



Partial Differential Equations. – *An isoperimetric result for an energy related to the p -capacity*, by PAOLO ACAMPORA and EMANUELE CRISTOFORONI, communicated on 10 November 2023.

ABSTRACT. – In this paper, we generalize the notion of the relative p -capacity of K with respect to Ω , by replacing the Dirichlet boundary condition with a Robin one. We show that, under volume constraints, our notion of p -capacity is minimal when K and Ω are concentric balls. We use the H -function (see Bossel (1986) and Daners (2006)) and a derearrangement technique.

KEYWORDS. – Robin, p -capacity, free boundary, isocapacitary inequality.

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1. INTRODUCTION

Let $p > 1$, $\beta > 0$ be real numbers. For every open bounded set $\Omega \subset \mathbb{R}^n$ with Lipschitz boundary, and every compact set $K \subseteq \bar{\Omega}$ with Lipschitz boundary, we define

$$(1.1) \quad E_{\beta,p}(K, \Omega) = \inf_{\substack{v \in W^{1,p}(\Omega) \\ v=1 \text{ in } K}} \left(\int_{\Omega} |\nabla v|^p dx + \beta \int_{\partial\Omega} |v|^p d\mathcal{H}^{n-1} \right).$$

We notice that it is sufficient to minimize among all functions $v \in H^1(\Omega)$ with $v = 1$ in K and $0 \leq v \leq 1$ a.e. Moreover, if K, Ω are sufficiently smooth, a minimizer u satisfies

$$(1.2) \quad \begin{cases} u = 1 & \text{in } K, \\ \Delta_p u = 0 & \text{in } \Omega \setminus K, \\ |\nabla u|^{p-2} \frac{\partial u}{\partial \nu} + \beta |u|^{p-2} u = 0 & \text{on } \partial\Omega \setminus \partial K, \end{cases}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$ is the p -Laplacian of u and ν is the outer unit normal to $\partial\Omega$. If $\overset{\circ}{K} = \Omega$, equation (1.2) has to be understood as $u = 1$ in Ω , and the energy is

$$E_{\beta,p}(\Omega, \Omega) = \beta \mathcal{H}^{n-1}(\partial\Omega).$$

In general, equation (1.2) has to be interpreted in the weak sense; that is, for every $\varphi \in W^{1,p}(\Omega)$ such that $\varphi \equiv 0$ in K ,

$$(1.3) \quad \int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla \varphi d\mathcal{L}^n + \beta \int_{\partial\Omega} u^{p-1} \varphi d\mathcal{H}^{n-1} = 0.$$

In particular, if u is a minimizer, letting $\varphi = u - 1$, we have that

$$E_{\beta,p}(K, \Omega) = \int_{\Omega} |\nabla u|^p dx + \beta \int_{\partial\Omega} u^p d\mathcal{H}^{n-1} = \beta \int_{\partial\Omega} u^{p-1} d\mathcal{H}^{n-1}.$$

Moreover, from the strict convexity of the functional, the minimizer is the unique solution to (1.3).

This problem is related to the so-called *relative p -capacity of K with respect to Ω* , defined as

$$\text{Cap}_p(K, \Omega) := \inf_{\substack{v \in W_0^{1,p}(\Omega) \\ v=1 \text{ in } K}} \left(\int_{\Omega} |\nabla v|^p dx \right).$$

In the case $p = 2$, it represents the electrostatic capacity of an annular condenser consisting of a conducting surface $\partial\Omega$, and a conductor K , where the electrostatic potential is prescribed to be 1 inside K and 0 outside Ω . Let ω_n be the measure of the unit sphere in \mathbb{R}^n , and let $M > \omega_n$; then, it is well known that there exists some $r \geq 1$ such that

$$\min_{\substack{|K|=\omega_n \\ |\Omega|\leq M}} \text{Cap}_p(K, \Omega) = \text{Cap}_p(B_1, B_r).$$

This is an immediate consequence of the Pólya–Szegő inequality for the Schwarz rearrangement (see, for instance, [11, 14]). We are interested in studying the same problem for the energy defined in (1.1), which corresponds to changing the Dirichlet boundary condition on $\partial\Omega$ into a Robin boundary condition; namely, we consider the following problem:

$$(1.4) \quad \inf_{\substack{|K|=\omega_n \\ |\Omega|\leq M}} E_{\beta,p}(K, \Omega).$$

In this case, the previous symmetrization techniques cannot be employed anymore.

Problem (1.4) has been studied in the linear case $p = 2$ in [7], with more general boundary conditions on $\partial\Omega$; namely,

$$\frac{\partial u}{\partial \nu} + \frac{1}{2} \Theta'(u) = 0,$$

where Θ is a suitable increasing function vanishing at 0. This kind of problem has also been addressed in the context of thermal insulation (see, for instance, [1, 2, 8]). Our main result reads as follows.

THEOREM 1.1. *Let $\beta > 0$ such that*

$$\beta^{\frac{1}{p-1}} > \frac{n-p}{p-1}.$$

Then, for every $M > \omega_n$, the solution to problem (1.4) is given by two concentric balls (B_1, B_r) ; that is,

$$\min_{\substack{|K|=\omega_n \\ |\Omega|\leq M}} E_{\beta,p}(K, \Omega) = E_{\beta,p}(B_1, B_r).$$

In particular, we have that either $r = 1$ or $M = \omega_n r^n$.

Moreover, if $K_0 \subseteq \bar{\Omega}_0$ are such that

$$E_{\beta,p}(K_0, \Omega_0) = \min_{\substack{|K|=\omega_n \\ |\Omega|\leq M}} E_{\beta,p}(K, \Omega),$$

and u is the minimizer of $E_{\beta,p}(K_0, \Omega_0)$, then the sets $\{u = 1\}$ and $\{u > 0\}$ coincide with two concentric balls up to a \mathcal{H}^{n-1} -negligible set.

REMARK 1.2. In the case

$$\beta^{\frac{1}{p-1}} \leq \frac{n-p}{p-1},$$

adapting the symmetrization techniques used in [7], it can be proved that a solution to problem (1.4) is always given by the pair (B_1, B_1) .

We point out that the proof of the theorem relies on the techniques involving the H -function introduced in [5, 9].

The case in which Ω is the Minkowski sum $\Omega = K + B_r(0)$, with the energy $E_{\beta,p}(K, \Omega)$, has been studied in [4] under suitable geometrical constraints (see also [10]).

2. PROOF OF THE THEOREM

To prove Theorem 1.1, we start by studying the function

$$R \mapsto E_{\beta,p}(B_1, B_R).$$

A similar study of the previous function can also be found in [4]. Let

$$\Phi_{p,n}(\rho) = \begin{cases} \log(\rho) & \text{if } p = n, \\ -\frac{p-1}{n-p} \frac{1}{\rho^{\frac{n-p}{p-1}}} & \text{if } p \neq n. \end{cases}$$

For every $R > 1$, consider

$$(2.1) \quad u^*(x) = 1 - \frac{\beta^{\frac{1}{p-1}} (\Phi_{p,n}(|x|) - \Phi_{p,n}(1))_+}{\Phi'_{p,n}(R) + \beta^{\frac{1}{p-1}} (\Phi_{p,n}(R) - \Phi_{p,n}(1))},$$

the solution to

$$\begin{cases} u^* = 1 & \text{in } B_1, \\ \Delta_p u^* = 0 & \text{in } B_R \setminus B_1, \\ |\nabla u^*|^{p-2} \frac{\partial u^*}{\partial \nu} + \beta |u^*|^{p-2} u^* = 0 & \text{on } \partial B_R. \end{cases}$$

We have that

$$(2.2) \quad E_{\beta,p}(B_1, B_R) = \int_{B_R} |\nabla u^*|^p dx + \beta \int_{\partial B_R} |u^*|^p d\mathcal{H}^{n-1} \\ = \frac{n\omega_n \beta}{[\Phi'_{p,n}(R) + \beta^{\frac{1}{p-1}} (\Phi_{p,n}(R) - \Phi_{p,n}(1))]^{p-1}}.$$

Notice that $E_{\beta,p}(B_1, B_R)$ is decreasing in $R > 0$ if and only if

$$\frac{d}{dR} (\Phi'_{p,n}(R) + \beta^{\frac{1}{p-1}} \Phi_{p,n}(R)) \geq 0,$$

that is, if and only if

$$R \geq \frac{n-1}{p-1} \frac{1}{\beta^{\frac{1}{p-1}}} =: \alpha_{\beta,p}.$$

Moreover,

$$E_{\beta,p}(B_1, B_1) = n\omega_n \beta, \\ \lim_{R \rightarrow \infty} E_{\beta,p}(B_1, B_R) = \begin{cases} n\omega_n \left(\frac{n-p}{p-1}\right)^{p-1} & \text{if } p < n, \\ 0 & \text{if } p \geq n. \end{cases}$$

Therefore, there are three cases:

$$(i) \quad \text{if } \beta^{\frac{1}{p-1}} \geq \frac{n-1}{p-1},$$

$$R \in [1, +\infty) \mapsto E_{\beta,p}(B_1, B_R)$$

is decreasing;

$$(ii) \quad \text{if } \frac{n-p}{p-1} < \beta^{\frac{1}{p-1}} < \frac{n-1}{p-1},$$

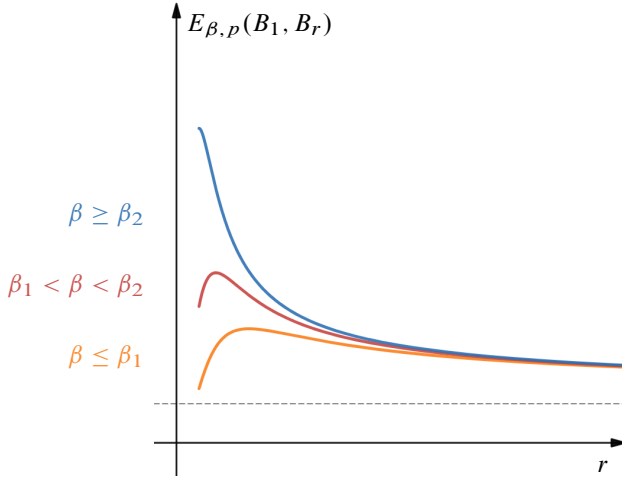
$$R \in [1, +\infty) \mapsto E_{\beta,p}(B_1, B_R)$$

increases on $[1, \alpha_{\beta,p}]$ and decreases on $[\alpha_{\beta,p}, +\infty)$, with the existence of a unique $R_{\beta,p} > \alpha_{\beta,p}$ such that $E_{\beta,p}(B_1, B_{R_{\beta,p}}) = E_{\beta,p}(B_1, B_1)$;

$$(iii) \quad \text{if } \beta^{\frac{1}{p-1}} \leq \frac{n-p}{p-1},$$

$$R \in [1, +\infty) \mapsto E_{\beta,p}(B_1, B_R)$$

reaches its minimum at $R = 1$.

FIGURE 1. $E_{\beta,p}(B_1, B_r)$ depending on the value of β .

See, for instance, Figure 1, where

$$\beta_1 = \left(\frac{n-p}{p-1}\right)^{p-1}, \quad \beta_2 = \left(\frac{n-1}{p-1}\right)^{p-1}, \quad p = 2.5, \quad n = 3.$$

We will need the following.

LEMMA 2.1. *Let $R > 1$, $\beta > 0$ and let u^* be the solution of the problem on (B_1, B_R) . Then,*

$$\frac{|\nabla u^*|}{u^*} \leq \beta^{\frac{1}{p-1}}$$

in $B_R \setminus B_1$ if and only if

$$E_{\beta,p}(B_1, B_\rho) \geq E_{\beta,p}(B_1, B_R)$$

for every $\rho \in [1, R]$.

PROOF. Recalling the expressions of u^* in (2.1), by straightforward computations, we have that

$$\frac{|\nabla u^*|}{u^*} \leq \beta^{\frac{1}{p-1}}$$

in $B_R \setminus B_1$ if and only if

$$(2.3) \quad \begin{aligned} & \Phi'_{p,n}(R) + \beta^{\frac{1}{p-1}} (\Phi_{p,n}(R) - \Phi_{p,n}(1)) \\ & \geq \Phi'_{p,n}(\rho) + \beta^{\frac{1}{p-1}} (\Phi_{p,n}(\rho) - \Phi_{p,n}(1)) \end{aligned}$$

for every $\rho \in [1, R]$. Using the expression of $E_{\beta,p}(B_1, B_\rho)$ in (2.2), (2.3) is equivalent to

$$E_{\beta,p}(B_1, B_\rho) \geq E_{\beta,p}(B_1, B_R)$$

for every $\rho \in [1, R]$. ■

DEFINITION 2.2. Let $\Omega \subseteq \mathbb{R}^n$ be an open set, and let $U \subseteq \Omega$ be another set. We define the *internal boundary* of U as

$$\partial_i U = \partial U \cap \Omega$$

and the *external boundary* of U as

$$\partial_e U = \partial U \cap \partial \Omega.$$

Let $K \subseteq \bar{\Omega} \subseteq \mathbb{R}^n$ be open bounded sets, and let u be the minimizer of $E_{\beta,p}(K, \Omega)$. In the following, we denote

$$U_t = \{x \in \Omega \mid u(x) > t\}.$$

DEFINITION 2.3 (*H*-function). Let $\varphi \in W^{1,p}(\Omega)$. We define

$$H(t, \varphi) = \int_{\partial_i U_t} |\varphi|^{p-1} d\mathcal{H}^{n-1} - (p-1) \int_{U_t} |\varphi|^p d\mathcal{L}^n + \beta \mathcal{H}^{n-1}(\partial_e U_t).$$

Notice that this definition is slightly different from the one given in [6].

LEMMA 2.4. Let $K \subseteq \Omega \subseteq \mathbb{R}^n$ be open bounded sets, and let u be the minimizer of $E_{\beta,p}(K, \Omega)$. Then, for a.e. $t \in (0, 1)$, we have

$$H\left(t, \frac{|\nabla u|}{u}\right) = E_{\beta,p}(K, \Omega).$$

PROOF. Recall that

$$\begin{aligned} (2.4) \quad E_{\beta,p}(K, \Omega) &= \int_{\Omega} |\nabla u|^p d\mathcal{L}^n + \beta \int_{\partial \Omega} u^p \\ &= \beta \int_{\partial \Omega} u^{p-1} d\mathcal{H}^{n-1}. \end{aligned}$$

Let $t \in (0, 1)$. We construct the following test functions: let $\varepsilon > 0$, and let

$$\varphi_\varepsilon(x) = \begin{cases} -1 & \text{if } u(x) \leq t, \\ \frac{u(x)-t}{\varepsilon u(x)^{p-1}} - 1 & \text{if } t < u(x) \leq t + \varepsilon, \\ \frac{1}{u(x)^{p-1}} - 1 & \text{if } u(x) > t + \varepsilon, \end{cases}$$

so that φ_ε is an approximation to the function $(u^{1-p}\chi_{U_t} - 1)$, and

$$\nabla\varphi_\varepsilon(x) = \begin{cases} 0 & \text{if } u(x) \leq t, \\ \frac{1}{\varepsilon} \left(\frac{\nabla u(x)}{u(x)^{p-1}} - (p-1) \frac{\nabla u(x)(u(x)-t)}{u(x)^p} \right) & \text{if } t < u(x) \leq t + \varepsilon, \\ -(p-1) \frac{\nabla u(x)}{u(x)^p} & \text{if } u(x) > t + \varepsilon. \end{cases}$$

We have that φ_ε is an admissible test function for the Euler–Lagrange equation (1.3), which entails

$$\begin{aligned} 0 &= \frac{1}{\varepsilon} \int_{\{t < u \leq t + \varepsilon\} \cap \Omega} \frac{|\nabla u|^{p-1}}{u^{p-1}} |\nabla u| d\mathcal{L}^n - (p-1) \int_{\{t < u \leq t + \varepsilon\} \cap \Omega} \frac{|\nabla u|^p}{u^p} \frac{u-t}{\varepsilon} d\mathcal{L}^n \\ &\quad - (p-1) \int_{\{u > t + \varepsilon\} \cap \Omega} \frac{|\nabla u|^p}{u^p} d\mathcal{L}^n + \beta \int_{\{t < u \leq t + \varepsilon\} \cap \partial\Omega} \frac{u-t}{\varepsilon} d\mathcal{H}^{n-1} \\ &\quad + \beta \mathcal{H}^{n-1}(\partial\Omega \cap \{u > t + \varepsilon\}) - \beta \int_{\partial\Omega} u^{p-1} d\mathcal{H}^{n-1}. \end{aligned}$$

Letting now ε go to 0, by the coarea formula, we get that for a.e. $t \in (0, 1)$,

$$(2.5) \quad \begin{aligned} \beta \int_{\partial\Omega} u^{p-1} d\mathcal{H}^{n-1} &= \int_{\partial_t U_t} \left(\frac{|\nabla u|}{u} \right)^{p-1} d\mathcal{H}^{n-1} \\ &\quad - (p-1) \int_{U_t} \left(\frac{|\nabla u|}{u} \right)^p d\mathcal{L}^n + \beta \mathcal{H}^{n-1}(\partial_e U_t). \end{aligned}$$

Joining (2.4) and (2.5), the lemma is proven. \blacksquare

REMARK 2.5. Notice that if K and Ω are two concentric balls, the minimizer u is the one written in (2.1), for which the statement of the above lemma holds for every $t \in (0, 1)$.

LEMMA 2.6. *Let $\varphi \in L^\infty(\Omega)$. Then, there exists $t \in (0, 1)$ such that*

$$H(t, \varphi) \leq E_{\beta, p}(K, \Omega).$$

PROOF. Let

$$w = |\varphi|^{p-1} - \left(\frac{|\nabla u|}{u} \right)^{p-1}.$$

Then, we evaluate

$$\begin{aligned} H(t, \varphi) - H\left(t, \frac{|\nabla u|}{u}\right) &= \int_{\partial_t U_t} w d\mathcal{H}^{n-1} - (p-1) \int_{U_t} \left(|\varphi|^p - \left(\frac{|\nabla u|}{u} \right)^p \right) d\mathcal{L}^n \end{aligned}$$

$$\begin{aligned} &\leq \int_{\partial_t U_t} w \, d\mathcal{H}^{n-1} - p \int_{U_t} \frac{|\nabla u|}{u} w \, d\mathcal{L}^n \\ &= -\frac{1}{t^{p-1}} \frac{d}{dt} \left(t^p \int_{U_t} \frac{|\nabla u|}{u} w \, d\mathcal{L}^n \right), \end{aligned}$$

where we used the inequality

$$(2.6) \quad a^p - b^p \leq \frac{p}{p-1} a (a^{p-1} - b^{p-1}) \quad \forall a, b \geq 0.$$

Multiplying by t^{p-1} and integrating, we get

$$(2.7) \quad \int_0^1 t^{p-1} \left(H(t, \varphi) - H\left(t, \frac{|\nabla u|}{u}\right) \right) dt \leq -\left[t^p \int_{U_t} \frac{|\nabla u|}{u} w \, d\mathcal{L}^n \right]_0^1 = 0,$$

from which we obtain the conclusion of the proof. \blacksquare

REMARK 2.7. Notice that inequality (2.6) holds as equality if and only if $a = b$. Therefore, if $\varphi \neq \frac{|\nabla u|}{u}$ on a set of positive measure, then inequality (2.7) is strict since

$$\left| \left\{ \varphi \neq \frac{|\nabla u|}{u} \right\} \cap U_t \right| > 0$$

for small enough t . Therefore, there exists $S \subset (0, 1)$ such that $\mathcal{L}^1(S) > 0$, and for every $t \in S$,

$$H(t, \varphi) < E_{\beta, p}(K, \Omega).$$

In the following, we fix a radius R such that $|B_R| \geq |\Omega|$, u^* the minimizer of $E_{\beta, p}(B_1, B_R)$, and

$$\begin{aligned} H^*(t, \varphi) &= \int_{\partial\{u^* > t\} \cap B_R} |\varphi|^{p-1} \, d\mathcal{H}^{n-1} - (p-1) \int_{\{u^* > t\}} |\varphi|^p \, d\mathcal{L}^n \\ &\quad + \beta \mathcal{H}^{n-1}(\partial\{u^* < t\} \cap \partial B_R). \end{aligned}$$

Here, we recall the isoperimetric inequality (see, for example, [12]), which will be useful in what follows.

DEFINITION 2.8. Let E be a measurable set. For every $t \in [0, 1]$, let

$$E^{(t)} = \left\{ x \in \mathbb{R}^n \mid \lim_{r \rightarrow 0} \frac{\mathcal{L}^n(E \cap B_r(x))}{\mathcal{L}^n(B_r(x))} = t \right\}$$

be the set of all points where E has density t . We define the *essential boundary* of E as the set

$$\partial^{\text{ess}} E = \mathbb{R}^n \setminus (E^{(0)} \cup E^{(1)}).$$

Notice that $\partial^{\text{ess}} E \subseteq \partial E$.

THEOREM 2.9 (Isoperimetric inequality). *Let E be a measurable set, and let B be the ball such that $\mathcal{L}^n(B) = \mathcal{L}^n(E)$. Then,*

$$\mathcal{H}^{n-1}(\partial^{\text{ess}} E) \geq \mathcal{H}^{n-1}(\partial B),$$

and the equality holds if and only if $E = B$ up to a set of measure 0.

PROPOSITION 2.10. *Let $\beta > 0$. Assume that*

$$(2.8) \quad \frac{|\nabla u^*|}{u^*} \leq \beta^{\frac{1}{p-1}}.$$

Then, we have that

$$E_{\beta,p}(K, \Omega) \geq E_{\beta,p}(B_1, B_R).$$

PROOF. In the following, if v is a radial function on B_R and $r \in (0, R)$, we denote with abuse of notation

$$v(r) = v(x),$$

where x is any point on ∂B_r . By Lemma 2.4, we know that for every $t \in (0, 1)$,

$$(2.9) \quad H^* \left(t, \frac{|\nabla u^*|}{u^*} \right) = E_{\beta,p}(B_1, B_R),$$

while by Lemma 2.6, for every $\varphi \in L^\infty(\Omega)$, there exists $t \in (0, 1)$ such that

$$(2.10) \quad E_{\beta,p}(K, \Omega) \geq H(t, \varphi).$$

We aim to find a suitable φ such that, for some t ,

$$(2.11) \quad H(t, \varphi) \geq H^* \left(t, \frac{|\nabla u^*|}{u^*} \right),$$

so that combining (2.10), (2.11), and (2.9), we conclude the proof. In order to construct φ , for every $t \in (0, 1)$, we define

$$(2.12) \quad r(t) = \left(\frac{|U_t|}{\omega_n} \right)^{\frac{1}{n}};$$

then, we set, for every $x \in \Omega$,

$$\varphi(x) = \frac{|\nabla u^*|}{u^*} (r(u(x))).$$

CLAIM. The functions $\varphi \chi_{U_t}$ and $\frac{|\nabla u^*|}{u^*} \chi_{B_{r(t)}}$ are equi-measurable; in particular,

$$(2.13) \quad \int_{U_t} \varphi^p d\mathcal{L}^n = \int_{B_{r(t)}} \left(\frac{|\nabla u^*|}{u^*} \right)^p d\mathcal{L}^n.$$

Indeed, let $g(r) = \frac{|\nabla u^*|}{u^*}(r)$, and by the coarea formula,

$$\begin{aligned}
 (2.14) \quad & |U_t \cap \{\varphi > s\}| \\
 &= \int_{U_t \cap \{g(r(u(x))) > s\}} d\mathcal{L}^n \\
 &= \int_t^{+\infty} \int_{\partial^* U_\tau \cap \{g(r(\tau)) > s\}} \frac{1}{|\nabla u(x)|} d\mathcal{H}^{n-1}(x) d\tau \\
 &= \int_0^{r(t)} \int_{\partial^* U_{r^{-1}(\sigma)}} \frac{1}{|\nabla u(x)| |r'(r^{-1}(\sigma))|} \chi_{\{g(\sigma) > s\}} d\mathcal{H}^{n-1}(x) d\sigma.
 \end{aligned}$$

Notice now that since

$$\omega_n r(\tau)^n = |U_\tau|,$$

then

$$(2.15) \quad r'(\tau) = -\frac{1}{n\omega_n r(\tau)^{n-1}} \int_{\partial^* U_\tau} \frac{1}{|\nabla u(x)|} d\mathcal{H}^{n-1}(x).$$

Therefore, substituting in (2.14), we get

$$|U_t \cap \{\varphi > s\}| = \int_0^{r(t)} n\omega_n \sigma^{n-1} \chi_{\{g(\sigma) > s\}} d\sigma = \left| B_{r(t)} \cap \left\{ \frac{|\nabla u^*|}{u^*} > s \right\} \right|,$$

where we have used polar coordinates to get the last equality. Thus, the claim is proved.

Recalling the definition of φ , (2.8) reads

$$\beta \geq \varphi^{p-1},$$

and then using (2.13) and the definition of H (see Definition 2.3), we have

$$\begin{aligned}
 (2.16) \quad H(t, \varphi) &= \beta \mathcal{H}^{n-1}(\partial_e U_t) + \int_{\partial_i U_t} \varphi^{p-1} d\mathcal{H}^{n-1} - (p-1) \int_{U_t} \varphi^p d\mathcal{L}^n \\
 &\geq \int_{\partial U_t} \varphi^{p-1} d\mathcal{H}^{n-1} - (p-1) \int_{B_{r(t)}} \left(\frac{|\nabla u^*|}{u^*} \right)^p d\mathcal{L}^n \\
 &\geq \int_{\partial B_{r(t)}} \left(\frac{|\nabla u^*|}{u^*} \right)^{p-1} d\mathcal{H}^{n-1} - (p-1) \int_{B_{r(t)}} \left(\frac{|\nabla u^*|}{u^*} \right)^p d\mathcal{L}^n \\
 &= H^* \left(u^*(r(t)), \frac{|\nabla u^*|}{u^*} \right) \\
 &= E_{\beta, p}(B_1, B_R),
 \end{aligned}$$

where in the last inequality we have used the isoperimetric inequality and the fact that φ is constant on ∂U_t . \blacksquare

REMARK 2.11. By Remark 2.7, we have that if K and Ω are such that

$$E_{\beta,p}(K, \Omega) = E_{\beta,p}(B_1, B_R),$$

then

$$\varphi = \frac{|\nabla u|}{u} \quad \text{for a. e. } x \in \Omega,$$

so that, by Lemma 2.4, we have equality in (2.16) for a.e. $t \in (0, 1)$. Thus, by the rigidity of the isoperimetric inequality, we get that U_t coincides with a ball up to a \mathcal{H}^{n-1} -negligible set for a.e. $t \in (0, 1)$. In particular,

$$\{u > 0\} = \bigcup_t U_t \quad \text{and} \quad \{u = 1\} = \bigcap_t U_t$$

coincide with two balls up to a \mathcal{H}^{n-1} -negligible set.

PROOF OF THEOREM 1.1. Fix $M = \omega_n R^n$ with $R > 1$. We divide the proof of the minimality of balls into two cases, and subsequently, we study the equality case.

Let us assume that

$$\beta^{\frac{1}{p-1}} \geq \frac{n-1}{p-1},$$

and recall that in this case the function

$$\rho \in [1, +\infty) \mapsto E_{\beta,p}(B_1, B_\rho)$$

is decreasing. Let u^* be the minimizer of $E_{\beta,p}(B_1, B_R)$; by Lemma 2.1, condition (2.8) holds and, by Proposition 2.10, we have that a solution to (1.4) is given by the concentric balls (B_1, B_R) .

Assume now that

$$\frac{n-p}{p-1} < \beta^{\frac{1}{p-1}} < \frac{n-1}{p-1};$$

then, in this case, letting

$$\alpha_{\beta,p} = \frac{(n-1)}{(p-1)\beta^{\frac{1}{p-1}}},$$

the function

$$\rho \in [1, +\infty) \mapsto E_{\beta,p}(B_1, B_\rho)$$

increases on $[1, \alpha_{\beta,p}]$ and decreases on $[\alpha_{\beta,p}, +\infty)$, and there exists a unique $R_{\beta,p} > \alpha_{\beta,p}$ such that

$$E_{\beta,p}(B_1, B_{R_{\beta,p}}) = E_{\beta,p}(B_1, B_1).$$

If $R \geq R_{\beta,p}$, the function u^* , the minimizer of $E_{\beta,p}(B_1, B_R)$, still satisfies condition (2.8), and, as in the previous case, a solution to (1.4) is given by the concentric balls

(B_1, B_R) . On the other hand, if $R < R_{\beta,p}$, we can consider $u_{\beta,p}^*$ the minimizer of $E_{\beta,p}(B_1, B_{R_{\beta,p}})$. By Lemma 2.1, we have that, for the function $u_{\beta,p}^*$, condition (2.8) holds, and, by Proposition 2.10, we have that if K and Ω are open bounded Lipschitz sets with $K \subseteq \Omega$, $|K| = \omega_n$, and $|\Omega| \leq M$, then

$$E_{\beta,p}(K, \Omega) \geq E_{\beta,p}(B_1, B_{R_{\beta,p}}) = E_{\beta,p}(B_1, B_1)$$

and a solution to (1.4) is given by the pair (B_1, B_1) .

For what concerns the equality case, we will follow the outline of the rigidity problem given in [13, Section 3] (see also [3, Section 2]). Let $K_0 \subseteq \bar{\Omega}_0$ be such that

$$E_{\beta,p}(K_0, \Omega_0) = \min_{\substack{|K|=\omega_n \\ |\Omega|\leq M}} E_{\beta,p}(K, \Omega).$$

Let u be the minimizer of $E_{\beta,p}(K_0, \Omega_0)$. If $\overset{\circ}{K}_0 = \Omega_0$, then $|\Omega_0| = |B_1|$ and the isoperimetric inequality yields

$$\mathcal{H}^{n-1}(\partial\Omega_0) \geq \mathcal{H}^{n-1}(\partial B_1),$$

while, from the minimality of (K_0, Ω_0) , we have that

$$E_{\beta,p}(K_0, \Omega_0) = \beta \mathcal{H}^{n-1}(\partial\Omega_0) \leq E_{\beta,p}(B_1, B_1) = \beta \mathcal{H}^{n-1}(\partial B_1),$$

so that $\mathcal{H}^{n-1}(\overset{\circ}{\Omega}_0) = \mathcal{H}^{n-1}(\partial B_1)$. Hence, by the rigidity of the isoperimetric inequality, we have that $\overset{\circ}{K}_0 = \overset{\circ}{\Omega}_0$ are balls of radius 1. On the other hand, if $\overset{\circ}{K}_0 \neq \overset{\circ}{\Omega}_0$, from the first part of the proof, there exists $R_0 > 1$ such that $|B_{R_0}| \geq M$ and

$$E_{\beta,p}(K_0, \Omega_0) = E_{\beta,p}(B_1, B_{R_0}).$$

Therefore, by Remark 2.11, we have that for a.e. $t \in (0, 1)$, the superlevel sets U_t coincide with balls up to \mathcal{H}^{n-1} -negligible sets, and $\{u = 1\}$ and $\{u > 0\}$ coincide with balls, up to \mathcal{H}^{n-1} -negligible sets, as well. We only have to show that $\{u = 1\}$ and $\{u > 0\}$ are concentric balls. To this aim, let us denote by $x(t)$ the center of the ball U_t and by $r(t)$ the radius of U_t , as already done in (2.12). In addition, we also have that

$$\frac{|\nabla u^*|}{u^*}(r(u(x))) = \varphi(x) = \frac{|\nabla u|}{u}(x),$$

so that if $u(x) = t$, then $|\nabla u(x)| = C_t > 0$. This ensures that we can write

$$\begin{aligned} x(t) &= \frac{1}{|U_t|} \int_{U_t} x \, d\mathcal{L}^n(x) \\ &= \frac{1}{|U_t|} \left(\int_t^1 \int_{\partial U_s} \frac{x}{|\nabla u(x)|} \, d\mathcal{H}^{n-1}(x) \, ds + \int_K x \, d\mathcal{L}^n(x) \right), \end{aligned}$$

and we can infer that $x(t)$ is an absolutely continuous function since $|\nabla u| > 0$ implies that $|U_t|$ is an absolutely continuous function as well. Moreover, on ∂U_t , we have that for every $v \in \mathbb{S}^{n-1}$,

$$(2.17) \quad u(x(t) + r(t)v) = t,$$

from which

$$(2.18) \quad \nabla u(x(t) + r(t)v) = -C_t v.$$

Differentiating (2.17), and using (2.18), we obtain

$$(2.19) \quad -C_t x'(t) \cdot v - C_t r'(t) = 1.$$

Finally, joining (2.19) and (2.15), along with the fact that $|\nabla u| = C_t$ on ∂U_t , we get

$$x'(t) \cdot v = 0$$

for every $v \in \mathbb{S}^{n-1}$, so that $x(t)$ is constant and U_t are concentric balls for a.e. $t \in (0, 1)$. In particular, $\{u = 1\} = \bigcap_t U_t$ and $\{u > 0\} = \bigcup_t U_t$ share the same center. ■

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REFERENCES

- [1] P. ACAMPORA – E. CRISTOFORONI, [A free boundary problem for the \$p\$ -Laplacian with nonlinear boundary conditions](#). *Ann. Mat. Pura Appl.* **203** (2024), 1–20.
- [2] P. ACAMPORA – E. CRISTOFORONI – C. NITSCH – C. TROMBETTI, [A free boundary problem in thermal insulation with a prescribed heat source](#). *ESAIM Control Optim. Calc. Var.* **29** (2023), article no. 3. Zbl 1509.35387 MR 4531396
- [3] G. ARONSSON – G. TALENTI, Estimating the integral of a function in terms of a distribution function of its gradient. *Boll. Un. Mat. Ital. B (5)* **18** (1981), no. 3, 885–894. Zbl 0476.49030 MR 641744
- [4] R. BARBATO, [Shape optimization for a nonlinear elliptic problem related to thermal insulation](#). 2022, arXiv:2207.03775.
- [5] M.-H. BOSSEL, [Longueurs extrémales et fonctionnelles de domaine](#). *Complex Variables Theory Appl.* **6** (1986), no. 2–4, 203–234. Zbl 0623.31002 MR 871731
- [6] D. BUCUR – D. DANERS, [An alternative approach to the Faber-Krahn inequality for Robin problems](#). *Calc. Var. Partial Differential Equations* **37** (2010), no. 1–2, 75–86. Zbl 1186.35118 MR 2564398

- [7] D. BUCUR – M. NAHON – C. NITSCH – C. TROMBETTI, [Shape optimization of a thermal insulation problem](#). *Calc. Var. Partial Differential Equations* **61** (2022), no. 5, article no. 186. Zbl [1496.35383](#) MR [4457936](#)
- [8] L. A. CAFFARELLI – D. KRIVENTSOV, [A free boundary problem related to thermal insulation](#). *Comm. Partial Differential Equations* **41** (2016), no. 7, 1149–1182. Zbl [1351.35268](#) MR [3528530](#)
- [9] D. DANERS, [A Faber-Krahn inequality for Robin problems in any space dimension](#). *Math. Ann.* **335** (2006), no. 4, 767–785. Zbl [1220.35103](#) MR [2232016](#)
- [10] F. DELLA PIETRA – C. NITSCH – C. TROMBETTI, [An optimal insulation problem](#). *Math. Ann.* **382** (2022), no. 1-2, 745–759. Zbl [1496.35189](#) MR [4377316](#)
- [11] S. KESAVAN, [Symmetrization & applications](#). Ser. Anal. 3, World Scientific Publishing, Hackensack, NJ, 2006. Zbl [1110.35002](#) MR [2238193](#)
- [12] F. MAGGI, [Sets of finite perimeter and geometric variational problems. An introduction to geometric measure theory](#). Cambridge Stud. Adv. Math. 135, Cambridge University Press, Cambridge, 2012. Zbl [1255.49074](#) MR [2976521](#)
- [13] A. L. MASIELLO – G. PAOLI, [A rigidity result for the Robin torsion problem](#). *J. Geom. Anal.* **33** (2023), no. 5, article no. 149. Zbl [1512.35176](#) MR [4554054](#)
- [14] G. PÓLYA – G. SZEGÖ, [Isoperimetric inequalities in mathematical physics](#). Ann. of Math. Stud. 27, Princeton University Press, Princeton, NJ, 1951. Zbl [0044.38301](#) MR [43486](#)

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