Measure and Integration. – *On the properties of the set where a generalized function of bounded variation takes an infinite value*, by Alessandro Cucinotta, communicated on 10 November 2023.

Abstract. – We study the properties of the set where a generalized function of bounded variation has infinite approximate limit, highlighting in this way the main geometric difference with functions of bounded variation. To this aim, we prove a new result on strict approximation of the sets of finite perimeter from the outside with open sets.

Keywords. – Generalized functions of bounded variation, approximate limits, sets of finite perimeter, density of a set.

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1. Introduction

The main geometric difference between the functions of bounded variation and the generalized functions of bounded variation is that the latter may have an infinite approximate limit on bigger sets. More precisely, the set of points where a function of bounded variation defined on $\Omega \subset \mathbb{R}^N$ has an infinite approximate limit is negligible w.r.t. the $N-1$ dimensional Hausdorff measure, while it is easy to see that the generalized functions of bounded variation may have an infinite approximate limit on the sets with a Hausdorff dimension greater than $N-1$. In the paper, we study this discrepancy.

Let $L^0(\Omega)$ be the collection of Lebesgue-measurable functions that are finite valued almost everywhere. Given $u \in L^0(\Omega)$, we denote by $\tilde{u}(x)$ its approximate limit at x whenever it exists, and we say that a set $C \subset \Omega$ is L^0-polar if there exists a function $v \in L^0(\Omega)$ such that $C \subset \{x \in \Omega : |\tilde{v}(x)| = +\infty\}$. Analogous definitions hold if u is in the space BV(Ω) of functions of bounded variation or in the space GBV(Ω) of generalized functions of bounded variation, and we denote the collections of polar sets, respectively, with $\mathcal{P}_{L^0}$, $\mathcal{P}_{\mathrm{BV}}$, and $\mathcal{P}_{\mathrm{GBV}}$ (Definition 3.1).

Employing results from the general theory of approximate limits and functions of bounded variation, we show that $\mathcal{P}_{L^0}$ is the class of sets that are negligible w.r.t. the Lebesgue measure λ^N, while $\mathcal{P}_{\mathrm{BV}}$ is the class of sets that are negligible w.r.t. the Hausdorff measure H^{N-1}, and we prove that all inclusions in the following chain are strict:

$$\mathcal{P}_{\mathrm{BV}} \subset \mathcal{P}_{\mathrm{GBV}} \subset \mathcal{P}_{L^0}.$$

The class $\mathcal{P}_{\mathrm{GBV}}$ cannot be characterized in terms of Hausdorff measures since one can show that GBV-polar sets may have any Hausdorff dimension smaller than the dimension of the ambient space.

To introduce our characterization of the class $\mathcal{P}_{\mathrm{GBV}}$, we observe that the previous results may be reformulated saying that a set $C \subset \Omega$ belongs to $\mathcal{P}_{L^0}$ if and only if there exists a collection of open sets $\{U_k\}_{k \in \mathbb{N}}$ containing C such that $\lambda^N(U_k) \to 0$, while it belongs to $\mathcal{P}_{\mathrm{BV}}$ if and only if in addition to the previous conditions we also have that the perimeters $P(U_k, \Omega)$ of the sets U_k tend to zero as k goes to infinity. We then prove that a subset $C \subset \Omega$ belongs to $\mathcal{P}_{\mathrm{GBV}}$ if and only if there exists a collection $\{U_k\}_{k \in \mathbb{N}}$ of open sets of a locally finite perimeter containing C whose Lebesgue measure decreases to zero as k goes to infinity. The proof of this fact cannot be obtained through the same tools used for $\mathcal{P}_{L^0}$ and $\mathcal{P}_{\mathrm{BV}}$ and relies on a new approximation result for the sets of finite perimeter which is interesting in itself.

More precisely, we prove that a set of finite perimeter $C \subset \Omega$ can be approximated in a strict sense (i.e., in measure and with the perimeters of the approximating sets approaching the perimeter of C) by open sets containing the points of Ω where C has density 1. We also show that one cannot replace the set of points where C has density 1 with C itself and that the approximating sets in general cannot have smooth boundary. This approximation theorem is not directly implied by existing ones ([1, Theorem 3.42], [2, 8]) and its proof relies on tools from the capacity theory combined with a strong approximation result proved by Quentin de Gromard in [7].

The first section fixes the notation and contains preliminaries, and in Section 3, we analyze the properties of polar sets, while the last section deals with the proof of the aforementioned approximation result.

2. Notation and preliminaries

Let Ω be an open subset of $\mathbb{R}^N$. Given $A \subset \mathbb{R}^N$, we denote its indicator function by 1_A. We denote by ω_N the volume of the N-dimensional unit ball, and given a Borel set $A \subset \Omega$, we will denote by $\lambda^N(A)$ its Lebesgue measure, while $H^{N-1}(A)$ will be its $N - 1$ dimensional Hausdorff measure. Whenever we apply λ^N or H^{N-1} to a set, we assume implicitly its λ^N- or H^{N-1}-measurability. We say that a sequence $\{\Omega_k\}_{k \in \mathbb{N}}$ of open subset of Ω is an exhaustion of Ω if for every k we have $\bar{\Omega}_k \subset \Omega_{k+1}$ and the union of the sets Ω_k is the whole Ω. It is well known that there exists an exhaustion of Ω made of smooth sets.

Given $x \in \Omega$, the upper and lower density of A at x are defined, respectively, as

$$\limsup_{\rho \to 0^+} \frac{\lambda^N(A \cap B_\rho(x))}{\lambda^N(B_\rho(x))} \quad \text{and} \quad \liminf_{\rho \to 0^+} \frac{\lambda^N(A \cap B_\rho(x))}{\lambda^N(B_\rho(x))},$$

while we say that A has density t at x (and we write $\theta(A, x) = t$) if t is both the upper and the lower density of A at x. The set of points in Ω where A has density t will be denoted by $A^{(t)}$, while the set of points where A has strictly positive upper density will be denoted by A^+. We will indicate with $L^0(\Omega)$ the collection of λ^N-a.e. real-valued Lebesgue measurable functions, and for any $u \in L^0(\Omega)$, its approximate upper limit at x is

$$u^+(x) := \inf \left\{ t \in \mathbb{R} : \theta(\{y \in \Omega : u(y) > t\}, x) = 0 \right\},$$

with the convention that $\inf(\emptyset) = +\infty$. Similarly, the approximate lower limit of u at x is the value

$$u^-(x) := \sup \left\{ t \in \mathbb{R} : \theta(\{y \in \Omega : u(y) < t\}, x) = 0 \right\},$$

with the convention that $\sup(\emptyset) = -\infty$. If $u^+(x) = u^-(x)$, their common value is called the approximate limit of u at x and is denoted by $\tilde{u}(x)$. It follows from the definition that for a continuous function u, we have $\tilde{u}(x) = u(x)$ for every $x \in \Omega$. It is also easy to see that if $f : \bar{\mathbb{R}} \to \bar{\mathbb{R}}$ is a continuous function and $u \in L^0(\Omega)$ has approximate limit $\tilde{u}(x)$ at x, then the approximate limit of $f \circ u$ at x is $f(\tilde{u}(x))$. We will use the following theorem.

THEOREM 2.1 ([4, Theorem 2.9.13]). *If $u \in L^0(\Omega)$, then for λ^N-almost every $x \in \Omega$, the approximate limit of u at x exists and is finite.*

For the properties of the space $BV(\Omega)$ of the functions of bounded variation, we refer to [1, Chapter 3] and [3, Chapter 5]. Given $u \in L^1_{\text{loc}}(\Omega)$, we denote its total variation [1, Definition 3.4] by $V(u, \Omega)$. If $u \in L^1(\Omega)$, then $u \in BV(\Omega)$ if and only if $V(u, \Omega) < +\infty$. If $u \in BV(\Omega)$, then Du will be its distributional derivative (which is a bounded Radon measure with values in $\mathbb{R}^N$) and $|Du|$ will be the variation of Du, which satisfies $|Du|(\Omega) = V(u, \Omega)$. It is known that the total variation is lower semicontinuous with respect to convergence in $L^1_{\text{loc}}(\Omega)$. Moreover, if $u \in BV(\Omega)$ and $f : \mathbb{R} \to \mathbb{R}$ is Lipschitz with $f(0) = 0$, then $f \circ u \in BV(\Omega)$ and $|Df(u)|(\Omega)| \le \text{Lip}(f)|Du|(\Omega)$. We define the norm $\|u\|_{BV} := \|u\|_{L^1} + |Du|(\Omega)$, and we refer to convergence in the BV norm as strong convergence.

We will indicate the perimeter of A in Ω [1, Definition 3.35] with $P(A, \Omega)$, and we say that A has a locally finite perimeter if its perimeter is finite in every precompact open subset of Ω; in this case, $\partial^* A$ will be its reduced boundary [1, Definition 3.54]. If $u \in BV(\Omega)$, the coarea formula [1, Theorem 3.40] implies that

$$\int_{-\infty}^{+\infty} P(\{u > t\}, \Omega) \, dt = |Du|(\Omega).$$

By the blow-up properties of the reduced boundary, we have the following theorems.

THEOREM 2.2 ([1, Theorem 3.59]). *Let A be a set of finite perimeter in Ω. Then,*
$P(A, \Omega) = H^{N-1}(\partial^* A \cap \Omega)$.

THEOREM 2.3 ([1, Theorem 3.61]). *Let A be a set of finite perimeter in Ω. Then,*
$\partial^* A \subset A^{(1/2)}$.

We will use also the following fine properties of BV functions.

THEOREM 2.4 ([3, Theorems 2–3, Section 5.9]). *If $u \in BV(\Omega)$, then for H^{N-1}-almost
every $x \in \Omega$, we have that $u^+(x)$ and $u^-(x)$ are finite and*

$$\lim_{\rho \to 0} \fint_{B_\rho(x)} u(y) \, dy = \frac{u^+(x) + u^-(x)}{2},$$

where the slashed integral denotes the mean value.

The next result is due to Quentin de Gromard and will be crucial in Section 4.

THEOREM 2.5 ([7, Theorem 3.1]). *Let $B \subset \Omega$ be a set of finite perimeter in Ω and
let $\varepsilon > 0$ be fixed. There exists a relatively closed set $L \subset \Omega$ such that the following
properties hold:*

(1) $\lambda^N(B \Delta L) < \varepsilon$;

(2) $|D(1_B - 1_L)|(\Omega) < \varepsilon$;

(3) $H^{N-1}((\partial L \cap \Omega) \setminus \partial^* L) < \varepsilon$.

We now recall the definition of the generalized functions of bounded variation.

DEFINITION 2.6. A function $u : \Omega \to \mathbb{R}$ is a generalized function of bounded variation
if for every $m \in \mathbb{R}_+$, the truncated function at the level m, i.e., $m \wedge u \vee -m$, belongs
to $BV_{loc}(\Omega)$. We denote such truncation by u^m and the space of these functions by
$GBV(\Omega)$.

If $u : \Omega \to \overline{\mathbb{R}}$ is real-valued λ^N-a.e. and satisfies the truncation condition of Defini-
tion 2.6, we say that $u \in GBV(\Omega)$, implicitly referring to its real-valued representative.
Some properties of GBV functions can be found in [1, Section 4.5]. Even if $GBV(\Omega)$
is not a vector space, the sum of positive GBV functions is still in $GBV(\Omega)$. We will
also use the following result.

PROPOSITION 2.7 ([1, Theorem 4.34]). *Let $u \in GBV(\Omega)$. Then, for λ^1-a.e. $t \in \mathbb{R}$, the
set $\{u > t\}$ has a locally finite perimeter.*

We now introduce the notation needed to apply the slicing techniques. Let ν be a
vector in S^{N-1} and $C \subset \mathbb{R}^N$. We denote by π_ν the hyperplane of $\mathbb{R}^N$ orthogonal to

v and by C_v the orthogonal projection of C on this hyperplane. For any $y \in \pi_v$, the (possibly empty) set $\{t \in \mathbb{R} : y + tv \in C\}$ is denoted by C_v^y.

Given $u : C \to \overline{\mathbb{R}}$, for every $y \in C_v$ such that $C_v^y \neq \emptyset$, the function $u_v^y : C_v^y \to \overline{\mathbb{R}}$ is defined by $u_v^y(t) := u(y + tv)$.

PROPOSITION 2.8 ([1, Proposition 4.35]). *Let* $u \in \mathrm{GBV}(\Omega)$ *and* $v \in S^{N-1}$. *Then, for* H^{N-1}-*a.e.* $y \in \Omega_v$, *we have that* $u_v^y \in \mathrm{GBV}(\Omega_v^y)$ *and*

$$(u^\pm)_v^y(t) = (u_v^y)^\pm(t),$$

for every $t \in \Omega_v^y$.

Finally, we need some results about the 1-capacity of a set. For these results, we refer to [3, Section 4.7] and [5]. Using the notation of [3], we say that a positive function u belongs to K^1 if it is in $L^{\frac{N}{N-1}}(\mathbb{R}^N)$ and its distributional derivative is a vector-valued function ∇u whose modulus is in $L^1(\mathbb{R}^N)$.

DEFINITION 2.9. The 1-capacity of a set $E \subset \mathbb{R}^N$ is defined as the quantity

$$\mathrm{Cap}_1(E) := \inf \left\{ \int_{\mathbb{R}^N} |\nabla u(x)| \, dx : u \in K^1, \ u \geq 1 \ \lambda^N\text{-a.e. on a neighborhood of } E \right\},$$

with the usual convention that $\inf \emptyset := +\infty$.

We will often refer to the 1-capacity simply as the capacity. The next propositions summarize some of the main properties of the capacity.

PROPOSITION 2.10 ([3, Section 4.7, Theorem 1]). *The set function* $\mathrm{Cap}_1(\cdot)$, *defined on the power set of* $\mathbb{R}^N$, *is increasing and countably subadditive.*

PROPOSITION 2.11 ([3, Section 5.6, Theorem 3]). *A set* $E \subset \mathbb{R}^N$ *satisfies* $\mathrm{Cap}_1(E) = 0$ *if and only if* $H^{N-1}(E) = 0$.

The next proposition gives a characterization of the capacity that will be used in Proposition 2.13.

PROPOSITION 2.12. *Given* $E \subset \mathbb{R}^N$, *the capacity of* E *coincides with the following quantities:*

$$a(E) = \inf \left\{ P(B, \mathbb{R}^N) : B \ \lambda^N\text{-measurable}, \ \lambda^N(B) < +\infty, \ E \subset \overset{\circ}{B} \right\},$$
$$b(E) = \inf \left\{ |Du|(\mathbb{R}^N) : u \in \mathrm{BV}(\mathbb{R}^N), \ u \geq 1 \ \lambda^N\text{-a.e. on a neigh. of } E \right\}.$$

PROOF. The equivalence of $\mathrm{Cap}_1(E)$ with $a(E)$ follows by [5, p. 145], so we only prove the equivalence of $a(E)$ and $b(E)$. We only prove that $b(E) \geq a(E)$ since the other

implication is trivial. To do this, we prove that if $u \in \mathrm{BV}(\mathbb{R}^N)$ and $u \geq 1$ λ^N-a.e. on a neighborhood of E, then there exists B as in $a(E)$ such that $P(B, \mathbb{R}^N) \leq |Du|(\mathbb{R}^N)$. To this aim, we set $v \in \mathrm{BV}(\mathbb{R}^N)$ as $v := 0 \vee u \wedge 1$, and we observe that $|Dv|(\mathbb{R}^N) \leq |Du|(\mathbb{R}^N)$. Moreover, by the coarea formula, we have that

$$\int_0^1 P(\{v > t\}, \mathbb{R}^N) \, dt = |Dv|(\mathbb{R}^N) \leq |Du|(\mathbb{R}^N),$$

so that there exists $t_0 \in (0, 1)$ such that $P(\{v > t_0\}, \mathbb{R}^N) \leq |Du|(\mathbb{R}^N)$.

We now note that $\{v > t_0\} \supset \{u \geq 1\}$; since $u \geq 1$ at λ^N-a.e. every point of a neighborhood of E, there exists a λ^N-null set A such that $\{v > t_0\} \cup A$ contains a neighborhood of E. Observe then that $P(\{v > t_0\} \cup A, \mathbb{R}^N) = P(\{v > t_0\}, \mathbb{R}^N)$ so that setting $B := \{v > t_0\} \cup A$, we conclude. ∎

PROPOSITION 2.13. *There exists a dimensional constant $c > 0$ such that for every $u \in \mathrm{BV}(\mathbb{R}^N)$ and $\varepsilon > 0$, the following estimate is satisfied:*

$$\mathrm{Cap}_1 \left\{ x : \exists \rho \in \mathbb{R}_+ : \fint_{B_\rho(x)} u(y) \, dy > \varepsilon \right\} \leq \frac{c}{\varepsilon} |Du|(\mathbb{R}^N).$$

PROOF. The desired estimate is proved for every $u \in K^1$ in [3, Lemma 1, Section 4.8]. To prove our version, one simply repeats the exact same argument of [3] using the equivalence between $\mathrm{Cap}_1(E)$ and $b(E)$ proved in Proposition 2.12. ∎

3. POLAR SETS

In this section, we introduce the classes of polar sets $\mathcal{P}_{L^0}$, $\mathcal{P}_{\mathrm{BV}}$, and $\mathcal{P}_{\mathrm{GBV}}$. First, we characterize $\mathcal{P}_{L^0}$ and $\mathcal{P}_{\mathrm{BV}}$, respectively, as the class of λ^N- and H^{N-1}-negligible sets, and we observe that these conditions can both be expressed in terms of intersections of open sets (with a perimeter constraint in the BV case). Then, we use these characterizations to prove that the inclusions in the chain $\mathcal{P}_{\mathrm{BV}} \subset \mathcal{P}_{\mathrm{GBV}} \subset \mathcal{P}_{L^0}$ are strict. To this aim, we prove that any relatively closed λ^N-negligible set $C \subset \Omega$ belongs to $\mathcal{P}_{\mathrm{GBV}}$ and that GBV-polar sets behave well w.r.t. one-dimensional slicings.

In the final part of the section, assuming a result which will be later proved in Section 4, we characterize $\mathcal{P}_{\mathrm{GBV}}$ in terms of intersections of open sets with a locally finite perimeter (Theorem 3.11), completing the picture on polar sets.

DEFINITION 3.1. A set $C \subset \Omega$ is called L^0 polar (respectively, BV polar or GBV polar) if there exists a function u in $L^0(\Omega)$ (respectively, in $\mathrm{BV}(\Omega)$ or in $\mathrm{GBV}(\Omega)$) such that $C \subset \{x \in \Omega : |\tilde{u}(x)| = +\infty\}$. We denote the collection of these sets with $\mathcal{P}_{L^0}$ (respectively, with $\mathcal{P}_{\mathrm{BV}}$ or $\mathcal{P}_{\mathrm{GBV}}$).

Replacing u with $|u|$ in the previous definition, we obtain that a set $C \subset \Omega$ is (L^0, BV or GBV) polar if and only if there exists a positive function v (in $L^0(\Omega)$, BV(Ω) or GBV(Ω)) such that $C \subset \{x \in \Omega : \tilde{v}(x) = +\infty\}$.

PROPOSITION 3.2. *Let $C \subset \Omega$. The following conditions are equivalent:*

(a) $C \in \mathcal{P}_{L^0}$;

(b) $\lambda^N(C) = 0$;

(c) *there exists a sequence of open sets $\{U_k\}_{k \in \mathbb{N}}$ containing C such that $\lambda^N(U_k) \to 0$.*

PROOF. If $C \in \mathcal{P}_{L^0}$, then $\lambda^N(C) = 0$ by Theorem 2.1.

If $\lambda^N(C) = 0$, then there exists the desired sequence of open sets by the outer regularity of the Lebesgue measure.

Suppose now that we have a sequence as in (c). Passing to a (not relabeled) subsequence $\{U_k\}_{i \in \mathbb{N}}$, we may suppose that $\lambda^N(U_k) \leq k^{-2}$. Now, define

$$v : \Omega \to \mathbb{R} \cup \{+\infty\}$$

by

$$v(x) := \sum_{k \in \mathbb{N}} 1_{U_k}(x),$$

and observe that by the monotone convergence theorem,

$$\int_\Omega |v(x)| \, dx = \sum_{k \in \mathbb{N}} \lambda^N(U_k) \leq \sum_{k \in \mathbb{N}} k^{-2} < +\infty,$$

so that v is real-valued λ^N-almost everywhere. Moreover, for every $k_0 \in \mathbb{N}$, we have that $v \geq k_0$ on $\bigcap_{k=1}^{k_0} U_k$, and since this is an open set containing C, we deduce that for every $x \in C$, we have $u^-(x) \geq k_0$. By the arbitrariness of k_0, we conclude. ∎

The next lemma is needed to characterize $\mathcal{P}_{\mathrm{BV}}$ and will be used also in Section 4. We denote by $S^N - 1$ the unit sphere in $\mathbb{R}^N$ and by σ_{N-1} its surface area.

LEMMA 3.3. *Let $Z \subset \Omega$ be such that $H^{N-1}(Z) < +\infty$. There exists a dimensional constant τ such that for every $\varepsilon > 0$, there exists an open set $V_\varepsilon \supset Z$ with $\lambda^N(V_\varepsilon) \leq \varepsilon$ and $P(V_\varepsilon, \Omega) \leq \tau(H^{N-1}(Z) + \varepsilon)$.*

PROOF. To lighten the notation, we set $c := H^{N-1}(Z)$. Let $\varepsilon > 0$ and $\delta > 0$; by the definition of Hausdorff measure, there exists a sequence of open balls $\{B_i\}_{i \in \mathbb{N}}$ each with radius r_i less than δ, such that their union contains Z and

$$(3.1) \qquad \frac{\omega_{N-1}}{2^{N-1}} \sum_{i \in \mathbb{N}} r_i^{N-1} \leq c + \varepsilon.$$

Define now $U_\delta := \bigcup_{i \in \mathbb{N}} B_i \cap \Omega$ and note that this set is open and contains Z. Moreover, since each r_i is less than δ, taking (3.1) into consideration, we get

$$\lambda^N(U_\delta) \leq \sum_{i \in \mathbb{N}} \omega_N \, r_i^N \leq \delta \sum_{i \in \mathbb{N}} \omega_N \, r_i^{N-1} \leq \delta \, \frac{\omega_N 2^{N-1}}{\omega_{N-1}} (c + \varepsilon).$$

Hence, choosing δ small enough, we have $\lambda^N(U_\delta) < \varepsilon$. Reasoning similarly, we obtain

$$P(U_\delta, \Omega) \leq \sum_{i \in \mathbb{N}} P(B_i \cap \Omega, \Omega) \leq \sigma_{N-1} \sum_{i \in \mathbb{N}} r_i^{N-1} \leq \frac{2^{N-1} \sigma_{N-1}}{\omega_{N-1}} (c + \varepsilon).$$

In conclusion, if δ is chosen small enough, we can define $V_\varepsilon := U_\delta$. ∎

We are now ready to characterize the class $\mathcal{P}_{\mathrm{BV}}$.

PROPOSITION 3.4. *Let $C \subset \Omega$. The following conditions are equivalent:*

(a) $C \in \mathcal{P}_{\mathrm{BV}}$;

(b) $H^{N-1}(C) = 0$;

(c) *there exists a sequence of open sets $\{U_k\}_{k \in \mathbb{N}}$ containing C such that $\lambda^N(U_k) \to 0$ and $P(U_k, \Omega) \to 0$.*

PROOF. If $C \in \mathcal{P}_{\mathrm{BV}}$, then $H^{N-1}(C) = 0$ by Theorem 2.4, while if $H^{N-1}(C) = 0$, there exists a sequence $\{U_k\}_{k \in \mathbb{N}}$ of open sets as in c by Lemma 3.3.

Suppose now that there exists a sequence of open sets as in (c). Passing to a (not relabeled) subsequence $\{U_k\}_{k \in \mathbb{N}}$, we may suppose that

$$\lambda^N(U_k) \leq k^{-2} \quad \text{and} \quad P(U_k, \Omega) \leq k^{-2}.$$

Reasoning as in Proposition 3.2, we define $v : \Omega \to \mathbb{R} \cup \{+\infty\}$ by

$$v(x) := \sum_{k \in \mathbb{N}} 1_{U_k}(x),$$

and we have that $v \in L^1(\Omega)$ and $C \subset \{x : \tilde{v}(x) = +\infty\}$. Moreover, v is the limit in $L^1(\Omega)$ of its partial sums, whose total variations are equibounded by the perimeter condition on $\{U_k\}_{k \in \mathbb{N}}$. By the lower semicontinuity of the variation, we deduce that $v \in \mathrm{BV}(\Omega)$. ∎

We now turn our attention to $\mathcal{P}_{\mathrm{GBV}}$.

PROPOSITION 3.5. *Let $C \subset \Omega$ be a relatively closed set such that $\lambda^N(C) = 0$. Then, $C \in \mathcal{P}_{\mathrm{GBV}}$.*

PROOF. Consider $u : \Omega \to \bar{\mathbb{R}}$ defined by

$$u(x) := \frac{1}{d(x, C)}.$$

Since C is a λ^N-null set, then u is real-valued λ^N-almost everywhere. Fix $m > 0$ and note that the truncated function u^m satisfies

$$u^m(x) = \frac{1}{d(x, C) \vee m^{-1}}.$$

Being the reciprocal of a Lipschitz function strictly greater than $\frac{1}{m}$, the function u^m is itself Lipschitz and belongs to $\mathrm{BV}_{\mathrm{loc}}(\Omega)$, implying that $u \in \mathrm{GBV}(\Omega)$. Moreover, since u is continuous, we have that $\{y : |\tilde{u}(y)| = +\infty\} = u^{-1}(+\infty) = C$. ∎

The previous proposition implies that GBV-polar sets may have any Hausdorff dimension smaller than the dimension of the ambient space, so that the inclusion $\mathcal{P}_{\mathrm{BV}} \subset \mathcal{P}_{\mathrm{GBV}}$ is strict. The next proposition concerns GBV-polar sets in dimension 1 and will be used to find a necessary condition for a set to be GBV-polar by means of a slicing argument.

PROPOSITION 3.6. *Let $\Omega \subset \mathbb{R}$ and let C be a subset of Ω. Then, $C \in \mathcal{P}_{\mathrm{GBV}}$ if and only if $\lambda^1(\bar{C} \cap \Omega) = 0$.*

PROOF. By Proposition 3.5, we know that if $\lambda^1(\bar{C} \cap \Omega) = 0$, then $C \in \mathcal{P}_{\mathrm{GBV}}$.

Vice versa, suppose that $C \in \mathcal{P}_{\mathrm{GBV}}$ and let $u \in \mathrm{GBV}(\Omega)$ be a positive function such that $C \subset \{y : \tilde{u}(y) = +\infty\}$. Denote by R_u the set of points where the approximate limit of u exists, and recall that $\lambda^1(\Omega \setminus R_u) = 0$. Fix $m \in \mathbb{N}$ and observe that $u^m := m \wedge u$ has approximate limit m at every point of C. We claim that its approximate limit is m at every point of $\bar{C} \cap R_u$.

To this aim, consider the left continuous representative of u^m [1, Theorem 3.28] and observe that it must have approximate limit m at any point of R_u (here, u^m has the approximate limit) that can be approximated from the left with points in C (so that the approximate limit of u^m must be greater than m). Analogously, the right continuous representative of u^m will have approximate limit m at any point of R_u that can be approximated from the right with points in C. Since the approximate limit does not depend on the representative, we deduce that u^m has approximate limit m at every point of $\bar{C} \cap R_u$. Hence, u must have an infinite approximate limit on every such point, implying by Theorem 2.1 that $\lambda^1(\bar{C} \cap \Omega) = \lambda^1(\bar{C} \cap R_u) = 0$. ∎

PROPOSITION 3.7. *Let $\nu \in S^{N-1}$ and $C \in \mathcal{P}_{\mathrm{GBV}}$. Then, for H^{N-1}-a.e. $y \in C_\nu$, we have $\lambda^1(\overline{C_\nu^y} \cap \Omega_\nu^y) = 0$.*

PROOF. By Proposition 2.8 for H^{N-1}-a.e. $y \in C_\nu$, we have $u_\nu^y \in \mathrm{GBV}(\Omega_\nu^y)$ and

$$\left\{ t \in \Omega_\nu^y : \widetilde{u_\nu^y}(t) = +\infty \right\} = \left\{ x \in \Omega : \tilde{u}(x) = +\infty \right\}_\nu^y \supset C_\nu^y,$$

so that Proposition 3.6 implies that $\lambda^1(\overline{C_\nu^y} \cap \Omega_\nu^y) = 0$. ∎

The previous proposition implies that the inclusion $\mathcal{P}_{\mathrm{GBV}} \subset \mathcal{P}_{L^0}$ is strict. An example of a L^0-polar set which is not GBV-polar is the cartesian product of $\mathbb{Q}$ with any set $A \subset \mathbb{R}^{N-1}$ such that $\lambda^{N-1}(A) > 0$, taking $\Omega := \mathbb{R}^N$.

The remaining part of the section is devoted to proving Theorem 3.11, which characterizes $\mathcal{P}_{\mathrm{GBV}}$ along the lines of conditions (c) in Propositions 3.2 and 3.4. We will use Proposition 3.8, which follows by Theorem 4.7 (whose proof is postponed to Section 4) by a standard localization argument.

PROPOSITION 3.8. *Let $A \subset \Omega$ be a set having a locally finite perimeter in Ω. Then, for every $\varepsilon > 0$, there exists $U \subset \Omega$, open set with a locally finite perimeter in Ω, such that $U \supset A^{(1)}$ and $\lambda^N(U \setminus A) \le \varepsilon$.*

The following lemma is needed in view of Proposition 3.10.

LEMMA 3.9. *Let $C \in \mathcal{P}_{\mathrm{GBV}}$. Then, there exists a positive function $u \in L^1(\Omega) \cap \mathrm{GBV}(\Omega)$ such that $C \subset \{x \in \Omega : \tilde{u}(x) = +\infty\}$.*

PROOF. Let $v \in \mathrm{GBV}(\Omega)$ be a positive function such that

$$C \subset \left\{ x \in \Omega : \tilde{v}(x) = +\infty \right\}.$$

Let then $\{\Omega_k\}_{k \in \mathbb{N}}$ be an exhaustion of Ω made of smooth sets and define $\Omega_0 := \varnothing$. Now, consider for every $k \ge 2$, the sets $B_k := \Omega_k \setminus \overline{\Omega}_{k-2}$ and note that the product $v 1_{B_k}$ is in $\mathrm{GBV}(\Omega)$ for every k and that $\lambda^N(B_k) < +\infty$. As a consequence for every $k \in \mathbb{N}$, we can choose a sequence $\{t_i^k\}_{i \in \mathbb{N}} \subset \mathbb{R}_+$ such that $t_i \uparrow +\infty$ as i goes to infinity and

$$(3.2) \qquad P\left(\{v > t_i^k\} \cap B_k, \Omega\right) < +\infty, \quad \lambda^N\left(\{v > t_i^k\} \cap B_k\right) < \frac{1}{2^i}.$$

We now define $v_k : \Omega \to \mathbb{R} \cup \{+\infty\}$ by

$$v_k(x) := \frac{1}{2^k} 1_{B_k}(x) \sum_{i \in \mathbb{N}} 1_{\{v > t_i^k\}}(x),$$

and we note that because of (3.2), we have $v_k \in \mathrm{GBV}(\Omega)$ and

$$(3.3) \qquad \int_\Omega \left| v_k(x) \right| dx \le \frac{1}{2^k}.$$

Moreover, by construction,

$$C \cap B_k \subset \{x \in \Omega : \widetilde{v_k}(x) = +\infty\}.$$

Define now $u : \Omega \to \mathbb{R} \cup \{+\infty\}$ by

$$u := \sum_{k \in \mathbb{N}} v_k$$

and observe that this function is in $L^1(\Omega)$ by (3.3) and that $u \in \mathrm{GBV}(\Omega)$ as it is a locally finite sum of positive GBV functions. Moreover, if $x \in C$, then $x \in C \cap B_k$ for some $k \in \mathbb{N}$ so that $\tilde{u}(x) = +\infty$ by construction. ∎

The next proposition provides a first characterization of $\mathcal{P}_{\mathrm{GBV}}$.

PROPOSITION 3.10. *Let $C \subset \Omega$. Then, $C \in \mathcal{P}_{\mathrm{GBV}}$ if and only if there exists a sequence $\{A_k\}_{k \in \mathbb{N}}$ of sets with a locally finite perimeter in Ω such that*

$$\lambda^N(A_k) \to 0 \quad and \quad \theta(A_k, x) = 1$$

for every $x \in C$ and for every $k \in \mathbb{N}$.

PROOF. Suppose first that we have a collection $\{A_k\}_{k \in \mathbb{N}}$ as in the statement. It is not restrictive to assume $A_{k+1} \subset A_k$ for every $k \in \mathbb{N}$ (taking the intersection of the first k sets). Then, define $u : \Omega \to \mathbb{R} \cup +\infty$ by

$$u(x) := \sum_{k \in \mathbb{N}} 1_{A_k}(x)$$

and note that u is real-valued λ^N-a.e. and that $u \in \mathrm{GBV}(\Omega)$ since its truncations are a finite sum of characteristic functions of sets having a locally finite perimeter. If $x \in C$, for every $k \in \mathbb{N}$, we have

$$\theta\big(\{u < k\}, x\big) \leq \theta(\Omega \setminus A_k, x) = 0,$$

proving that $\tilde{u}(x) = +\infty$. Hence, $C \in \mathcal{P}_{\mathrm{GBV}}$.

Vice versa, let $C \in \mathcal{P}_{\mathrm{GBV}}$. By Lemma 3.9, there exists a positive function $u \in \mathrm{GBV}(\Omega) \cap L^1(\Omega)$ such that $\tilde{u}(x) = +\infty$ for every $x \in C$. Note that this means that for every $t \in \mathbb{R}$ and for every $x \in C$, we have $\theta(\{u > t\}, x) = 1$. Since $u \in \mathrm{GBV}(\Omega)$, by Proposition 2.7, we can find a sequence $\{t_k\}_{k \in \mathbb{N}}$ with $t_k \uparrow +\infty$ such that $\{u > t_k\}$ has a locally finite perimeter in Ω, and since $u \in L^1(\Omega)$, we also obtain that $\lambda^N(\{u > t_k\}) \downarrow 0$. Defining $A_k := \{u > t_k\}$, we conclude. ∎

The next theorem, together with Propositions 3.2 and 3.4, completes the picture of polar sets.

THEOREM 3.11. *Let $C \subset \Omega$. Then, $C \in \mathcal{P}_{\text{GBV}}$ if and only if there exists a sequence of open sets $\{U_k\}_{k \in \mathbb{N}}$ having a locally finite perimeter in Ω and containing C such that $\lambda^N(U_k) \to 0$.*

PROOF. If we have such a collection $\{U_k\}_{k \in \mathbb{N}}$, the statement follows by Proposition 3.10 since we have stronger hypotheses.

Vice versa, suppose that $C \in \mathcal{P}_{\text{GBV}}$. By Proposition 3.10, there exists a sequence $\{A_k\}_{k \in \mathbb{N}}$ of sets with a locally finite perimeter in Ω such that

$$\lambda^N(A_k) \to 0 \quad \text{and} \quad \theta(A_k, x) = 1$$

for every $x \in C$ and every $k \in \mathbb{N}$. By Proposition 3.8, for every $k \in \mathbb{N}$, there exists an open set U_k with a locally finite perimeter in Ω such that $U_k \supset A_k^{(1)}$ and $\lambda^N(U_k \setminus A_k) \le \frac{1}{k}$. This last fact, since $\lambda^N(A_k) \to 0$, implies that $\lambda^N(U_k) \to 0$. Finally, for every $k \in \mathbb{N}$, we have that $U_k \supset A_k^{(1)} \supset C$, concluding the proof. ∎

4. OUTER APPROXIMATION OF SETS OF FINITE PERIMETER WITH OPEN SETS

The goal of the section is to prove that if $A \subset \Omega$ is a set of finite perimeter in Ω, then for every $\varepsilon > 0$, there exists an open set U such that $U \supset A^{(1)}$, $\lambda^N(U \setminus A) < \varepsilon$, and $|P(U, \Omega) - P(A, \Omega)| < \varepsilon$ (Theorem 4.7). This fact then implies the result that we used in the previous section (Proposition 3.8). To prove the aforementioned theorem, we first need to show that strong convergence in BV implies (up to passing to a subsequence) convergence H^{N-1}-almost everywhere (Theorem 4.2). This fact seems to be known (for example, it is considered in the general setting of metric spaces in [6]) but does not appear in the standard references about functions of bounded variation. Since it is an easy consequence of Proposition 2.13, we give a complete proof. The next proposition is a preliminary version of Theorem 4.2.

PROPOSITION 4.1. *Let $\{u_k\}_{k \in \mathbb{N}}$ be a sequence of positive functions of bounded variation such that $u_k \to 0$ strongly in $\text{BV}(\mathbb{R}^N)$. Then, there is a (not relabeled) subsequence such that $u_k^+(x) \to 0$ for H^{N-1}-almost every $x \in \mathbb{R}$.*

PROOF. Passing to a subsequence, we may suppose that for every $k \in \mathbb{N}$, we have $\|u_k\|_{\text{BV}} < 2^{-k}$, and with this extra hypothesis, we fix $\varepsilon > 0$ and we prove that the set

$$A := \left\{ x \in \mathbb{R}^N : \forall n \in \mathbb{N} \, \exists k \in \mathbb{N} : k \ge n \text{ and } u_k^+(x) > 2\varepsilon \right\}$$

has H^{N-1} zero measure. If we are able to do this, then the statement follows by the arbitrariness of ε. Since $H^{N-1}(A) = 0$ if and only if $\text{Cap}_1(A) = 0$, by Proposition 2.11,

we prove the latter. To this aim, observe that

$$A = \bigcap_{n \in \mathbb{N}} \bigcup_{k \geq n} \{x : u_k^+(x) > 2\varepsilon\},$$

so that the monotonicity and the subadditivity of the capacity (Proposition 2.10) give that for every $n \in \mathbb{N}$, we have

(4.1) $$\text{Cap}_1(A) \leq \sum_{k \geq n} \text{Cap}_1(\{x : u_k^+(x) > 2\varepsilon\}).$$

Taking into account that u is positive together with Theorem 2.4, we get that

$$\{x : u_k^+(x) > 2\varepsilon\} \subset \left\{x : \frac{u_k^+(x) + u_k^-(x)}{2} > \varepsilon\right\}$$
$$\subset \left\{x : \exists \rho > 0 : \fint_{B_\rho(x)} u_k(y)\, dy > \varepsilon\right\},$$

which implies by Proposition 2.13 that there exists a constant $C > 0$ such that

$$\text{Cap}_1(\{x : u_k^+(x) > 2\varepsilon\}) \leq C|Du_k|(\mathbb{R}^n).$$

Combining this with (4.1), we deduce that for every $n \in \mathbb{N}$, we have

$$\text{Cap}_1(A) \leq C \sum_{k \geq n} \|u_k\|_{\text{BV}} \leq C \sum_{k \geq n} 2^{-k},$$

and letting n increase to infinity, we get that $\text{Cap}_1(A) = 0$. ∎

THEOREM 4.2. *Let $\{u_k\}_{k \in \mathbb{N}}$ be a sequence converging strongly in $\text{BV}(\mathbb{R}^N)$ to $u \in \text{BV}(\mathbb{R}^N)$. Then, there exists a (not relabeled) subsequence such that $u_k^+(x) \to u^+(x)$ for H^{N-1}-almost every $x \in \mathbb{R}^N$.*

PROOF. We will prove that $|u_k^+ - u^+| \leq |u_k - u|^+$, and applying Proposition 4.1 to (a subsequence of) $|u - u_k|$, we will then obtain that for H^{N-1}-almost every $x \in \Omega$,

$$\limsup_{k \to +\infty} |u_k^+(x) - u^+(x)| \leq \lim_{k \to +\infty} |u_k - u|^+(x) = 0,$$

which implies the statement of the theorem.

We now prove that $|u_k^+ - u^+| \leq |u_k - u|^+$. By the triangle inequality, we have $u_k \leq u + |u_k - u|$, and passing to the approximate upper limit on both sides, we get $u_k^+ \leq (u + |u_k - u|)^+ \leq u^+ + |u_k - u|^+$, giving $u_k^+ - u^+ \leq |u_k - u|^+$. By symmetry, we deduce that $|u_k^+ - u^+| \leq |u_k - u|^+$. ∎

The next corollary is obtained from the previous theorem by a standard localization argument.

COROLLARY 4.3. *Let $\{u_k\}_{k\in\mathbb{N}} \subset \mathrm{BV}_{\mathrm{loc}}(\Omega)$ be a sequence converging strongly in $\mathrm{BV}_{\mathrm{loc}}(\Omega)$ to $u \in \mathrm{BV}_{\mathrm{loc}}(\Omega)$. There exists a (not relabeled) subsequence such that $u_k^+(x) \to u^+(x)$ for H^{N-1}-almost every $x \in \Omega$.*

We recall that given $E \subset \Omega$, we denote by $E^{(1)}$ the set of points in Ω where E has density 1 and by E^+ the set of points where E has positive upper density. By the Lebesgue differentiation theorem, for λ^N-almost every x in Ω, we have

$$1_E(x) = 1_{E^+}(x) = 1_{E^{(1)}}(x).$$

Moreover, from the definition of upper density, it easily follows that $1_E^+(x) = 1_{E^+}(x)$ and that $E^{(1)} = {}^c(({}^c E)^+)$, where ${}^c E := \Omega \setminus E$. The next three lemmas provide the intermediate steps needed to prove Theorem 4.7.

LEMMA 4.4. *Let $C \subset \Omega$ be a relatively closed set of finite perimeter in Ω. Then,*

$$C^+ \cup \left[(\Omega \cap \partial C) \setminus \partial^* C\right] = C.$$

PROOF. Note that by Theorem 2.3, we have $C^+ \supset \overset{\circ}{C} \cup \partial^* C$, so that

$$C^+ \cup \left[(\Omega \cap \partial C) \setminus \partial^* C\right] \supset \overset{\circ}{C} \cup (\Omega \cap \partial C) \supset C.$$

Vice versa, since C is relatively closed, we have $C^+ \cup (\Omega \cap \partial C) \subset C$, so that also $C^+ \cup [(\Omega \cap \partial C) \setminus \partial^* C] \subset C$. ∎

LEMMA 4.5. *Let $B \subset \Omega$ be a set of finite perimeter in Ω. For every $\varepsilon > 0$, there exist a sequence $\{C_k\}_{k\in\mathbb{N}}$ of relatively closed sets of finite perimeter in Ω and a Borel set $Z \subset \Omega$ with the following properties:*

(a) $|D(1_B - 1_{C_k})|(\Omega) \to 0$ and $\lambda^N(B \triangle C_k) \to 0$;

(b) *for every $x \notin Z$, we have that $1_{C_k}(x) \to 1_{B^+}(x)$;*

(c) $H^{N-1}(Z) < \varepsilon$.

PROOF. By Theorem 2.5, there exists a sequence $\{C_k\}_{k\in\mathbb{N}}$ of relatively closed sets with finite perimeter in Ω such that $\lambda^N(C_k \triangle B) \to 0$, $|D(1_{C_k} - 1_B)|(\Omega) \to 0$, and

$$(4.2) \qquad\qquad H^{N-1}\left((\partial C_k \cap \Omega) \setminus \partial^* C_k\right) < \frac{\varepsilon}{2^k}.$$

By Corollary 4.3, there exists a H^{N-1}-null set $\widetilde{Z} \subset \Omega$ and a (not relabeled) subsequence of $\{C_k\}_{k\in\mathbb{N}}$ such that for every $x \notin \widetilde{Z}$, we have $1_{C_k}^+(x) \to 1_B^+(x)$, and, taking into account that $1_A^+ = 1_{A^+}$, we deduce that for every such x, we have

$$(4.3) \qquad\qquad 1_{C_k^+}(x) \to 1_{B^+}(x).$$

Define now

$$Z := \tilde{Z} \cup \bigcup_{k \in \mathbb{N}} \big[(\partial C_k \cap \Omega) \setminus \partial^* C_k \big],$$

and note that by (4.2), we have that $H^{N-1}(Z) < \varepsilon$. Since every C_k is relatively closed in Ω, by Lemma 4.4, we have that $C_k^+ \setminus Z = C_k \setminus Z$, which together with (4.3) implies that for every $x \notin Z$, we have $1_{C_k}(x) \to 1_{B^+}(x)$. ∎

LEMMA 4.6. *Let $A \subset \Omega$ be a set of finite perimeter in Ω. Then, for every $\varepsilon > 0$, there exist $Z \subset \Omega$ such that $H^{N-1}(Z) < \varepsilon$ and an open set U such that $U \cup Z \supset A^{(1)}$, $\lambda^N(U \setminus A) < \varepsilon$, and $P(U, \Omega) < P(A, \Omega) + \varepsilon$.*

PROOF. Let $B := \Omega \setminus A$ and note that $P(B, \Omega) = P(A, \Omega) < +\infty$. By Lemma 4.5, there exists a sequence $\{C_k\}_{k \in \mathbb{N}}$ of relatively closed sets with finite perimeter in Ω and a Borel set $Z \subset \Omega$ such that

(a) $|D(1_B - 1_{C_k})|(\Omega) < \varepsilon 2^{-k}$ and $\lambda^N(B \triangle C_k) < \varepsilon 2^{-k}$;

(b)   for every $x \notin Z$, we have $1_{C_k}(x) \to 1_{B^+}(x)$;

(c) $H^{N-1}(Z) < \varepsilon$.

We define

$$C := \bigcap_{k \in \mathbb{N}} C_k$$

and $U := \Omega \setminus C$, and we claim that U has the required properties. By (b), we get that $B^+ \setminus Z \supset C \setminus Z$, and passing to the complement in this inclusion, keeping in mind that $A^{(1)} = {}^c(B^+)$, we get $A^{(1)} \subset Z \cup U$. Moreover,

$$\lambda^N(U \setminus A) \leq \sum_{k \in \mathbb{N}} \lambda^N\big((\Omega \setminus C_k) \setminus A\big) \leq \sum_{k \in \mathbb{N}} \lambda^N\big((\Omega \setminus A) \triangle C_k\big) = \sum_{k \in \mathbb{N}} \lambda^N(C_k \triangle B),$$

and the last term of the chain is less than ε by (a).

To conclude the proof, we only need to show that $P(U, \Omega) < P(A, \Omega) + \varepsilon$. To achieve this, we prove that the function $1_U - 1_A$ is in $BV_{loc}(\Omega)$ and satisfies $|D(1_U - 1_A)|(\Omega) < \varepsilon$, so that we obtain

$$P(U, \Omega) = |D1_U|(\Omega) \leq \big|D(1_U - 1_A)\big|(\Omega) + |D1_A|(\Omega) < P(A, \Omega) + \varepsilon.$$

Observe that $1_U - 1_A = (1 - 1_C) - (1 - 1_B) = 1_B - 1_C$ so that it is sufficient to show that $1_B - 1_C \in BV_{loc}(\Omega)$ and satisfies $|D(1_B - 1_C)|(\Omega) < \varepsilon$.

To this aim, let $f : \mathbb{R} \to \mathbb{R}$ be the truncation function at the levels zero and one, i.e., $f(t) := 0 \vee t \wedge 1$, and we claim that the following identity holds λ^N-almost everywhere in Ω:

(4.4)
$$1_B - 1_C = f\Big(\sum_{k \in \mathbb{N}} (1_B - 1_{C_k}) \Big).$$

First of all, observe that for every $x \notin Z$, the term $1_{B^+}(x) - 1_{C_k}(x)$ is eventually zero by the condition (b), so that since $\lambda^N(Z) = 0$ and $\lambda^N(B \Delta B_+) = 0$, we have that the series in the r.h.s. of (4.4) is well-defined λ^N-almost everywhere. Another consequence of (b) is that λ^N-almost every point of C belongs to B^+ so that λ^N-almost every such point is in B. Taking this into account, (4.4) follows. As a consequence, the function $1_B - 1_C$ is the limit in $L^1_{\mathrm{loc}}(\Omega)$ of the sequence $\{f_j\}_{j \in \mathbb{N}}$ given by

$$f_j := f\left(\sum_{k=1}^{j} (1_B - 1_{C_k}) \right),$$

and since f is Lipschitz with Lipschitz constant one, each function f_j belongs to $\mathrm{BV}_{\mathrm{loc}}(\Omega)$ and satisfies

$$|Df_j|(\Omega) \le \sum_{k \in \mathbb{N}} \left| D(1_B - 1_{C_k}) \right|(\Omega) < \varepsilon.$$

This implies that also $1_B - 1_C$ belongs to $\mathrm{BV}_{\mathrm{loc}}(\Omega)$ and satisfies $|D(1_B - 1_C)|(\Omega) < \varepsilon$, concluding the proof. ∎

THEOREM 4.7. *Let $A \subset \Omega$ be a set of finite perimeter in Ω. Then, for every $\varepsilon > 0$, there exists an open set U such that $U \supset A^{(1)}$, $\lambda^N(U \setminus A) < \varepsilon$, and $|P(U, \Omega) - P(A, \Omega)| < \varepsilon$.*

PROOF. By Lemma 4.6, for every $n \in \mathbb{N}$, we find an open set V_n and a set Z_n such that $V_n \cup Z_n \supset A^{(1)}$, $\lambda^N(V_n \setminus A) < \frac{1}{n}$, $P(V_n, \Omega) < P(A, \Omega) + \frac{1}{n}$, and $H^{N-1}(Z) < \frac{1}{\tau n}$, where τ is the dimensional constant of Lemma 3.3.

By the aforementioned lemma, we then find an open set $W_n \supset Z_n$ such that $\lambda^N(W_n) < \frac{1}{n}$ and $P(W_n, \Omega) < \frac{1}{n}$, and we define $U_n := V_n \cup W_n$. In this way, we have a sequence $\{U_n\}_{n \in \mathbb{N}}$ of open sets containing $A^{(1)}$, converging to A in measure such that

$$P(U_n, \Omega) \le P(V_n, \Omega) + P(W_n, \Omega) \le P(A, \Omega) + \frac{2}{n}.$$

Then, by the lower semicontinuity of the perimeter w.r.t. convergence in measure, it is sufficient to define $U := U_n$ for n sufficiently large. ∎

REMARK 4.8. The previous theorem makes sense even if the set A is defined modulo λ^N-negligible sets, as all the terms involved are invariant under modifications on λ^N-null sets.

REMARK 4.9. Theorem 4.7 fails if we replace the condition that $U \supset A^{(1)}$ with $U \supset A$. We show this by considering the case when $\lambda^N(A) = 0$. Indeed, if we could find a sequence of open sets $\{U_n\}$ containing A with Lebesgue measure and perimeter decreasing to zero, we would obtain by Proposition 3.4 that $A \in \mathcal{P}_{\mathrm{BV}}$, while the same proposition shows that this is in general not possible.

The next example shows that we cannot require the smoothness of the approximating set in Theorem 4.7 when $N \geq 2$.

EXAMPLE 4.10. Let $\Omega := \mathbb{R}^N$ and let $A \subset \mathbb{R}^N$ be an open dense set with a finite perimeter and a finite Lebesgue measure (for example, a countable union of balls with dense centers and sufficiently small radii). Suppose by contradiction that there exists a smooth open set U such that $U \supset A^{(1)} \supset A$ and $\lambda^N(U) < +\infty$. If we had such a set U, we would then get that $\bar{U} = \mathbb{R}^N$, so that

$$\lambda^N(\partial U) = \lambda^N(\bar{U}) - \lambda^N(U) = +\infty.$$

On the other hand, the smoothness of U implies that $\lambda^N(\partial U) = 0$, giving the desired contradiction.

REFERENCES

[1] L. AMBROSIO – N. FUSCO – D. PALLARA, *Functions of bounded variation and free discontinuity problems*. Oxford Math. Monogr., Oxford University Press, New York, 2000. Zbl 0957.49001 MR 1857292

[2] G. E. COMI – M. TORRES, One-sided approximation of sets of finite perimeter. *Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl.* **28** (2017), no. 1, 181–190. Zbl 1362.28006 MR 3621776

[3] L. C. EVANS – R. F. GARIEPY, *Measure theory and fine properties of functions*. Revised edn., Textb. Math., CRC Press, Boca Raton, FL, 2015. Zbl 1310.28001 MR 3409135

[4] H. FEDERER, *Geometric measure theory*. Grundlehren Math. Wiss. 153, Springer, New York, 1969. Zbl 0176.00801 MR 257325

[5] H. FEDERER – W. P. ZIEMER, The Lebesgue set of a function whose distribution derivatives are p-th power summable. *Indiana Univ. Math. J.* **22** (1972/73), 139–158. Zbl 0238.28015 MR 435361

[6] P. LAHTI – N. SHANMUGALINGAM, Fine properties and a notion of quasicontinuity for BV functions on metric spaces. *J. Math. Pures Appl. (9)* **107** (2017), no. 2, 150–182. Zbl 1359.30055 MR 3597372

[7] T. QUENTIN DE GROMARD, Strong approximation of sets in BV(Ω). *Proc. Roy. Soc. Edinburgh Sect. A* **138** (2008), no. 6, 1291–1312. Zbl 1172.49027 MR 2488060

[8] T. SCHMIDT, Strict interior approximation of sets of finite perimeter and functions of bounded variation. *Proc. Amer. Math. Soc.* **143** (2015), no. 5, 2069–2084. Zbl 1311.28005 MR 3314116

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