(\partial\Delta)}\big(\overline{\varphi}(V_s^3), \overline{\varphi}(V_s^2)\big)ds \\
&\quad + \int_{(0,(H_{t^*}^1)_1)\cup((H_{t^*}^2)_1,\ell)} \rho_{\overline{\varphi}(\partial\Delta)}\big(\overline{\varphi}(V_s^1), \overline{\varphi}(V_s^2)\big)ds \\
&\le \int_0^h \rho_{\varphi(\partial\mathcal{Q})}\big(\varphi(V_t^1), \varphi(V_t^2)\big)dt + \int_0^\ell \rho_{\varphi(\partial\mathcal{Q})}\big(\varphi(V_s^1), \varphi(V_s^2)\big)ds \\
&\quad + (\ell + h)\frac{\varepsilon}{2(h+\ell)} \\
&\le \Psi_0(\varphi) + \frac{\varepsilon}{2}.
\end{aligned}
$$

To finish the proof, it suffices to prove the claim in the case that $\mathcal{Q}$ is class (iii). However, this is just a question of repeating the argument above for the opposing vertex since the horizontal side of $\mathcal{Q}$ was not used at any point. Finally, in every case, $\partial\Delta$ has two sides parallel to α. ■

The next result that we present concerns the extension of the boundary values inside a convex polygon Δ having two non-consecutive parallel sides. This is an opportune generalization of the analogous results on rectangles proved in [13, Lemma 2.11]. Loosely speaking, there are two differences between the current setting and the one considered in [13, Lemma 2.11]. In [13], the rectangular domain is partitioned in rectangular strips, while here the polygon Δ is partitioned into strips that are not necessarily rectangular, which we later split further into a rectangle and two triangles, one at either end.

LEMMA 3.3 (Skeleton-strips). *Let $h, \ell > 0$, let $\Delta \subset [0,\ell] \times [0,h]$ be a convex polygon and let $[0,\ell] \times [0,h]$ be the smallest rectangle containing Δ. Further, assume that $\partial\Delta$ has a pair of horizontal sides, one of which lies in $[0,\ell] \times \{0\}$ and the other in $[0,\ell] \times \{h\}$. For every $\varphi : \partial\Delta \to \mathbb{R}^2$ continuous piecewise linear injective map and for every $\varepsilon > 0$, there exist $M \in \mathbb{N}$ and values*

$$0 = t_0 < t_1 < \cdots < t_{M-1} < t_M = h$$

such that the following properties hold:

(i) $t_{i+1} - t_i < \varepsilon$ *for every* $i = 0, \ldots, M-1$,

(ii) *for every* $i = 0, \dots, M-1$, *call* $S_i := \Delta \cap (\mathbb{R} \times [t_i, t_{i+1}])$ *(which we call a horizontal strip). Then,* φ *is linear on* I_i^1 *and* I_i^2, *where* I_i^1, I_i^2 *are the two non-horizontal segments of* $\partial S_i \cap \partial\Delta$. *Moreover,*

$$\mathcal{H}^1\big(\varphi(I_i^1)\big) + \mathcal{H}^1\big(\varphi(I_i^2)\big) < \varepsilon, \tag{3.5}$$

(iii) *there exists* $\overline{\varphi} : \bigcup_{i=0}^{M-1} \partial S_i \to \mathbb{R}^2$ *a continuous, piecewise linear, injective map such that* $\overline{\varphi} = \varphi$ *on* $\partial\Delta$ *and*

$$\sum_{i=0}^{M-1} \Psi_0(\overline{\varphi}_{\rceil \partial S_i}) \le \Psi_0(\varphi) + \varepsilon, \tag{3.6}$$

(iv) *for every* $i = 0, \dots, M-1$, *the quadrilateral* S_i *can be decomposed in the essentially disjoint union* $T_i^1 \cup R_i \cup T_i^2$, *where* R_i *is a rectangle with horizontal and vertical sides and* T_i^1, T_i^2 *are right angle triangles whose hypotenuses are* I_i^1, I_i^2 *the two segments of* $\partial\Delta \cap \partial S_i$,

(v) *there exists* $\widetilde{\varphi} : \bigcup_{i=0}^{M-1} \partial T_i^1 \cup \partial R_i \cup \partial T_i^2 \to \mathbb{R}^2$ *a continuous, piecewise linear, injective map such that* $\widetilde{\varphi} = \overline{\varphi}$ *on* $\bigcup_{i=0}^{M-1} \partial S_i$, *and*

$$\sum_{i=0}^{M-1} \big(\Psi_0(\widetilde{\varphi}_{\rceil \partial T_i^1}) + \Psi_0(\widetilde{\varphi}_{\rceil \partial R_i}) + \Psi_0(\widetilde{\varphi}_{\rceil \partial T_i^2})\big) \le \Psi_0(\varphi) + \varepsilon. \tag{3.7}$$

PROOF. Let $\varepsilon > 0$ be fixed. Throughout the proof, we denote by $\mathcal{P}$ the polygon of boundary $\varphi(\partial\Delta)$.

Step I. Finding the value M *and fixing* $(t_i)_{i=0,\dots,M}$. There are a finite number of vertices of Δ; call it M_1. There are similarly a finite number of vertices of $\mathcal{P}$; call it M_2. Since $|D_\tau\varphi| \in L^\infty(\partial\Delta)$, we have that any segment on $\partial\Delta$ of length at most $\varepsilon(1 + \|D_\tau\varphi\|_\infty)^{-1}$ has image whose length is at most ε. Further, since $\mathcal{H}^1(\partial\Delta) < \infty$, we find a number M_3 bounded by $\varepsilon^{-1}\mathcal{H}^1(\partial\Delta)(1 + \|D_\tau\varphi\|_\infty)$ splitting $\partial\Delta$ into M_3 segments; then, the length of the segments and their images is bounded by ε.

We take the set of all $\tilde{t} \in (0, h)$ as the t-coordinates such that Δ has a vertex with t-coordinate equal to $\tilde{t}$ (their number is bounded by M_1); further, all $\tilde{t} \in (0, h)$ such that $\partial_\tau\varphi$ does not exist (their number is bounded by M_2), and then add a finite number (bounded by M_3) of $\tilde{t}$ such that whenever $\tilde{t}, \tilde{t}^*$ are a pair of neighbors (with respect to the order $<$), we have

$$|H_{\tilde{t}}^1 - H_{\tilde{t}^*}^1| < \varepsilon > |H_{\tilde{t}}^2 - H_{\tilde{t}^*}^2| \quad \text{and} \quad |\mathbf{H}_{\tilde{t}}^1 - \mathbf{H}_{\tilde{t}^*}^1| < \varepsilon > |\mathbf{H}_{\tilde{t}}^2 - \mathbf{H}_{\tilde{t}^*}^2|,$$

where we use the convention that the letters in bold style $\mathbf{H}$ correspond to $\varphi(H)$. Indexing this set from $\{1, \dots, M-1\}$ and calling $t_0 = 0$ and $t_M = h$, we determine t_i and the

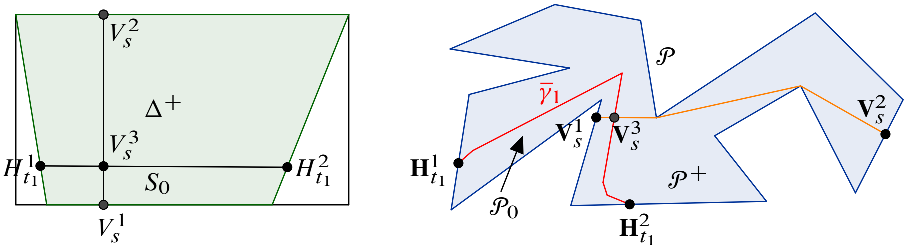

FIGURE 3. The figure shows the slicing of the set Δ into $\Delta^+ \cup S_0$ by the horizontal line $\mathbb{R} \times \{t_1\}$ and $\mathcal{P}$ into $\mathcal{P}^+ \cup \mathcal{P}_0$ by the modified geodesic called $\overline{\gamma}_1$.

number M, where $M \leq M_1 + M_2 + M_3$. We define the strips $S_i = (\mathbb{R} \times [t_i, t_{i+1}]) \cap \Delta$. They are all convex quadrilaterals with two horizontal sides.

Step II. Definition of the curve $\overline{\gamma}_1 = \overline{\varphi}(\Delta \cap (\mathbb{R} \times \{t_1\}))$ and the polygons $\Delta^+ \cup S_0 = \Delta$ and $\mathcal{P}^+ \cup \mathcal{P}_0 = \mathcal{P}$. The goal of this step is to define the piecewise linear curve $\overline{\gamma}_1$, internal to $\mathcal{P}$, which will be the image of the segment $H_{t_1}^1 H_{t_1}^2$ in a map $\overline{\varphi}$ extending φ. The precise parametrization of $\overline{\varphi}$ will be presented in the next step; here we only aim to define the curve $\overline{\gamma}_1 \subset \mathcal{P}$.

Our argument is recursive and so we deal with the first curve $\overline{\gamma}_1$ defined on $H_{t_1}^1 H_{t_1}^2$ separating Δ into S_0 and $\Delta \cap \mathbb{R} \times [t_1, h] = \Delta^+$ (see Figure 3). Similarly, the curve $\overline{\gamma}_1$ means dividing the polygon $\mathcal{P}$ into two further polygons: a polygon $\mathcal{P}_0$ (which will be the image of S_0) containing the curve $\varphi(H_{t_0}^1 H_{t_0}^2)$ and another polygon $\mathcal{P}^+$ (which will be the image of Δ^+) (see Figure 3).

Since $\mathcal{P}$ is a non-degenerate polygon, let $\overline{\delta}(\mathcal{P}) > 0$ be the parameter of Definition 2.2 and let $\delta_1 > 0$ be so small that

$$\delta_1 < \min\left\{\overline{\delta}(\mathcal{P}), \frac{\varepsilon(t_1 - t_0)}{8h\mathcal{H}^1(\partial\mathcal{P})}, \frac{h}{2^3}, \frac{\varepsilon}{2}\right\}$$

and Lemma 2.4 applies with δ_1 for $\mathcal{P}$ and $\dfrac{\varepsilon(t_1 - t_0)}{8h(\ell + h)}$.

We define

$$\overline{\gamma}_1 \text{ as a } \delta_1\text{-modification of the geodesic in } \mathcal{P} \text{ connecting } \mathbf{H}_{t_1}^1 \text{ and } \mathbf{H}_{t_1}^2.$$

Step III. Definition of $\overline{\varphi}$ on ∂S_0. In this step, we care about the definition of $\overline{\varphi}$ on ∂S_0. More precisely, we let $\overline{\varphi} = \varphi$ on $\partial\Delta$ and we specify the parametrization

$$\overline{\varphi} : \partial\Delta^+ \cap \big(\mathbb{R} \times \{t_1\}\big) \to \overline{\gamma}_1$$

so that $\overline{\varphi}$ is continuous, injective and piecewise linear, and

$$\Psi_0(\overline{\varphi}_{\rceil \partial S_0}) + \Psi_0(\overline{\varphi}_{\rceil \partial\Delta^+}) \leq \Psi_0(\varphi) + \frac{\varepsilon}{2h}(t_1 - t_0). \tag{3.8}$$

Let us observe that thanks to Lemma 2.7 it is enough to look for a continuous and injective parametrization $\psi : \partial\Delta^+ \cup \partial S_0 \to \mathbb{R}^2$ coinciding with φ on $\partial\Delta$ such that (3.8) holds for ψ with error $\frac{\varepsilon}{4h}(t_1 - t_0)$; namely,

$$\Psi_0(\psi_{\rceil \partial S_0}) + \Psi_0(\psi_{\rceil \partial\Delta^+}) < \Psi_0(\varphi) + \frac{\varepsilon}{4h}(t_1 - t_0).$$

Indeed, the correct $\overline{\varphi}$ can be found as a δ-linearization of ψ for some δ small enough depending on $\frac{\varepsilon}{4h}(t_1 - t_0)$ such that

$$\Psi_0(\overline{\varphi}_{\rceil \partial S_0}) + \Psi_0(\overline{\varphi}_{\rceil \partial\Delta^+}) < \Psi_0(\psi_{\rceil \partial S_0}) + \Psi_0(\psi_{\rceil \partial\Delta^+}) + \frac{\varepsilon}{4h}(t_1 - t_0).$$

Thanks to our choice of δ_1 and the fact that $\overline{\gamma}_1$ is a δ_1-modification with variable endpoints of the geodesic connecting $\mathbf{H}^1_{t_1}$ and $\mathbf{H}^2_{t_1}$, hence splitting $\mathscr{P}$ into the two polygons $\mathscr{P}_0$ and $\mathscr{P}^+$, we can apply Lemma 2.4 to get that

$$\begin{aligned} \rho_{\mathscr{P}_0}\big(\mathbf{H}^1_t, \mathbf{H}^2_t\big) &\le \rho_{\mathscr{P}}\big(\mathbf{H}^1_t, \mathbf{H}^2_t\big) + \frac{\varepsilon(t_1 - t_0)}{8h(\ell + h)} \quad \text{for any } t_0 < t < t_1, \\ \rho_{\mathscr{P}^+}\big(\mathbf{H}^1_t, \mathbf{H}^2_t\big) &\le \rho_{\mathscr{P}}\big(\mathbf{H}^1_t, \mathbf{H}^2_t\big) + \frac{\varepsilon(t_1 - t_0)}{8h(\ell + h)} \quad \text{for any } t_1 < t < t_M. \end{aligned} \tag{3.9}$$

For short, denote $c_1 = (H^1_{t_1})_1$ and $c_2 = (H^2_{t_1})_1$. Then, $0 \le c_1 < c_2 \le \ell$. For every $0 < s < \ell$, we call γ_s the geodesic inside $\mathscr{P}$ connecting $\mathbf{V}^1_s$ and $\mathbf{V}^2_s$. Moreover, whenever $c_1 < s < c_2$, we also set $V^3_s := (s, t_1)$ the point in the intersection of $H^1_{t_1} H^2_{t_1}$ with $V^1_s V^2_s$. For every $s \in (0, c_1) \cup (c_2, \ell)$, we have that either $V^1_s, V^2_s \in S_0$ and using Lemma 2.4,

$$\rho_{\mathscr{P}_0}(\mathbf{V}^1_s, \mathbf{V}^2_s) \le \rho_{\mathscr{P}}(\mathbf{V}^1_s, \mathbf{V}^2_s) + \frac{\varepsilon(t_1 - t_0)}{8h(\ell + h)},$$

or $V^1_s, V^2_s \in \Delta^+$ and by Lemma 2.4,

$$\rho_{\mathscr{P}^+}(\mathbf{V}^1_s, \mathbf{V}^2_s) \le \rho_{\mathscr{P}}(\mathbf{V}^1_s, \mathbf{V}^2_s) + \frac{\varepsilon(t_1 - t_0)}{8h(\ell + h)}.$$

The two equations above can be expressed simultaneously as

$$\max\big\{\rho_{\mathscr{P}_0}(\mathbf{V}^1_s, \mathbf{V}^2_s), \rho_{\mathscr{P}^+}(\mathbf{V}^1_s, \mathbf{V}^2_s)\big\} \le \rho_{\mathscr{P}}(\mathbf{V}^1_s, \mathbf{V}^2_s) + \frac{\varepsilon(t_1 - t_0)}{8h(\ell + h)} \tag{3.10}$$

for all $s \in (0, c_1) \cup (c_2, \ell)$.

On the other hand, whenever $s \in (c_1, c_2)$, the points $\mathbf{V}^1_s \in \mathscr{P}_0$ and $\mathbf{V}^2_s \in \mathscr{P}^+$; thus, the geodesic γ_s necessarily intersects $\overline{\gamma}_1$. Let κ be the (injective and continuous) constant-speed parametrization of $\overline{\gamma}_1$ from $[0, \mathscr{H}^1(\overline{\gamma}_1)]$, $\kappa(0) = \mathbf{H}^1_{t_1}$ and $\kappa(\mathscr{H}^1(\overline{\gamma}_1)) = \mathbf{H}^2_{t_1}$. For every $s \in (c_1, c_2)$, we then let $\mathbf{X}(s)$ be the point in $\gamma_s \cap \overline{\gamma}_1$ such that

$$\mathbf{X}(s) = \kappa\big(\max\big\{x \in \big[0, \mathscr{H}^1(\overline{\gamma}_1)\big] : \kappa(x) \in \gamma_s \cap \overline{\gamma}_1\big\}\big).$$

Then, thanks to Lemma 2.3, it is easy to see that the map $s \to \kappa^{-1}(\mathbf{X}(s))$ is non-decreasing; therefore, if $c_1 < s < s' < c_2$, then

$$\mathbf{X}(s') \in \kappa\big(\big[\kappa^{-1}(\mathbf{X}(s)), \mathcal{H}^1(\overline{\gamma}_1)\big]\big).$$

Notice that, in general, the function $s \to \kappa^{-1} \circ \mathbf{X}(s)$ is not continuous, nor injective nor surjective. However, any one-dimensional monotone function can be approximated uniformly by strictly monotone functions. Further, it is always possible to slightly modify these strictly monotone approximations in such a way that they become continuous and the price of this is loosing the control on the uniform distance from the original function on a subset whose measure can be made as small as desired. Then, for every $\sigma > 0$, it is always possible to find a continuous bijection $\mathbf{X}_\sigma$ of $[c_1, c_2]$ onto $\overline{\gamma}_1$ such that

$$\mathcal{H}^1(J_\sigma) < \sigma \quad \text{where } J_\sigma := \big\{s \in (c_1, c_2) : \big|\mathbf{X}_\sigma(s) - \mathbf{X}(s)\big| > \sigma\big\}. \tag{3.11}$$

We can then fix $\sigma = \frac{\delta_1}{2}$ and apply Lemma 2.4 to get that

$$\rho_{\mathcal{P}_0}\big(\mathbf{V}_s^1, \mathbf{X}_{\frac{\delta_1}{2}}(s)\big) + \rho_{\mathcal{P}^+}\big(\mathbf{X}_{\frac{\delta_1}{2}}(s), \mathbf{V}_s^2\big) \le \rho_{\mathcal{P}}(\mathbf{V}_s^1, \mathbf{V}_s^2) + \frac{\varepsilon(t_1 - t_0)}{8h(\ell + h)} \tag{3.12}$$

for all $s \in (c_1, c_2) \setminus J_{\frac{\delta_1}{2}}$. On the other hand, we have the trivial estimate

$$\rho_{\mathcal{P}_0}\big(\mathbf{V}_s^1, \mathbf{X}_{\frac{\delta_1}{2}}(s)\big) + \rho_{\mathcal{P}^+}\big(\mathbf{X}_{\frac{\delta_1}{2}}(s), \mathbf{V}_s^2\big) \le \mathcal{H}^1(\partial\mathcal{P}) + \mathcal{H}^1(\overline{\gamma}_1) < 2\mathcal{H}^1(\partial\mathcal{P}) \tag{3.13}$$

for all $s \in J_{\frac{\delta_1}{2}}$.

We define $\psi : \partial\Delta^+ \cup \partial S_0 \to \mathbb{R}^2$ as $\psi = \varphi$ on $\partial\Delta$ and $\psi(V_s^3) = \mathbf{X}_{\frac{\delta_1}{2}}(s)$ for every $s \in (c_1, c_2)$. In particular, ψ is continuous and injective and fails to be piecewise linear only on the segment $H_{t_1}^1 H_{t_1}^2$. Moreover, gathering together (3.9), (3.10), (3.12) and (3.13), we deduce that

$$\begin{aligned}
&\Psi_0(\psi_{\rceil \partial S_0}) + \Psi_0(\psi_{\rceil \partial\Delta^+}) \\
&\quad = \int_0^{t_1} \rho_{\mathcal{P}_0}\big(\psi(H_t^1), \psi(H_t^2)\big)dt + \int_{t_1}^h \rho_{\mathcal{P}^+}\big(\psi(H_t^1), \psi(H_t^2)\big)dt \\
&\quad\quad + \int_{(0,c_1)\cup(c_2,\ell)} \min\big\{\rho_{\mathcal{P}_0}(\mathbf{V}_s^1, \mathbf{V}_s^2) + \rho_{\mathcal{P}^+}(\mathbf{V}_s^1, \mathbf{V}_s^2)\big\} \\
&\quad\quad + \int_{(c_1,c_2)\setminus J_{\frac{\delta_1}{2}}} \rho_{\mathcal{P}_0}\big(\psi(V_s^1), \psi(V_s^3)\big) + \rho_{\mathcal{P}^+}\big(\psi(V_s^3), \psi(V_s^2)\big) \\
&\quad\quad + \int_{J_{\frac{\delta_1}{2}}} \rho_{\mathcal{P}_0}\big(\psi(V_s^2), \psi(V_s^3)\big) + \rho_{\mathcal{P}^+}\big(\psi(V_s^3), \psi(V_s^2)\big)
\end{aligned}$$

$$
\begin{aligned}
&\leq \int_0^h \rho_{\mathcal{P}}\big(\varphi(H_t^1), \varphi(H_t^2)\big)dt + \int_{(0,\ell)\setminus J_{\frac{\delta_1}{2}}} \rho_{\mathcal{P}}\big(\varphi(V_s^1), \varphi(V_s^2)\big)ds \\
&\quad + \frac{\varepsilon(t_1 - t_0)}{8h(\ell + h)}\big(\ell + h - \mathcal{H}^1(J_{\frac{\delta_1}{2}})\big) + \mathcal{H}^1(J_{\frac{\delta_1}{2}})2\mathcal{H}^1(\partial\mathcal{P}) \\
&\leq \Psi_0(\varphi) + \frac{\varepsilon}{4h}(t_1 - t_0),
\end{aligned}
$$

where in the last inequality we used (3.11) and the fact that $\delta_1 < \frac{\varepsilon(t_1 - t_0)}{8h\mathcal{H}^1(\partial\mathcal{P})}$.

Finally, thanks to Lemma 2.7 and the considerations of the first part of the step, we can find a function $\overline{\varphi} : \partial\Delta^+ \cup \partial S_0 \to \mathbb{R}^2$ that is continuous, injective and piecewise linear and such that (3.8) holds.

Step IV. Definition of $\widetilde{\varphi}$ on $\partial T_0^1 \cup \partial R_0 \cup \partial T_0^2$. In this step, we further subdivide the strip S_0 in the essentially disjoint union of two triangles T_0^1, T_0^2 and a rectangle R_0 with the following properties. The rectangle R_0 is the biggest rectangle with horizontal and vertical sides inside S_0, such that the horizontal sides are contained in ∂S_0, while T_0^1, T_0^2 are the two disjoint right-angle triangles containing I_0^1, I_0^2, respectively.

We continue to define some new

$$\widetilde{\varphi} : \partial\Delta^+ \cup \partial T_0^1 \cup \partial R_0 \cup \partial T_0^2 \to \mathbb{R}^2$$

coinciding with $\overline{\varphi}$ on $\partial\Delta^+ \cup \partial S_0$ such that $\widetilde{\varphi}$ is injective, continuous and piecewise linear and satisfies the following estimate:

$$\Psi_0(\widetilde{\varphi}_{\rceil\partial T_0^1}) + \Psi_0(\widetilde{\varphi}_{\rceil\partial R_0}) + \Psi_0(\widetilde{\varphi}_{\rceil\partial T_0^2}) \leq \Psi_0(\overline{\varphi}_{\rceil\partial S_0}) + \frac{\varepsilon}{2h}(t_1 - t_0). \tag{3.14}$$

Let us emphasize that $\widetilde{\varphi}$ will be defined so that $\widetilde{\varphi}(\partial T_0^1 \cup \partial R_0 \cup \partial T_0^2) \subset \mathcal{P}_0$.

We denote by d_1 and d_2 the two values such that the projection of ∂S_0 onto $\mathbb{R} \times \{0\}$ is exactly $[d_1, d_2] \times \{0\}$, and we call x_1 and x_2 those values for which the projection of ∂R_0 onto $\mathbb{R} \times \{0\}$ is the segment $[x_1, x_2] \times \{0\}$. Notice that $d_1 \leq c_1 \leq x_1 < x_2 \leq c_2 \leq d_2$.

Since $\mathcal{P}_0$ is a non-degenerate polygon, we let $\overline{\delta}(\mathcal{P}_0)$ be the parameter of Definition 2.2 and take

$$\delta_1' < \min\left\{\overline{\delta}(\mathcal{P}_0), \frac{\varepsilon(t_1 - t_0)}{32h\mathcal{H}^1(\partial\mathcal{P}_0)}\right\}$$

and Lemma 2.4 applies with δ_1' for $\mathcal{P}_0$ and $\frac{\varepsilon(t_1 - t_0)}{16h(\ell + h)}$.

Let now ν_{x_1} be the geodesic inside $\mathcal{P}_0$ connecting $\overline{\varphi}(V_{x_1}^1)$ and $\overline{\varphi}(V_{x_1}^3)$ and let $\overline{\nu}_{x_1}$ be its δ_1'-modification in the sense of Definition 2.2. In particular, $\overline{\nu}_{x_1}$ splits $\mathcal{P}_0$ into two non-degenerate polygons $\mathcal{P}_0^1$ and $\mathcal{U}$, where $\mathcal{P}_0^1$ contains $\overline{\varphi}(I_0^1)$ and $\mathcal{U}$ contains $\overline{\varphi}(I_0^2)$, where $I_0^{1,2}$ are the two non-horizontal segments of $\partial S_0 \cap \partial\Delta$. This situation is depicted in Figure 4.

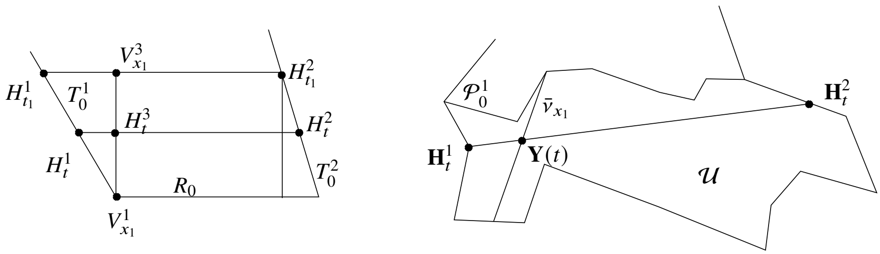

FIGURE 4. The sets T^1_0, T^2_0, R_0, $\mathcal{P}^1_0$ and $\mathcal{U}$ in Step IV.

Thanks to Lemma 2.4, we have

$$\begin{aligned}\rho_{\mathcal{U}}\big(\overline{\varphi}(V^1_s),\overline{\varphi}(V^3_s)\big)&\le\rho_{\mathcal{P}_0}\big(\overline{\varphi}(V^1_s),\overline{\varphi}(V^3_s)\big)+\frac{\varepsilon(t_1-t_0)}{16h(\ell+h)},\ \text{for all } s\in(x_1,x_2),\\ \rho_{\mathcal{U}}\big(\overline{\varphi}(V^1_s),\overline{\varphi}(V^2_s)\big)&\le\rho_{\mathcal{P}_0}\big(\overline{\varphi}(V^1_s),\overline{\varphi}(V^2_s)\big)+\frac{\varepsilon(t_1-t_0)}{16h(\ell+h)},\ \text{for all } s\in(x_2,d_2),\end{aligned}\tag{3.15}$$

while for all $s\in(d_1,x_1)$,

$$\rho_{\mathcal{P}^1_0}\big(\overline{\varphi}(V^1_s),\overline{\varphi}(V^2_s)\big)\le\rho_{\mathcal{P}_0}\big(\overline{\varphi}(V^1_s),\overline{\varphi}(V^2_s)\big)+\frac{\varepsilon(t_1-t_0)}{16h(\ell+h)}.\tag{3.16}$$

We continue similarly as described in Step III. For every $t\in(0,t_1)$, we denote the point $\overline{\varphi}(H^1_t)=\mathbf{H}^1_t\in\mathcal{P}^1_0$ and $\overline{\varphi}(H^2_t)=\mathbf{H}^2_t\in\mathcal{U}$; thus, the geodesic ζ_t connecting $\mathbf{H}^1_t$ and $\mathbf{H}^2_t$ inside $\mathcal{P}_0$ must intersect $\overline{v}_{x_1}$. So also in this case, for every $t\in(0,t_1)$, we can find a map $\mathbf{Y}(t)$ identifying the last point of the intersection $\zeta_t\cap\overline{v}_{x_1}$ running $\overline{v}_{x_1}$ from $\overline{\varphi}(V^1_{x_1})$ to $\overline{\varphi}(V^3_{x_1})$. Moreover, exactly as explained in Step III, we can find a continuous and injective approximation $\mathbf{Y}_{\frac{\delta'_1}{2}}$ such that

$$\big|\mathbf{Y}(t)-\mathbf{Y}_{\frac{\delta'_1}{2}}(t)\big|<\frac{\delta'_1}{2}\quad\text{for all } t\in(0,t_1)\setminus J^1_{\frac{\delta'_1}{2}}\ \text{and}\ \mathcal{H}^1\big(J^1_{\frac{\delta'_1}{2}}\big)<\frac{\delta'_1}{2}.$$

So if we now call $H^3_t:=(t,x_1)$, then we can define $\psi^1:\partial T^1_0\cup\partial S_0\to\mathbb{R}^2$ in this way: $\psi^1=\overline{\varphi}$ on ∂S_0 and

$$\psi^1(H^3_t)=\mathbf{Y}_{\frac{\delta'_1}{2}}(t)\quad\text{for all } t\in(0,t_1).$$

Then, the map ψ^1 is continuous and injective and fails to be piecewise linear only on the segment $\{x_1\}\times[0,t_1]$. Using Lemma 2.4, for all $t\in(0,t_1)\setminus J^1_{\frac{\delta'_1}{2}}$, we can estimate

$$\begin{aligned}&\rho_{\mathcal{P}^1_0}\big(\psi^1(H^1_t),\psi^1(H^3_t)\big)+\rho_{\mathcal{U}}\big(\psi^1(H^3_t),\psi^1(H^2_t)\big)\\&\qquad\le\rho_{\mathcal{P}_0}\big(\overline{\varphi}(H^1_t),\overline{\varphi}(H^2_t)\big)+\frac{\varepsilon(t_1-t_0)}{16h(\ell+h)},\end{aligned}\tag{3.17}$$

while for the remaining $t \in J^l_{\frac{\delta_1'}{2}}$, we get

$$\rho_{\mathcal{P}_0^1}\big(\psi^1(H_t^1), \psi^1(H_t^3)\big) + \rho_{\mathcal{U}}\big(\psi^1(H_t^3), \psi^1(H_t^2)\big) \leq \mathcal{H}^1(\overline{\nu}_{x_1}) + \mathcal{H}^1(\partial\mathcal{P}_0) \leq 2\mathcal{H}^1(\partial\mathcal{P}_0). \tag{3.18}$$

Then, (3.15), (3.16), (3.17) and (3.18) give

$$\begin{aligned}
&\Psi_0\big(\psi^1_{\rceil\partial T_0^1}\big) + \Psi_0\big(\psi^1_{\rceil\partial(S_0\setminus T_0^1)}\big)\\
&\leq \int_{d_1}^{c_1} \rho_{\mathcal{P}_0}\big(\overline{\varphi}(V_s^1), \overline{\varphi}(V_s^2)\big) + \frac{\varepsilon(t_1-t_0)}{16h(\ell+h)}ds\\
&\quad + \int_{c_1}^{c_2} \rho_{\mathcal{P}_0}\big(\overline{\varphi}(V_s^1), \overline{\varphi}(V_s^3)\big) + \frac{\varepsilon(t_1-t_0)}{16h(\ell+h)}ds\\
&\quad + \int_{c_2}^{d_2} \rho_{\mathcal{P}_0}\big(\overline{\varphi}(V_s^1), \overline{\varphi}(V_s^2)\big) + \frac{\varepsilon(t_1-t_0)}{16h(\ell+h)}ds\\
&\quad + \int_{(0,t_1)\setminus J^1_{\frac{\delta_1'}{2}}} \rho_{\mathcal{P}_0}\big(\overline{\varphi}(H_t^1), \overline{\varphi}(H_t^2)\big) + \frac{\varepsilon(t_1-t_0)}{16h(\ell+h)}dt + 2\mathcal{H}^1\big(J^1_{\frac{\delta_1'}{2}}\big)\mathcal{H}^1(\partial\mathcal{P}_0)\\
&\leq \Psi_0(\overline{\varphi}_{\rceil\mathcal{P}_0}) + \frac{\varepsilon}{8h}(t_1-t_0),
\end{aligned}$$

and, as in Step III, Lemma 2.7 ensures that there is some continuous, injective and piecewise linear map $\widetilde{\varphi}^1 : \partial T_0^1 \cup \partial S_0 \to \mathbb{R}^2$ such that $\widetilde{\varphi}^1 = \psi^1 = \overline{\varphi}$ on ∂S_0 and

$$\Psi_0\big(\widetilde{\varphi}^1_{\rceil\partial T_0^1}\big) + \Psi_0\big(\widetilde{\varphi}^1_{\rceil\partial(S_0\setminus T_0^1)}\big) \leq \Psi_0(\overline{\varphi}_{\rceil\mathcal{P}_0}) + \frac{\varepsilon}{4h}(t_1-t_0).$$

To conclude the step, we need to repeat the very same argument on $\widetilde{\varphi}^1_{\rceil\partial(S_0\setminus T_0^1)}$ by replacing $\mathcal{P}_0$ with $\mathcal{U}$ and considering a δ_1''-modification of ν_{x_2}, where δ_1'' is chosen so that

$$\delta_1'' < \min\left\{\overline{\delta}(\mathcal{U}), \frac{\varepsilon(t_1-t_0)}{32\ell\mathcal{H}^1(\partial\mathcal{U})}\right\}$$

and Lemma 2.4 applies with δ_1'' for $\mathcal{U}$ and $\frac{\varepsilon(t_1-t_0)}{16h(\ell+h)}$.

This would provide a continuous, injective and piecewise linear map $\widetilde{\varphi} : \partial T_0^1 \cup \partial R_0 \cup \partial T_0^2 \to \mathbb{R}^2$ extending $\widetilde{\varphi}^1$ (hence, ultimately, $\overline{\varphi}$) such that

$$\begin{aligned}
\Psi_0(\widetilde{\varphi}_{\rceil\partial T_0^1}) + \Psi_0(\widetilde{\varphi}_{\rceil\partial R_0}) + \Psi_0(\widetilde{\varphi}_{\rceil\partial T_0^2}) &\leq \Psi_0\big(\widetilde{\varphi}^1_{\rceil\partial(S_0\setminus T_0^1)}\big) + \frac{\varepsilon}{4h}(t_1-t_0)\\
&\leq \Psi_0(\overline{\varphi}_{\rceil\mathcal{P}_0}) + \frac{\varepsilon}{2h}(t_1-t_0)
\end{aligned}$$

thus proving (3.14) and concluding the step.

Step V. Recursion and conclusion. In this final step, we want to conclude our construction by recursion. In Steps III and IV, we divided Δ into a new convex polygon with two horizontal sides Δ^+ and a horizontal strip S_0 given by a rectangle R_0 and two triangles T_0^1, T_0^2.

Then, we defined continuous, injective and piecewise linear functions $\overline{\varphi}, \widetilde{\varphi}$ such that $\overline{\varphi} = \widetilde{\varphi}$ on $\partial\Delta^+$ satisfies (by (3.8), (3.14) and our choice of δ_1)

$$\begin{aligned}\Psi_0(\overline{\varphi}_{\rceil \partial S_0}) + \Psi_0(\overline{\varphi}_{\rceil \partial \Delta^+}) &\le \Psi_0(\varphi) + \frac{\varepsilon}{2h}(t_1 - t_0 + 2\delta_1)\\ &\le \Psi_0(\varphi) + \frac{\varepsilon}{2h}(t_1 - t_0) + \frac{\varepsilon}{4}\frac{1}{2},\\ \Psi_0(\widetilde{\varphi}_{\rceil \partial T_0^1}) + \Psi_0(\widetilde{\varphi}_{\rceil \partial R_0}) + \Psi_0(\widetilde{\varphi}_{\rceil \partial T_0^2}) &\le \Psi_0(\overline{\varphi}_{\rceil \partial S_0}) + \frac{\varepsilon}{2h}(t_1 - t_0 + 2\delta_1)\\ &\le \Psi_0(\overline{\varphi}_{\rceil \partial S_0}) + \frac{\varepsilon}{2h}(t_1 - t_0) + \frac{\varepsilon}{4}\frac{1}{2}.\end{aligned}$$

Iterating the construction and choosing for every i the parameter δ_i suitably small depending on t_{i-1}, $\frac{h}{2^{i+1}}$ and the polygon $\mathcal{P}^+ = \mathcal{P} \setminus \bigcup_{j=0}^{i-1} \mathcal{P}_j$, we then find

$$\sum_{i=0}^{M-1} \Psi_0(\overline{\varphi}_{\rceil \partial S_i}) \le \Psi_0(\varphi) + \frac{\varepsilon}{2h}\sum_{i=0}^{M-1}(t_{i+1} - t_i) + \frac{\varepsilon}{4}\sum_{i=0}^{M-1}\frac{1}{2^i}$$

and

$$\begin{aligned}&\sum_{i=0}^{M-1} \Psi_0(\widetilde{\varphi}_{\rceil \partial T_i^1}) + \Psi_0(\widetilde{\varphi}_{\rceil \partial R_i}) + \Psi_0(\widetilde{\varphi}_{\rceil \partial T_i^2})\\ &\qquad \le \sum_{i=0}^{M-1} \Psi_0(\overline{\varphi}_{\rceil \partial S_i}) + \frac{\varepsilon}{2h}\sum_{i=0}^{M-1}(t_{i+1} - t_i) + \frac{\varepsilon}{4}\sum_{i=0}^{M-1}\frac{1}{2^i},\end{aligned}$$

which finally imply (3.6) and (3.7), respectively. ∎

4. PIECEWISE AFFINE EXTENSION

In this section, we investigate two possible finitely piecewise affine homeomorphic extension inside triangles.

LEMMA 4.1 (Extension-direct). *Let $T \subset \mathbb{R}^2$ be a triangle of corners A, B, C such that BC is horizontal and A^* is the intersection of BC with the bisector of the angle at A. If $\varphi : \partial T \to \mathbb{R}^2$ is continuous, injective and linear on each of the segments AB, AC, BA^* and A^*C, then there exists a bi-affine homeomorphism $v : T \to \mathbb{R}^2$ such that $v = \varphi$ on ∂T and*

$$\|Dv\|_0(T) \le \mathcal{H}^1(\varphi(\partial T))\mathcal{H}^1(\partial T).$$

Proof. The proof is immediate, and indeed it is enough to consider the continuous map v which is affine on each of the triangles $T_1 := ABA^*$, $T_2 := BA^*C$. Then, we can compute

$$D_1 v\rceil_{T_1} = \frac{\varphi(A^*) - \varphi(B)}{(A^*)_1 - (B)_1}\mathfrak{L}^2, \quad D_1 v\rceil_{T_2} = \frac{\varphi(C) - \varphi(A^*)}{(C)_1 - (A^*)_1}\mathfrak{L}^2$$

and

$$D_2 v\rceil_{T_1} = \frac{\varphi(A) - \varphi(A^*) - D_1 v\rceil_{T_1}\big((A)_1 - (A^*)_1\big)}{(A)_2 - (A^*)_2}\mathfrak{L}^2,$$
$$D_2 v\rceil_{T_2} = \frac{\varphi(A) - \varphi(A^*) - D_1 v\rceil_{T_2}\big((A)_1 - (A^*)_1\big)}{(A)_2 - (A^*)_2}\mathfrak{L}^2.$$

In particular, one can estimate

$$\begin{aligned}
\|Dv\|_0(T) &= |D_1 v|(T_1) + |D_2 v|(T_1) + |D_1 v|(T_2) + |D_2 v|(T_2)\\
&\le \frac{1}{2}\big|\varphi(A^*) - \varphi(B)\big|\big[\big|(A)_2 - (A^*)_2\big| + \big|(A)_1 - (A^*)_1\big|\big]\\
&\quad + \frac{1}{2}\big|\varphi(A^*) - \varphi(A)\big|\big|(B)_1 - (A^*)_1\big|\\
&\quad + \frac{1}{2}\big|\varphi(C) - \varphi(A^*)\big|\big[\big|(A)_2 - (A^*)_2\big| + \big|(A)_1 - (A^*)_1\big|\big]\\
&\quad + \frac{1}{2}\big|\varphi(A^*) - \varphi(A)\big|\big|(C)_1 - (A^*)_1\big|\\
&\le \mathcal{H}^1\big(\varphi(\partial T)\big)\frac{1}{2}\big(4|A - A^*| + |B - C|\big)\\
&\le \mathcal{H}^1\big(\varphi(\partial T)\big)\frac{1}{2}\big(2|A - B| + 2|A - C| + 2|B - C|\big)\\
&\le \mathcal{H}^1\big(\varphi(\partial T)\big)\mathcal{H}^1(\partial T).
\end{aligned}$$
■

Lemma 4.2 (Extension-indirect). *Let $T \subset \mathbb{R}^2$ be a triangle of corners A, B, C such that AB is horizontal, BC is vertical and let $\varphi : \partial T \to \mathbb{R}^2$ be a continuous, piecewise linear, injective map such that φ is linear on the hypotenuse AC. For every $\varepsilon > 0$, there exists a finitely piecewise affine homeomorphism $v : T \to \mathbb{R}^2$ such that $v = \varphi$ on ∂T and*

$$\|Dv\|_0(T) \le \Psi_0(\varphi) + 242\mathcal{H}^1(\partial T)\mathcal{H}^1\big(\varphi(AC)\big) + \varepsilon.$$

Proof. Let $\varepsilon > 0$ be fixed arbitrary small.

For simplicity of notation, through the proof we refer to β as the internal angle of the corner $A = (0, 0)$ and we will denote $\mathbf{d} := |\varphi(A) - \varphi(C)| = \mathcal{H}^1(\varphi(AC))$ and $d = |A - C|$. Clearly, since the internal angle in B is $\pi/2$, then $\beta \in (0, \pi/2)$.

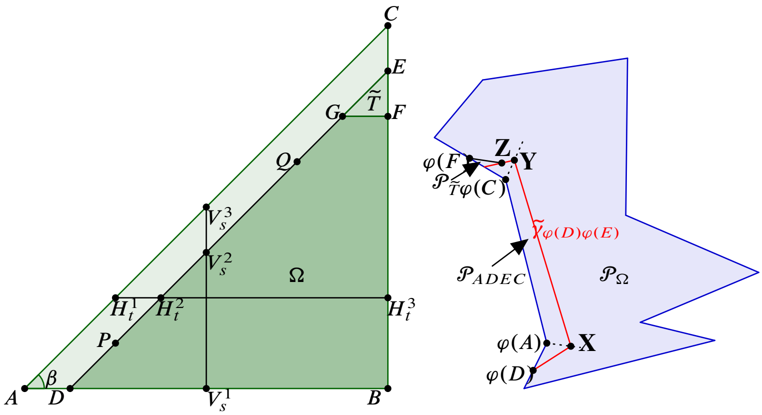

FIGURE 5. The decomposition of T into $T = \Omega \cup \widetilde{T} \cup ADEC$, and the corresponding decomposition of $\operatorname{int}\varphi(\partial T)$ into $\mathcal{P}_\Omega \cup \mathcal{P}_{\widetilde{T}} \cup \mathcal{P}_{ADEC}$.

Since the polygon of boundary $\varphi(\partial T)$ is non-degenerate, then Definition 2.2 provides some constant $\overline{\delta} > 0$, and then we consider

$$\eta < \left\{1, \frac{d}{2}, \frac{\overline{\delta}}{4}, \frac{\mathbf{d}}{14}, \frac{\varepsilon}{4}, \left(1 + \frac{1}{2\tan\beta}\right)^{-1}, \|D\varphi\|_\infty^{-1}\right\}. \tag{4.1}$$

The basic idea of the proof is to find a suitable one-dimensional skeleton Υ inside T, construct a continuous, piecewise linear and injective map $\widetilde{\varphi} : \Upsilon \to \mathbb{R}^2$ coinciding with φ on ∂T and finally perform a suitable piecewise affine extension inside each component of the partition of T identified by Υ.

For clarity, we present the proof in three separate steps.

Step I. Definition of a first skeleton Ξ *and a continuous piecewise linear injective map* $\varphi^1 : \Xi \to \mathbb{R}^2$. In this step, we would like to construct a one-dimensional skeleton of the form $\Xi = \partial T \cup DE \cup FG$, for some suitably chosen points D, E, F, G, and we will define an extension φ^1 of φ on $DE \cup FG$ that is still continuous, piecewise linear and injective. See Figure 5 for an illustration.

We will select D, E, F, G so that

$$D \in AB, \quad E, F \in BC, \quad G \in DE, \quad FG \parallel AB \quad \text{and} \quad DE \parallel AC,$$

satisfying the following estimates:

$$\begin{aligned} &|A - D| < \eta \quad \text{and} \quad |C - E| < \eta, \\ &\mathcal{H}^1\big(\varphi^1(\partial\widetilde{T})\big) < 4\eta, \\ &\Psi_0(\varphi^1_{\rceil\partial\Omega}) \le \Psi_0(\varphi) + (\mathbf{d} + 14\eta)\mathcal{H}^1(\partial T), \end{aligned} \tag{4.2}$$

where $\widetilde{T} \subset T$ is the triangle of corners E, F, G and $\Omega \subset T$ is the trapezoid of corners D, B, F, G.

Having fixed η, by the assumptions on φ, we can choose $D \in AB$ and $E \in BC$ so that

(i) $|A - D| < \eta$ and $|C - E| < \eta$;

(ii) $|\varphi(A) - \varphi(D)| < \eta$ and $|\varphi(C) - \varphi(E)| < \eta$;

(iii) the restriction of φ is linear on AD and CE;

(iv) DE is parallel to AC;

(v) the point $\mathbf{X}$ is on the internal bisector of $\varphi(A)$ and $\mathbf{Y}$ on the internal bisector of $\varphi(C)$ such that $|\varphi(A) - \mathbf{X}| < 2\eta$ and $|\varphi(C) - \mathbf{Y}| < 2\eta$ and the piecewise linear path $\widetilde{\gamma}_{\varphi(D)\varphi(E)} := \varphi(D)\mathbf{X}\mathbf{Y}\varphi(E)$ lies in the interior of $\varphi(\partial T)$ and is a $\overline{\delta}/2$-modification of the geodesic $\gamma_{\varphi(D)\varphi(E)}$ in the sense of Definition 2.2.

Observe that (ii) and (v) imply that $|\varphi(D) - \mathbf{X}|, |\varphi(E) - \mathbf{Y}| < 3\eta$ and also

$$\begin{aligned} \mathcal{H}^1(\widetilde{\gamma}_{\varphi(D)\varphi(E)}) &< |\varphi(D) - \mathbf{X}| + |\mathbf{X} - \mathbf{Y}| + |\varphi(E) - \mathbf{Y}| \\ &< 6\eta + |\varphi(A) - \mathbf{X}| + |\varphi(A) - \varphi(C)| + |\varphi(C) - \mathbf{Y}| \\ &< \mathbf{d} + 10\eta. \end{aligned} \tag{4.3}$$

We find a point F on the segment EB, a point $G \in DE$ and a point $\mathbf{Z} \in [\mathbf{Y}\varphi(E)]$ such that

(vi) $|F - E| < \eta$ and $|\varphi(F) - \varphi(E)| < \eta$;

(vii) φ is linear on EF;

(viii) $|\varphi(E) - \mathbf{Z}| < \eta$ and $[\varphi(F)\mathbf{Z}]$ lies in $\operatorname{int}\varphi(\partial T)$, and as a consequence, $|\varphi(F) - \mathbf{Z}| < 2\eta$;

(ix) $G_2 = F_2$, $|G - E| < \eta(\sin\beta)^{-1}$ and $|G - F| < \eta(\tan\beta)^{-1}$.

This concludes the definition of Ξ. Indeed, (i) ensures the first equation of (4.2), then $\widetilde{T}$ is a right-angle triangle and Ω is a trapezoid inside T.

We now proceed to construct a function $\varphi^1 : \Xi \to \mathbb{R}^2$ extending φ such that the second and third estimates of (4.2) are satisfied. In order to do that, we consider two further auxiliary points $P, Q \in DG$ so that

(x) $|P - D| < \eta$ and $|Q - G| < \eta$;

and we set

$$\varphi^1(P) := \mathbf{X}, \quad \varphi^1(Q) := \mathbf{Y} \quad \text{and} \quad \varphi^1(G) := \mathbf{Z}.$$

We then define $\varphi^1 : \Xi \to \mathbb{R}^2$ so that $\varphi^1 = \varphi$ on ∂T, $\varphi^1_{\rceil DP}$ is the parametrization at constant speed of the segment $\varphi(D)\mathbf{X}$, $\varphi^1_{\rceil PQ}$ is the parametrization at constant speed of the segment $\mathbf{XY}$, $\varphi^1_{\rceil QG}$ is the parametrization at constant speed of the segment $\mathbf{YZ}$, $\varphi^1_{\rceil GE}$ is the parametrization at constant speed of the segment $\mathbf{Z}\varphi(E)$ and, finally, $\varphi^1_{\rceil GF}$ is the parametrization at constant speed of the segment $\mathbf{Z}\varphi(F)$. A sketch of the situation is presented in Figure 5.

The second estimate of (4.2) is a direct consequence of (vi) and (viii); indeed, φ^1 is linear in each of the segments EF, FG, EG, and by triangular inequality, we have

$$\begin{aligned}\mathcal{H}^1\big(\varphi^1(\partial\widetilde{T})\big) &= \big|\varphi(E)-\varphi(F)\big| + \big|\varphi(E)-\mathbf{Z}\big| + \big|\mathbf{Z}-\varphi(F)\big| \\ &\le 2\big(\big|\varphi(E)-\varphi(F)\big| + \big|\varphi(E)-\mathbf{Z}\big|\big) \le 4\eta.\end{aligned}$$

The remaining part of the step is devoted to the proof of the third estimate of (4.2). In the following, X_i is the i-th coordinate of the point X. For every $t \in [0, E_1]$, we denote by H_t^1, H_t^2, H_t^3 the intersections between the horizontal line $\mathbb{R}\times\{t\}$ and the curves AC, DE, BF, respectively. Clearly, $H_0^1 = A$, $H_0^2 = D$, $H_0^3 = B$, $H_{F_2}^2 = G$ and $H_{F_2}^3 = F$. Similarly, for every $s \in [D_1, B_1]$, we denote by V_s^1, V_s^2, V_s^3 the intersections between the vertical line $\{s\}\times\mathbb{R}$ and the sets DB, $DG\cup GF$, AC, respectively. Then,

$$V_{D_1}^1 = V_{D_1}^2 = D, \quad V_{B_1}^3 = C, \quad V_{B_1}^2 = F \quad \text{and} \quad V_{B_1}^1 = B.$$

We recall that, by definition, $\Psi_0(\varphi^1_{\rceil\partial\Omega})$ corresponds to the following quantity:

$$\int_0^{F_1} \rho_{\mathcal{P}_\Omega}\big(\varphi^1(H_t^2), \varphi^1(H_t^3)\big)dt + \int_{D_1}^{B_1} \rho_{\mathcal{P}_\Omega}\big(\varphi^1(V_s^1), \varphi^1(V_s^2)\big)ds,$$

where $\mathcal{P}_\Omega$ is the non-degenerate polygon identified by $\varphi^1(\partial\Omega)$.

Notice that, by construction, the curve $\varphi^1(DG)$ is exactly the piecewise linear curve $\varphi(D)\mathbf{XYZ}$, so $\varphi^1(H_t^2)$ and $\varphi^1(V_s^2)$ will lie on $\varphi(D)\mathbf{XYZ}$ for every $t \in [0, F_2]$ and $s \in [D_1, G_1]$. On the other hand, when $s \in [G_1, B_1]$, we get that $\varphi^1(V_s^2)$ lies on the segment $\mathbf{Z}\varphi(F)$.

Obviously, one can construct a path in $\mathcal{P}_\Omega$ from $\varphi^1(H_t^3)$ to $\varphi^1(H_t^2)$ by following $\gamma_{\varphi(H_t^1)\varphi(H_t^3)} \cap \mathcal{P}_\Omega$ and when necessary going around $\mathcal{P}_{\widetilde{T}}$ on its boundary and travelling along $\widetilde{\gamma}_{\varphi(D)\varphi(E)}$ till one gets to $\varphi^1(H_t^2)$. The length of this curve bounds the length of the geodesic in $\mathcal{P}_\Omega$ between $\varphi^1(H_t^2)$ and $\varphi^1(H_t^3)$. Further,

$$\begin{aligned}(4.4)\qquad &\rho_{\mathcal{P}_\Omega}\big(\varphi^1(H_t^2), \varphi^1(H_t^3)\big) \\ &\le \mathcal{H}^1\big(\gamma_{\varphi(H_t^1)\varphi(H_t^3)} \cap \mathcal{P}_\Omega\big) + \mathcal{H}^1(\partial\mathcal{P}_{\widetilde{T}}) + \mathcal{H}^1(\widetilde{\gamma}_{\varphi(D)\varphi(E)}) \\ &\le \rho_{\varphi(\partial T)}\big(\varphi(H_t^1), \varphi(H_t^3)\big) + \mathbf{d} + 14\eta,\end{aligned}$$

where in the last inequality we used the second of (4.2) and (4.3). Analogously, for every $s \in (D_1, B_1)$, it holds that

$$\begin{aligned}(4.5)\qquad &\rho_{\mathcal{P}_\Omega}\big(\varphi^1(V_s^1), \varphi^1(V_s^2)\big) \\ &\le \mathcal{H}^1\big(\gamma_{\varphi(V_s^1)\varphi(V_s^3)} \cap \mathcal{P}_\Omega\big) + \mathcal{H}^1(\partial\mathcal{P}_{\widetilde{T}}) + \mathcal{H}^1(\widetilde{\gamma}_{\varphi(D)\varphi(E)}) \\ &\le \rho_{\varphi(\partial T)}\big(\varphi(V_s^1), \varphi(V_s^3)\big) + \mathbf{d} + 14\eta.\end{aligned}$$

Gathering the last two estimates together, we obtain

$$
\begin{aligned}
(4.6)\quad \Psi_0(\varphi^1_{\rceil\partial\Omega}) &= \int_0^{F_2} \rho_{\mathcal{P}_\Omega}\big(\varphi^1(H_t^2), \varphi^1(H_t^3)\big)dt + \int_{D_1}^{B_1} \rho_{\mathcal{P}_\Omega}\big(\varphi^1(V_s^1), \varphi^1(V_s^2)\big)ds \\
&\le \int_0^{F_2} \rho_{\varphi(\partial T)}\big(\varphi(H_t^1), \varphi(H_t^3)\big)dt + \int_{D_1}^{B_1} \rho_{\varphi(\partial T)}\big(\varphi(V_s^1), \varphi(V_s^3)\big)ds \\
&\quad + (\mathbf{d} + 14\eta)\big(|F - B| + |D - B|\big) \\
&\le \Psi_0(\varphi) + (\mathbf{d} + 14\eta)\mathcal{H}^1(\partial T),
\end{aligned}
$$

which is exactly the third estimate of (4.2).

Step II. Definition of the final skeleton Υ *and the continuous piecewise linear injective map* $\widetilde{\varphi} : \Upsilon \to \mathbb{R}^2$. The aim of this step is to define a set Υ depicted in Figure 6 and a map $\widetilde{\varphi}$ satisfying the following properties:

(1) the set Υ subdivides T in the essentially disjoint union $\widehat{T} \cup \widetilde{T} \cup \mathbb{P} \cup \bigcup_{i=0}^{M-1} R_i$, where $\mathbb{P}$ is a polygon which is $\widetilde{C}$-bi-Lipschitz ($\widetilde{C}$ is a universal constant) equivalent to the rectangle $[0, |A - C|] \times [0, |A - D| \sin\beta]$, $\widehat{T}$ is a triangle near either A or C, $\widetilde{T}$ is the triangle defined in Step I of corners E, F, G and R_i are M pairwise essentially disjoint rectangles having horizontal and vertical sides.

(2) $\widetilde{\varphi} : \Upsilon \to \mathbb{R}^2$ is piecewise linear, continuous and injective; moreover, it coincides with φ^1 on Ξ (where Ξ is the set defined in Step I) and satisfies the following estimates:

$$
(4.7)\qquad \sum_{i=0}^{M-1} \Psi_0\big(\widetilde{\varphi}_{\rceil\partial R_i}\big) \le \Psi_0(\varphi) + (\mathbf{d} + 14\eta)\mathcal{H}^1(\partial T) + \frac{\eta}{50}.
$$

We use Lemma 3.3 on φ^1 and on Ω with

$$
\xi < \frac{1}{100} \min\left\{ \sin\beta \operatorname{dist}(D, AC), \frac{\eta}{2d} \right\}
$$

and get a number $M \in \mathbb{N}$ and M values

$$
0 = t_0 < t_1 < \cdots < t_{M-1} < t_M = F_2
$$

such that $|t_{i+1} - t_i| < \xi$ for every $i = 1, \ldots, M-1$. We remark that it is also possible to choose $\xi = \varepsilon$. Moreover, it is possible to decompose Ω into the essentially disjoint union of $M - 1$ horizontal strips $S_i = \mathbb{R} \times [t_{i-1}, t_i] \cap \Omega$ such that φ^1 is linear on $I_i := \partial S_i \cap DG$ and on $\partial S_i \cap FB$ and $\mathcal{H}^1(\varphi^1(I_i)) \le \xi$. Further, we deduce from Lemma 3.3 the existence of a continuous, piecewise linear and injective

$$
\varphi^2 : \partial T \cup DE \cup FG \cup \bigcup_{i=1}^{M-1} \partial S_i \to \mathbb{R}^2
$$

such that $\varphi^2 = \varphi^1$ on $\partial T \cup DE \cup FG$ and

$$\sum_{i=0}^{M-1} \Psi_0\big(\varphi^2_{\rceil \partial S_i}\big) \le \Psi_0\big(\varphi^1_{\rceil \partial \Omega}\big) + \xi.$$

A consequence of the above inequality, the choice of ξ and the third estimate of (4.2) is that

$$\begin{aligned}\sum_{i=0}^{M-1} \Psi_0\big(\varphi^2_{\rceil \partial S_i}\big) &\le \Psi_0(\varphi) + (\mathbf{d} + 14\eta)\mathcal{H}^1(\partial T) + \xi \\ &< \Psi_0(\varphi) + (\mathbf{d} + 14\eta)\mathcal{H}^1(\partial T) + \frac{\eta}{100}.\end{aligned} \tag{4.8}$$

We now proceed to construct the skeleton Υ and the final map $\widetilde{\varphi}$. Observe that for every $i = 0, \dots, M-1$, the right-angle triangle T_i of hypotenuse I_i constructed at the exterior of Ω is still contained in T and does not intersect AC. Indeed, by construction, one has that $\mathcal{H}^1(\partial S_i \cap FB) = |t_{i+1} - t_i| < \xi$ and hence $\mathcal{H}^1(I_i) < \frac{\xi}{\sin\beta}$ and the distance between any point of T_i and the segment AC must be at least $\operatorname{dist}(D, AC) - \frac{\xi}{\sin\beta} > \frac{99}{100}\operatorname{dist}(D, AC)$, thus ensuring that $T_i \subset T$. Clearly, $T_i \cap T_j$ is either empty or containing at most one corner (this corresponds to the case where $j = i \pm 1$).

Then, connecting the horizontal and vertical sides of T_i for every $i = 0, \dots, M-1$, we obtain a continuous piecewise linear path Γ connecting D and G that lies inside T and, at the same time, outside the trapezoid Ω. Moreover, by construction, we have that

$$\operatorname{dist}(AC, \Gamma) \ge \frac{99}{100}\operatorname{dist}(D, AC) = \frac{99}{100}|A - D|\sin\beta$$

and

$$\mathcal{H}^1(\Gamma) \le 2|D - G| + |E - G| \le 2|D - E| \le 2h. \tag{4.9}$$

Assuming that β is bounded away from 0 and $\frac{\pi}{2}$, we have that the non-degenerate polygon $\mathbb{P} \subset T$ of boundary $AC \cup AD \cup \Gamma \cup GE \cup EC$ is $\widetilde{C}$-bi-Lipschitz equivalent to a rectangle of side-lengths $|A - C| = d$ and $|A - D|\sin\beta$. On the other hand, if β is very close to 0, it suffices to take away the triangle $\widehat{T}$ with vertexes at A, $A/2 + D/2$ and a third vertex on AC with angle $\pi/2$. Then, the remaining part of $\mathbb{P}$ is again $\widetilde{C}$-bi-Lipschitz equivalent to a rectangle of side-lengths $|A - C| = d$ and $|A - D|\sin\beta$. Similarly, if β is close to $\pi/2$, we subtract a triangle $\widehat{T}$ close to C and then the remaining set is $\widetilde{C}$-bi-Lipschitz equivalent to a rectangle of side-lengths $|A - C| = d$ and $|A - D|\sin\beta$.

Let us now observe that, by construction, the polygon of boundary $DB \cup FB \cup FG \cup \Gamma$ can be seen as the essentially disjoint union of M rectangles R_i of horizontal and vertical sides such that $S_i = R_i \cap \Omega$ and $T_i = R_i \setminus \Omega$.

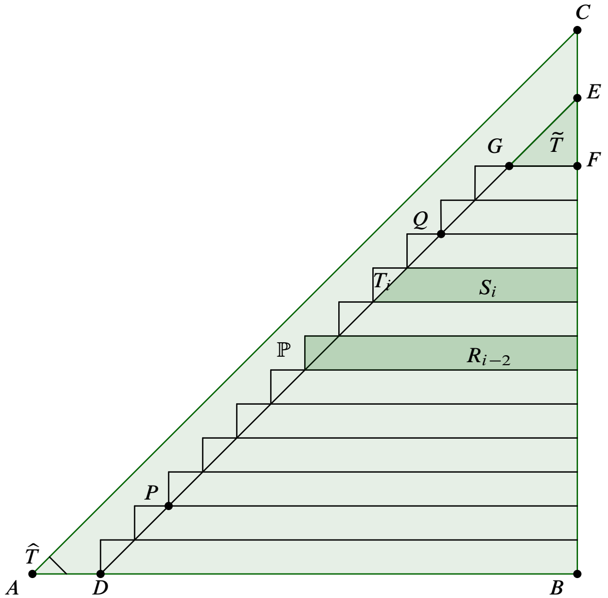

FIGURE 6. The division of T into $T = \mathbb{P} \cup \widetilde{T} \cup \bigcup_i R_i$ and $R_i = T_i \cup S_i$ where one vertex of R_i lies on DE. The set $\mathbb{P}$ is uniformly bi-Lipschitz equivalent to the rectangle $[0, |A - C|] \times [0, |A - D|]$. The usage of $\widehat{T}$ is optional and is used near either A or C when the hypotenuse is close to being either horizontal or vertical.

Notice that the rectangles R_i are pairwise essentially disjoint; moreover, they are essentially disjoint from $\mathbb{P}$ and $\widetilde{T}$. We are finally in position to define the set

$$\Upsilon := \partial\mathbb{P} \cup \partial\widehat{T} \cup \partial\widetilde{T} \cup \bigcup_{i=0}^{M-1} \partial R_i,$$

satisfying all conditions of property (1).

We now focus on the definition of $\widetilde{\varphi} : \Upsilon \to \mathbb{R}^2$. We set

$$\widetilde{\varphi} := \varphi^2 \quad \text{on} \left(\Xi \cup \bigcup_{i=0}^{M-1} \partial S_i \right) \setminus DG.$$

Then, we only need to care about the definition of $\widetilde{\varphi}$ on Γ. The idea is to let $\widetilde{\varphi}(\partial T_i \setminus I_i) = \varphi^2(I_i)$ and decide the parametrization in such a way that $\Psi_0(\widetilde{\varphi}_{\rceil \partial R_i}) \lessapprox \Psi_0(\varphi^2_{\rceil \partial S_i})$.

For this reason, for every $i = 0, \dots, M-1$, we define $\widetilde{\varphi} : \partial T_i \setminus I_i \to \mathbb{R}^2$ as the bi-linear map that parametrizes the segment $\varphi^2(I_i)$ at constant speed. We recall that Lemma 3.3 gives a φ^2 that is linear and equal to φ^1 on each I_i, and $\mathcal{H}^1(\varphi^2(I_i)) < \xi$. As a consequence, for any choice of $X \in \Gamma \cap \partial T_i$ and $Y \in I_i$, we get that $|\widetilde{\varphi}(X) - \varphi^2(Y)| < \xi$, and since any horizontal/vertical slice of R_i intersects ∂S_i, it is immediate to deduce that

$$\Psi_0(\widetilde{\varphi}_{\rceil \partial R_i}) \le \Psi_0\big(\varphi^2_{\rceil \partial S_i}\big) + \xi \mathscr{H}^1(\Gamma \cap \partial T_i)$$

for every $i = 0, \dots, M-1$.

Recalling the first of (4.2), (4.9) and the bound on ξ, thanks to (4.8), we deduce

$$\begin{aligned}\sum_{i=0}^{M-1} \Psi_0(\widetilde{\varphi}_{\rceil \partial R_i}) &\le \sum_{i=0}^{M-1} \big(\Psi_0\big(\varphi^2_{\rceil \partial S_i}\big) + \xi \mathscr{H}^1(\Gamma \cap \partial T_i)\big) \\ &\le \Psi_0(\varphi) + (\mathbf{d} + 14\eta)\mathscr{H}^1(\partial T) + \frac{\eta}{100} + \xi \mathscr{H}^1(\Gamma) \\ &\le \Psi_0(\varphi) + (\mathbf{d} + 14\eta)\mathscr{H}^1(\partial T) + \frac{\eta}{50}\end{aligned}$$

which is exactly (4.7). Then, the fact that $\widetilde{\varphi}$ is continuous and injective and coincides with $\varphi^2 = \varphi^1$ on $\Xi \setminus DG$ implies property (2), hence the conclusion of the step.

Step III. Piecewise affine extensions in the components of $T \setminus \Upsilon$ and conclusion. In this conclusive step, we perform independently piecewise affine extensions in the different components of $T \setminus \Upsilon$. Indeed, thanks to Step II, we have that T is the disjoint union of $\mathbb{P}$, $\widetilde{T}$, possibly $\widehat{T}$ and M rectangles R_i and $\widetilde{\varphi}$ is continuous, injective and piecewise linear on the respective boundaries. This allows us to work separately in each component with piecewise affine extensions that coincide with the restriction of $\widetilde{\varphi}$ on the boundary of the considered component.

Let us first consider the rectangle R_i for some $i = 0, \dots, M-1$. Proposition 2.8 applied to R_i and $\widetilde{\varphi}_{\rceil \partial R_i}$ with parameter $\frac{\varepsilon}{2^i}$ provides a finitely piecewise affine homeomorphism $v_i : R_i \to \mathbb{R}^2$ extending $\widetilde{\varphi}$ on ∂R_i such that

$$\|Dv_i\|_0(R_i) \le \Psi_0(\widetilde{\varphi}_{\rceil \partial R_i}) + \frac{\varepsilon}{2^i}. \tag{4.10}$$

We pass now to consider the extension inside the triangle $\widetilde{T}$. Since $\widetilde{\varphi}$ is linear on each side of $\partial \widetilde{T}$, then we define $w : \widetilde{T} \to \mathbb{R}^2$ as the unique affine extension of the boundary value $\widetilde{\varphi}$. We recall that whenever one considers two triples of distinct points in the plane, then there is only one affine function mapping the first triple into the second one. We can then directly compute

$$D_1 w = \frac{\widetilde{\varphi}(F) - \widetilde{\varphi}(G)}{|F - G|} \quad \text{and} \quad D_2 w = \frac{\widetilde{\varphi}(E) - \widetilde{\varphi}(F)}{|E - F|}.$$

Therefore, recalling (vi), (vii), (viii), (ix) and the fact that

$$\widetilde{\varphi}(G) = \mathbf{Z}, \widetilde{\varphi}(E) = \varphi(E), \widetilde{\varphi}(F) = \varphi(F),$$

from Step I, we get

$$\|Dw\|_0(\widetilde{T}) = \frac{1}{2}\big(|E-F|\big|\widetilde{\varphi}(G)-\widetilde{\varphi}(F)\big| + |F-G|\big|\widetilde{\varphi}(E)-\widetilde{\varphi}(F)\big|\big) \le \frac{\eta}{2}\big(\big|\mathbf{Z}-\widetilde{\varphi}(F)\big| + |F-G|\big) \le \eta^2\Big(1+\frac{1}{2\tan\beta}\Big). \tag{4.11}$$

To extend in $\widehat{T}$, we use Lemma 4.1. We have $\mathscr{H}^1(\partial\widehat{T}) \le \widetilde{C}\eta$ and $\mathscr{H}^1(\widetilde{\varphi}(\partial\widehat{T})) \le \widetilde{C}\|D\varphi\|_\infty\eta$. Recalling (4.1), we get a bi-affine homeomorphism v equal to $\widetilde{\varphi}$ on $\partial\widehat{T}$ and $\|Dv\|_0(T) \le \eta \le \varepsilon$.

At last, we discuss the extension inside $\mathbb{P}$. Since $\mathbb{P}$ is $\widetilde{C}$-bi-Lipschitz equivalent to $\mathcal{R} := [0,|A-C|] \times [0,|A-D|\sin\beta]$, then there exists a $\widetilde{C}$-bi-Lipschitz finitely piecewise affine homeomorphism $\varphi : \mathbb{P} \to \mathcal{R}$. We now observe that by construction, $\partial\mathcal{R}$ is a rectangle and $\psi := \widetilde{\varphi}\circ\varphi^{-1}$ is a piecewise linear injective map defined on $\partial\mathcal{R}$; thus, we are in position to apply Corollary 2.10 to $\psi : \partial\mathcal{R} \to \mathbb{R}^2$ to get a finitely piecewise affine homeomorphism $\widetilde{\omega} : \mathcal{R} \to \mathbb{R}^2$ coinciding with ψ on $\partial\mathcal{R}$ such that

$$\|D\widetilde{\omega}\|_{L^1(\mathcal{R})} \le \widetilde{C}\mathscr{H}^1(\partial\mathcal{R})\mathscr{H}^1\big(\psi(\partial\mathcal{R})\big).$$

Then, the map $\omega := \widetilde{\omega}\circ\varphi : \mathbb{P} \to \mathbb{R}^2$ is a finitely piecewise affine homeomorphism coinciding with $\widetilde{\varphi}$ on $\partial\mathbb{P}$ and satisfying

$$\begin{aligned}\|D\omega\|_{L^1(\mathbb{P})} &\le \widetilde{C}\|D\widetilde{\omega}\|_{L^1(\mathcal{R})} \le \widetilde{C}\mathscr{H}^1\big(\varphi(\partial\mathbb{P})\big)\mathscr{H}^1\big(\widetilde{\varphi}\circ\varphi^{-1}(\partial\mathcal{R})\big)\\ &\le \widetilde{C}\mathscr{H}^1(\partial\mathbb{P})\mathscr{H}^1\big(\widetilde{\varphi}(\partial\mathbb{P})\big).\end{aligned}$$

Once here we recall (4.9) and the first of (4.2) to get that

$$\mathscr{H}^1(\partial\mathbb{P}) \le |A-C| + |D-A| + |C-E| + \mathscr{H}^1(\Gamma) + |E-G| \le 4d + 2\eta,$$

while from (ii), the fact that $\widetilde{\varphi}(\Gamma\cup EG) = \varphi^1(DE) = \widetilde{\gamma}_{\varphi(D)\varphi(E)}$ and (4.3), we deduce

$$\begin{aligned}\mathscr{H}^1\big(\widetilde{\varphi}(\partial\mathbb{P})\big) &\le \mathscr{H}^1\big(\varphi(AC)\big) + \mathscr{H}^1\big(\varphi(AD)\big) + \mathscr{H}^1\big(\varphi(CE)\big) + \mathscr{H}^1\big(\varphi^1(ED)\big)\\ &\le \big|\varphi(A)-\varphi(C)\big| + \big|\varphi(A)-\varphi(D)\big| + \big|\varphi(C)-\varphi(E)\big| + \mathscr{H}^1(\widetilde{\gamma}_{\varphi(D)\varphi(E)})\\ &\le 2\mathbf{d} + 12\eta.\end{aligned}$$

Then, the last two observations together with the estimate on $D\omega$ imply

$$\|D\omega\|_0(\mathbb{P}) \le \widetilde{C}\|D\omega\|_{L^1(\mathbb{P})} \le \widetilde{C}(d+\eta)(\mathbf{d}+\eta) \le \widetilde{C}d\mathbf{d}. \tag{4.12}$$

We finally define $v : T \to \mathbb{R}^2$ the finitely piecewise affine map such that

$$v_{\rceil\mathbb{P}} = \omega, \quad v_{\rceil\widetilde{T}} = w \quad \text{and} \quad v_{\rceil R_i} = v_i \text{ for every } i = 0,\dots,M-1.$$

We observe that v is continuous and injective, hence a homeomorphism, since $v = \widetilde{\varphi}$ on Υ. From the same observation, we also deduce that $v = \varphi$ on ∂T because $\widetilde{\varphi} = \varphi$ there. Gathering together (4.10), (4.11), (4.12) and (4.7), we find

$$\begin{aligned}
\|Dv\|_0(T) &= \|D\omega\|_0(\mathbb{P}) + \|Dw\|_0(\widetilde{T}) + \sum_{i=0}^{M-1} \|Dv_i\|_0(R_i) + \|Dv\|_0(\widehat{T}) \\
&\leq \widetilde{C} d\mathbf{d} + \eta^2\Big(1 + \frac{1}{2\tan\beta}\Big) + \sum_{i=0}^{M-1}\Big(\Psi_0(\widetilde{\varphi}\rceil_{\partial R_i}) + \frac{\varepsilon}{2^i}\Big) + d\varepsilon \\
&\leq \widetilde{C} d\mathbf{d} + \eta^2\Big(1 + \frac{1}{2\tan\beta}\Big) + \Psi_0(\varphi) + C\mathbf{d}\mathcal{H}^1(\partial T) + \widetilde{C}\varepsilon \\
&\leq \Psi_0(\varphi) + \widetilde{C}\mathcal{H}^1(\partial T)\mathcal{H}^1\big(\varphi(AC)\big) + \widetilde{C}\varepsilon,
\end{aligned}$$

where in the last inequality, we used that

$$d = |A - C| \leq \mathcal{H}^1(\partial T) \quad \text{and} \quad \mathbf{d} = \big|\varphi\big(A - \varphi(C)\big)\big|. \qquad \blacksquare$$

5. Proof of Theorems 1.1 and 1.2

Proof of Theorem 1.1. Let $\varepsilon > 0$ be arbitrary fixed and let

$$0 < \eta < \min\left\{1, \frac{\varepsilon}{\widetilde{C} + \widetilde{C}\mathcal{H}^1(\mathcal{Q})}\right\} \tag{5.1}$$

for some large appropriate but fixed geometric constant $\widetilde{C}$. We describe in detail the proof when $\mathcal{Q}$ is of class (iii) in the sense of Remark 3.1, while the other cases are an obvious modification of the current argument.

Applying Lemma 3.2 to $\mathcal{Q}, \varphi, \alpha$ and the parameter η, we can partition $\mathcal{Q}$ in two triangles T_1, T_2 and a convex polygon Δ and find a continuous, piecewise linear, injective map $\overline{\varphi} : \partial T_1 \cup \partial T_2 \cup \partial\Delta \to \mathbb{R}^2$ with the properties listed in the statement of Lemma 3.2. In particular, from (3.3), it follows that

$$\Psi_\alpha(\overline{\varphi}\rceil_{\partial\Delta}) \leq \Psi_\alpha(\varphi) + \eta, \tag{5.2}$$

where (3.1) and (3.2) ensure that

$$\mathcal{H}^1(\partial T_1) + \mathcal{H}^1(\partial T_2) < \eta, \quad \mathcal{H}^1\big(\overline{\varphi}(\partial T_1)\big) + \mathcal{H}^1\big(\overline{\varphi}(\partial T_2)\big) < \eta. \tag{5.3}$$

Furthermore, since Δ is a convex polygon with two parallel sides in direction α, we can apply the α-rotated version of Lemma 3.3 to Δ, $\overline{\varphi}^1$ and the parameter η to find M increasing values $(t_i)_{i=0}^{M-1}$ and α-rotated strips S_i, which can be seen as the union of a rectangle R_i and two triangles T_i^1 and T_i^2, and a continuous piecewise linear injective map $\hat{\varphi} : \bigcup_{i=0}^{M-1} \partial T_i^1 \cup \partial R_i \cup \partial T_i^2 \to \mathbb{R}^2$ coinciding with $\overline{\varphi}$ on $\partial\Delta$ with the properties of Lemma 3.3. In particular, thanks to (3.7), we deduce

$$\sum_{i=0}^{M-1} \big(\Psi_\alpha(\hat{\varphi}_{\rceil \partial T_i^1}) + \Psi_\alpha(\hat{\varphi}_{\rceil \partial R_i}) + \Psi_\alpha(\hat{\varphi}_{\rceil \partial T_i^2})\big) \le \Psi_\alpha(\overline{\varphi}^1_{\rceil \partial\Delta}) + \eta, \tag{5.4}$$

while (3.5) ensures that

$$\mathcal{H}^1\big(\hat{\varphi}(I_i^1)\big) + \mathcal{H}^1\big(\hat{\varphi}(I_i^2)\big) < \eta \tag{5.5}$$

where $I_i^1 = \partial T_i^1 \cap \partial\Delta$ and $I_i^2 = \partial T_i^2 \cap \partial\Delta$.

We are finally in position to define a function

$$\widetilde{\varphi} : \partial T_1 \cup \left(\bigcup_{i=0}^{M-1} \partial T_i^1 \cup \partial R_i \cup \partial T_i^2 \right) \cup \partial T_2 \to \mathbb{R}^2$$

that is continuous, injective, finitely piecewise linear and such that $\widetilde{\varphi} = \overline{\varphi}^1$ on $\partial T_1 \cup \partial T_2$ and $\widetilde{\varphi} = \hat{\varphi}$ on $\bigcup_{i=0}^{M-1} \partial T_i^1 \cup \partial R_i \cup \partial T_i^2$.

Once here we will perform the homeomorphic piecewise affine extension on T_1, T_2, T_i^1, T_i^2 and R_i independently for every $i = 0, \dots, M-1$. We first focus on the extension inside the strips S_i. Let $i \in \{0, \dots, M-1\}$ be fixed; we then apply the α-rotated version of Proposition 2.8 to R_i, $\widetilde{\varphi}_{\rceil \partial R_i}$ and parameter $\eta(t_{i+1} - t_i)$ to find finitely piecewise affine homeomorphisms $v_i : R_i \to \mathbb{R}^2$ coinciding with $\widetilde{\varphi}$ on ∂R_i such that

$$\|Dv_i\|_\alpha(R_i) \le \Psi_\alpha(\widetilde{\varphi}_{\rceil \partial R_i}) + \eta(t_{i+1} - t_i). \tag{5.6}$$

By construction, we have that T_i^1, T_i^2 are right-angle triangles whose hypotenuse is contained in $\partial\Delta \cap \partial\mathcal{Q}$, and we can apply the α-rotated version of Lemma 4.2 to $T_i^{1,2}$, $\widetilde{\varphi}_{\rceil \partial T_i^{1,2}}$ and parameter $\eta(t_{i+1} - t_i)$ to find finitely piecewise affine homeomorphisms

$$w_i^{1,2} : T_i^{1,2} \to \mathbb{R}^2$$

coinciding with $\widetilde{\varphi}$ on $\partial T_i^{1,2}$ such that

$$\|Dw_i^{1,2}\|_\alpha(T_i^{1,2}) \le \Psi_\alpha\big(\widetilde{\varphi}_{\rceil \partial T_i^{1,2}}\big) + \widetilde{C}\mathcal{H}^1(\partial T_i^{1,2})\mathcal{H}^1\big(\widetilde{\varphi}(I_i^{1,2})\big) + \widetilde{C}\eta(t_{i+1} - t_i),$$

which thanks to (5.5) implies

$$\begin{aligned} &\|Dw_i^1\|_\alpha(T_i^1) + \|Dw_i^2\|_\alpha(T_i^2) \\ &\quad \le \Psi_\alpha\big(\widetilde{\varphi}_{\rceil \partial T_i^1}\big) + \Psi_\alpha\big(\widetilde{\varphi}_{\rceil \partial T_i^2}\big) + \big[\widetilde{C}\big(\mathcal{H}^1(\partial T_i^1) + \mathcal{H}^1(\partial T_i^2)\big) + \widetilde{C}(t_{i+1} - t_i)\big]\eta. \end{aligned} \tag{5.7}$$

Let us also notice that, for future need, since the triangles $T_i^{1,2}$ have one angle equal to $\pi/2$ and, by construction, their hypotenuse is $I_i^{1,2}$, then one has $\mathcal{H}^1(\partial T_i^{1,2}) \leq 3\mathcal{H}^1(I_i^{1,2})$ for every i. Moreover, having $I_i^{1,2} \subset (\partial\Delta \cap \partial\mathcal{Q})$ and the triangles pairwise essentially disjoint, we deduce

$$\sum_{i=0}^{M-1} \big(\mathcal{H}^1(\partial T_i^1) + \mathcal{H}^1(\partial T_i^2)\big) \leq 3 \sum_{i=0}^{M-1} \big(\mathcal{H}^1(I_i^1) + \mathcal{H}^1(I_i^2)\big) \leq 3\mathcal{H}^1(\partial\mathcal{Q}). \tag{5.8}$$

Let us now consider the extension inside the triangles T_1, T_2. In this case, by construction, we are in position to apply the α-rotated version of Lemma 4.1 to $T_{1,2}$ and $\widetilde{\varphi}_{\rceil\partial T_{1,2}}$ to find bi-affine homeomorphisms $w_{1,2} : T_{1,2} \to \mathbb{R}^2$ such that

$$\|Dw_{1,2}\|_\alpha(T_{1,2}) \leq \mathcal{H}^1\big(\widetilde{\varphi}(\partial T_{1,2})\big)\mathcal{H}^1\big(\partial T_{1,2}\big)$$

which, thanks to (5.3), gives

$$\|Dw_1\|_\alpha(T_1) + \|Dw_2\|_\alpha(T_2) < 2\eta^2. \tag{5.9}$$

We can finally define $v : \mathcal{Q} \to \mathbb{R}^2$ to be the piecewise affine function such that

$$v = w_{1,2} \text{ on } T_{1,2}, \quad v = w_i^{1,2} \text{ on } T_i^{1,2} \quad \text{and} \quad v = v_i \text{ on } R_i \text{ for every } i = 0, \dots, M-1.$$

By construction, v is continuous because $v = \widetilde{\varphi}$ on the one-dimensional skeleton $\partial T_1 \cup (\bigcup_{i=0}^{M-1} \partial T_i^1 \cup \partial R_i \cup \partial T_i^2) \cup \partial T_2$, and, moreover, v coincides with $\widetilde{\varphi} = \varphi$ on $\partial\mathcal{Q}$. To conclude, it is only left to verify the validity of (1.4), but this is now a straightforward consequence of (5.6), (5.7), (5.9), (5.4), (5.8) and (5.2). Indeed, we get

$$\begin{aligned}
\|Dv\|_\alpha(\mathcal{Q}) &= \|Dw_1\|_\alpha(T_1) + \|Dw_2\|_\alpha(T_2) \\
&\quad + \sum_{i=0}^{M-1} \big(\|Dw_i^1\|_\alpha(T_i^1) + \|Dv_i\|_\alpha(R_i) + \|Dw_i^2\|_\alpha(T_i^2)\big) \\
&\leq \sum_{i=0}^{M-1} \big(\Psi_\alpha\big(\hat{\varphi}_{\rceil\partial T_i^1}\big) + \Psi_\alpha\big(\hat{\varphi}_{\rceil\partial R_i}\big) + \Psi_\alpha\big(\hat{\varphi}_{\rceil\partial T_i^2}\big)\big) \\
&\quad + 4\eta^2 + \widetilde{C}\eta \sum_{i=0}^{M-1} \big[\big(\mathcal{H}^1(\partial T_i^1) + \mathcal{H}^1(\partial T_i^2)\big) + (t_{i+1} - t_i)\big] \\
&\leq \Psi_\alpha(\overline{\varphi}^1_{\rceil\partial\Delta}) + \eta + 4\eta^2 + \widetilde{C}\eta\big(\mathcal{H}^1(\partial\mathcal{Q}) + \operatorname{diam}\mathcal{Q}\big) \\
&\leq \Psi_\alpha(\overline{\varphi}_{\rceil\partial\Delta}) + 6\eta + \widetilde{C}\eta\big(\mathcal{H}^1(\partial\mathcal{Q})\big), \\
&\leq \Psi_\alpha(\varphi) + \widetilde{C}\eta\big(1 + \mathcal{H}^1(\partial\mathcal{Q})\big),
\end{aligned}$$

and then estimate (1.4) follows since η has been chosen as in (5.1). ∎

5.1. *Proof of Theorem* 1.2

To prove the claim, it suffices to repeat the proof of the above lemmas as before but using the estimates from Theorem 2.9 instead of from Proposition 2.8. In all of our calculations, we estimate $\int \rho_{\mathcal{P}_\Omega}(\varphi^1(H_t^2), \varphi^1(H_t^3))dt$ and $\int_{D_1}^{B_1} \rho_{\mathcal{P}_\Omega}(\varphi^1(V_s^1), \varphi^1(V_s^2))ds$ separately. Now it suffices to keep them separate instead of summing them.

The key estimates in Lemma 3.2 are (3.4), in Lemma 3.3 they are (3.12), (3.13) and the calculation following, and in Lemma 4.2 they are (4.4), (4.5), (4.6).

We can then repeat the proof of Theorem 1.1 with the difference that in (5.4), (5.6) and so on we use the separate estimates, rather than the summed estimates expressed using Ψ_α. ■

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