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Algebra. — *On a theorem of Schmid*, by FRANCESCO ESPOSITO and ANDREA MAFFEI, communicated on 14 December 2007.

ABSTRACT. — We establish for which parabolic subgroups P of a simply connected and semisimple algebraic group G with unipotent radical U and Levi factor H the rings $\Bbbk[G/H]^U$ and $\Bbbk[U^-]$ are isomorphic as H-algebras. We show a relation of this problem with a theorem of Schmid and we compare the multiplications in the rings $\Bbbk[U^-]$ and $\Bbbk[G/H]$.

KEY WORDS: Semisimple algebraic groups; homogeneous spaces; symmetric spaces; regular functions.

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Let G be a simply connected and semisimple algebraic group over an algebraically closed field $\Bbbk$. Let P be a parabolic subgroup of G, U its unipotent radical and H a Levi factor. Let also U^- be the unipotent radical of a parabolic opposite to P.

We want to describe the relation between the coordinate rings of G/H and U^-. We notice that since $U^- \cap H = \{1\}$ the inclusion of U^- in G induces an H-equivariant immersion $\imath : U^- \hookrightarrow G/H$. Moreover, since every orbit of a unipotent group acting on an affine variety is closed, $\imath$ is a closed immersion. So we have a surjective morphism of H-algebras $\imath^* : \Bbbk[G/H] \to \Bbbk[U^-]$. Let $\varphi : \Bbbk[G/H]^U \to \Bbbk[U^-]$ be the restriction of $\imath^*$. Then our main result is the following

THEOREM. *Let G be simple and $\Bbbk$ of characteristic 0. The map φ is an isomorphism if and only if either U is commutative, or G is of type F_4 and P is the maximal parabolic with semisimple Levi of type C_3, or G is of type B_n and P is the maximal parabolic with semisimple Levi of type A_{n-1}.*

The paper is organized as follows. In Section 1 we study the case of U commutative, in which we can give a simpler proof which partly holds in positive characteristic. This case is related to symmetric varieties. Indeed, for every such U, the Levi H is the subgroup of points fixed by an involution of G. In particular, we can use the theorem above to show that to determine the decomposition of $\Bbbk[U^-]$ into H-modules is equivalent to determining the decomposition of $\Bbbk[G/H]$ into G-modules. This relates a theorem of Schmid [5] to a theorem of Helgason [3]. In the recent years there has been some interest in the products of irreducible modules in these two rings [1, 4, 2] and we compare these two products. In the second section we prove the theorem in the general case.

1. THE SYMMETRIC CASE

We keep the notation introduced above. Let also $T \subset H$ be a maximal torus, $T \subset B_G \subset P$ a Borel subgroup and B_G^- the opposite Borel. Notice that H acts on U^- by conjugation.

Notice that if V is a representation of G then the set V^U of points fixed by U is H-stable, in particular $\Bbbk[G/H]^U$ is an H-algebra and φ is a morphism of H-algebras. Notice also that since the image of $U \times U^-$ is dense in G/H the morphism φ is certainly injective.

Let also U_G be the unipotent radical of B_G and notice that $B_H := B_G \cap H$ is a Borel of H and that $U_H = U_G \cap H$ is the unipotent radical of B_H.

Finally, we will denote the Lie algebra of a group by the corresponding gothic letter.

THEOREM 1. *Assume that the Lie algebra $\mathfrak{g}$ of G admits an invariant nondegenerate bilinear form and that U is commutative. Then:*

(i) $\varphi : \Bbbk[G/H]^U \to \Bbbk[U^-]$ *is an isomorphism of H-algebras;*
(ii) *the restriction of ι to the U_G-invariants induces an isomorphism of algebras ψ : $\Bbbk[G/H]^{U_G} \to \Bbbk[U^-]^{U_H}$.*

PROOF. (i) We have already noticed that φ is injective; we now prove that it is surjective. Notice that $\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{z} \oplus \mathfrak{h}' \oplus \mathfrak{u}^-$ where $\mathfrak{u}$ is the Lie algebra of U, $\mathfrak{u}^-$ of U^-, and $\mathfrak{z}$ and $\mathfrak{h}'$ are respectively the center and commutator of the Lie algebra of H. Let $z \in \mathfrak{z}$ be an element whose centralizer is equal to H. Since z is semisimple we have a closed immersion $\jmath$ of G/H in $\mathfrak{g}$. Hence we have the following surjective morphisms of rings:

$$S[\mathfrak{g}^*] \twoheadrightarrow \Bbbk[G/H] \twoheadrightarrow \Bbbk[U^-].$$

Notice that $\jmath\iota(U^-) = U^- \cdot z$ is contained in $z \oplus \mathfrak{u}^-$. Let now $\mathfrak{g}^* = \mathfrak{u}^* \oplus \mathfrak{z}^* \oplus (\mathfrak{h}')^* \oplus (\mathfrak{u}^-)^*$ be the decomposition $\mathfrak{g}^*$ induced by the decomposition $\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{z} \oplus \mathfrak{h}' \oplus \mathfrak{u}^-$. Notice that $\jmath\iota(U^-) = U^- \cdot z$ is contained in $z \oplus \mathfrak{u}^-$. Hence $(\mathfrak{h}')^*$ and $\mathfrak{u}^*$ vanish on $z \oplus \mathfrak{u}^-$ and $\mathfrak{z}^*$ gives constant functions. So we have a surjective map from $S[(\mathfrak{u}^-)^*]$ to $\Bbbk[U^-]$. Notice now that since $\mathfrak{h}' \oplus \mathfrak{z} \oplus \mathfrak{u}$ is a U-stable subspace, so is $(\mathfrak{u}^-)^*$, and the restriction map from $S(\mathfrak{g}^*)$ to $S[(\mathfrak{u}^-)^*]$ is U-equivariant. Finally, notice that U^- commutative implies that the action of U on $\mathfrak{u}$ is trivial, and using the invariant form we see that this implies that the action of U on $(\mathfrak{u}^-)^*$ is trivial. Hence any function on U^- can be lifted to a U-invariant function on $\mathfrak{g}$, hence also on G/H.

(ii) follows immediately from (i). □

In Theorem 2 we will characterize all the possible H such that φ is an isomorphism in the case of characteristic zero.

From now on we assume the characteristic to be zero.

Let us introduce a little bit of notation. Let $\Lambda = \mathrm{Hom}(T, \Bbbk^*)$ be the set of weights of T and let Λ_H^+ be the subset of weights dominant with respect to B_H, and Λ_G^+ those dominant with respect to B_G. For each $\lambda \in \Lambda_G^+$ (resp. Λ_H^+) let V_λ (resp. W_λ) be the irreducible representation of G (resp. H) of highest weight λ; notice that $\Lambda_G^+ \subset \Lambda_H^+$ and $V_\lambda^U \simeq W_\lambda$ as H-modules for each $\lambda \in \Lambda_G^+$.

For each $\lambda \in \Lambda_G^+$ let $m_\lambda = \dim (V_\lambda^*)^H$ and let Ω be the set of dominant weights such that $m_\lambda > 0$. So from $\Bbbk[G] \simeq V \otimes V^*$ we obtain $\Bbbk[G/H] \simeq \bigoplus V_\lambda \otimes \Bbbk^{m_\lambda}$.

Before studying the general situation we want to explain the relation between Theorem 1 and a theorem of Schmid. We want to characterize first the cases in which U is commutative. We fix some notations. Let Φ be the root system of $\mathfrak{g}$ with respect to the torus T, and Δ the simple roots corresponding to the choice of the Borel B. Moreover, for all $\alpha \in \Phi$ let x_α be a root vector of weight α. Let also $\Psi \subset \Phi$ be the root system of H.

LEMMA 2. *Assume that G is simple. Then the following conditions are equivalent:*

(i) *U is commutative;*
(ii) $\mathfrak{u}$ *is irreducible as an H-module;*
(iii) *P is the maximal parabolic associated to a root α which appears with multiplicity* 1 *in the highest root;*
(iv) *there is an involution σ such that $H = G^\sigma$.*

PROOF. (ii)⇒(i). Indeed, $[\mathfrak{u}, \mathfrak{u}]$ is an H-submodule of $\mathfrak{u}$ different from $\mathfrak{u}$.

(i)⇒(iii). Assume first that P is not maximal and let $\alpha, \beta \in \Delta \smallsetminus \Psi$ be distinct simple roots. Since G is simple, the Dynkin diagram is connected; let $\gamma_1, \ldots, \gamma_n$ with $\gamma_1 = \alpha$ and $\gamma_n = \beta$ be a path in the Dynkin diagram from α to β. Let $\gamma = \gamma_1 + \cdots + \gamma_{n-1}$. Then $x_\gamma, x_\beta \in \mathfrak{u}$ and $[x_\gamma, x_\beta] \neq 0$. Let P be the maximal parabolic corresponding to the simple root α and assume that α appears with multiplicity greater than or equal to 2 in the highest root. Then there is a root β such that α appears with multiplicity 1 in β and $\beta + \alpha \in \Phi$. So $x_\beta, x_\alpha \in \mathfrak{u}$ and $[x_\alpha, x_\beta] \neq 0$.

(iii)⇒(i). We prove that x_α generates $\mathfrak{u}$ as an H-module so $\mathfrak{u}$ is an irreducible H-module of lowest weight α. Let $x_\beta \in \mathfrak{u}$. Then $\beta = \alpha + \sum_{i=1}^n \gamma_i$ with $\gamma_i \in \Delta \smallsetminus \Psi$. We proceed by induction on n. If κ is the Killing form then $\kappa(\beta, \beta) > 0$ so either $\kappa(\beta, \alpha) > 0$ or $\kappa(\beta, \gamma_i) > 0$ for some i. In the first case $\gamma = \beta - \alpha \in \Psi$ so $x_\gamma \in \mathfrak{h}$ and x_β is a scalar multiple of $[x_\gamma, x_\alpha]$. In the second case $x_{\beta - \gamma_i} \in \mathfrak{u}$ and we conclude by induction.

(iii)⇒(iv). We can define an involution σ acting trivially on the root vectors x_β and $x_{-\beta}$ for $\beta \in \Delta \smallsetminus \{\alpha\}$ and acting as -1 on the root vectors e_α and $e_{-\alpha}$.

(iv)⇒(i). Let $H = G^\sigma$. Then σ fixes the torus T and the roots Φ. So the action is given by multiplication by -1 on x_β for $\beta \in \Phi \smallsetminus \Psi$ and by multiplication by 1 on x_β for $\beta \in \Psi$. Now we can argue as in (i)⇒(iii). □

It follows immediately that for G semisimple, U is commutative if and only if H is the subgroup of points fixed by an involution. In particular, B_G has an open orbit in G/H in this case, in particular $m_\lambda \leq 1$ for all λ. Hence if U is commutative we have

$$\Bbbk[G/H] = \bigoplus_{\lambda \in \Omega} V_\lambda$$

and from Theorem 1 by taking U-invariants we get

$$\Bbbk[U^-] = \bigoplus_{\lambda \in \Omega} W_\lambda.$$

In particular, using Helgason's description [3] of spherical representations (irreducible representations of G which have a nonzero vector fixed by H) we see how we can deduce the description of $\Bbbk[U]$ as an H-module given by Schmid [5] for hermitian symmetric varieties.

REMARK 3. We want to compare the product of two G-submodules of $\Bbbk[G/H]$ under the usual multiplication in the ring with the product of H-submodules of $\Bbbk[U^-]$ in the case where U is commutative. Recently these products have been studied: in [4, 1] for the case of G-submodules of $\Bbbk[G/H]$ and in [2] for the case of H-submodules of $\Bbbk[U^-]$.

Given $\lambda, \mu \in \Omega$ let $\mathcal{V}(\lambda, \mu) = \{\nu \in \Omega \,:\, V_\nu \subset V_\lambda \cdot V_\mu \subset \Bbbk[G/H]\}$ and similarly define $\mathcal{W}(\lambda, \mu) = \{\nu \in \Omega \,:\, W_\nu \subset W_\lambda \cdot W_\mu \subset \Bbbk[U^-]\}$.

From the previous constructions we obtain $\mathcal{W}(\lambda, \mu) \subset \mathcal{V}(\lambda, \mu)$. We now give an example where equality does not hold.

Let $X_1 = X_2 = \Bbbk^n$ and $X = X_1 \oplus X_2$. Let $G = \mathrm{SL}(X)$ and let H be the Levi subgroup which stabilizes X_1 and X_2. The center of H is the one-parameter subgroup $\gamma(t) = t\,\mathrm{id}_{X_1} \oplus t^{-1}\mathrm{id}_{X_2}$ and its semisimple part is $\mathrm{SL}(X_1) \times \mathrm{SL}(X_2)$. Let also $V = \mathfrak{sl}(X)$ be the representation of highest weight θ, the highest root of G, and notice that $W = V^U = X_1 \otimes X_2^*$ and that $\gamma(t)$ acts by multiplication by t^2 on W and so by multiplication by t^4 on $W \cdot W$. In particular, $0 \notin \mathcal{W}(\theta, \theta)$, while it is easy to see that $0 \in \mathcal{V}(\theta, \theta)$ (see for example [1]).

2. The general case

We now go back to the general situation and we classify all H such that the map φ introduced in the first section is an isomorphism.

LEMMA 4. *Assume that G is simple. If $\varphi : \Bbbk[G/H]^U \to \Bbbk[U^-]$ is an isomorphism then either $\mathfrak{u}$ is irreducible or G is not simply laced and $\mathfrak{u} \simeq W_\theta \oplus W_{\theta'}$ as H-modules where θ is the highest root of $\mathfrak{g}$ and θ' is the highest short root.*

PROOF. From $\Bbbk[G/H]^U \simeq \Bbbk[U^-]$ as H-modules we deduce that

$$\Bbbk[U^-] \simeq \bigoplus_{\lambda \in \Lambda_G^+} W_\lambda \otimes \Bbbk^{m_\lambda};$$

in particular, all the H-modules which appear in $\Bbbk[U^-]$ have a highest weight which is dominant with respect to B_G.

Notice also that the exponential map gives an algebraic H-equivariant isomorphism between $\mathfrak{u}^-$ and U^- and in particular $\Bbbk[U^-] \simeq \Bbbk[\mathfrak{u}^-] \simeq S(\mathfrak{u})$ as H-algebras. In particular, the representation $\mathfrak{u} = S^1(\mathfrak{u})$ must appear in the sum $\bigoplus_{\lambda \in \Lambda_G^+} W_\lambda \otimes \Bbbk^{m_\lambda}$. But $\mathfrak{u} \subset \mathfrak{u}_G$, the Lie algebra of U_G whose weights are given by positive roots. Now if G is simply laced there is only one such root which is dominant with respect to B_G and that is the highest root. So in this case $\mathfrak{u} = W_\theta$ and it is irreducible. On the other hand, if G is not simply laced there can be two such roots: θ and θ', and $\mathfrak{u}$ can be either irreducible or isomorphic to $W_\theta \oplus W_{\theta'}$. $\square$

When $\mathfrak{u}$ is an irreducible H-module, we are back in the commutative case, so we will have to analyze the other case.

LEMMA 5. *Assume that G is a simple Lie group not simply laced. If $\mathfrak{u} \simeq W_\theta \oplus W_{\theta'}$ as H-modules then P is a maximal parabolic corresponding to a simple root α which appears with multiplicity 2 in the highest root and with multiplicity 1 in θ'.*

PROOF. Assume first P is not maximal and consider simple roots α, β which are not in the root system of H. Let $\mathfrak{u}_1$ be the span of the root spaces $\mathfrak{g}_\gamma$ with γ a positive root where

α appears with multiplicity 1 and β with multiplicity 0; similarly, let $\mathfrak{u}_2$ be the span of the root spaces $\mathfrak{g}_\gamma$ with γ a positive root where β appears with multiplicity 1 and α with multiplicity 0; finally, let $\mathfrak{u}_3$ be the span of the root spaces $\mathfrak{g}_\gamma$ with γ a positive root where both α and β appear with multiplicity 1. These spaces are not zero and H-stable, contrary to the fact that $\mathfrak{u}$ is the sum of two irreducible H-modules. Similarly one can treat the case in which P is the maximal parabolic corresponding to a root α which appears with multiplicity at least 3 in the highest root or with multiplicity 2 in both θ and θ' (this last case actually cannot occur under our hypothesis). □

PROPOSITION 6. *Assume that G is a simple Lie group not simply laced and that P is a maximal parabolic corresponding to a simple root α which appears with multiplicity 2 in the highest root and with multiplicity 1 in θ' and that $\mathfrak{u} \simeq W_\theta \oplus W_{\theta'}$. Then φ is an isomorphism if and only if $m_{\theta'} = 1$ and W_θ is not an irreducible factor of $S^2(W_{\theta'})$.*

PROOF. We notice first that $(V_\theta^*)^H = \mathfrak{g}^H = Z(\mathfrak{h}) = \mathfrak{z}$, the center of the Lie algebra of H, is one-dimensional and spanned by the fundamental coweight $z = \omega_\alpha^\vee$ relative to the root α. In particular, $m_\theta = 1$.

Notice also that z acts as $\langle\lambda, z\rangle$ times the identity on the representation of W_λ. In particular, it acts as the identity on $W_{\theta'}$ and twice the identity on W_θ. So, since $S^1(\mathfrak{u}) = W_\theta \oplus W_{\theta'}$ generates $S(\mathfrak{u})$ and the multiplication is equivariant, if we look at the action of z on $S(\mathfrak{u})$ we see that $W_{\theta'}$ can appear only in $S^1(\mathfrak{u})$ and W_θ can appear only as a factor of $S^1(\mathfrak{u})$ and in $S^2(\mathfrak{u})$ as a factor of $S^2(W_{\theta'})$.

In particular, if W_θ is not an irreducible factor of $S^2(W_{\theta'})$ then W_θ and $W_{\theta'}$ appear with multiplicity one in $S(\mathfrak{u})$ and their sum is equal to $\mathfrak{u} = S^1(\mathfrak{u})$. Moreover, if $m_{\theta'} = 1$ then W_θ and $W_{\theta'}$ also appear with multiplicity 1 in $\mathbb{k}[G/H]^U$. So since φ is always injective its image contains $S^1(\mathfrak{u})$ which generates $S(\mathfrak{u})$, hence it is surjective.

Conversely, assume that φ is an isomorphism. Then $S[\mathfrak{u}] \simeq \bigoplus_{\lambda\in\Omega} W_\lambda \otimes \mathbb{k}^{m_\lambda}$. We have already noticed that $m_\theta = 1$ in this case and W_θ indeed appears in degree one. So it cannot appear also in degree two as a factor of $S^2(W_{\theta'})$, hence W_θ is not an irreducible factor of $S^2(W_{\theta'})$. Also we have already noticed that $W_{\theta'}$ can appear only in degree one where it appears with multiplicity 1, so we must have $m_{\theta'} = 1$. □

THEOREM 7. *Let G be simple. Then φ is an isomorphism if and only if either U is commutative, or G is of type F_4 and P is the maximal parabolic with semisimple Levi of type C_3, or G is of type B_n and P is the maximal parabolic with semisimple Levi of type A_{n-1}.*

PROOF. From Lemma 1 we must analyze the following cases (the simple roots α_i and the fundamental weights ω_i are numbered as in Bourbaki):

CASE 1: G of type B_n and P a maximal parabolic corresponding to the i-th simple root with $i = 2, \ldots, n$. We have $\theta' = \sum \alpha_i = \omega_1$ and $\theta = \alpha_1 + 2\sum_{i>1} \alpha_i = \omega_2$. In particular, $V_{\theta'}^H = 0$ if $i \neq n$ and is one-dimensional for $i = n$. Finally, $S^2(W_{\theta'}) = S^2(\mathbb{k}^n)$ is irreducible in this case and does not contain $W_\theta = \Lambda^2(\mathbb{k}^n)$ and Proposition 6 applies.

CASE 2: G of type C_n and P a maximal parabolic corresponding to the n-th simple root. We have $\theta' = \alpha_1 + 2\sum_{1<i<n} \alpha_i + \alpha_n = \omega_2$ and $\theta = 2\sum_{i<n} \alpha_i + \alpha_n$. In particular $S^2(W_{\theta'}) = S^2(\Lambda^2(\mathbb{k}^{2n})/\mathbb{k}) \supset W_\theta = S^2(\mathbb{k}^{2n})$ and Proposition 6 applies.

CASE 3: G of type F_4 and P a maximal parabolic corresponding to the first simple root. We have $\theta' = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 = \omega_4$ and $\theta = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4 = \omega_1$. Both conditions of Proposition 6 can be easily checked.

CASE 4: G of type G_2 and P a maximal parabolic corresponding to the second simple root. We have $\theta' = 2\alpha_1 + \alpha_2 = \omega_1$ and $\theta = 3\alpha_1 + 2\alpha_2 = \omega_2$. In this case $W^H_{\theta'} = 0$ and Proposition 6 applies. □

As already noticed in the introduction, for a general semisimple G it immediately follows that φ is an isomorphism if and only if $G \supset H$ is a product of the cases listed in the Theorem.

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Andrea Maffei
Dipartimento di Matematica
Università di Roma “La Sapienza”
piazzale Aldo Moro 5
00185 ROMA, Italy
amaffei@mat.uniroma1.it

Francesco Esposito
Dipartimento di Matematica Pura e Applicata
Università degli Studi di Padova
via Trieste 63
35121 PADOVA, Italy
esposito@math.unipd.it