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Partial Differential Equations. – *An isoperimetric result for an energy related to the p-capacity*, by Paolo Acampora and Emanuele Cristoforoni, communicated on 10 November 2023.

Abstract. – In this paper, we generalize the notion of the relative p-capacity of K with respect to Ω, by replacing the Dirichlet boundary condition with a Robin one. We show that, under volume constraints, our notion of p-capacity is minimal when K and Ω are concentric balls. We use the H-function (see Bossel (1986) and Daners (2006)) and a derearrangement technique.

Keywords. – Robin, p-capacity, free boundary, isocapacitary inequality.

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1. Introduction

Let $p > 1$, $\beta > 0$ be real numbers. For every open bounded set $\Omega \subset \mathbb{R}^n$ with Lipschitz boundary, and every compact set $K \subseteq \bar{\Omega}$ with Lipschitz boundary, we define

$$E_{\beta,p}(K,\Omega) = \inf_{\substack{v \in W^{1,p}(\Omega) \\ v=1 \text{ in } K}} \left(\int_\Omega |\nabla v|^p \, dx + \beta \int_{\partial\Omega} |v|^p \, d\mathcal{H}^{n-1} \right). \tag{1.1}$$

We notice that it is sufficient to minimize among all functions $v \in H^1(\Omega)$ with $v = 1$ in K and $0 \le v \le 1$ a.e. Moreover, if K, Ω are sufficiently smooth, a minimizer u satisfies

$$\begin{cases} u = 1 & \text{in } K, \\ \Delta_p u = 0 & \text{in } \Omega \setminus K, \\ |\nabla u|^{p-2} \frac{\partial u}{\partial \nu} + \beta |u|^{p-2} u = 0 & \text{on } \partial\Omega \setminus \partial K, \end{cases} \tag{1.2}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$ is the p-Laplacian of u and ν is the outer unit normal to $\partial\Omega$. If $\mathring{K} = \Omega$, equation (1.2) has to be understood as $u = 1$ in Ω, and the energy is

$$E_{\beta,p}(\Omega,\Omega) = \beta \mathcal{H}^{n-1}(\partial\Omega).$$

In general, equation (1.2) has to be interpreted in the weak sense; that is, for every $\varphi \in W^{1,p}(\Omega)$ such that $\varphi \equiv 0$ in K,

$$\int_\Omega |\nabla u|^{p-2} \nabla u \nabla \varphi \, d\mathcal{L}^n + \beta \int_{\partial\Omega} u^{p-1} \varphi \, d\mathcal{H}^{n-1} = 0. \tag{1.3}$$

In particular, if u is a minimizer, letting $\varphi = u - 1$, we have that

$$E_{\beta,p}(K,\Omega) = \int_\Omega |\nabla u|^p \, dx + \beta \int_{\partial\Omega} u^p \, d\mathcal{H}^{n-1} = \beta \int_{\partial\Omega} u^{p-1} \, d\mathcal{H}^{n-1}.$$

Moreover, from the strict convexity of the functional, the minimizer is the unique solution to (1.3).

This problem is related to the so-called *relative p-capacity of K with respect to Ω*, defined as

$$\mathrm{Cap}_p(K,\Omega) := \inf_{\substack{v \in W_0^{1,p}(\Omega) \\ v=1 \text{ in } K}} \left(\int_\Omega |\nabla v|^p \, dx \right).$$

In the case $p = 2$, it represents the electrostatic capacity of an annular condenser consisting of a conducting surface $\partial\Omega$, and a conductor K, where the electrostatic potential is prescribed to be 1 inside K and 0 outside Ω. Let ω_n be the measure of the unit sphere in $\mathbb{R}^n$, and let $M > \omega_n$; then, it is well known that there exists some $r \geq 1$ such that

$$\min_{\substack{|K|=\omega_n \\ |\Omega|\leq M}} \mathrm{Cap}_p(K,\Omega) = \mathrm{Cap}_p(B_1, B_r).$$

This is an immediate consequence of the Pólya–Szegö inequality for the Schwarz rearrangement (see, for instance, [11, 14]). We are interested in studying the same problem for the energy defined in (1.1), which corresponds to changing the Dirichlet boundary condition on $\partial\Omega$ into a Robin boundary condition; namely, we consider the following problem:

$$\inf_{\substack{|K|=\omega_n \\ |\Omega|\leq M}} E_{\beta,p}(K,\Omega). \tag{1.4}$$

In this case, the previous symmetrization techniques cannot be employed anymore.

Problem (1.4) has been studied in the linear case $p = 2$ in [7], with more general boundary conditions on $\partial\Omega$; namely,

$$\frac{\partial u}{\partial \nu} + \frac{1}{2}\Theta'(u) = 0,$$

where Θ is a suitable increasing function vanishing at 0. This kind of problem has also been addressed in the context of thermal insulation (see, for instance, [1, 2, 8]). Our main result reads as follows.

THEOREM 1.1. *Let $\beta > 0$ such that*

$$\beta^{\frac{1}{p-1}} > \frac{n-p}{p-1}.$$

Then, for every $M > \omega_n$, the solution to problem (1.4) is given by two concentric balls (B_1, B_r); that is,

$$\min_{\substack{|K|=\omega_n \\ |\Omega|\le M}} E_{\beta,p}(K,\Omega) = E_{\beta,p}(B_1, B_r).$$

In particular, we have that either $r = 1$ or $M = \omega_n r^n$.

Moreover, if $K_0 \subseteq \overline{\Omega}_0$ are such that

$$E_{\beta,p}(K_0,\Omega_0) = \min_{\substack{|K|=\omega_n \\ |\Omega|\le M}} E_{\beta,p}(K,\Omega),$$

and u is the minimizer of $E_{\beta,p}(K_0,\Omega_0)$, then the sets $\{u = 1\}$ and $\{u > 0\}$ coincide with two concentric balls up to a $\mathcal{H}^{n-1}$-negligible set.

Remark 1.2. In the case

$$\beta^{\frac{1}{p-1}} \le \frac{n-p}{p-1},$$

adapting the symmetrization techniques used in [7], it can be proved that a solution to problem (1.4) is always given by the pair (B_1, B_1).

We point out that the proof of the theorem relies on the techniques involving the H-function introduced in [5, 9].

The case in which Ω is the Minkowski sum $\Omega = K + B_r(0)$, with the energy $E_{\beta,p}(K,\Omega)$, has been studied in [4] under suitable geometrical constraints (see also [10]).

2. Proof of the theorem

To prove Theorem 1.1, we start by studying the function

$$R \mapsto E_{\beta,p}(B_1, B_R).$$

A similar study of the previous function can also be found in [4]. Let

$$\Phi_{p,n}(\rho) = \begin{cases} \log(\rho) & \text{if } p = n, \\ -\frac{p-1}{n-p}\frac{1}{\rho^{\frac{n-p}{p-1}}} & \text{if } p \neq n. \end{cases}$$

For every $R > 1$, consider

$$u^*(x) = 1 - \frac{\beta^{\frac{1}{p-1}}\big(\Phi_{p,n}(|x|) - \Phi_{p,n}(1)\big)_+}{\Phi'_{p,n}(R) + \beta^{\frac{1}{p-1}}\big(\Phi_{p,n}(R) - \Phi_{p,n}(1)\big)}, \tag{2.1}$$

the solution to

$$\begin{cases} u^* = 1 & \text{in } B_1, \\ \Delta_p u^* = 0 & \text{in } B_R \setminus B_1, \\ |\nabla u^*|^{p-2} \frac{\partial u^*}{\partial \nu} + \beta |u^*|^{p-2} u^* = 0 & \text{on } \partial B_R. \end{cases}$$

We have that

$$\begin{aligned} E_{\beta,p}(B_1, B_R) &= \int_{B_R} |\nabla u^*|^p \, dx + \beta \int_{\partial B_R} |u^*|^p \, d\mathcal{H}^{n-1} \\ &= \frac{n\omega_n \beta}{\left[\Phi'_{p,n}(R) + \beta^{\frac{1}{p-1}} \left(\Phi_{p,n}(R) - \Phi_{p,n}(1)\right)\right]^{p-1}}. \end{aligned} \tag{2.2}$$

Notice that $E_{\beta,p}(B_1, B_R)$ is decreasing in $R > 0$ if and only if

$$\frac{d}{dR}\left(\Phi'_{p,n}(R) + \beta^{\frac{1}{p-1}} \Phi_{p,n}(R)\right) \geq 0,$$

that is, if and only if

$$R \geq \frac{n-1}{p-1} \frac{1}{\beta^{\frac{1}{p-1}}} =: \alpha_{\beta,p}.$$

Moreover,

$$E_{\beta,p}(B_1, B_1) = n\omega_n \beta,$$

$$\lim_{R\to\infty} E_{\beta,p}(B_1, B_R) = \begin{cases} n\omega_n \left(\frac{n-p}{p-1}\right)^{p-1} & \text{if } p < n, \\ 0 & \text{if } p \geq n. \end{cases}$$

Therefore, there are three cases:

(i) if $\beta^{\frac{1}{p-1}} \geq \frac{n-1}{p-1}$,

$$R \in [1, +\infty) \mapsto E_{\beta,p}(B_1, B_R)$$

is decreasing;

(ii) if $\frac{n-p}{p-1} < \beta^{\frac{1}{p-1}} < \frac{n-1}{p-1}$,

$$R \in [1, +\infty) \mapsto E_{\beta,p}(B_1, B_R)$$

increases on $[1, \alpha_{\beta,p}]$ and decreases on $[\alpha_{\beta,p}, +\infty)$, with the existence of a unique $R_{\beta,p} > \alpha_{\beta,p}$ such that $E_{\beta,p}(B_1, B_{R_{\beta,p}}) = E_{\beta,p}(B_1, B_1)$;

(iii) if $\beta^{\frac{1}{p-1}} \leq \frac{n-p}{p-1}$,

$$R \in [1, +\infty) \mapsto E_{\beta,p}(B_1, B_R)$$

reaches its minimum at $R = 1$.

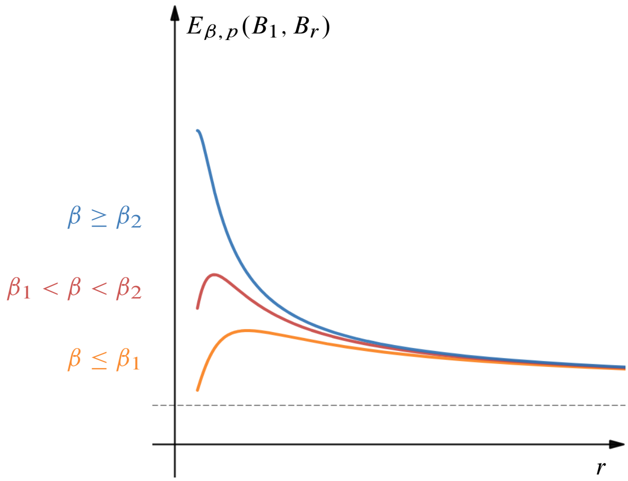

FIGURE 1. $E_{\beta,p}(B_1, B_r)$ depending on the value of β.

See, for instance, Figure 1, where

$$\beta_1 = \left(\frac{n-p}{p-1}\right)^{p-1}, \quad \beta_2 = \left(\frac{n-1}{p-1}\right)^{p-1}, \quad p = 2.5,\ n = 3.$$

We will need the following.

LEMMA 2.1. *Let $R > 1$, $\beta > 0$ and let u^* be the solution of the problem on (B_1, B_R). Then,*

$$\frac{|\nabla u^*|}{u^*} \leq \beta^{\frac{1}{p-1}}$$

in $B_R \setminus B_1$ if and only if

$$E_{\beta,p}(B_1, B_\rho) \geq E_{\beta,p}(B_1, B_R)$$

for every $\rho \in [1, R]$.

PROOF. Recalling the expressions of u^* in (2.1), by straightforward computations, we have that

$$\frac{|\nabla u^*|}{u^*} \leq \beta^{\frac{1}{p-1}}$$

in $B_R \setminus B_1$ if and only if

$$\begin{aligned} \Phi'_{p,n}(R) &+ \beta^{\frac{1}{p-1}}\big(\Phi_{p,n}(R) - \Phi_{p,n}(1)\big) \\ &\geq \Phi'_{p,n}(\rho) + \beta^{\frac{1}{p-1}}\big(\Phi_{p,n}(\rho) - \Phi_{p,n}(1)\big) \end{aligned} \tag{2.3}$$

for every $\rho \in [1, R]$. Using the expression of $E_{\beta,p}(B_1, B_\rho)$ in (2.2), (2.3) is equivalent to

$$E_{\beta,p}(B_1, B_\rho) \geq E_{\beta,p}(B_1, B_R)$$

for every $\rho \in [1, R]$. ■

Definition 2.2. Let $\Omega \subseteq \mathbb{R}^n$ be an open set, and let $U \subseteq \Omega$ be another set. We define the *internal boundary* of U as

$$\partial_{\mathrm{i}} U = \partial U \cap \Omega$$

and the *external boundary* of U as

$$\partial_{\mathrm{e}} U = \partial U \cap \partial\Omega.$$

Let $K \subseteq \overline{\Omega} \subseteq \mathbb{R}^n$ be open bounded sets, and let u be the minimizer of $E_{\beta,p}(K, \Omega)$. In the following, we denote

$$U_t = \left\{x \in \Omega \mid u(x) > t\right\}.$$

Definition 2.3 (H-function). Let $\varphi \in W^{1,p}(\Omega)$. We define

$$H(t, \varphi) = \int_{\partial_{\mathrm{i}} U_t} |\varphi|^{p-1} \, d\mathcal{H}^{n-1} - (p-1) \int_{U_t} |\varphi|^p \, d\mathcal{L}^n + \beta \mathcal{H}^{n-1}(\partial_{\mathrm{e}} U_t).$$

Notice that this definition is slightly different from the one given in [6].

Lemma 2.4. *Let $K \subseteq \Omega \subseteq \mathbb{R}^n$ be open bounded sets, and let u be the minimizer of $E_{\beta,p}(K, \Omega)$. Then, for a.e. $t \in (0, 1)$, we have*

$$H\left(t, \frac{|\nabla u|}{u}\right) = E_{\beta,p}(K, \Omega).$$

Proof. Recall that

$$E_{\beta,p}(K, \Omega) = \int_\Omega |\nabla u|^p \, d\mathcal{L}^n + \beta \int_{\partial\Omega} u^p = \beta \int_{\partial\Omega} u^{p-1} \, d\mathcal{H}^{n-1}. \tag{2.4}$$

Let $t \in (0, 1)$. We construct the following test functions: let $\varepsilon > 0$, and let

$$\varphi_\varepsilon(x) = \begin{cases} -1 & \text{if } u(x) \leq t, \\ \frac{u(x)-t}{\varepsilon u(x)^{p-1}} - 1 & \text{if } t < u(x) \leq t + \varepsilon, \\ \frac{1}{u(x)^{p-1}} - 1 & \text{if } u(x) > t + \varepsilon, \end{cases}$$

so that φ_ε is an approximation to the function $(u^{1-p}\chi_{U_t} - 1)$, and

$$\nabla\varphi_\varepsilon(x) = \begin{cases} 0 & \text{if } u(x) \le t, \\ \frac{1}{\varepsilon}\left(\frac{\nabla u(x)}{u(x)^{p-1}} - (p-1)\frac{\nabla u(x)(u(x)-t)}{u(x)^p}\right) & \text{if } t < u(x) \le t+\varepsilon, \\ -(p-1)\frac{\nabla u(x)}{u(x)^p} & \text{if } u(x) > t+\varepsilon. \end{cases}$$

We have that φ_ε is an admissible test function for the Euler–Lagrange equation (1.3), which entails

$$\begin{aligned} 0 = \frac{1}{\varepsilon}\int_{\{t<u\le t+\varepsilon\}\cap\Omega} \frac{|\nabla u|^{p-1}}{u^{p-1}}|\nabla u|\,d\mathcal{L}^n - (p-1)\int_{\{t<u\le t+\varepsilon\}\cap\Omega} \frac{|\nabla u|^p}{u^p}\frac{u-t}{\varepsilon}\,d\mathcal{L}^n \\ - (p-1)\int_{\{u>t+\varepsilon\}\cap\Omega}\frac{|\nabla u|^p}{u^p}\,d\mathcal{L}^n + \beta\int_{\{t<u\le t+\varepsilon\}\cap\partial\Omega}\frac{u-t}{\varepsilon}\,d\mathcal{H}^{n-1} \\ + \beta\mathcal{H}^{n-1}\big(\partial\Omega\cap\{u>t+\varepsilon\}\big) - \beta\int_{\partial\Omega} u^{p-1}\,d\mathcal{H}^{n-1}. \end{aligned}$$

Letting now ε go to 0, by the coarea formula, we get that for a.e. $t \in (0,1)$,

$$\begin{aligned} \beta\int_{\partial\Omega} u^{p-1}\,d\mathcal{H}^{n-1} = \int_{\partial_{\mathrm{i}} U_t}\left(\frac{|\nabla u|}{u}\right)^{p-1}d\mathcal{H}^{n-1} \\ - (p-1)\int_{U_t}\left(\frac{|\nabla u|}{u}\right)^{p}d\mathcal{L}^n + \beta\mathcal{H}^{n-1}(\partial_{\mathrm{e}} U_t). \end{aligned} \tag{2.5}$$

Joining (2.4) and (2.5), the lemma is proven. ∎

REMARK 2.5. Notice that if K and Ω are two concentric balls, the minimizer u is the one written in (2.1), for which the statement of the above lemma holds for every $t \in (0,1)$.

LEMMA 2.6. *Let $\varphi \in L^\infty(\Omega)$. Then, there exists $t \in (0,1)$ such that*

$$H(t,\varphi) \le E_{\beta,p}(K,\Omega).$$

PROOF. Let

$$w = |\varphi|^{p-1} - \left(\frac{|\nabla u|}{u}\right)^{p-1}.$$

Then, we evaluate

$$\begin{aligned} H(t,\varphi) - H\left(t,\frac{|\nabla u|}{u}\right) \\ = \int_{\partial_{\mathrm{i}} U_t} w\,d\mathcal{H}^{n-1} - (p-1)\int_{U_t}\left(|\varphi|^p - \left(\frac{|\nabla u|}{u}\right)^p\right)d\mathcal{L}^n \end{aligned}$$

$$\leq \int_{\partial_i U_t} w\, d\mathcal{H}^{n-1} - p\int_{U_t} \frac{|\nabla u|}{u} w\, d\mathcal{L}^n$$
$$= -\frac{1}{t^{p-1}}\frac{d}{dt}\left(t^p \int_{U_t} \frac{|\nabla u|}{u} w\, d\mathcal{L}^n\right),$$

where we used the inequality

$$a^p - b^p \leq \frac{p}{p-1} a\,(a^{p-1} - b^{p-1}) \quad \forall a, b \geq 0. \tag{2.6}$$

Multiplying by t^{p-1} and integrating, we get

$$\int_0^1 t^{p-1}\left(H(t,\varphi) - H\left(t, \frac{|\nabla u|}{u}\right)\right) dt \leq -\left[t^p \int_{U_t} \frac{|\nabla u|}{u} w\, d\mathcal{L}^n\right]_0^1 = 0, \tag{2.7}$$

from which we obtain the conclusion of the proof. ■

REMARK 2.7. Notice that inequality (2.6) holds as equality if and only if $a = b$. Therefore, if $\varphi \neq \frac{|\nabla u|}{u}$ on a set of positive measure, then inequality (2.7) is strict since

$$\left|\left\{\varphi \neq \frac{\nabla u}{u}\right\} \cap U_t\right| > 0$$

for small enough t. Therefore, there exists $S \subset (0,1)$ such that $\mathcal{L}^1(S) > 0$, and for every $t \in S$,

$$H(t,\varphi) < E_{\beta,p}(K,\Omega).$$

In the following, we fix a radius R such that $|B_R| \geq |\Omega|$, u^* the minimizer of $E_{\beta,p}(B_1, B_R)$, and

$$H^*(t,\varphi) = \int_{\partial\{u^*>t\}\cap B_R} |\varphi|^{p-1}\, d\mathcal{H}^{n-1} - (p-1)\int_{\{u^*>t\}} |\varphi|^p\, d\mathcal{L}^n$$
$$+ \beta\mathcal{H}^{n-1}\big(\partial\{u^* < t\} \cap \partial B_R\big).$$

Here, we recall the isoperimetric inequality (see, for example, [12]), which will be useful in what follows.

DEFINITION 2.8. Let E be a measurable set. For every $t \in [0,1]$, let

$$E^{(t)} = \left\{x \in \mathbb{R}^n \mid \lim_{r\to 0} \frac{\mathcal{L}^n\big(E \cap B_r(x)\big)}{\mathcal{L}^n\big(B_r(x)\big)} = t\right\}$$

be the set of all points where E has density t. We define the *essential boundary* of E as the set

$$\partial^{\text{ess}} E = \mathbb{R}^n \setminus (E^{(0)} \cup E^{(1)}).$$

Notice that $\partial^{\text{ess}} E \subseteq \partial E$.

THEOREM 2.9 (Isoperimetric inequality). *Let E be a measurable set, and let B be the ball such that $\mathcal{L}^n(B) = \mathcal{L}^n(E)$. Then,*

$$\mathcal{H}^{n-1}(\partial^{\text{ess}} E) \geq \mathcal{H}^{n-1}(\partial B),$$

and the equality holds if and only if $E = B$ up to a set of measure 0.

PROPOSITION 2.10. *Let $\beta > 0$. Assume that*

$$\frac{|\nabla u^*|}{u^*} \leq \beta^{\frac{1}{p-1}}. \tag{2.8}$$

Then, we have that

$$E_{\beta,p}(K,\Omega) \geq E_{\beta,p}(B_1, B_R).$$

PROOF. In the following, if v is a radial function on B_R and $r \in (0, R)$, we denote with abuse of notation

$$v(r) = v(x),$$

where x is any point on ∂B_r. By Lemma 2.4, we know that for every $t \in (0, 1)$,

$$H^*\left(t, \frac{|\nabla u^*|}{u^*}\right) = E_{\beta,p}(B_1, B_R), \tag{2.9}$$

while by Lemma 2.6, for every $\varphi \in L^\infty(\Omega)$, there exists $t \in (0, 1)$ such that

$$E_{\beta,p}(K,\Omega) \geq H(t,\varphi). \tag{2.10}$$

We aim to find a suitable φ such that, for some t,

$$H(t,\varphi) \geq H^*\left(t, \frac{|\nabla u^*|}{u^*}\right), \tag{2.11}$$

so that combining (2.10), (2.11), and (2.9), we conclude the proof. In order to construct φ, for every $t \in (0, 1)$, we define

$$r(t) = \left(\frac{|U_t|}{\omega_n}\right)^{\frac{1}{n}}; \tag{2.12}$$

then, we set, for every $x \in \Omega$,

$$\varphi(x) = \frac{|\nabla u^*|}{u^*}\big(r\big(u(x)\big)\big).$$

CLAIM. The functions $\varphi\chi_{U_t}$ and $\frac{|\nabla u^*|}{u^*}\chi_{B_{r(t)}}$ are equi-measurable; in particular,

$$\int_{U_t} \varphi^p \, d\mathcal{L}^n = \int_{B_{r(t)}} \left(\frac{|\nabla u^*|}{u^*}\right)^p d\mathcal{L}^n. \tag{2.13}$$

Indeed, let $g(r) = \frac{|\nabla u^*|}{u^*}(r)$, and by the coarea formula,

$$\begin{aligned}
\left|U_t \cap \{\varphi > s\}\right| &= \int_{U_t \cap \{g(r(u(x)))>s\}} d\mathcal{L}^n \\
&= \int_t^{+\infty} \int_{\partial^* U_\tau \cap \{g(r(\tau))>s\}} \frac{1}{\left|\nabla u(x)\right|} \, d\mathcal{H}^{n-1}(x)\, d\tau \\
&= \int_0^{r(t)} \int_{\partial^* U_{r^{-1}(\sigma)}} \frac{1}{\left|\nabla u(x)\right| \left|r'\left(r^{-1}(\sigma)\right)\right|} \chi_{\{g(\sigma)>s\}} \, d\mathcal{H}^{n-1}(x)\, d\sigma.
\end{aligned} \tag{2.14}$$

Notice now that since

$$\omega_n r(\tau)^n = |U_\tau|,$$

then

$$r'(\tau) = -\frac{1}{n\omega_n r(\tau)^{n-1}} \int_{\partial^* U_\tau} \frac{1}{\left|\nabla u(x)\right|} \, d\mathcal{H}^{n-1}(x). \tag{2.15}$$

Therefore, substituting in (2.14), we get

$$\left|U_t \cap \{\varphi > s\}\right| = \int_0^{r(t)} n\omega_n \sigma^{n-1} \chi_{\{g(\sigma)>s\}} \, d\sigma = \left| B_{r(t)} \cap \left\{ \frac{|\nabla u^*|}{u^*} > s \right\} \right|,$$

where we have used polar coordinates to get the last equality. Thus, the claim is proved.

Recalling the definition of φ, (2.8) reads

$$\beta \geq \varphi^{p-1},$$

and then using (2.13) and the definition of H (see Definition 2.3), we have

$$\begin{aligned}
H(t,\varphi) &= \beta \mathcal{H}^{n-1}(\partial_e U_t) + \int_{\partial_i U_t} \varphi^{p-1} \, d\mathcal{H}^{n-1} - (p-1) \int_{U_t} \varphi^p \, d\mathcal{L}^n \\
&\geq \int_{\partial U_t} \varphi^{p-1} \, d\mathcal{H}^{n-1} - (p-1) \int_{B_{r(t)}} \left(\frac{|\nabla u^*|}{u^*} \right)^p d\mathcal{L}^n \\
&\geq \int_{\partial B_{r(t)}} \left(\frac{|\nabla u^*|}{u^*} \right)^{p-1} d\mathcal{H}^{n-1} - (p-1) \int_{B_{r(t)}} \left(\frac{|\nabla u^*|}{u^*} \right)^p d\mathcal{L}^n \\
&= H^*\left(u^*\left(r(t)\right), \frac{|\nabla u^*|}{u^*} \right) \\
&= E_{\beta,p}(B_1, B_R),
\end{aligned} \tag{2.16}$$

where in the last inequality we have used the isoperimetric inequality and the fact that φ is constant on ∂U_t. ■

REMARK 2.11. By Remark 2.7, we have that if K and Ω are such that

$$E_{\beta,p}(K,\Omega) = E_{\beta,p}(B_1, B_R),$$

then

$$\varphi = \frac{|\nabla u|}{u} \quad \text{for a. e. } x \in \Omega,$$

so that, by Lemma 2.4, we have equality in (2.16) for a.e. $t \in (0,1)$. Thus, by the rigidity of the isoperimetric inequality, we get that U_t coincides with a ball up to a $\mathcal{H}^{n-1}$-negligible set for a.e. $t \in (0,1)$. In particular,

$$\{u > 0\} = \bigcup_t U_t \quad \text{and} \quad \{u = 1\} = \bigcap_t U_t$$

coincide with two balls up to a $\mathcal{H}^{n-1}$-negligible set.

PROOF OF THEOREM 1.1. Fix $M = \omega_n R^n$ with $R > 1$. We divide the proof of the minimality of balls into two cases, and subsequently, we study the equality case.

Let us assume that

$$\beta^{\frac{1}{p-1}} \geq \frac{n-1}{p-1},$$

and recall that in this case the function

$$\rho \in [1, +\infty) \mapsto E_{\beta,p}(B_1, B_\rho)$$

is decreasing. Let u^* be the minimizer of $E_{\beta,p}(B_1, B_R)$; by Lemma 2.1, condition (2.8) holds and, by Proposition 2.10, we have that a solution to (1.4) is given by the concentric balls (B_1, B_R).

Assume now that

$$\frac{n-p}{p-1} < \beta^{\frac{1}{p-1}} < \frac{n-1}{p-1};$$

then, in this case, letting

$$\alpha_{\beta,p} = \frac{(n-1)}{(p-1)\beta^{\frac{1}{p-1}}},$$

the function

$$\rho \in [1, +\infty) \mapsto E_{\beta,p}(B_1, B_\rho)$$

increases on $[1, \alpha_{\beta,p}]$ and decreases on $[\alpha_{\beta,p}, +\infty)$, and there exists a unique $R_{\beta,p} > \alpha_{\beta,p}$ such that

$$E_{\beta,p}(B_1, B_{R_{\beta,p}}) = E_{\beta,p}(B_1, B_1).$$

If $R \geq R_{\beta,p}$, the function u^*, the minimizer of $E_{\beta,p}(B_1, B_R)$, still satisfies condition (2.8), and, as in the previous case, a solution to (1.4) is given by the concentric balls

(B_1, B_R). On the other hand, if $R < R_{\beta,p}$, we can consider $u^*_{\beta,p}$ the minimizer of $E_{\beta,p}(B_1, B_{R_{\beta,p}})$. By Lemma 2.1, we have that, for the function $u^*_{\beta,p}$, condition (2.8) holds, and, by Proposition 2.10, we have that if K and Ω are open bounded Lipschitz sets with $K \subseteq \Omega$, $|K| = \omega_n$, and $|\Omega| \le M$, then

$$E_{\beta,p}(K, \Omega) \ge E_{\beta,p}(B_1, B_{R_{\beta,p}}) = E_{\beta,p}(B_1, B_1)$$

and a solution to (1.4) is given by the pair (B_1, B_1).

For what concerns the equality case, we will follow the outline of the rigidity problem given in [13, Section 3] (see also [3, Section 2]). Let $K_0 \subseteq \overline{\Omega}_0$ be such that

$$E_{\beta,p}(K_0, \Omega_0) = \min_{\substack{|K|=\omega_n \\ |\Omega|\le M}} E_{\beta,p}(K, \Omega).$$

Let u be the minimizer of $E_{\beta,p}(K_0, \Omega_0)$. If $\mathring{K}_0 = \Omega_0$, then $|\Omega_0| = |B_1|$ and the isoperimetric inequality yields

$$\mathcal{H}^{n-1}(\partial\Omega_0) \ge \mathcal{H}^{n-1}(\partial B_1),$$

while, from the minimality of (K_0, Ω_0), we have that

$$E_{\beta,p}(K_0, \Omega_0) = \beta\mathcal{H}^{n-1}(\partial\Omega_0) \le E_{\beta,p}(B_1, B_1) = \beta\mathcal{H}^{n-1}(\partial B_1),$$

so that $\mathcal{H}^{n-1}(\Omega_0) = \mathcal{H}^{n-1}(\partial B_1)$. Hence, by the rigidity of the isoperimetric inequality, we have that $\mathring{K}_0 = \Omega_0$ are balls of radius 1. On the other hand, if $\mathring{K}_0 \ne \Omega_0$, from the first part of the proof, there exists $R_0 > 1$ such that $|B_{R_0}| \ge M$ and

$$E_{\beta,p}(K_0, \Omega_0) = E_{\beta,p}(B_1, B_{R_0}).$$

Therefore, by Remark 2.11, we have that for a.e. $t \in (0,1)$, the superlevel sets U_t coincide with balls up to $\mathcal{H}^{n-1}$-negligible sets, and $\{u = 1\}$ and $\{u > 0\}$ coincide with balls, up to $\mathcal{H}^{n-1}$-negligible sets, as well. We only have to show that $\{u = 1\}$ and $\{u > 0\}$ are concentric balls. To this aim, let us denote by $x(t)$ the center of the ball U_t and by $r(t)$ the radius of U_t, as already done in (2.12). In addition, we also have that

$$\frac{|\nabla u^*|}{u^*}\big(r\big(u(x)\big)\big) = \varphi(x) = \frac{|\nabla u|}{u}(x),$$

so that if $u(x) = t$, then $|\nabla u(x)| = C_t > 0$. This ensures that we can write

$$\begin{aligned} x(t) &= \frac{1}{|U_t|}\int_{U_t} x \, d\mathcal{L}^n(x) \\ &= \frac{1}{|U_t|}\left(\int_t^1 \int_{\partial U_s} \frac{x}{|\nabla u(x)|} \, d\mathcal{H}^{n-1}(x)\, ds + \int_K x \, d\mathcal{L}^n(x)\right), \end{aligned}$$

and we can infer that $x(t)$ is an absolutely continuous function since $|\nabla u| > 0$ implies that $|U_t|$ is an absolutely continuous function as well. Moreover, on ∂U_t, we have that for every $\nu \in \mathbb{S}^{n-1}$,

$$u\big(x(t) + r(t)\nu\big) = t, \tag{2.17}$$

from which

$$\nabla u\big(x(t) + r(t)\nu\big) = -C_t \nu. \tag{2.18}$$

Differentiating (2.17), and using (2.18), we obtain

$$-C_t\, x'(t) \cdot \nu - C_t\, r'(t) = 1. \tag{2.19}$$

Finally, joining (2.19) and (2.15), along with the fact that $|\nabla u| = C_t$ on ∂U_t, we get

$$x'(t) \cdot \nu = 0$$

for every $\nu \in \mathbb{S}^{n-1}$, so that $x(t)$ is constant and U_t are concentric balls for a.e. $t \in (0,1)$. In particular, $\{u = 1\} = \bigcap_t U_t$ and $\{u > 0\} = \bigcup_t U_t$ share the same center. ∎

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