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On the interpolation of the spaces $W^{l,1}(\mathbb{R}^d)$ and $W^{r,\infty}(\mathbb{R}^d)$

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Abstract. We study some properties of spaces obtained by interpolation of the Sobolev spaces $W^{k,1}(\mathbb{R}^d)$ and $W^{l,\infty}(\mathbb{R}^d)$, where l and r are nonnegative integers, and $d \geq 2$. We are concerned with the standard real and complex methods of interpolation. In the case of the real method, an old result of De Vore and Scherer (1979) gives that

$$(W^{l,1}(\mathbb{R}^d), W^{l,\infty}(\mathbb{R}^d))_{\theta,p_\theta} = W^{l,p_\theta}(\mathbb{R}^d),$$

where $\theta \in (0,1)$ and $1/p_\theta = 1-\theta$. We complement this result by considering the case $l \neq r$. We prove that, when $l \neq r$,

$$(\star) \qquad (W^{l,1}(\mathbb{R}^d), W^{r,\infty}(\mathbb{R}^d))_{\theta,q} = B_q^{\sigma,q}(\mathbb{R}^d),$$

where $\sigma := (1-\theta)l + \theta r$ and $1/q = 1-\theta$, if and only if $l - r \in \mathbb{R} \setminus [1,d]$. Also, we prove a similar fact when $W^{l,1}$ is replaced in $(\star)$ by a space $W^{s,p}$ where $s \neq r$ is a real number and $p \in (1,\infty)$. Several other problems like the boundedness of the Riesz transforms on interpolation spaces are also considered.

In the case of the complex method, it was proved by M. Milman (1983) that, for any $1 < p < \infty$,

$$(\star\star) \qquad (W^{l,1}(\mathbb{R}^d), W^{l,p}(\mathbb{R}^d))_\theta = W^{l,p_\theta}(\mathbb{R}^d),$$

where $1/p_\theta = (1-\theta) + \theta/p$. We show by simple arguments that $(\star\star)$ fails when $p = \infty$ and $l \geq 1$, answering a question of P. W. Jones (1984). As an immediate consequence of these arguments, we show that certain closed subspaces of $(C(\mathbb{T}^d))^N$ (with $N \in \mathbb{N}^*$) that are described by Fourier multipliers are not complemented in $(C(\mathbb{T}^d))^N$.

Contents

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1. Introduction

In this paper, we study interpolation properties of classical function spaces such as the Sobolev spaces $W^{l,p}(\mathbb{R}^d)$, where l is a nonnegative integer and $p \in [1,\infty]$. Here, as usual, $W^{l,p}(\mathbb{R}^d)$ is the space of all distributions f on $\mathbb{R}^d$ for which the quantity

$$\|f\|_{W^{l,p}} := \|f\|_{L^p} + \|\nabla^l f\|_{L^p}$$

is finite.

We will first consider the real interpolation method (for details, see Chapter 3 of [5] or Chapter 5 of [4]). Recall that for a compatible couple (X_0, X_1) of quasi-normed spaces, given the parameters $\theta \in (0,1)$ and $q \in [1,\infty]$, we define the interpolation space $(X_0, X_1)_{\theta,q}$ as being the quasi-normed space of all the elements $f \in X_0 + X_1$ for which the quantity

$$\|f\|_{(X_0,X_1)_{\theta,q}} := \Big(\int_0^\infty (t^{-\theta} K_t(f, X_0, X_1))^q \,\frac{dt}{t} \Big)^{1/q}$$

is finite, where K_t is the K-functional, defined by

$$K_t(f, X_0, X_1) := \inf\{\|f_0\|_{X_0} + t\|f_1\|_{X_1} \mid f = f_0 + f_1\},$$

for any $t > 0$.

In 1972, De Vore and Scherer [14] explicitly computed the K-functional that corresponds to the couple $(W^{l,1}, W^{l,\infty})$, where l is a nonnegative integer. This allowed them to interpolate between the Sobolev spaces W^{l_j,p_j} in the case where $l_0 = l_1 = l$. Indeed, by reiteration, it suffices to have the interpolation result in the case $p_0 = 1$, $p_1 = \infty$:

$$(W^{l,1}(\mathbb{R}^d), W^{l,\infty}(\mathbb{R}^d))_{\theta,q} = W^{l,q}(\mathbb{R}^d), \tag{1.1}$$

where $1/q = 1 - \theta$ (see also Corollary 5.13 in [4]). The arguments used by De Vore and Scherer are based on a careful analysis involving combinatorial ideas and spline-functions techniques. For a proof based on the Whitney covering lemma, see Section 5 in Chapter 5 of [4]. (For a version of (1.1) on more general domains, see Theorem 9 in [27].) Also, Bourgain (see Theorem 3 in [6]) gave a short elegant proof of (1.1) using the Calderón–Zygmund decomposition and elementary interpolation theory.

On the other hand, we have the following result that corresponds to the case $l_0 \neq l_1$, $p_0 = p_1 = p \in [1,\infty]$ (see Theorem 4.17 in [4]):

$$(W^{l_0,p}(\mathbb{R}^d), W^{l_1,p}(\mathbb{R}^d))_{\theta,q} = B_q^{\alpha,q}(\mathbb{R}^d), \tag{1.2}$$

for any $q \in [1,\infty]$, where $\alpha = (1-\theta)l_0 + \theta l_1$ and $B_q^{\alpha,q}$ is a Besov space. The proof of this result is based on A. Marchaud's inequalities [4], Theorem 4.4.

Note that the results (1.1) and (1.2) do not cover most of the cases where $l_0 \neq l_1$ and $p_0 \neq p_1$. In the nonlimiting case, i.e., $p_0, p_1 \in (1,\infty)$, one can use the characterization of the spaces W^{l_j,p_j} provided by the Littlewood–Paley theory and prove that for any two different nonnegative integers l_0 and l_1,

$$(W^{l_0,p_0}(\mathbb{R}^d), W^{l_1,p_1}(\mathbb{R}^d))_{\theta,q} = B_q^{\alpha,q}(\mathbb{R}^d), \tag{1.3}$$

where $\alpha = (1-\theta)l_0 + \theta l_1$ and $1/q = (1-\theta)/p_0 + \theta/p_1$ (see Lemma 2.6 below). This nonlimiting case is less interesting. More interesting are the limiting cases when at least one of the parameters p_0 or p_1 equals 1 or ∞ (and $p_0 \neq p_1$). Here, one cannot use the same arguments as for (1.3). This is due to the fact that one cannot describe efficiently spaces as $W^{l,1}$ and $W^{l,\infty}$ by means of Littlewood–Paley theory.

The interpolation problem in the limiting cases was partially solved in 2003 by Cohen, Dahmen, Daubechies and De Vore in [11]. (In fact, some partial results were proved earlier. See, for instance, [12].) They proved that, as long as $p \in (1,\infty)$ and $s \in \mathbb{R} \setminus [1-1/p', 1]$, we have

$$(\mathrm{BV}(\mathbb{R}^d), B_p^{s,p}(\mathbb{R}^d))_{\theta,q} = B_q^{\sigma,q}(\mathbb{R}^d), \tag{1.4}$$

where $\sigma = (1-\theta) + \theta s$ and $1/q = 1-\theta+\theta/p$. It is not hard to see that in this result we can replace the spaces BV with the Sobolev space $W^{1,1}$ and, since $B_2^{r,2} = W^{r,2}$ when r is a nonnegative integer, we obtain

$$(W^{1,1}(\mathbb{R}^d), W^{r,2}(\mathbb{R}^d))_{\theta,q} = B_q^{\sigma,q}(\mathbb{R}^d),$$

where $\sigma = (1-\theta) + \theta r$ and $1/q = 1-\theta+\theta/2$ as long as $r=0$ or $r \geq 2$. With this we have at least one result in the limiting case $p_0 = 1$ that cannot be covered by (1.1) and (1.2). The interpolation result (1.4) relies on an "almost" characterization of the space BV by means of wavelets. Note that unlike the Besov spaces or the Sobolev spaces $W^{r,p}$, for $p \in (1,\infty)$, the space BV does not even have an unconditional basis (see, for instance, the discussion in the introduction of [11]). Hence, it cannot be completely described via wavelets. However, the partial description provided by the authors of [11] is sufficient for establishing the interpolation result in (1.4).

All the above interpolation results remain true in the case of the corresponding homogeneous spaces.

In what follows, we will study the interpolation spaces $(X_0, X_1)_{\theta,q}$ where at least one of the spaces X_j is of the form $W^{l,1}$ or $W^{r,\infty}$, where l and r are different integers. As we have seen, the cases where the differential regularity of X_0 coincide with the differential regularity of X_1 are well studied. Hence, we will consider only the case where X_0 and X_1 are of different differential regularity. We will also consider the homogeneous versions of the function spaces and in some situations, for the sake of simplicity, we provide proofs only for the homogeneous version if the situation for the inhomogeneous case is similar.

Note that in all the known situations in which we have an explicit description of the interpolation space, as in (1.1), (1.2), (1.3) and (1.4), the result of the interpolation is a Triebel–Lizorkin space. For instance, the interpolation space in (1.1) is the space $W^{l,q} = F_2^{l,q}$ and in (1.2), (1.3) and (1.4), we have the space $B_q^{\sigma,q} = F_q^{\sigma,q}$. For this reason, it is natural to compare the spaces obtained by interpolation to Triebel–Lizorkin spaces of the form $F_\tau^{\sigma,q}$, where the parameters σ and q are the "right" ones and τ is any number in the interval $[1,\infty]$. In the case the interpolation space is not a Triebel–Lizorkin space, we will call it pathological. The "pathologies" we find while interpolating function spaces will give rise to some other properties of the classical Sobolev spaces. There are however some situations in which we prove only the homogeneous versions (see, for instance, Corollary 3.24).

Throughout the entire paper, the dimension d of $\mathbb{R}^d$ will be always at least 2.

Real interpolation. In the case where both endpoints are in a limiting situation, we prove the following.

Theorem 1.1. *Suppose l and r, with $l \neq r$, are some nonnegative integers and fix $\theta \in (0,1)$. Let $\sigma := (1-\theta)l + \theta r$ and $q := 1/(1-\theta)$. Let X be the interpolation space*

$$X := (W^{l,1}(\mathbb{R}^d), W^{r,\infty}(\mathbb{R}^d))_{\theta,q}.$$

If $l \in \mathbb{R} \setminus [r, r+d]$, then $X = B_q^{\sigma,q}(\mathbb{R}^d)$. In the case $l \in (r, r+d]$, no space $F_\tau^{\sigma,q}$, with $\tau \in [1,\infty]$, embeds in X. The same result holds for the homogeneous spaces.

Remark 1.2. Suppose that X and Y are two quasi-normed function spaces. When we say that Y does not embed in X, we also mean that we have the noninequality

$$\|f\|_X \lesssim\!\!\!\!\!/\;\, \|f\|_Y,$$

for $f \in X \cap Y$. Similar quantitative facts are taken into account when we say $X \neq Y$.

In order to prove the above theorem, we will need the following generalization of (1.4) where the space BV is replaced by $W^{l,1}$, where l is any integer, possible negative. (See Section 2.1 for a definition of the spaces $W^{l,1}$ when l is a negative integer.)

Proposition 1.3. *Consider some parameters $l \in \mathbb{Z}$, $s \in \mathbb{R}$, with $s \neq l$, and $p \in (1,\infty)$, $t \in [1,\infty]$. If $l \geq 1$ and $s \in \mathbb{R} \setminus [l - 1/p', l]$, or $l \leq 0$ and $s \in \mathbb{R} \setminus [l - d/p', l]$, then for any $\theta \in (0,1)$, we have*

$$(W^{l,1}(\mathbb{R}^d), F_t^{s,p}(\mathbb{R}^d))_{\theta,q} = B_q^{\sigma,q}(\mathbb{R}^d),$$

where $\sigma = (1-\theta)l + \theta s$ and $1/q = 1 - \theta + \theta/p$. The same result holds for the homogeneous spaces.

In the situation $F_t^{s,p} = B_p^{s,p}$, Proposition 1.3 has been already proven in the case $l = 1$ (with BV instead of $W^{1,1}$) by Cohen, Dahmen, Daubechies and De Vore [11], and in the case $l = 0$ by Cohen [10]. We prove Proposition 1.3 by using the technique introduced in [11] and [10]. We use the "almost" characterization via wavelets of the space $\dot{\text{BV}}$ and L^1 that was given in [11] and [10], respectively, in order to give similar "almost" characterizations for the spaces $\dot{W}^{l,1}$. When $l \geq 1$, we simply deduce our characterization of $\dot{W}^{l,1}$ directly from that of $\dot{\text{BV}}$ given in [11]. When $l = 0$, we use instead a result from [10], and then we easily deduce the characterization of $\dot{W}^{l,1}$ when $l \leq 0$.

We mention that, according to equations (16) and (17) on p. 17 of [20], it was proved (by different methods) by N. Kruglyak in[1] 1996, see [24], that

$$(L^1(\mathbb{R}^d), B_p^{s,p}(\mathbb{R}^d))_{\theta,q} = B_q^{\sigma,q}(\mathbb{R}^d), \tag{1.5}$$

where $s > 0$, $\sigma = (1-\theta) + \theta s$ and $1/q = 1 - \theta + \theta/p$. This covers the particular case $l = 0$ in Proposition 1.3, when $t = p$ and $s > 0$. An explicit computation of the K-functional for the couples $(L^p, \dot{W}^{k,q})$, with $p, q \in [1,\infty)$ and k a nonnegative integer, can be

[1]It seems that this is the first apparition of (1.5) in the literature. See also [25].

found in Chapter 9, Part II, of [21] (see also [25]). Using the theory in [21], [33], and (1.5), it is possible to prove Proposition 1.3 in the case $l < s$, $t = p$. We do not consider this approach here.

Unfortunately, we do not know whether or not the condition imposed to s in Proposition 1.3 is sharp, unless we are in the case $l \leq 0$ (see Corollary 3.21). This remains an *open* question. However, when one of the endpoints is a space of the form $W^{r,\infty}$ and the other one is a Triebel–Lizorkin space, we can identify all the pathological cases.

Theorem 1.4. *Consider some parameters $s \in \mathbb{R}$, $r \in \mathbb{N}$, with $s \neq r$, and let $p, q \in (1,\infty)$, $\theta \in (0,1)$, $t \in [1,\infty]$, $\sigma \in \mathbb{R}$ be such that $1/q = (1-\theta)/p$ and $\sigma = (1-\theta)s + \theta r$. Let X be the interpolation space*

$$X := (F_t^{s,p}(\mathbb{R}^d), W^{r,\infty}(\mathbb{R}^d))_{\theta,q}.$$

If $s \in \mathbb{R} \setminus (r, r + d/p]$, then $X = B_q^{\sigma,q}$. In the case $s \in (r, r + d/p]$, no space $F_\tau^{\sigma,q}$, with $\tau \in [1,\infty]$, embeds in X. The same result holds for the homogeneous spaces.

Theorem 1.1 and Theorem 1.4 seem to be new even in the nonpathological case. The nonpathological cases of Theorem 1.1 and Theorem 1.4 are deduced from Proposition 1.3 by simple arguments that involve duality and the celebrated theorem of T. Wolff proved in [40] concerning the real interpolation method. The proof of Theorem 1.1 and Theorem 1.4 in the pathological cases rests on trace theory and the interpolation theorem of Wolff. Roughly speaking, we show that (in the pathological cases) the space obtained by interpolation is "closer" to have a trace on a particular subset $\Gamma \subset \mathbb{R}^d$ than its corresponding Triebel–Lizorkin space $F_\tau^{\sigma,q}$. For instance, in the case of Theorem 1.1, when $r = 0$ and $l \in (0, d/p]$, the space $(W^{l,1}, L^\infty)_{\theta,q}$ has a trace on $\mathbb{R}^{d-l} \simeq \mathbb{R}^{d-l} \times \{0\}^l$, while $F_\tau^{\sigma,q} = F_\tau^{l/q,q}$ has no trace on $\mathbb{R}^{d-l}$.

It is remarkable that this trace argument covers all the pathological cases in Theorem 1.1 and, combined with Wolff's theorem, all the pathological cases in Theorem 1.4. The main point of the paper is this power of the simple trace argument rather than the result of Proposition 1.3 or the nonpathological parts of Theorem 1.1 and Theorem 1.4.

Theorem 1.1, Proposition 1.3 and Theorem 1.4 above are the subject of Sections 3.2.1 and 3.2.2.

Sums of spaces. One particular property of any interpolation space $(X_0, X_1)_{\theta,q}$, where (X_0, X_1) is a given compatible couple, is that it embeds in the sum of the endpoint spaces. In other words,

$$(X_0, X_1)_{\theta,q} \hookrightarrow X_0 + X_1. \tag{1.6}$$

In view of this property, one can refine Theorem 1.4. In the most of the pathological cases, as long as we are dealing with homogeneous spaces, one can prove much more. Namely, we have that if $r, p, q, \theta, s, \sigma$ are as in Theorem 1.4 and $r < s < r + d/p$, then (see Proposition 3.25),

$$\dot{F}_\tau^{\sigma,q}(\mathbb{R}^d) \not\hookrightarrow \dot{F}_t^{s,p}(\mathbb{R}^d) + \dot{W}^{r,\infty}(\mathbb{R}^d). \tag{1.7}$$

This easily follows from Theorem 1.4 and some dilation arguments that are possible thanks to the fact that we work with the homogeneous version of the spaces. Of a particular interest is also the inhomogeneous version of this result. Restricting ourselves to the case of the inhomogeneous Sobolev–Slobodeckii spaces one can deduce from (1.7) the following proposition.

Proposition 1.5. *Let r be a nonnegative integer and let $p, q \in [1,\infty)$, $\theta \in (0,1)$, $s, \sigma \in \mathbb{R}$ be some parameters such that $1/q = (1-\theta)/p$ and $\sigma = (1-\theta)s + \theta r$. If $r < s < r + d/p$, then*

$$W^{\sigma,q}(\mathbb{R}^d) \not\hookrightarrow W^{s,p}(\mathbb{R}^d) + W^{r,\infty}(\mathbb{R}^d).$$

This is in contrast with the following fact, proved by Mironescu in [30] (see Theorem 1.4 in [30]). Suppose that $s_0, s_1 > 0$, with $s_0 \neq s_1$, $1 \le p_0 < p_1 < \infty$, and $1/q = (1-\theta)/p_0 + \theta/p_1$, $\sigma = (1-\theta)s_0 + \theta s_1$ for some $\theta \in (0,1)$. Then

$$W^{\sigma,q}(\mathbb{R}^d) \hookrightarrow W^{s_0,p_0}(\mathbb{R}^d) \cap W^{\sigma,q}(\mathbb{R}^d) + W^{s_1,p_1}(\mathbb{R}^d) \cap W^{\sigma,q}(\mathbb{R}^d).$$

In particular, we have

$$W^{\sigma,q}(\mathbb{R}^d) \hookrightarrow W^{s_0,p_0}(\mathbb{R}^d) + W^{s_1,p_1}(\mathbb{R}^d). \tag{1.8}$$

As it was shown in [30], if we drop the condition $p_1 < \infty$, the embedding (1.8) may fail. The example given in [30] is the nonembedding

$$W^{1-\theta,1/(1-\theta)}(\mathbb{R}) \not\hookrightarrow W^{1,1}(\mathbb{R}) + L^\infty(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R}).$$

Proposition 1.5 above enlarges the number of examples of this kind. In fact, supposing the condition $p_1 < \infty$ fails, Proposition 1.5, together with the nonpathological part of Theorem 1.4 and formula (1.6), can decide in most of the cases whether or not the embedding (1.8) holds. More precisely, if $p_1 = \infty$, $s_1 \in \mathbb{N}$, $s_0 \notin \{s_1, d/p_0\}$ and σ is not an integer,[2] then (1.8) holds if and only if $s_0 \in \mathbb{R} \setminus (s_1, s_1 + d/p_0)$. If $s_0 = s_1$, then (1.8) still holds thanks to (1.1) and (1.6). The case $s_0 = d/p$ remains *open*.

The Riesz transforms. There is yet another aspect of the pathological situations that deserves attention: the boundedness of some common singular integral operators such as the Riesz transforms on spaces obtained by interpolation. Here, the Riesz transforms $R_1, \dots, R_d$ on $\mathbb{R}^d$ are the operators defined by the equality

$$\widehat{R_j f}(\xi) := \frac{i\xi_j}{|\xi|}\,\hat{f}(\xi),$$

for any Schwartz function f on $\mathbb{R}^d$ and any $j \in \{1, \dots, d\}$. (Here, as usual, $\hat{f}$ is the Fourier transform of f.) We study the Riesz transforms on the interpolation spaces that appear in the pathological case of Proposition 1.4, when $r = 0$ and $s < d/p$. More precisely, we prove that, as long as $s \in (0, d/p)$, none of the Riesz transforms R_j is bounded on the space

$$X := (F_t^{s,p}(\mathbb{R}^d), L^\infty(\mathbb{R}^d))_{\theta,q},$$

where the parameters p, t, θ and q are as in the statement of Theorem 1.4 (see Proposition 3.26). Even worse, no R_j is bounded from $(F_t^{s,p}, L^\infty)_{\theta,1}$ to $(F_t^{s,p}, L^\infty)_{\theta,\infty}$ (see

[2] Presumably the assertion remains true when σ is an integer.

Proposition 3.30). This is in contrast to the fact that the Riesz transforms are bounded on the interpolation space

$$(F_2^{0,p}(\mathbb{R}^d), L^\infty(\mathbb{R}^d))_{\theta,q} = (L^p(\mathbb{R}^d), L^\infty(\mathbb{R}^d))_{\theta,q} = L^q(\mathbb{R}^d).$$

We prove this nonboundedness result by combining Theorem 1.4 with the remarkable result of Adams and Frazier obtained in 1988, see [1], that

$$\text{BMO} \cap F_t^{s,p} = L^\infty \cap F_t^{s,p} + R_1(L^\infty \cap F_t^{s,p}) + \cdots + R_d(L^\infty \cap F_t^{s,p}), \tag{1.9}$$

as long as $s > 0$ and $p, t \in (1, \infty)$. This in turn rests on a celebrated construction of Uchiyama and some ideas of Baernstein (see [1] for the references therein), also making use of the atomic decomposition for the spaces $F_t^{s,p}$.

It is natural to expect that R_j are unbounded on X in the most general pathological situation described by the statement of Theorem 1.4. However, this problem remains *open*.

Interpolation functors. Due to the extremal properties of the real interpolation method, one can easily improve our result concerning the pathological situation of Theorem 1.4.

Proposition 1.6. *Consider the parameters $s \in \mathbb{R}$, $r \in \mathbb{N}$, with $s \neq r$, and let $p, q \in (1, \infty)$, $\theta \in (0, 1)$, $t \in [1, \infty]$, $\sigma \in \mathbb{R}$ be as in the statement of Theorem 1.4. Moreover, suppose that $s \in (r, r + d/p]$ and fix some $\tau \in [1, \infty]$. Then $F_\tau^{\sigma,q}$ is not an interpolation space of exponent θ with respect to the couple $(F_t^{s,p}, W^{r,\infty})$. The same result holds for the homogeneous spaces.*

Proposition 1.6 has the following immediate consequence for the homogeneous version of the spaces. With the same notation as in Proposition 1.6, if $s \in (r, r + d/p)$, then there exists a linear operator

$$T : \dot{F}_t^{s,p} + \dot{W}^{r,\infty} \to \dot{F}_t^{s,p} + \dot{W}^{r,\infty},$$

that is bounded on $\dot{F}_t^{s,p}$ and on $\dot{W}^{r,\infty}$, and not bounded on $\dot{F}_\tau^{\sigma,q}$ (see Corollary 3.24).

Complex interpolation. In Section 4 we study some aspects of the complex interpolation of Sobolev type spaces. (See, for instance, Chapter 4 of [5] for a description of the complex method.) By means of the Littlewood–Paley theory and the retraction method, it is easy to obtain that, for any $\theta \in (0, 1)$ and any $l \in \mathbb{N}^*$,

$$(W^{l,p_0}, W^{l,p_1})_\theta = W^{l,p_\theta}, \tag{1.10}$$

as long as $p_0, p_1 \in (1, \infty)$ and $1/p_\theta = (1 - \theta)/p_0 + \theta/p_1$ (see [5], Chapter 6). We cannot handle the case $p_0 = 1$ by the Littlewood–Paley theory. However, as it was proved by Milman in 1983 (see Theorem B in [29]), the equality (1.10) continues to hold even in the case $p_0 = 1$, $p_1 \in (1, \infty)$. The main tool used by Milman rests on a result of Peetre that makes a connection between the complex and the real interpolation method via the concept of Fourier type of a Banach space. The case where $p_1 = \infty$ is also of interest. Here, neither the Littlewood–Paley theory nor Milman's method can be applied. In 1984, P. W. Jones (see Section 8.22 on page 519 of [18]) asked if (1.10) continues to hold when $1 \leq p_0 < p_1 = \infty$. In Section 4.1, we give a negative answer to this question. The fact that (1.10) fails in this extreme case is a consequence of the following slightly more general negative result (see Remark 4.2 and Corollary 4.3).

Proposition 1.7. *Let $l \geq 0$ be an integer and consider some parameters $1 \leq p, q < \infty$ and $s > 1/p$. Fix some $\theta \in (0,1)$ and define $\sigma := (1-\theta)s + \theta l$, $\rho := p/(1-\theta)$. Then, for any $1 \leq t < \infty$, we have that*

$$F_t^{\sigma,\rho}(\mathbb{R}^d) \not\hookrightarrow (F_q^{s,p}(\mathbb{R}^d), W^{l,\infty}(\mathbb{R}^d))_\theta.$$

As in the case of the real method, we prove Proposition 1.7 by a simple trace argument. This time, we show that $(F_q^{s,p}(\mathbb{R}^d), W^{l,\infty}(\mathbb{R}^d))_\theta$ has a trace on $\mathbb{R}^{d-1}$ that is embedded in a space $F_{\rho_1}^{\sigma-1/\rho,\rho}(\mathbb{R}^{d-1})$, where $\rho_1 < \rho$, while the trace of $F_t^{\sigma,\rho}(\mathbb{R}^d)$ is the space $F_\rho^{\sigma-1/\rho,\rho}(\mathbb{R}^{d-1})$. It remains to notice that $F_\rho^{\sigma-1/\rho,\rho}(\mathbb{R}^{d-1})$ is strictly larger than $F_{\rho_1}^{\sigma-1/\rho,\rho}(\mathbb{R}^{d-1})$.

Noncomplemented subspaces. The trace technique also allows us to easily see that some closed subspaces of $(C(\mathbb{T}^d))^N$ (where N is a positive integer) are not complemented. Recall the following result obtained by G. M. Henkin in 1967, see [16].

Proposition 1.8. *Suppose $l \geq 1$ and $d \geq 2$ are two integers. Then the space $C^l(\mathbb{T}^d)$ is not an isomorphic copy of a complemented subspace of $C(S)$, for any compact space S.*

This result was improved by S. V. Kislyakov in 1975, see [19], who showed, using the Grothendieck theorem on absolutely summing operators, that $C^l(\mathbb{T}^d)$ $(l \geq 1)$ is not an isomorphic copy of a quotient space of $C(S)$. Note that Proposition 1.8 implies the fact that the closed subspace $G_l(C)$ of $(C(\mathbb{T}^d))^N$ (here, $N = |\{\alpha \in \mathbb{N}^d \mid |\alpha| = l\}|$) formed by the l-gradients (elements of the form $\nabla^l f$) is not complemented in $(C(\mathbb{T}^d))^N$. This result was generalized to other differential expressions (than l-gradients) by the authors of [22] in 2015, using Grothendieck's theorem and some Sobolev type embeddings. We will prove that $G_l(C)$ is not complemented in $(C(\mathbb{T}^d))^N$ using trace theory. In fact, we will prove in Section 4.2 a more general result (see Proposition 4.5) that involves Fourier multipliers instead of differential expressions. Our result, Proposition 4.5, has a more restricted range of applications than the main result of [22]. However, it has a shorter proof and it covers some cases that are not covered by the results in [22]. We mention that Proposition 4.5 is not the most general result that can be obtained by our method, however, for simplicity we do not provide here further generalizations.

General remark. Several general observations are in order. The main point of the paper is the study of some of the pathological situations that arise in the interpolation theory of Sobolev type spaces. One of our main tools is trace theory. When we consider traces on subspaces of $\mathbb{R}^d$, we are using standard trace theory as presented, for instance, in Section 4.4 of Chapter 4 in [38]. In some cases we need to consider traces on fractal type subsets of $\mathbb{R}^d$. In these cases we use the more recent trace theory developed (between others) by Bricchi, Caetano and Haroske (see [8] and the reference therein). We will recall in Appendix A several results from the trace theory that will be used in the paper.

Apart from the nontrivial results given in [11] and [1] (see (1.4) and (1.9) above), we use standard facts from the interpolation theory as can be found in [5] and [4]. Also, we use standard facts from the theory of function spaces as can be found, for instance, in [38], Chapter 3 of [36], or [15].

Notation. Throughout the paper, we use mainly standard notation. For example, $\mathbb{N} = \{0, 1, 2, \ldots\}$ is the set all natural numbers, $\mathbb{N}^* = \{1, 2, \ldots\}$ is the set of the positive natural numbers and $\mathbb{Z}$ is the set of integers.

Often, we use the symbols $\lesssim$ and $\sim$. For two nonnegative variable quantities a and b we write $a \lesssim b$ if there exists a constant $C > 0$ such that $a \leq Cb$. If $a \lesssim b$ and $b \lesssim a$, then we write $a \sim b$. Other notation will be introduced when needed.

Some notation will be used only in one subsection. For instance, we will denote by φ and ψ some functions related to the wavelets that appear in Section 3.1. Outside of Section 3.1, φ and ψ will have a different meaning. This re-use of the notation should be clear from the context.

2. Function spaces and interpolation

2.1. Homogeneous and inhomogeneous spaces

We quickly recall here the definition of some standard function spaces. We begin by recalling the definition of the Sobolev spaces. As we have already mentioned, when $l \geq 0$ is an integer and $p \in [1, \infty]$, $W^{l,p}(\mathbb{R}^d)$ is the space of all distributions f on $\mathbb{R}^d$ for which the quantity

$$\|f\|_{W^{l,p}} := \|f\|_{L^p} + \|\nabla^l f\|_{L^p}$$

is finite. The homogeneous spaces will be defined here in a slightly nonstandard way.[3] Let $\mathcal{S}_\sharp$ be space of all Schwartz functions f on $\mathbb{R}^d$ such that $\widehat{f}$ vanishes in a neighborhood of 0. When $1 \leq p < \infty$ the homogeneous space $\dot{W}^{l,p}(\mathbb{R}^d)$ is obtained by completion of $\mathcal{S}_\sharp$ under the norm

$$\|f\|_{\dot{W}^{l,p}} := \|\nabla^l f\|_{L^p}.$$

We can see that we can also define the above homogeneous spaces $\dot{W}^{l,p}$ by completing the normed function spaces $\dot{W}^{l,p}_c(\mathbb{R}^d)$. Here, $\dot{W}^{l,p}_c(\mathbb{R}^d)$ is the space of all the compactly supported functions whose $\dot{W}^{l,p}$-norm is finite. The spaces $\dot{W}^{l,p}$ as defined here are complete. The main advantage of the above definition of the homogeneous Sobolev spaces is that we have the embedding

$$\dot{W}^{l_1,p_1}(\mathbb{R}^d) \hookrightarrow \dot{W}^{l_2,p_2}(\mathbb{R}^d), \tag{2.1}$$

for any $l_1, l_2 \in \mathbb{N}$, $p_1, p_2 \in (1, \infty)$, with $l_1 > l_2$ and $l_1 - d/p_1 = l_1 - d/p_1$. (This follows, for instance, from Theorem 4.31 on p. 102 of [2] by a dilation argument.)

In the case where $l = -r \leq 0$ and $1 \leq p < \infty$, we define $W^{l,p}(\mathbb{R}^d)$ and $\dot{W}^{l,p}(\mathbb{R}^d)$ by completion of $\mathcal{S}_\sharp$ under the norms

$$\|f\|_{W^{l,p}} := \inf\Big\{ \sum_{|\alpha| \leq r} \|f_\alpha\|_{L^p} \;\Big|\; f = \sum_{|\alpha| \leq r} \nabla^\alpha f_\alpha \Big\}, \quad \text{and}$$

$$\|f\|_{\dot{W}^{l,p}} := \inf\Big\{ \sum_{|\alpha| = r} \|f_\alpha\|_{L^p} \;\Big|\; f = \sum_{|\alpha| = r} \nabla^\alpha f_\alpha \Big\},$$

[3] Usually the elements of a homogeneous Sobolev space are defined as distributions (factorized to polynomials) whose homogeneous Sobolev seminorms are finite. See also Chapter 6 of [15]. However, most of the standard properties (as interpolation or duality) translates to our case without difficulties.

respectively. Note that when $l \leq 0$ the spaces $W^{l,p}$ and $\dot{W}^{l,p}$ are Banach spaces whose elements are distributions. When $l \geq 1$, we can also define the space $\dot{W}^{l,p}(\mathbb{R}^d)$ as being isomorphic to the dual of $\dot{W}^{-l,p}(\mathbb{R}^d)$, where the isomorphism is chosen so that $\dot{W}^{l,p}_c(\mathbb{R}^d) \hookrightarrow \dot{W}^{l,p}(\mathbb{R}^d)$. Also, when $p \in [1,\infty)$, we have that $(W^{-r,p}(\mathbb{R}^d))^* = W^{r,p'}(\mathbb{R}^d)$ and when $p \in (1,\infty)$, we have $(\dot{W}^{-r,p}(\mathbb{R}^d))^* = \dot{W}^{r,p'}(\mathbb{R}^d)$. When $l \geq 1$, we define the space $\dot{W}^{l,\infty}(\mathbb{R}^d)$ as being isomorphic to the dual of $\dot{W}^{-l,1}(\mathbb{R}^d)$ and the isomorphism is chosen such that $\dot{W}^{l,\infty}_c(\mathbb{R}^d) \hookrightarrow \dot{W}^{l,\infty}(\mathbb{R}^d)$.

In some situations it will be convenient to use the spaces $C^l(\mathbb{R}^d)$, $C^l_0(\mathbb{R}^d)$ and its homogeneous version $\dot{C}^l_0(\mathbb{R}^d)$. We define $C^l_0(\mathbb{R}^d)$ as being the closure of the space of Schwartz functions in the $W^{l,\infty}$-norm. Similarly, we define $\dot{C}^l_0(\mathbb{R}^d)$ as being the closure of the space of Schwartz functions in the $\dot{W}^{l,\infty}$-norm.

We continue by briefly recalling the definition of the Triebel–Lizorkin and Besov spaces (see [38] for details). Consider a radial function $\Phi \in C^\infty_c(\mathbb{R}^d)$ such that $\operatorname{supp}\Phi \subset B(0,2)$ and $\Phi \equiv 1$ on $B(0,1)$. For $k \in \mathbb{Z}$, we define the operators P_k, acting on the space of tempered distributions on $\mathbb{R}^d$, by the relation

$$\widehat{P_k f}(\xi) := \Big(\Phi\Big(\frac{\xi}{2^k}\Big) - \Phi\Big(\frac{\xi}{2^{k-1}}\Big)\Big)\hat{f}(\xi),$$

for any Schwartz function f on $\mathbb{R}^d$. We will also consider the operator $P_{\leq 0}$ defined by

$$\widehat{P_{\leq 0} f}(\xi) := \Phi(\xi)\hat{f}(\xi),$$

for any Schwartz function f on $\mathbb{R}^d$. The operators $P_{\leq 0}$, P_k will be called *Littlewood–Paley "projections"* adapted to $\mathbb{R}^d$. For any Schwartz function f, we have that

$$f = \sum_{k\in\mathbb{Z}} P_k f,$$

in the sense of tempered distributions. In some cases it is useful to consider another function $\tilde{\Phi} \in C^\infty_c(\mathbb{R}^d)$, similar to Φ, such that $\operatorname{supp}\tilde{\Phi} \subset B(0,4)$ and $\tilde{\Phi} \equiv 1$ on $B(0,2)$ and, starting from this to consider some Littlewood–Paley projections $\tilde{P}_k$ and $\tilde{P}_{\leq 0}$. We have the useful identities $\tilde{P}_k P_k = P_k$ and $\tilde{P}_{\leq 0} P_{\leq 0} = P_{\leq 0}$.

The inhomogeneous Triebel–Lizorkin space $F^{s,p}_q(\mathbb{R}^d)$ (with $1 \leq p, q < \infty$ and s a real number) is the space consisting of those tempered distributions f on $\mathbb{R}^d$ for which the following norm is finite:

$$\|f\|_{F^{s,p}_q} := \|P_{\leq 0} f\|_{L^p} + \Big\|\Big(\sum_{k\geq 0} 2^{skq}\,\|P_k f\|^q\Big)^{1/q}\Big\|_{L^p}.$$

A remarkable fact is that if $l \geq 0$ is an integer and $1 < p < \infty$, then $F^{l,p}_2(\mathbb{R}^d) = W^{l,p}(\mathbb{R}^d)$ with equivalent norms (see, for instance, item (iii) of Theorem on p. 29 of [38]).

The Besov space $B^{s,p}_q(\mathbb{R}^d)$ (with $1 \leq p, q \leq \infty$ and s a real number) is the inhomogeneous Besov space consisting of those tempered distributions f on $\mathbb{R}^d$ for which the following norm is finite:

$$\|f\|_{B^{s,p}_q} := \|P_{\leq 0} f\|_{L^p} + \Big(\sum_{k\geq 0} 2^{skq}\,\|P_k f\|^q_{L^p}\Big)^{1/q}.$$

If $s > 0$ is not an integer and $1 < p < \infty$, then $B_p^{s,p}(\mathbb{R}^d) = W^{s,p}(\mathbb{R}^d)$ with equivalent norms. Here, $W^{s,p}$ is a Sobolev–Slobodeckii space (see, for instance, [35], p. 12, for a definition).

The homogeneous space $\dot{F}_q^{s,p}(\mathbb{R}^d)$ (with $1 \leq p, q < \infty$ and s a real number) is obtained by completion of $\mathcal{S}_\sharp$ under the norm

$$\|f\|_{\dot{F}_q^{s,p}} := \Big\| \Big(\sum_{k\in\mathbb{Z}} 2^{skq} \|P_k f\|^q \Big)^{1/q} \Big\|_{L^p}.$$

When $k \geq 0$ is an integer and $1 < p < \infty$, then $\dot{F}_2^{k,p}(\mathbb{R}^d) = \dot{W}^{k,p}(\mathbb{R}^d)$ with equivalent norms.

The space $\dot{B}_q^{s,p}(\mathbb{R}^d)$ (with $1 \leq p, q \leq \infty$ and s a real number) is obtained by completion of $\mathcal{S}_\sharp$ under the norm

$$\|f\|_{\dot{B}_q^{s,p}} := \Big(\sum_{j\in\mathbb{Z}} 2^{skq} \|P_k f\|_{L^p}^q \Big)^{1/q}.$$

In the above definitions of the Triebel–Lizorkin and Besov spaces, one can use also the “projections” $\tilde{P}_k$, $\tilde{P}_{\leq 0}$ instead of P_k, $P_{\leq 0}$ without changing the spaces we obtain.

If $s > 0$ is not an integer and $1 < p < \infty$, then $\dot{B}_p^{s,p}(\mathbb{R}^d) = \dot{W}^{s,p}(\mathbb{R}^d)$ with equivalent norms (for simplicity, we may consider that this is the definition of $\dot{W}^{s,p}(\mathbb{R}^d)$). One can also define $\dot{W}^{s,p}$ as being the completion of $\mathcal{S}_\sharp$ or $\dot{W}_c^{s,p}(\mathbb{R}^d)$ under the $\dot{W}^{s,p}(\mathbb{R}^d)$-norm.

We have that $\dot{F}_2^{0,1}(\mathbb{R}^d)$ can be identified with the Hardy space $H^1(\mathbb{R}^d)$ (see, for instance, Remark 6.5.2 on p. 70 of [15]). The space $\dot{F}_2^{0,\infty}(\mathbb{R}^d)$ is defined as the dual of $\dot{F}_2^{0,1}(\mathbb{R}^d) = H^1(\mathbb{R}^d)$, i.e., $\dot{F}_2^{0,\infty}(\mathbb{R}^d) = \mathrm{BMO}(\mathbb{R}^d)$. In the inhomogeneous case, we have that $F_2^{0,1}$ can be identified with the local Hardy space $h^1(\mathbb{R}^d)$ (see, for instance, Proposition 2.1.2 on p. 14 of [35]). The space $F_2^{0,\infty}(\mathbb{R}^d)$ is defined as the dual of $F_2^{0,1}(\mathbb{R}^d) = h^1(\mathbb{R}^d)$, i.e., $F_2^{0,\infty}(\mathbb{R}^d) = \mathrm{bmo}(\mathbb{R}^d)$, the local bounded mean oscillation space (see, for instance, [35], p. 13, for a definition). We will use several times the embeddings $L^\infty(\mathbb{R}^d) \hookrightarrow \mathrm{BMO}(\mathbb{R}^d)$ and $L^\infty(\mathbb{R}^d) \hookrightarrow \mathrm{bmo}(\mathbb{R}^d)$.

Notice that for the homogeneous spaces on $\mathbb{R}^d$, we have the following dilation properties:

$$\|f^\lambda\|_{\dot{F}_q^{s,p}} \sim \lambda^{s-d/p} \|f\|_{\dot{F}_q^{s,p}}, \tag{2.2}$$

for any $f \in \dot{F}_q^{s,p}(\mathbb{R}^d)$, respectively, and any $\lambda > 0$, where $f^\lambda(\cdot) := f(\lambda\,\cdot)$. A similar equivalence holds for $\dot{B}_q^{s,p}$ instead of $\dot{F}_q^{s,p}$.

Also,

$$\|f^\lambda\|_{\dot{W}^{l,1}} \sim \lambda^{l-d} \|f\|_{\dot{W}^{l,1}} \quad \text{and} \quad \|f^\lambda\|_{\dot{W}^{l,\infty}} \sim \lambda^{l} \|f\|_{\dot{W}^{l,\infty}}, \tag{2.3}$$

when $f \in \dot{W}^{l,1}(\mathbb{R}^d)$ or $f \in \dot{W}^{l,\infty}(\mathbb{R}^d)$, respectively.[4]

Remark 2.1. The properties of the homogeneous spaces that we define here are in most cases deduced from the properties of their inhomogeneous versions (see, for instance, (2.1) or the tool provided by Lemma 2.3 below). Apart from this, we also need some interpola-

[4]In the case of $\dot{W}^{l,\infty}$, the linear operator $f \to f^\lambda$ is defined by duality.

tion identities that are similar to those of the standard spaces (see, for instance, Lemma 2.6 and its proof). The standard definition for the Hardy space H^1 coincides with our definition of $\dot{F}_2^{0,1}$ and we can use the well-known properties of H^1 in this case. Other easy facts can be checked by standard arguments.

Remark 2.2. In fact, we could choose to define the homogeneous spaces only as normed spaces endowed with the corresponding norm. This is due to the fact that in order to define a real interpolation space for a compatible couple the completeness of the involved spaces is not required. However, in order to avoid some technical details appearing in the proofs of our results, we prefer the definition given in this subsection.

In a similar way we can define Triebel–Lizorkin and Besov spaces on $\mathbb{T}^d$. The properties of the spaces defined on the $\mathbb{T}^d$ are similar to those of the spaces defined on $\mathbb{R}^d$ (see Chapter 3 of [36] for details). When working on $\mathbb{T}^d$, it is sometimes convenient to consider functions whose integral on $\mathbb{T}^d$ vanishes. In general, if $X(\mathbb{T}^d)$ is a function space on $\mathbb{T}^d$ (in this case all the elements of $X(\mathbb{T}^d)$ will be taken to be distributions), we denote by $X_\sharp(\mathbb{T}^d)$ the subspace of $X(\mathbb{T}^d)$ generated by the distributions $f \in X(\mathbb{T}^d)$ for which $\widehat{f}(0) = 0$.

Let Ω be a Lipschitz bounded domain in $\mathbb{R}^d$. Then $F_q^{s,p}(\Omega)$ (with $1 \le p, q < \infty$ and s a real number) is the space consisting of restrictions to Ω of elements from $F_q^{s,p}(\mathbb{R}^d)$, normed with

$$\|f\|_{F_q^{s,p}(\Omega)} := \inf\bigl\{\|g\|_{F_q^{s,p}(\mathbb{R}^d)} \mid g \in F_q^{s,p}(\mathbb{R}^d),\ g = f \text{ on } \Omega\bigr\}.$$

In a similar way, $B_q^{s,p}(\Omega)$ (with $1 \le p, q \le \infty$ and s a real number) is the space consisting of restrictions to Ω of elements from $B_q^{s,p}(\mathbb{R}^d)$, normed with

$$\|f\|_{B_q^{s,p}(\Omega)} := \inf\bigl\{\|g\|_{B_q^{s,p}(\mathbb{R}^d)} \mid g \in B_q^{s,p}(\mathbb{R}^d),\ g = f \text{ on } \Omega\bigr\}.$$

Analogously, we can define the spaces $\dot{F}_q^{s,p}(\Omega)$, $\dot{B}_q^{s,p}(\Omega)$, $\dot{W}^{l,\infty}(\Omega)$ or other similar spaces.

When $s > 0$, we have that $F_q^{s,p} = L^p \cap \dot{F}_q^{s,p}$ and $B_q^{s,p} = L^p \cap \dot{B}_q^{s,p}$. In what follows, we will introduce a tool (possibly well-known) that will enable us to make another connection between the homogeneous spaces and their inhomogeneous counterpart.

Consider the family of open cubes $(Q_k)_{k\in\mathbb{Z}^d}$, where $Q_k := k + (-1,1)^d$ for all $k \in \mathbb{Z}^d$, and let $\chi \in C_c^\infty((-1,1)^d)$ with $\chi \equiv 1$ on $[-3/2, 3/2]^d$. For each $k \in \mathbb{Z}^d$, we define the function χ_k by

$$\chi_k(x) := \frac{\chi(x-k)}{\sum_{\nu\in\mathbb{Z}^d} \chi(x-\nu)},$$

for any $x \in \mathbb{R}^d$. We observe that $\chi_k \in C_c^\infty(Q_k)$, $\chi_k \sim 1$ on $k + [-3/2, 3/2]^d$, and

$$\sum_{k\in\mathbb{Z}^d} \chi_k \equiv 1 \quad \text{on } \mathbb{R}^d.$$

By solving a linear system, one can find for each $m \in \mathbb{N}$ a unique single variable polynomial $p_m(t)$ of degree m such that

$$\int_{-1}^{1} \partial_t^j p_m(t)\, dt = \delta_{jm}, \tag{2.4}$$

for any $j \in \mathbb{N}$, where δ_{jm} is the Kronecker symbol. For any multiindex $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$, we consider the polynomial $p_\alpha(x) := p_{\alpha_1}(x_1) \cdots p_{\alpha_d}(x_d)$. By (2.4), we have

$$\int_{Q_0} \nabla^\beta p_\alpha(x)\, dx = \delta_{\beta\alpha} \quad \text{for any } \beta \in \mathbb{N}^d. \tag{2.5}$$

Fix now some $l \in \mathbb{N}^*$. For any Schwartz function f on $\mathbb{R}^d$ and any $k \in \mathbb{Z}^d$, we define some polynomials p_k^l (depending on f) of degree at most $l-1$ by

$$p_k^l(x) := \sum_{|\alpha| \le l-1} \Big(\int_{Q_k} \nabla^\alpha f(y)\, dy \Big) p_\alpha(x-k).$$

Thanks to (2.5), we have

$$\int_{Q_k} \nabla^\beta (f - p_k^l)(x)\, dx = 0,$$

for any $\beta \in \mathbb{N}^d$ with $|\beta| \le l-1$ and any $k \in \mathbb{Z}^d$. Hence, by Poincaré's inequality (see, for instance, Chapter 4 of [41]), for any integers a and j satisfying $0 \le a \le j \le l$ and any $k \in \mathbb{Z}^d$, we have that

$$\|\nabla^a (f - p_k^l))\|_{L^p(Q_k)} \lesssim \|\nabla^j (f - p_k^l))\|_{L^p(Q_k)}, \tag{2.6}$$

for any $p \in [1, \infty]$.

With this notation, we can now introduce the linear operator

$$L_l f := \sum_{k \in \mathbb{Z}^d} \chi_k (f - p_k^l),$$

for any Schwartz function f on $\mathbb{R}^d$.

Note that, by introducing a second operator $\tilde{L}$,

$$\tilde{L}_l f := \sum_{k \in \mathbb{Z}^d} \chi_k p_k^l,$$

we obtain the decomposition

$$f = L_l f + \tilde{L}_l f.$$

Lemma 2.3. *Let $p \in (1, \infty)$, $s > 0$ be some parameters and let l be the smallest integer with $l \ge s$.*

- (i) *The operator $L_l \colon \dot{W}^{s,p}(\mathbb{R}^d) \to W^{s,p}(\mathbb{R}^d)$ is bounded.*
- (ii) *The operator $\tilde{L}_l \colon \dot{W}^{s,p}(\mathbb{R}^d) \to \dot{W}^{s,p}(\mathbb{R}^d) \cap W^{r,\infty}(\mathbb{R}^d)$ is bounded for any $r \in \mathbb{N}$.*

Proof. First we prove Lemma 2.3 in the case $s \in \mathbb{N}^*$. In this case, we have $l = s$. For any Schwartz function f and any integer a with $0 \le a \le l$, we have

$$\begin{aligned} \|\nabla^a L_l f\|_{L^p(\mathbb{R}^d)}^p &\lesssim \sum_{k \in \mathbb{Z}^d} \|\nabla^a (\chi_k (f - p_k^l))\|_{L^p(Q_k)}^p \\ &\lesssim \sum_{k \in \mathbb{Z}^d} \sum_{j=0}^{a} \| |\nabla^{a-j} \chi_k| |\nabla^j (f - p_k^l)| \|_{L^p(Q_k)}^p \end{aligned}$$

$$\lesssim \sum_{j=0}^{a} \sum_{k\in\mathbb{Z}^d} \|\nabla^j (f - p_k^l)\|^p_{L^p(Q_k)}$$
$$\lesssim \sum_{k\in\mathbb{Z}^d} \|\nabla^l f\|^p_{L^p(Q_k)} \sim \|\nabla^l f\|^p_{L^p(\mathbb{R}^d)},$$

where we have used the Poincaré inequality (see (2.6)). This proves (i) in the case $l = s$. Let us observe that, in the case $0 \le a \le l-1$, we can also write

$$\begin{aligned} \|\nabla^a L_l f\|^p_{L^p(\mathbb{R}^d)} &\lesssim \sum_{j=0}^{a} \sum_{k\in\mathbb{Z}^d} \|\nabla^j (f - p_k^l)\|^p_{L^p(Q_k)} \\ &\lesssim \sum_{k\in\mathbb{Z}^d} \|\nabla^{l-1} (f - p_k^l)\|^p_{L^p(Q_k)} \\ &\lesssim \sum_{k\in\mathbb{Z}^d} \|\nabla^{l-1} f\|^p_{L^p(Q_k)} + \sum_{k\in\mathbb{Z}^d} \|\nabla^{l-1} p_k^l\|^p_{L^p(Q_k)} \\ &\lesssim \|\nabla^{l-1} f\|^p_{L^p(\mathbb{R}^d)} + \sum_{k\in\mathbb{Z}^d} \Big(\int_{Q_k} |\nabla^{l-1} f(x)|\, dx \Big)^p, \end{aligned} \tag{2.7}$$

where, in order to pass to the second "$\lesssim$" we have used (2.6).

By Jensen's inequality, we have

$$\Big(\int_{Q_k} |\nabla^{l-1} f(x)|\, dx \Big)^p \lesssim \int_{Q_k} \|\nabla^{l-1} f(x)\|^p\, dx,$$

and from (2.7), we obtain

$$\|\nabla^a L_l f\|^p_{L^p(\mathbb{R}^d)} \lesssim \|\nabla^{l-1} f\|^p_{L^p(\mathbb{R}^d)}.$$

We have now that L_l is bounded from $\dot{W}^{l,p}$ to $W^{l,p}$ and also from $\dot{W}^{l-1,p}$ to $W^{l-1,p}$. By real interpolation, one obtains that L_l is bounded from $\dot{B}^{s,p}_p$ to $B^{s,p}_p$. In a similar way, using the complex interpolation, we obtain that L_l is bounded from $\dot{F}^{s,p}_2$ to $F^{s,p}_2$. This proves (i).

In order to prove (ii), one observes that $\tilde{L}_l = \mathrm{id} - L_l$. This proves that $\tilde{L}_l$ is bounded from $\dot{W}^{s,p}$ to $\dot{W}^{s,p}$. On the other hand, we observe that $\tilde{L}_l$ is bounded from $\dot{W}^{l,p}$ to $W^{r,\infty}$ for any integer $r \ge 0$. Indeed, let us consider for each integer a with $0 \le a \le l-1$, the parameter $p_a \in (1,\infty)$ defined by the relation $1/p_a = 1/p - (l-a)/d$. Using Jensen's inequality and the Sobolev embedding $\dot{W}^{l,p} \hookrightarrow \dot{W}^{a,p_a}$ (see (2.1)), we can write

$$\begin{aligned} \|\nabla^j p_k^l\|_{L^\infty(Q_k)} &\lesssim \sum_{a=0}^{l-1} \int_{Q_k} |\nabla^a f(x)|\, dx \lesssim \sum_{a=0}^{l-1} \Big(\int_{Q_k} |\nabla^a f(x)|^{p_a}\, dx \Big)^{1/p_a} \\ &\le \sum_{a=0}^{l-1} \Big(\int_{\mathbb{R}^d} |\nabla^a f(x)|^{p_a}\, dx \Big)^{1/p_a} = \sum_{a=0}^{l-1} \|f\|_{\dot{W}^{a,p_a}(\mathbb{R}^d)} \lesssim \|f\|_{\dot{W}^{l,p}(\mathbb{R}^d)}, \end{aligned}$$

for each integer $0 \le j \le l-1$. Since p_k^l is a polynomial of degree at most $l-1$, we have $\nabla^j p_k^l \equiv 0$ for any $j \ge l$. Hence, we can write

$$\|\nabla^j p_k^l\|_{L^\infty(Q_k)} \lesssim \|f\|_{\dot{W}^{l,p}(\mathbb{R}^d)},$$

for each integer $j \geq 0$. Using this we get

$$\begin{aligned}\|\nabla^r \tilde{L}_l f\|_{L^\infty(\mathbb{R}^d)} &\lesssim \sup_{k\in\mathbb{Z}^d} \sum_{j=0}^{r} \| |\nabla^{r-j}\chi_k| |\nabla^j p_k^l| \|_{L^\infty(Q_k)} \\ &\lesssim \sup_{k\in\mathbb{Z}^d} \sum_{j=0}^{r} \|\nabla^j p_k^l\|_{L^\infty(Q_k)} \lesssim \|f\|_{\dot{W}^{l,p}(\mathbb{R}^d)}.\end{aligned}$$

As in the case of (ii), we can now prove by interpolation that $\tilde{L}_l$ is bounded from $\dot{W}^{s,p}$ to $W^{r,\infty}$. This proves Lemma 2.3. ■

2.2. Some auxiliary interpolation facts

Note that the trace of $L^\infty(\mathbb{R}^d)$ on $\mathbb{R}^{d-1}$ is not well defined. Since we are going to use trace theory, it is convenient to replace the space L^∞ with C (and $W^{r,\infty}$ with C^r). The following easy lemma ensures us that, when interpolating the couple $(F_t^{s,p}, W^{r,\infty})$, changing $W^{r,\infty}$ with C^r does not affect the result of the interpolation.

Lemma 2.4. *Let r be a nonnegative integer and consider the parameters $p, t \in (1,\infty)$, $s > 0$. Then, for any fixed $\theta \in (0,1)$,*

$$(F_t^{s,p}(\mathbb{R}^d), W^{r,\infty}(\mathbb{R}^d))_\theta = (F_t^{s,p}(\mathbb{R}^d), C_0^r(\mathbb{R}^d))_\theta, \tag{2.8}$$

with equivalence of norms. Also, for any $q \in [1,\infty]$,

$$(F_t^{s,p}(\mathbb{R}^d), W^{r,\infty}(\mathbb{R}^d))_{\theta,q} = (F_t^{s,p}(\mathbb{R}^d), C_0^r(\mathbb{R}^d))_{\theta,q}, \tag{2.9}$$

with equivalence of norms.

The same fact holds for the homogeneous spaces or for the spaces defined on $\mathbb{T}^d$.

Remark 2.5. Since $C_0^r \hookrightarrow C^r$, in the case of the inhomogeneous spaces we have, by (2.8) and (2.9), that

$$(F_t^{s,p}(\mathbb{R}^d), W^{r,\infty}(\mathbb{R}^d))_\theta = (F_t^{s,p}(\mathbb{R}^d), C^r(\mathbb{R}^d))_\theta \tag{2.10}$$

and

$$(F_t^{s,p}(\mathbb{R}^d), W^{r,\infty}(\mathbb{R}^d))_{\theta,q} = (F_t^{s,p}(\mathbb{R}^d), C^r(\mathbb{R}^d))_{\theta,q},$$

respectively.

Proof. Since $(C_0)^* = \mathcal{M}$, where $\mathcal{M}$ is the space of the Radon measures on $\mathbb{R}^d$, we get that $(C_0^r)^* = \mathcal{M}^{-r}$, where $\mathcal{M}^{-r}$ is the space of all distributions f of the form

$$f = \sum_{|\alpha|\leq r} \nabla^\alpha \mu_\alpha,$$

with each μ_α from $\mathcal{M}$ (see, for instance, Section 4.3 in [41]). The norm on $\mathcal{M}^{-r}$ is given by

$$\|f\|_{\mathcal{M}^{-r}} := \inf\Big\{ \sum_{|\alpha|\leq r} \|\mu_\alpha\|_{\mathcal{M}} \mid f = \sum_{|\alpha|\leq r} \nabla^\alpha \mu_\alpha \Big\}.$$

We have now that (see Corollary 4.5.2 on p. 98 of [5])

$$(F_t^{s,p}, C_0^r)_\theta^* = ((F_t^{s,p})^*, (C_0^r)^*)_\theta = (F_{t'}^{-s,p'}, \mathcal{M}^{-r})_\theta. \tag{2.11}$$

Consider some element $g \in (F_{t'}^{-s,p'}, \mathcal{M}^{-r})_\theta$ and some $f \in \mathcal{F}(F_{t'}^{-s,p'}, \mathcal{M}^{-r})$ such that $a = f(\theta)$ and

$$\|f\|_{\mathcal{F}(F_{t'}^{-s,p'}, \mathcal{M}^{-r})} \le 2\|g\|_{(F_{t'}^{-s,p'}, \mathcal{M}^{-r})_\theta}. \tag{2.12}$$

Consider some $\varphi \in C_c^\infty(\mathbb{R}^d)$ of integral 1, and for any $\varepsilon > 0$, denote by φ_ε the function $\varphi_\varepsilon(x) = \varepsilon^{-d}\varphi(\varepsilon^{-1}x)$. Define the function f_ε by

$$f_\varepsilon(z, x) := f(z) * \varphi_\varepsilon(x),$$

where the convolution is in the variable x. One can see readily that $f_\varepsilon \in \mathcal{F}(F_{t'}^{-s,p'}, W^{-r,1})$ and

$$\|f_\varepsilon\|_{\mathcal{F}(F_{t'}^{-s,p'}, W^{-r,1})} \le \|f\|_{\mathcal{F}(F_{t'}^{-s,p'}, \mathcal{M}^{-r})},$$

for any $\varepsilon > 0$. This, together with (2.12), shows that

$$\|g_\varepsilon\|_{(F_{t'}^{-s,p'}, W^{-r,1})_\theta} \le 2\|g\|_{(F_{t'}^{-s,p'}, \mathcal{M}^{-r})_\theta}, \tag{2.13}$$

for any $\varepsilon > 0$, where $g_\varepsilon := g * \varphi_\varepsilon$. Note that, since $F_{t'}^{-s,p'}$ is reflexive, by Calderón's reflexivity theorem (see Paragraph 12.2 on p. 121 of [9]), the space $(F_{t'}^{-s,p'}, W^{-r,1})_\theta$ is reflexive. One can easily see that $(F_{t'}^{-s,p'}, W^{-r,1})_\theta$ is separable (it suffices to see that $\mathcal{S}_\sharp$ is dense in $F_{t'}^{-s,p'} \cap W^{-r,1}$). Consequently, since $(F_{t'}^{-s,p'}, W^{-r,1})_\theta^*$ is reflexive, and its dual is $(F_{t'}^{-s,p'}, W^{-r,1})_\theta$, we get that $(F_{t'}^{-s,p'}, W^{-r,1})_\theta^*$ is separable.[5] The space $(F_{t'}^{-s,p'}, W^{-r,1})_\theta^*$ is also a predual of $(F_{t'}^{-s,p'}, W^{-r,1})_\theta$. By applying the sequential Banach–Alaoglu theorem, one can find some $\tilde{g} \in (F_{t'}^{-s,p'}, W^{-r,1})_\theta$ such that $g_{1/n} \to \tilde{g}$ when $n \to \infty$ in the sense of distributions, up to a subsequence. Since $g_\varepsilon \to g$ in the sense of distributions when $\varepsilon \to 0$ and $\tilde{g}, g \in F_{t'}^{-s,p'} + W^{-r,1}$, we get $g = \tilde{g} \in (F_{t'}^{-s,p'}, W^{-r,1})_\theta$. Also, by (2.13), one gets

$$\|g\|_{(F_{t'}^{-s,p'}, W^{-r,1})_\theta} \le \liminf_{\varepsilon\to 0} \|g_\varepsilon\|_{(F_{t'}^{-s,p'}, W^{-r,1})_\theta} \le 2\|g\|_{(F_{t'}^{-s,p'}, \mathcal{M}^{-r})_\theta}.$$

With this we have

$$(F_{t'}^{-s,p'}, \mathcal{M}^{-r})_\theta \hookrightarrow (F_{t'}^{-s,p'}, W^{-r,1})_\theta.$$

Since we also have the trivial embedding

$$(F_{t'}^{-s,p'}, \mathcal{M}^{-r})_\theta \hookleftarrow (F_{t'}^{-s,p'}, W^{-r,1})_\theta,$$

we get

$$(F_{t'}^{-s,p'}, \mathcal{M}^{-r})_\theta = (F_{t'}^{-s,p'}, W^{-r,1})_\theta.$$

[5] We use here the fact that if the dual X^* of a Banach space X is separable, then X is separable (see, for instance, Theorem 4.6-8 on p. 245 of [23]).

By this and (2.11), we have

$$(F_t^{s,p}, C_0^r)_\theta^* = (F_{t'}^{-s,p'}, W^{-r,1})_\theta.$$

Calderón's reflexivity theorem and the duality theorem (see Corollary 4.5.2 on p. 98 of [5]) give now that

$$(F_t^{s,p}, C_0^r)_\theta = (F_t^{s,p}, C_0^r)_\theta^{**} = (F_{t'}^{-s,p'}, W^{-r,1})_\theta^* = (F_t^{s,p}, W^{r,\infty})_\theta,$$

which proves (2.8).

One can prove equality (2.9) directly, or one can deduce it from (2.8) (see Theorem 4.7.2 in [5]):

$$\begin{aligned}(F_t^{s,p}, W^{r,\infty})_{\theta,q} &= ((F_t^{s,p}, W^{r,\infty})_{1/2}, (F_t^{s,p}, W^{r,\infty})_{3/2})_{\eta,q}\\ &= ((F_t^{s,p}, C^r)_{1/2}, (F_t^{s,p}, C^r)_{3/2})_{\eta,q}\\ &= (F_t^{s,p}, C^r)_{\theta,q},\end{aligned}$$

where $\eta \in (0,1)$ is such that $\theta = (1-\eta)/2 + 3\eta/2$.

On the same lines we can prove the corresponding equalities in the case of the homogeneous spaces and the spaces on $\mathbb{T}^d$. ■

We will often need the following result concerning the real interpolation of Triebel–Lizorkin spaces.

Lemma 2.6. *Consider some parameters $p_0, p_1 \in [1,\infty)$, $s_0, s_1 \in \mathbb{R}$ and $t_0, t_1 \in [1,\infty]$ such that $p_0 \neq p_1$ and $s_0 \neq s_1$. Then, for any $\theta \in (0,1)$,*

$$(\dot F_{t_0}^{s_0,p_0}, \dot F_{t_1}^{s_1,p_1})_{\theta,p} = \dot F_p^{s,p},$$

where $s = (1-\theta)s_0 + \theta s_1$ and $1/p = (1-\theta)/p_0 + \theta/p_1$. Supposing $p_0 \in (1,\infty)$ and $s_0 \neq 0$, we also have

$$(\dot F_{t_0}^{s_0,p_0}, \dot F_2^{0,\infty})_{\theta,p} = \dot F_p^{(1-\theta)s_0,p}.$$

The same result holds for the inhomogeneous version of the Triebel–Lizorkin spaces.

Lemma 2.6 follows from standard facts in interpolation theory. Since it is hard to localize it in the literature, we give a proof below (however, see Theorem 5 in Chapter 5 of [32] for the case $t_0 = t_1 = 2$, and Theorem 6 in Chapter 5 of [32] for the case $t_0 = p_0$, $t_1 = p_1$).

Proof. Consider the retraction operator

$$P : L^{p_0}(\dot\ell_{t_0}^{s_0}) + L^{p_1}(\dot\ell_{t_1}^{s_1}) \to \dot F_{t_0}^{s_0,p_0} + \dot F_{t_1}^{s_1,p_1}$$

and the extension operator

$$E : \dot F_{t_0}^{s_0,p_0} + \dot F_{t_1}^{s_1,p_1} \to L^{p_0}(\dot\ell_{t_0}^{s_0}) + L^{p_1}(\dot\ell_{t_1}^{s_1}),$$

defined formally as

$$P(f_k)_{k\in\mathbb{Z}} := \sum_{k\in\mathbb{Z}} \tilde P_k f_k \quad \text{and} \quad Ef := (P_k f)_{k\in\mathbb{Z}},$$

where P_k and $\tilde{P}_k$ are Littlewood–Paley "projections" such that $\tilde{P}_k P_k = P_k$. We can see that $P \circ E = \mathrm{id}$ on $\dot{F}^{s_0,p_0}_{t_0} + \dot{F}^{s_1,p_1}_{t_1}$.

By using the retraction method (see Theorem 6.4.2 in [5]) for P and E, we see that it suffices to prove that

$$(L^{p_0}(\dot{\ell}^{s_0}_{t_0}), L^{p_1}(\dot{\ell}^{s_1}_{t_1}))_{\theta,p} = L^p(\dot{\ell}^s_p). \tag{2.14}$$

Indeed, by applying Theorem 5.7 (ii) on p. 129 of [34], we have that

$$(L^{p_0}(\dot{\ell}^{s_0}_{t_0}), L^{p_1}(\dot{\ell}^{s_1}_{t_1}))_{\theta,p} = L^p((\dot{\ell}^{s_0}_{t_0}, \dot{\ell}^{s_1}_{t_1})_{\theta,p}), \tag{2.15}$$

and now, applying Theorem 5.6.1 on p. 122 of [5],

$$(\dot{\ell}^{s_0}_{t_0}, \dot{\ell}^{s_1}_{t_1})_{\theta,p} = \dot{\ell}^s_p,$$

which, together with (2.15), gives (2.14).

The second assertion follows from the first assertion by duality. Indeed, if $t_0 > 1$, we have

$$(\dot{F}^{s_0,p_0}_{t_0}, \dot{F}^{0,\infty}_2)_{\theta,p} = (\dot{F}^{-s_0,p_0'}_{t_0'}, \dot{F}^{0,1}_2)^*_{\theta,p'} = (\dot{F}^{-(1-\theta)s_0,p'}_{p'})^* = \dot{F}^{(1-\theta)s_0,p}_p.$$

If $t_0 = 1$, we have

$$(\dot{F}^{-s_0,p_0'}_{(c_0)})^* = \dot{F}^{s_0,p_0}_1,$$

where $\dot{F}^{-s_0,p_0'}_{(c_0)}$ is defined by replacing the ℓ_∞ space in the definition of $\dot{F}^{-s_0,p_0'}_\infty$ by the c_0 space. We now interpolate the spaces $\dot{F}^{-s_0,p_0'}_{(c_0)}$, $\dot{F}^{0,1}_2$ as above by using the method of retraction. We get

$$(\dot{F}^{s_0,p_0}_1, \dot{F}^{0,\infty}_2)_{\theta,p} = (\dot{F}^{-s_0,p_0'}_{(c_0)}, \dot{F}^{0,1}_2)^*_{\theta,p'} = (\dot{F}^{-(1-\theta)s_0,p'}_{p'})^* = \dot{F}^{(1-\theta)s_0,p}_p,$$

which proves the second identity in Lemma 2.6. ∎

Another useful tool will be the following celebrated theorem of Wolff proved in Theorem 1 of [40]. Below we give the version that appears in Theorem 2.11, p. 317, of [4].

Theorem 2.7. *Suppose (X_1, X_4) is a compatible couple of quasi-normed spaces. Consider some parameters $\theta_2, \theta_3 \in (0,1)$, $q_2, q_3 \in [1,\infty]$, and let X_2, X_3 be some quasi-normed spaces such that*

$$X_2 = (X_1, X_3)_{\theta_2,q_2}, \quad X_3 = (X_2, X_4)_{\theta_3,q_3}.$$

Then, with equivalence of the quasi-norms, we have

$$X_2 = (X_1, X_4)_{\phi_2,q_2}, \quad X_3 = (X_1, X_4)_{\phi_3,q_3},$$

where

$$\phi_2 := \frac{\theta_2\theta_3}{1-\theta_2+\theta_2\theta_3}, \quad \phi_3 := \frac{\theta_3}{1-\theta_2+\theta_2\theta_3}.$$

3. The real method

3.1. Wavelets and the spaces $\dot{W}^{l,1}(\mathbb{R}^d)$

In this section we provide "almost" characterizations via wavelets for $\dot{W}^{l,1}(\mathbb{R}^d)$, when $l \in \mathbb{Z}$, similar to the ones obtained in [11] in the case of $\dot{\mathrm{BV}}(\mathbb{R}^d)$.

3.1.1. Some notation related to the wavelet system. Concerning the wavelet system,[6] we work essentially in the same setting as the authors of [11], Section 1 (see also Chapter 3 of [28]). Let us recall here some notation used in Section 1 of [11]. Let $N \in \mathbb{N}^*$ and let $\varphi \in C_c^N(\mathbb{R})$ be a scaling function associated with the orthogonal wavelet $\psi \in C_c^\infty(\mathbb{R})$. Consider also the set $E := \{0,1\}^d \setminus \{(0,\dots,0)\}$ and for each $e = (e_1,\dots,e_d) \in E$, define the function $\psi^e \in C_c^N(\mathbb{R}^d)$ by

$$\psi^e(x) := \psi^{e_1}(x_1)\cdots\psi^{e_d}(x_d),$$

for any $x \in \mathbb{R}^d$, where $\psi^0 := \varphi$ and $\psi^1 := \psi$. We assume that

$$\int_{\mathbb{R}^d} x^\alpha \psi^e(x)\,dx = 0,$$

for any multiindex $\alpha \in \mathbb{N}^d$ with $|\alpha| \le N$.

Let D be the set of all dyadic cubes in $\mathbb{R}^d$. We can now define the BV-normalized wavelets as follows. For each $e \in E$ and each dyadic cube $I = 2^{-j}((k + [0,1]^d)$ (where $j \in \mathbb{Z}$ and $k \in \mathbb{Z}^d$), we define the function

$$\psi_I^e(x) := 2^{j(d-1)}\psi^e(2^j x - k),$$

for any $x \in \mathbb{R}^d$. We will say that ψ_I^e are ("mother") wavelets of class N.

Remark 3.1. Even when we do not mention explicitly, we will always consider wavelets of class N sufficiently large. For instance, when we are describing spaces like $W^{l,p}$ via wavelets, we will consider that the wavelets involved in the description are of class $N > |l|$.

The family of wavelets $(\psi_I^e)_{I\in D, e\in E}$ is a complete orthogonal system in $L^2(\mathbb{R}^d)$. We define also a dual system $(\tilde{\psi}_I^e)_{I\in D, e\in E}$; each $\tilde{\psi}_I^e$ differs to ψ_I^e by a scaling factor, i.e., keeping the same notation as above,

$$\tilde{\psi}_I^e(x) := 2^j\psi^e(2^j x - k),$$

for any $x \in \mathbb{R}^d$. We have that

$$\langle \psi_I^e, \tilde{\psi}_{I'}^{e'}\rangle = \delta_{e,e'}\,\delta_{I,I'},$$

for any $e, e' \in E$ and any $I, I' \in D$, where δ is the Kronecker symbol.

As in Section 1 of [11], in order to simplify the notation, we use the vector valued functions

$$\psi_I := (\psi_I^e)_{e\in E} \quad \text{and} \quad \tilde{\psi}_I := (\tilde{\psi}_I^e)_{e\in E}.$$

[6] We will mainly consider here only the "homogeneous" wavelet systems. In other words, for simplicity, we give more attention to the "mother" wavelets.

The wavelets coefficients of a distribution f on $\mathbb{R}^d$ are defined by

$$f_I^e := \langle f, \tilde{\psi}_I^e \rangle$$

for each $e \in E$ and each dyadic cube I. In a more contracted way, we would write

$$f_I := \langle f, \tilde{\psi}_I \rangle = (f_I^e)_{e \in E}.$$

With this notation the corresponding wavelet decomposition of a function $f \in L^2(\mathbb{R}^d)$ can be expressed as

$$f = \sum_{I \in D} f_I \, \psi_I = \sum_{e \in E} \sum_{I \in D} f_I^e \, \psi_I^e,$$

where the convergence is in the sense of distributions.

One can also use "father" wavelets (see the introduction in [11]) in order to write the decomposition

$$f = \sum_{I \in D_+} f_I^+ \psi_I = \sum_{e \in E \cup \{0\}} \sum_{I \in D_+} f_I^{+,e} \, \psi_I^e, \tag{3.1}$$

where D_+ is the set of the dyadic cubes of side length at most 1 and $\psi^0 := \varphi \otimes \cdots \otimes \varphi$, $f_I^+ = f_I$ whenever $I \in D_+$.

In what follows we restrict to the use of the "mother" wavelets, since in this case the proofs are cleaner.

3.1.2. Description of $\dot{W}^{l,1}(\mathbb{R}^d)$ via wavelets. In Definition 1.2 of [11], the authors introduced the spaces $\ell_p^\gamma(D)$ and $w\ell_1^\gamma(D)$ by defining the norm

$$\|(c_I)_{I \in D}\|_{\ell_p^\gamma} := \Big(\sum_{I \in D} |I|^{(1-p)\gamma} |c_I|^p \Big)^{1/p},$$

and the quasi-norm

$$\|(c_I)_{I \in D}\|_{w\ell_1^\gamma} := \sup_{\lambda > 0} \lambda \sum_{|c_I| > \lambda |I|^\gamma} |I|^\gamma.$$

The spaces $\ell_p^\gamma(D)$ and $w\ell_1^\gamma(D)$ consist of those sequences $(c_I)_{I \in D}$ that have finite ℓ_p^γ-norm or $w\ell_1^\gamma$-quasi-norm, respectively.

Note that, in the case of $w\ell_1^\gamma(D)$, we have the quasi-triangle inequality

$$\|(c_I^1 + c_I^2)_{I \in D}\|_{w\ell_1^\gamma} \le 2 \|(c_I^1)_{I \in D}\|_{w\ell_1^\gamma} + 2 \|(c_I^2)_{I \in D}\|_{w\ell_1^\gamma},$$

for any two sequences $(c_I^1)_{I \in D}$ and $(c_I^2)_{I \in D}$.

Adapted to the case of $\dot{W}^{1,1}(\mathbb{R}^d)$ rather than to $\dot{\mathrm{BV}}(\mathbb{R}^d)$, the main result in [11] (see Theorem 1.3 in [11]) reads as follows.

Theorem 3.2. *Suppose $\gamma \in \mathbb{R} \setminus [1 - 1/d, 1]$. If $f \in \dot{W}^{1,1}(\mathbb{R}^d)$, then the sequence $(f_I)_{I \in D}$ belongs to $w\ell_1^\gamma$, and*

$$\|(f_I)_{I \in D}\|_{w\ell_1^\gamma} \lesssim \|f\|_{\dot{W}^{1,1}}.$$

Remark 3.3. The proof of Theorem 3.2 (with $\dot{\mathrm{BV}}$ in place $\dot{W}^{1,1}$) given in [11] splits in two cases. First, the authors of [11] show that the result holds in the case where $\gamma \in \mathbb{R} \setminus [0,1]$ (see Section 3 in [11]). Then they prove the result in the case where $\gamma \in [0, 1-1/d)$, which requires a more subtle analysis (see Section 4 in [11]).

By slightly adapting the arguments in Section 3 of [11], Cohen proved the following analogue of Theorem 3.2 for the space L^1 (see Théorème 2.1 in [10]). In the case where the wavelets are BV-normalized, this result reads as follows.

Theorem 3.4. *Suppose $\gamma \in \mathbb{R} \setminus [0,1]$. If $f \in L^1(\mathbb{R}^d)$, then the sequence $(|I|^{1/d} f_I)_{I\in D}$ belongs to $w\ell_1^\gamma$, and*

$$\|(|I|^{1/d} f_I)_{I\in D}\|_{w\ell_1^\gamma} \lesssim \|f\|_{L^1}.$$

Remark 3.5. By constructing counterexamples, the authors of [11] have shown that the range of the parameter γ in Theorem 3.2 cannot be improved (see Section 6 in [11]). As mentioned in Remarque 2.2 of [10], the range of γ in Theorem 3.4 is also optimal.

Fix a nonnegative integer $l \in \mathbb{N}$ and consider a function $g \in C_c((0,1)^d)$ such that all its moments of order $l-1$ vanish, i.e.,

$$\int_{\mathbb{R}^d} x^\alpha g(x)\,dx = 0, \tag{3.2}$$

for all $\alpha \in \mathbb{N}^d$ with $|\alpha| \le l-1$. (By convention, when $l = 0$, we impose no vanishing condition on g.) For any dyadic cube $I = 2^{-j}((0,1)^d + k) \in D$, where $j \in \mathbb{Z}$ and $k \in \mathbb{Z}^d$, we define the function g_I by

$$g_I(x) = 2^j g(2^j x - k) \quad \text{for all } x \in \mathbb{R}^d.$$

For any function $f \in L^1(\mathbb{R}^d)$ and any dyadic cube $I \in D$, we introduce the quantity

$$c_I(f) := |\langle f, g_I\rangle|.$$

In [11], the analogue of Theorem 3.2 was derived from the following more general result (see Theorem 2.5 in [11]):

Lemma 3.6. *Suppose $\gamma \in \mathbb{R} \setminus [1-1/d, 1]$ and let $g \in C_c((0,1)^d)$ be a function with zero integral. If $f \in \dot{W}^{1,1}(\mathbb{R}^d)$, then the sequence $(c_I(f))_{I\in D}$ belongs to $w\ell_1^\gamma$, and*

$$\|(c_I(f))_{I\in D}\|_{w\ell_1^\gamma} \lesssim \|f\|_{\dot{W}^{1,1}}. \tag{3.3}$$

We need a version of this result adapted to the case of the Sobolev spaces $\dot{W}^{l,1}(\mathbb{R}^d)$ when l is any positive integer. The following version can be easily deduced from the work in [11].

Lemma 3.7. *Suppose $l \ge 1$ and $\gamma \in \mathbb{R} \setminus [1-1/d, 1]$. Consider some $r \in \mathbb{N}$ and let $g \in C_c^1((0,1)^d)$ be a function such that all its moments of order $l-1$ are vanishing. If $f \in \dot{W}^{l,1}(\mathbb{R}^d)$, then the sequence $(|I|^{(1-l)/d} c_I(f))_{I\in D}$ belongs to $w\ell_1^\gamma$, and*

$$\|(|I|^{(1-l)/d} c_I(f))_{I\in D}\|_{w\ell_1^\gamma} \lesssim \|f\|_{\dot{W}^{l,1}}. \tag{3.4}$$

In order to prove Lemma 3.7, we need the following (well-known) fact.

Lemma 3.8. *Let $r \geq 1$ be an integer. Consider some function $g \in C_c^1((0,1)^d)$ such that all its moments of order r are vanishing. Then there exists a family of functions $G = (G_\alpha)_{|\alpha|=r}$ such that for each α, the function $G_\alpha \in C_c^r((0,1)^d)$ is of integral zero and*

$$g = \nabla^r \cdot G = \sum_{|\alpha|=r} \nabla^\alpha G_\alpha.$$

This lemma is a direct consequence of standard techniques in elliptic theory. For instance, we can observe that g belongs to the Hölder space $C^{1/2} = B_\infty^{1/2,\infty}$, and then we can apply repeatedly the Bogovskiĭ formula for the divergence operator (see Remark 4.12 in [13]).

Proof of Lemma 3.7. One can deduce Lemma 3.7 directly from Lemma 3.6. Since the case $l = 1$ is covered by Lemma 3.6, it remains to prove the statement in the case $l \geq 2$.

By Lemma 3.8, there exists a family of functions $G = (G_\alpha)_{|\alpha|=l-1}$ such that for each α, the function $G_\alpha \in C_c^{l-1}((0,2)^d)$ is of integral zero, and

$$g = \nabla^{l-1} \cdot G = \sum_{|\alpha|=l-1} \nabla^\alpha G_\alpha.$$

From this we immediately get

$$|I|^{(1-l)/d} g_I = \sum_{|\alpha|=l-1} \nabla^\alpha (G_\alpha)_I. \tag{3.5}$$

Using (3.3) for the functions $\nabla^\alpha f \in \dot{W}^{1,1}(\mathbb{R}^d)$, we have

$$\|(\langle f, \nabla^\alpha (G_\alpha)_I\rangle)_{I\in D}\|_{w\ell_1^\gamma} = \|(\langle \nabla^\alpha f, (G_\alpha)_I\rangle)_{I\in D}\|_{w\ell_1^\gamma} \lesssim \|\nabla\nabla^\alpha f\|_{L^1} \leq \|\nabla^l f\|_{L^1},$$

for every multiindex α with $|\alpha| = l - 1$. Hence, by adding up and using (3.5) together with the quasi-norm property of $w\ell_1^\gamma$, we get

$$\|(\langle f, g_I\rangle)_{I\in D}\|_{w\ell_1^\gamma} \lesssim \sum_{|\alpha|=r-1} \|(\langle f, \nabla^\alpha (G_\alpha)_I\rangle)_{I\in D}\|_{w\ell_1^\gamma} \lesssim \|\nabla^l f\|_{L^1},$$

which proves Lemma 3.7, when $l \geq 1$. ■

Theorem 3.9. *Fix some $l \in \mathbb{Z}$ and let $(\psi_I)_{I\in D}$ be a wavelet system on $\mathbb{R}^d$ of class at least $|l| + 1$. If $l \geq 1$ and $\gamma \in \mathbb{R} \setminus [1 - 1/d, 1]$, or $l \leq 0$ and $\gamma \in \mathbb{R} \setminus [0, 1]$, we have*

$$\|(|I|^{(1-l)/d} f_I)_{I\in D}\|_{w\ell_1^\gamma} \lesssim \|f\|_{\dot{W}^{l,1}} \lesssim \|(|I|^{(1-l)/d} f_I)_{I\in D}\|_{\ell_1}, \tag{3.6}$$

for all $f \in \dot{W}^{l,1}(\mathbb{R}^d)$.

Proof. We treat first the case where $l \geq 0$. The case $l = 0$ easily follows from Theorem 3.4. Suppose $l > 0$. Consider a positive integer p and some function $\zeta \in C_c^2((0, 2^p)^d)$ such that all its moments of order at most $l - 1$ are vanishing. Then there exist $M := 2^{pd+1}$

open cubes $Q_1, \ldots, Q_M$, each of them being a translation of the unit cube $(0,1)^d$, and some functions $\chi_1, \ldots, \chi_M \in C_c^\infty(\mathbb{R}^d)$ such that $(0,2^p)^d \subset Q_1 \cup \cdots \cup Q_M$, each χ_ν is supported in Q_ν, and

$$\sum_{\nu=1}^{M} \chi_\nu = 1 \quad \text{on } [0,2^p]^d.$$

Since all the moments of order at most $l-1$ of ζ are vanishing, by Lemma 3.8, we can find a family $\Psi = (\Psi_\alpha)_{|\alpha|=l}$, with $\Psi_\alpha \in C_c^{l-1}((0,2^p)^d)$ such that

$$\zeta = \nabla^r \cdot \Psi = \sum_{|\alpha|=l} \nabla^\alpha \Psi_\alpha.$$

(When $l=0$, we take by convention $\Psi = \zeta$, and the above formula becomes $\zeta = \nabla^0 \cdot \Psi$, where ∇^0 is by convention the identity operator.)

We decompose ζ as

$$\zeta = \nabla^l \cdot \Big(\sum_{\nu=1}^{M} \Psi \chi_\nu \Big) = \sum_{\nu=1}^{M} \nabla^l \cdot (\Psi \chi_\nu) = \sum_{\nu=1}^{M} g^\nu, \tag{3.7}$$

where $g^\nu := \nabla^l \cdot (\Psi\chi_\nu) \in L_c^\infty(Q_\nu)$, for all $\nu \in \{1,\ldots,M\}$ (here, $\Psi\chi_\nu = (\Psi_\alpha \chi_\nu)_{|\alpha|=r}$). Note that each g^ν satisfy the vanishing moments condition (3.2). Since the inequality (3.4) is translation invariant, by applying Lemma 3.7, we get

$$\|(|I|^{(1-l)/d} c_I^\nu(f))_{I\in D}\|_{w\ell_1^\gamma} \le \|g^\nu\|_{L^\infty} \|\nabla^l f\|_{L^1} \lesssim \|\nabla^l f\|_{L^1}, \tag{3.8}$$

for any $\nu \in \{1,\ldots,N\}$, where $c_I^\nu(f) := |\langle f, g_I^\nu \rangle|$.

Let ζ_I be the functions given by

$$\zeta_I(x) := 2^j \zeta(2^j x - k),$$

for each dyadic cube $I = 2^{-j}((0,1)^d + k)$. For $\widetilde{c}_I(f) := \langle f, \zeta_I \rangle$, thanks to (3.7) and the triangle inequality,

$$|\widetilde{c}_I(f)| = |\langle f, \zeta_I \rangle| \le \sum_{\nu=1}^{M} |\langle f, g_I^\nu \rangle| = \sum_{\nu=1}^{M} c_I^\nu(f). \tag{3.9}$$

Using (3.9), the quasi-triangle inequality for the $w\ell_1^\gamma$-quasi-norm and formula (3.8), we get

$$\|(|I|^{(1-l)/d} \widetilde{c}_I(f))_{I\in D}\|_{w\ell_1^\gamma} \lesssim \sum_{\nu=1}^{M} \|(|I|^{(1-l)/d} c_I^\nu(f))_{I\in D}\|_{w\ell_1^\gamma} \lesssim \|\nabla^l f\|_{L^1}. \tag{3.10}$$

Let $(\widetilde{\psi}^e)_{e\in E}$ be the generators of the dual wavelets. Applying (3.10) for $\zeta = \widetilde{\psi}^e$, for each $e \in E$, and using the quasi-triangle inequality for the $w\ell_1^\gamma$-quasi-norm, we obtain

$$\|(|I|^{(1-l)/d} f_I)_{I\in D}\|_{w\ell_1^\gamma} \lesssim \sum_{e\in E} \|(|I|^{(1-l)/d} f_I^e)_{I\in D}\|_{w\ell_1^\gamma} \lesssim \|\nabla^l f\|_{L^1},$$

which proves the first estimate in (3.6) in the case $l \ge 0$.

The second estimate in (3.6) follows immediately from the triangle inequality (by a limiting argument[7])

$$\|f\|_{\dot{W}^{l,1}} \le \sum_{I\in D}\sum_{e\in E} |f_I^e|\,\|\widetilde{\psi}_I^e\|_{\dot{W}^{l,1}},$$

and the fact that $\|\widetilde{\psi}_I^e\|_{\dot{W}^{l,1}} \sim |I|^{(1-l)/d}$, for all $I \in D$.

Now, we treat the case $l = -r < 0$. Let $f \in \dot{W}^{-r,1}(\mathbb{R}^d) \cap C_c^\infty(\mathbb{R}^d)$ and consider a family $F = (F_\alpha)_{|\alpha|=r} \in L^1(\mathbb{R}^d)$ such that

$$f = \nabla^a \cdot F = \sum_{|\alpha|=r-1} \nabla^\alpha F_\alpha, \tag{3.11}$$

in the sense of distributions, and

$$\sum_{|\alpha|=r} \|F_\alpha\|_{L^1} \le 2\|f\|_{\dot{W}^{-r,1}}.$$

By (3.11), we have

$$f_I^e = \langle f, \widetilde{\psi}_I^e\rangle = (-1)^r \sum_{|\alpha|=r} \langle F_\alpha, \nabla^\alpha \widetilde{\psi}_I^e\rangle = (-1)^r |I|^{-r/d} \sum_{|\alpha|=r} \langle F_\alpha, (\nabla^\alpha \widetilde{\psi}^e)_I\rangle,$$

and we get

$$|I|^{(1+r)/d}|f_I^e| \le \sum_{|\alpha|=r} |I|^{1/d} |\langle F_\alpha, (\nabla^\alpha \widetilde{\psi}^e)_I\rangle|. \tag{3.12}$$

Using (3.10) for the function $\zeta = \nabla^\alpha \widetilde{\psi}^e$ (in the case $l = 0$),

$$\|(|I|^{1/d}\langle F_\alpha, (\nabla^\alpha \widetilde{\psi}^e)_I\rangle)_{I\in D}\|_{w\ell_1^\gamma} \lesssim \|F_\alpha\|_{L^1},$$

for each $\alpha \in \mathbb{N}^d$ with $|\alpha| = r$. This, together with (3.12), the quasi-triangle inequality and (3.11), implies that

$$\begin{aligned}\|(|I|^{(1+r)/d} f_I)_{I\in D}\|_{w\ell_1^\gamma} &\lesssim \sum_{|\alpha|=r}\sum_{e\in E} \|(|I|^{1/d}\langle F_\alpha, (\nabla^\alpha \widetilde{\psi}^e)_I\rangle)_{I\in D}\|_{w\ell_1^\gamma}\\ &\lesssim \sum_{|\alpha|=r} \|F_\alpha\|_{L^1} \lesssim \|f\|_{\dot{W}^{-r,1}},\end{aligned}$$

which proves the first estimate in (3.6) in the case $l = -r < 0$.

As in the case $l \ge 0$, the second estimate in (3.6) follows immediately from the triangle inequality. Indeed, by a limiting argument, we have

$$\|f\|_{\dot{W}^{-r,1}} \le \sum_{I\in D}\sum_{e\in E} \|f_I^e\|\,\|\widetilde{\psi}_I^e\|_{\dot{W}^{-r,1}},$$

and it remains to see that $\|\widetilde{\psi}_I^e\|_{\dot{W}^{-a,1}} \lesssim |I|^{(1+r)/d}$, for all $I \in D$.

[7] We first consider functions f with finite wavelet expansion and then we pass to limit.

In order to estimate $\|\widetilde{\psi}_I^e\|_{\dot{W}^{l,1}}$, we note that, since all the moments of order at most $r-1$ of $\widetilde{\psi}^e$ are vanishing, one can use Lemma 3.8 to write

$$\widetilde{\psi}^e = \nabla^r \cdot \Psi = \sum_{|\alpha|=r} \nabla^\alpha \Psi_\alpha,$$

for some family $\Psi = (\Psi_\alpha)_{|\alpha|=r} \in C_c^r$. It follows that

$$\widetilde{\psi}_I^e = |I|^{r/d} \sum_{|\alpha|=r} \nabla^\alpha (\Psi_\alpha)_I$$

and

$$\|\widetilde{\psi}_I^e\|_{\dot{W}^{-r,1}} \le |I|^{r/d} \sum_{|\alpha|=r} \|(\Psi_\alpha)_I\|_{L^1} = |I|^{r/d}|I|^{1/d} \sum_{|\alpha|=r} \|\Psi_\alpha\|_{L^1} \lesssim |I|^{(1+r)/d},$$

for all $I \in D$. ∎

Remark 3.10. One can obtain a version of Theorem 3.9 for the inhomogeneous spaces $W^{l,1}$ by using essentially the same arguments as above and the decomposition (3.1) involving the "father" wavelets.

3.2. Interpolation results

3.2.1. The nonpathological case. Fix some parameters $s \in \mathbb{R}$ and $p \in (1,\infty)$. According to Section 10 in Chapter 6 of [28] (see also the discussion in [11], pp. 242–243), we have

$$\|f\|_{\dot{B}_p^{s,p}(\mathbb{R}^d)} \sim \|(f_I)_{I\in D}\|_{\ell_p^\mu}, \tag{3.13}$$

for any Schwartz function f on $\mathbb{R}^d$, where $\mu := 1 + (s-1)p'/d$. (Here, we suppose that the wavelets involved are of class at least $|s|+1$.) We can rewrite the quantity $\|(f_I)_{I\in D}\|_{\ell_p^\mu}$ using the weights $|I|^{(1-l)/d}$ as follows. For any $\gamma \in \mathbb{R}$ and any finitely supported sequence $(c_I)_{I\in D}$, we have

$$\begin{aligned}\|(|I|^{(1-l)/d} c_I)_{I\in D}\|_{\ell_p^\gamma} &= \Big(\sum_{I\in D_d} |I|^{(1-p)\gamma} |I|^{(1-l)p/d} |c_I|^p\Big)^{1/p}\\ &= \Big(\sum_{I\in D_d} |I|^{(1-p)\mu} |c_I|^p\Big)^{1/p}\\ &= \|(c_I)_{I\in D}\|_{\ell_p^\mu},\end{aligned}$$

where $\mu = \gamma + (l-1)p'/d$. Hence, from (3.13),

$$\|f\|_{\dot{B}_p^{s,p}(\mathbb{R}^d)} \sim \|(|I|^{(1-l)/d} f_I)_{I\in D}\|_{\ell_p^\gamma}, \tag{3.14}$$

for any Schwartz function f on $\mathbb{R}^d$, where $\gamma := 1 + (s-l)p'/d$.

Let us introduce the weighted spaces $\ell_p^\gamma(D, \omega_l)$ and $w\ell_1^\gamma(D, \omega_l)$ by defining their quasi-norms

$$\|(c_I)_{I\in D}\|_{\ell_p^\gamma(\omega_l)} := \|(|I|^{(1-l)/d}\, c_I)_{I\in D}\|_{\ell_p^\gamma}$$

and

$$\|(c_I)_{I\in D}\|_{w\ell_1^\gamma(\omega_l)} := \|(|I|^{(1-l)/d}\, c_I)_{I\in D}\|_{w\ell_1^\gamma}.$$

The spaces $\ell_p^\gamma(D, \omega_l)$ and $w\ell_1^\gamma(D, \omega_l)$ consist of those sequences $(c_I)_{I\in D}$ of finite $\ell_p^\gamma(\omega_l)$-norm or finite $w\ell_1^\gamma(\omega_l)$-quasi-norm, respectively.

For simplicity, we will denote the space $w\ell_1^0$ by $w\ell_1$. Notice that $w\ell_1(D)$ is the discrete Lorentz space $L^{1,\infty}(D)$ on the set of the dyadic cubes D endowed with the counting measure.

Proposition 3.11. *Consider some parameters $l \in \mathbb{Z}$, $s \in \mathbb{R}$, $p \in (1, \infty)$ and define $\gamma := 1 + (s - l)p'/d$. If $l \geq 1$ and $\gamma \in \mathbb{R} \setminus [1 - 1/d, 1]$, or $l \leq 0$ and $\gamma \in \mathbb{R} \setminus [0, 1]$, then, for any $\theta \in (0, 1)$, we have*

$$(\dot{W}^{l,1}(\mathbb{R}^d), \dot{B}_p^{s,p}(\mathbb{R}^d))_{\theta,q} = \dot{B}_q^{\sigma,q}(\mathbb{R}^d),$$

where $\sigma = (1 - \theta)l + \theta s$ and $1/q = 1 - \theta + \theta/p$. The same fact holds for the inhomogeneous version of the spaces.

(Here, we suppose that the wavelets involved are of class at least $|l| + |s| + 1$.)

Remark 3.12. Note that the conditions $\gamma \in \mathbb{R} \setminus [1 - 1/d, 1]$ and $\gamma \in \mathbb{R} \setminus [0, 1]$ are equivalent to the conditions $s \in \mathbb{R} \setminus [l - 1/p', l]$ and $s \in \mathbb{R} \setminus [l - d/p', l]$, respectively. In other words, Proposition 3.11 is a reformulation of Proposition 1.3, in the particular case $t = p$, in terms of the parameter γ.

Proof. The proof follows the same argument as in Theorem 1.4 of [11]. First, rewriting the estimates of Theorem 3.9 using the weighted spaces $\ell_p^\gamma(D, \omega_l)$ and $w\ell_p^\gamma(D, \omega_l)$, we have

$$\|(f_I)_{I\in D}\|_{w\ell_1^\gamma(\omega_l)} \lesssim \|f\|_{\dot{W}^{l,1}} \lesssim \|(f_I)_{I\in D}\|_{\ell_1^\gamma(\omega_l)}, \tag{3.15}$$

provided that $l \geq 1$ and $\gamma \in \mathbb{R} \setminus [1 - 1/d, 1]$, or $l \leq 0$ and $\gamma \in \mathbb{R} \setminus [0, 1]$.

Also, from (3.14),

$$\|f\|_{\dot{B}_p^{s,p}} \sim \|(f_I)_{I\in D}\|_{\ell_p^\gamma(\omega_l)}. \tag{3.16}$$

Let

$$P : \ell_1^\gamma(\omega_l) + \ell_p^\gamma(\omega_l) \to \dot{W}^{l,1} + \dot{B}_p^{s,p}$$

and

$$E : \dot{W}^{l,1} + \dot{B}_p^{s,p} \to w\ell_1^\gamma(\omega_l) + \ell_p^\gamma(\omega_l)$$

be defined formally as

$$P(c_I)_{I\in D} := \sum_{I\in D} c_I \psi_I \quad \text{and} \quad Ef := (f_I)_{I\in D}.$$

(Recall (3.15) and (3.16) in order to see that P and E are well defined.)

We have $P \circ E = \mathrm{id}$ on $\dot{W}^{l,1} + \dot{B}_p^{s,p}$. By the retraction method for P and E (see Theorem 6.4.2 in [5]), we observe that it suffices to prove that

$$(w\ell_1^\gamma(\omega_l), \ell_p^\gamma(\omega_l))_{\theta,q} = (\ell_1^\gamma(\omega_l), \ell_p^\gamma(\omega_l))_{\theta,q} = \ell_q^\gamma(\omega_l). \tag{3.17}$$

Indeed, from (3.15) and (3.16), we have the boundedness of the operators $P: \ell_1^\gamma(\omega_l) \to \dot{W}^{l,1}$, $P: \ell_p^\gamma(\omega_l) \to \dot{B}_p^{s,p}$ and $E: \dot{W}^{l,1} \to w\ell_1^\gamma(\omega_l)$, $E: \dot{B}_p^{s,p} \to \ell_p^\gamma(\omega_l)$. Consequently,

$$P : (\ell_1^\gamma(\omega_l), \ell_p^\gamma(\omega_l))_{\theta,q} = \ell_q^\gamma(\omega_l) \to (\dot{W}^{l,1}, \dot{B}_p^{s,p})_{\theta,q}$$

and

$$E : (\dot{W}^{l,1}, \dot{B}_p^{s,p})_{\theta,q} \to (\ell_1^\gamma(\omega_l), \ell_p^\gamma(\omega_l))_{\theta,q} = \ell_q^\gamma(\omega_l),$$

are bounded operators. This shows that

$$\|f\|_{(\dot{W}^{l,1}, \dot{B}_p^{s,p})_{\theta,q}} = \|P(Ef)\|_{(\dot{W}^{l,1}, \dot{B}_p^{s,p})_{\theta,q}} \lesssim \|(f_I)_{I\in D}\|_{\ell_q^\gamma(\omega_l)}$$

and

$$\|(f_I)_{I\in D}\|_{\ell_q^\gamma(\omega_l)} = \|Ef\|_{\ell_q^\gamma(\omega_l)} \lesssim \|f\|_{(\dot{W}^{l,1}, \dot{B}_p^{s,p})_{\theta,q}},$$

i.e., by (3.14),

$$\|f\|_{(\dot{W}^{l,1}, \dot{B}_p^{s,p})_{\theta,q}} \sim \|(f_I)_{I\in D}\|_{\ell_q^\gamma(\omega_l)} \sim \|f\|_{\dot{B}_p^{s,p}},$$

for any $f \in \mathcal{S}_\sharp$.

Let us see now that (3.17) holds. Note that, in all the sequence spaces we consider, we have the same weights involved. Hence, (3.17) is equivalent (by the retraction method) to the equality

$$(w\ell_1, \ell_p)_{\theta,q} = (\ell_1, \ell_p)_{\theta,q} = \ell_q,$$

which is known to hold (see, for instance, Theorem 5.3.1 in [5]). ■

Using Lemma 2.6 one can prove now Proposition 1.3 in full generality.

Proof of Proposition 1.3. The proof follows from Lemma 2.6 and Proposition 3.11 (see also Remark 3.12) via an application of Wolff's theorem (Theorem 2.7). Indeed, with the notation used in the statement of Proposition 1.3, by Proposition 3.11, we get

$$(\dot{W}^{l,1}, \dot{B}_q^{\sigma,q})_{1/2,q_1} = \dot{B}_{q_1}^{\sigma_1,q_1}, \tag{3.18}$$

where $\sigma_1 = l/2 + \sigma/2$ and $1/q_1 = 1/2 + 1/(2q)$. By Lemma 2.6, we also have

$$(\dot{B}_{q_1}^{\sigma_1,q_1}, \dot{F}_t^{s,p})_{\phi,q} = \dot{B}_q^{\sigma,q}, \tag{3.19}$$

where $\phi = \theta/(2-\theta)$. Using now (3.18), (3.19) together with Wolff's theorem (Theorem 2.7), we conclude the proof of Proposition 1.3. ■

One can now deduce the nonpathological case of Theorem 1.4 from Proposition 1.3. Since $(\dot{F}_{t'}^{-s,p'})^* \neq \dot{F}_t^{s,p}$, when $t = 1$, we start with the particular case $t = p$, and then we obtain the full Theorem 1.4 by using Lemma 2.6.

Proposition 3.13. *Let $r \in \mathbb{N}$. Suppose $p \in (1, \infty)$ and $\theta \in (0, 1)$. Then, for any $s \in \mathbb{R} \setminus [r, r + d/p]$, we have*

$$(\dot{B}^{s,p}_p(\mathbb{R}^d), \dot{W}^{r,\infty}(\mathbb{R}^d))_{\theta,q} = \dot{B}^{\sigma,q}_q(\mathbb{R}^d),$$

where $\sigma = (1-\theta)s + \theta r$ and $1/q = (1-\theta)/p$. A similar statement holds for the inhomogeneous spaces.

Proof. This follows by duality from Proposition 1.3. Indeed, applying Proposition 1.3 with $l = -r \leq 0$ and $t = p$, we get

$$(\dot{B}^{-s,p'}_{p'}, \dot{W}^{-r,1})_{\theta,q'} = \dot{B}^{-\sigma,q'}_{q'},$$

and, since the dual of $\dot{W}^{-r,1}$ is $\dot{W}^{r,\infty}$, we can write

$$\begin{aligned}(\dot{B}^{s,p}_p, \dot{W}^{r,\infty})_{\theta,q} &= ((\dot{B}^{-s,p'}_{p'})^*, (\dot{W}^{-r,1})^*)_{\theta,q'} \\ &= (\dot{B}^{-s,p'}_{p'}, \dot{W}^{-r,1})^*_{\theta,q'} = (\dot{B}^{-\sigma,q'}_{q'})^* = \dot{B}^{\sigma,q}_q,\end{aligned}$$

and Proposition 3.13 is proven. ∎

Proof of the nonpathological case of Theorem 1.4. The proof follows from Lemma 2.6 and Proposition 3.13 via Wolff's interpolation theorem (Theorem 2.7). (See the proof of Proposition 1.3.) ∎

We can now use Proposition 3.13 in order to prove the following.

Proposition 3.14. *Suppose r and l are some nonnegative integers such that $l \in \mathbb{R} \setminus [r, r + d]$. Then, for any $\theta \in (0, 1)$,*

$$(\dot{W}^{l,1}(\mathbb{R}^d), \dot{W}^{r,\infty}(\mathbb{R}^d))_{\theta,q} = \dot{B}^{\sigma,q}_q(\mathbb{R}^d),$$

where $\sigma = (1-\theta)l + \theta r$ and $1/q = 1 - \theta$.

Proof. Suppose $l < r$; the case $l > r + d$ being similar. Pick some $\eta_1 \in (\theta, 1)$. Proposition 3.11 gives us that

$$(\dot{W}^{l,1}, \dot{B}^{\sigma,q}_q)_{\eta_1,p} = \dot{B}^{s,p}_p, \tag{3.20}$$

where $s = (1-\eta_1)l + \eta_1\sigma$ and $1/p = 1 - \eta_1$.

Consider $\eta_2 := (\eta_1 - \theta)/(1 - \eta_1) \in (0, 1)$. Note that $1/q = (1-\eta_2)/p$ and $\sigma = (1-\eta_2)s + \eta_2 r$, hence, by applying Proposition 3.13, we obtain

$$(\dot{B}^{s,p}_p, \dot{W}^{r,\infty})_{\eta_2,q} = \dot{B}^{\sigma,q}_q. \tag{3.21}$$

By (3.20), (3.21) and Wolff's interpolation theorem (Theorem 2.7), we conclude the proof of Proposition 3.14. ∎

As we did in the case of Proposition 1.3, we can deduce the nonpathological case of Theorem 1.4 from its particular case $t = p$.

3.2.2. The pathological case. In this section we deal with the pathological case of Theorem 1.1 and Theorem 1.4. Our first result in this direction relies on classical trace theory (see Appendix A).

Proposition 3.15. *Suppose r and l are some integers such that $r < l \le r + d$. Fix some parameters $\theta \in (0,1)$ and $t \in (1,\infty)$, and let σ and q be some numbers such that $\sigma = (1-\theta)l + \theta r$ and $1/q = 1-\theta$. Then*

$$B_t^{\sigma,q}(\mathbb{R}^d) \not\hookrightarrow W^{l,1}(\mathbb{R}^d) + C^r(\mathbb{R}^d).$$

In particular,

$$B_t^{\sigma,q}(\mathbb{R}^d) \not\hookrightarrow (W^{l,1}(\mathbb{R}^d), W^{r,\infty}(\mathbb{R}^d))_{\theta,q}.$$

Proof. We first consider the case where $r = 0$. In this case, we have $1 \le l \le d$ and $\sigma = (1-\theta)l = l/q$. We argue by contradiction. Suppose

$$B_t^{\sigma,q}(\mathbb{R}^d) \hookrightarrow W^{l,1}(\mathbb{R}^d) + C(\mathbb{R}^d). \tag{3.22}$$

If $l = d$, then $\sigma = d/q$, and since $W^{d,1}(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d)$, (3.22) gives

$$B_t^{d/q,q}(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d).$$

However, it is known that there exist functions $f \in B_t^{d/q,q}(\mathbb{R}^d)$ that are not bounded. This[8] disproves (3.22) in the case $l = d$.

Suppose that $l < d$. Let Tr_{d-l} be the trace operator on the subspace $\mathbb{R}^{d-l} \times \{0\}^l \subset \mathbb{R}^d$. We have that $\mathrm{Tr}_{d-l}\colon W^{l,1}(\mathbb{R}^d) \to L^1(\mathbb{R}^{d-l})$ boundedly. Indeed, when $l = 1$, this is clear. When $l \ge 2$, by Uspenskiĭ's result (see [39] or [31]), we have that $\mathrm{Tr}_{d-l+1}\colon W^{l,1}(\mathbb{R}^d) \to B_1^{1,1}(\mathbb{R}^{d-l+1})$, and then we use the fact that $\mathrm{Tr}\colon B_1^{1,1}(\mathbb{R}^{d-l+1}) \to L^1(\mathbb{R}^{d-l})$. Moreover, $\mathrm{Tr}_{d-l}\colon C(\mathbb{R}^d) \to C(\mathbb{R}^{d-l})$ boundedly, and together with (3.22), we obtain

$$\mathrm{Tr}_{d-l}\, B_t^{\sigma,q}(\mathbb{R}^d) \hookrightarrow \mathrm{Tr}_{d-l}\, W^{l,1}(\mathbb{R}^d) + \mathrm{Tr}_{d-l}\, C(\mathbb{R}^d) \hookrightarrow L^1(\mathbb{R}^{d-l}) + C(\mathbb{R}^{d-l}).$$

However, this is contradicted by Proposition A.1 and consequently (3.22) cannot hold.

Suppose now that $r > 0$ and that

$$B_t^{\sigma,q}(\mathbb{R}^d) \hookrightarrow W^{l,1}(\mathbb{R}^d) + C^r(\mathbb{R}^d).$$

We introduce the operator

$$D := (\mathrm{id}, i\partial_1, \dots, i\partial_d),$$

where id is the identity operator.

By means of Calderón–Zygmund theory, for any Schwartz function f, one can find a family $F = (F_\alpha)_{|\alpha|=r} \in B_t^{\sigma,q}(\mathbb{R}^d)$ such that $f = D^r \cdot F$ and $\|F\|_{B_t^{\sigma,q}} \lesssim \|f\|_{B_t^{\sigma-r,q}}$. Indeed, one can consider $F := (D(\mathrm{id}-\triangle)^{-1})^r f$, where the operator $D(\mathrm{id}-\triangle)^{-1}$ acts component-wise. Noticing that $D^r\colon W^{l,1} \to W^{l-r,1}$ and $D^r\colon C^r \to C$, we can write

$$\|f\|_{W^{l-r,1}+C} = \|D^r \cdot F\|_{W^{l-r,1}+C} \lesssim \|F\|_{W^{l,1}+C^r} \lesssim \|F\|_{B_t^{\sigma,q}} \lesssim \|f\|_{B_t^{\sigma-r,q}},$$

which implies

$$B_t^{\sigma-r,q}(\mathbb{R}^d) \hookrightarrow W^{l-r,1}(\mathbb{R}^d) + C(\mathbb{R}^d).$$

However, as we have already seen, this embedding is false.

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[8] This argument is based on the example given in [30], p. 2, related to the irregular triples.

The last assertion of Proposition 3.15 follows from Lemma 2.4 and the intermediate space property:

$$(W^{l,1}(\mathbb{R}^d), W^{r,\infty}(\mathbb{R}^d))_{\theta,q} = (W^{l,1}(\mathbb{R}^d), C^r(\mathbb{R}^d))_{\theta,q} \hookrightarrow W^{l,1}(\mathbb{R}^d) + C^r(\mathbb{R}^d).$$

Proposition 3.15 is proved. ∎

As a corollary of the proof of Proposition 3.15, one can get a similar conclusion if we replace the Sobolev space $W^{l,1}$ with the Besov space $B_1^{l,1}$. Moreover, using more advanced trace theory, one can deal with the spaces $B_1^{s,1}$ when the parameter s is allowed to be any *real* (not necessarily integer) number in the interval $(r, r+d]$.

Proposition 3.16. *Suppose r is an integer and s is a real number such that $r < s \le r + d$. Fix some parameters $\theta \in (0,1)$, $t \in (1,\infty)$ and let σ and q be some numbers such that $\sigma = (1-\theta)s + \theta r$ and $1/q = 1-\theta$. Then*

$$B_t^{\sigma,q}(\mathbb{R}^d) \not\hookrightarrow B_1^{s,1}(\mathbb{R}^d) + C^r(\mathbb{R}^d).$$

In particular,

$$B_t^{\sigma,q}(\mathbb{R}^d) \not\hookrightarrow (B_1^{s,1}(\mathbb{R}^d), W^{r,\infty}(\mathbb{R}^d))_{\theta,q}.$$

Proof. The proof follows essentially the same strategy as the proof of Proposition 3.15. However, we use traces on more general subsets of $\mathbb{R}^d$ rather than subspaces. As in the proof of Proposition 3.15, it suffices to prove Proposition 3.16 only in the case where $r = 0$. Assume for contradiction that

$$B_t^{\sigma,q}(\mathbb{R}^d) \hookrightarrow B_1^{s,1}(\mathbb{R}^d) + C(\mathbb{R}^d). \tag{3.23}$$

Suppose $s \neq d$. Otherwise we can easily obtain the contradiction by mentioning that, since $B_1^{s,1}(\mathbb{R}^d) \hookrightarrow C(\mathbb{R}^d)$, the right-hand side of (3.23) is embedded in $C(\mathbb{R}^d)$. However, $B_t^{\sigma,q}(\mathbb{R}^d)$ is not embedded in $C(\mathbb{R}^d)$.

Suppose $0 < s < d$. Define the parameter $\delta := d - s \in (0,d)$ and consider a δ-full subset $\Gamma \subset \mathbb{R}^d$ (see Appendix A). By (3.23) and Theorem A.2 (i), we have

$$\operatorname{Tr}_\Gamma B_t^{\sigma,q}(\mathbb{R}^d) \hookrightarrow \operatorname{Tr}_\Gamma B_1^{s,1}(\mathbb{R}^d) + \operatorname{Tr}_\Gamma C(\mathbb{R}^d) \hookrightarrow L^1(\Gamma) + L^\infty(\Gamma), \tag{3.24}$$

where the L^p spaces on Γ are considered with the respect to the Hausdorff measure $\mathcal{H}^\delta$. However, since $\sigma = (d-\delta)/q$, by applying Theorem A.2 (ii) the space $B_t^{\sigma,q}(\mathbb{R}^d)$ has no trace on Γ. This disproves (3.24) and we have that (3.23) cannot hold. ∎

Proposition 3.16 can be used to give a partial converse to Proposition 1.4. First, we are concerned with the inhomogeneous version.

Corollary 3.17. *Let r be a nonnegative integer and let $p, q \in [1,\infty)$, $\theta \in (0,1)$, $s, \sigma \in \mathbb{R}$ be some parameters such that $1/q = (1-\theta)/p$ and $\sigma = (1-\theta)s + \theta r$. If $r < s \le r + d/p$, then*

$$(B_p^{s,p}(\mathbb{R}^d), W^{r,\infty}(\mathbb{R}^d))_{\theta,q} \neq B_q^{\sigma,q}(\mathbb{R}^d).$$

Proof. The case where $p = 1$ is already covered by Proposition 3.16, hence, we may take $p \in (1,\infty)$. Suppose by contradiction that we have

$$(B_p^{s,p}, W^{r,\infty})_{\theta,q} = B_q^{\sigma,q}. \tag{3.25}$$

Note that $1/q = (1-\theta)/p < 1/p$, hence, there exists a unique number $\eta_1 \in (0,1)$ such that $1/p = 1 - \eta_1 + \eta_1/q$. Define the real number

$$s_0 := \frac{s - \eta_1 \sigma}{1 - \eta_1},$$

which is positive thanks to the fact that $s > \sigma$ (this follows from the inequality $s > r$ and the formula $\sigma = (1-\theta)s + \theta r$). Since $1/p = 1 - \eta_1 + \eta_1/q$ and $s = (1-\eta_1)s_0 + \eta_1 \sigma$,

$$(B_1^{s_0,1}, B_q^{\sigma,q})_{\eta_1,p} = B_p^{s,p},$$

which together with (3.25), implies via Wolff's theorem (Theorem 2.7) that

$$(B_1^{s_0,1}, W^{r,\infty})_{\eta_2,p} = B_p^{s,p}, \tag{3.26}$$

where

$$\eta_2 := \frac{\eta_1 \theta}{1 - \eta_1 + \eta_1 \theta}.$$

One can check that $1/p = 1 - \eta_2$ and $s = (1-\eta_2)s_0 + \eta_2 r$. From this, as long as $r < s \le r + d/p$, we have

$$s_0 = \frac{s - \eta_2 r}{1 - \eta_2} = p\Big(s - \frac{r}{p'}\Big) = p\Big(s - r + \frac{r}{p}\Big) = p(s-r) + r \in (r, r+d].$$

However, it follows from Proposition 3.16 that (3.26) cannot hold for this range of the parameter s_0. ∎

We can generalize Corollary 3.17 as follows.

Proposition 3.18. *Let r be a nonnegative integer and let $p, q \in [1,\infty)$, $\theta \in (0,1)$, $t \in [1,\infty]$, $s, \sigma \in \mathbb{R}$ be some parameters such that $1/q = (1-\theta)/p$ and $\sigma = (1-\theta)s + \theta r$. If $r < s \le r + d/p$, then, for any $\tau \in [1,\infty]$,*

$$F_\tau^{\sigma,q}(\mathbb{R}^d) \not\hookrightarrow (F_t^{s,p}(\mathbb{R}^d), W^{r,\infty}(\mathbb{R}^d))_{\theta,q}.$$

Proof. Suppose by contradiction that

$$F_\tau^{\sigma,q} \hookrightarrow (F_t^{s,p}, W^{r,\infty})_{\theta,q}. \tag{3.27}$$

Introducing the spaces $X_0 := F_t^{s,p}$ and $X_1 := (F_t^{s,p}, W^{r,\infty})_{\theta,q} = F_\tau^{\sigma,q}$, we observe that X_0 is in the class $\mathcal{C}(0, F_t^{s,p}, W^{r,\infty})$ and X_1 is in the class $\mathcal{C}(\theta, F_t^{s,p}, W^{r,\infty})$ (see Definition 3.51 and Theorem 3.5.2 in pp. 48–49 of [5]), and we can apply the reiteration theorem (see Theorem 3.5.3 on p. 50 of [5]). Therefore, for any $\rho \in [1,\infty]$,

$$(F_t^{s,p}, F_\tau^{\sigma,q})_{1/2,\rho} \hookrightarrow (X_0, X_1)_{1/2,\rho} = (F_t^{s,p}, W^{r,\infty})_{\phi,\rho},$$

where $\phi = \theta/2$. Setting $\rho \in (1,\infty)$ such that $1/\rho = 1/(2p) + 1/(2q)$, we have (using also Lemma 2.6)

$$B_\rho^{\sigma_1,\rho} = (F_t^{s,p}, F_\tau^{\sigma,q})_{1/2,\rho},$$

and hence

$$B_\rho^{\sigma_1,\rho} \hookrightarrow (F_t^{s,p}, W^{r,\infty})_{\phi,\rho}, \tag{3.28}$$

where $\sigma_1 = (1-\phi)s + \phi r$. Notice also that from Lemma 2.6, we get (since $L^\infty \hookrightarrow \mathrm{bmo} = F_2^{0,\infty}$ and $W^{r,\infty} \hookrightarrow F_2^{r,\infty}$)

$$(F_t^{s,p}, W^{r,\infty})_{\phi,\rho} \hookrightarrow (F_t^{s,p}, F_2^{r,\infty})_{\phi,\rho} = B_\rho^{\sigma_1,\rho},$$

which together with (3.28) gives us the equality

$$B_\rho^{\sigma_1,\rho} = (F_t^{s,p}, W^{r,\infty})_{\phi,\rho}.$$

Now, using again the reiteration theorem as above, we have

$$(3.29)\qquad \begin{aligned}(B_\rho^{\sigma_1,\rho}, W^{r,\infty})_{\theta_1,q} &= ((F_t^{s,p}, W^{r,\infty})_{\phi,\rho}, W^{r,\infty})_{\theta_1,q} \\ &= (F_t^{s,p}, W^{r,\infty})_{\theta,q} = B_q^{\sigma,q},\end{aligned}$$

where $\theta_1 := \theta/(2-\theta)$. We can easily check that $r < \sigma_1 \le r + d/\rho$. Hence, Corollary 3.17 shows that (3.29) cannot hold. By this we contradict (3.27). ∎

Let Q be the cube $[-1,1]^d$. With minor modifications, we can prove a version of Proposition 3.18 for the cube Q (and for C^r instead $W^{r,\infty}$). More precisely, when the parameters p, q, t, s, σ are as in the statement of Proposition 3.18, then

$$(3.30)\qquad (F_t^{s,p}(Q), C^r(Q))_{\theta,q} \neq B_q^{\sigma,q}(Q).$$

We can use this fact to prove a similar result that concerns the homogeneous version of the spaces.

Proposition 3.19. *Let r be a nonnegative integer and let $p, q \in [1,\infty)$, $\theta \in (0,1)$, $t \in [1,\infty]$, $s, \sigma \in \mathbb{R}$ be some parameters such that $1/q = (1-\theta)/p$ and $\sigma = (1-\theta)s + \theta r$. If $r < s \le r + d/p$, then, for any $\tau \in [1,\infty]$,*

$$\dot{F}_\tau^{\sigma,q}(\mathbb{R}^d) \not\hookrightarrow (\dot{F}_t^{s,p}(\mathbb{R}^d), \dot{W}^{r,\infty}(\mathbb{R}^d))_{\theta,q}.$$

Before passing to the proof of Proposition 3.19 let us recall some facts from the theory of subcouples introduced by Pisier in [33] that will be useful in what follows. Let (X_0, X_1) be a compatible couple of Banach spaces and let (Y_0, Y_1) be a subcouple of (X_0, X_1). In other words, we have that for any $j = 0, 1$, Y_j is a (Banach) subspace of X_j. According to Pisier [33], we say that (Y_0, Y_1) is *K-closed* in (X_0, X_1) if, for any $f \in Y_0 + Y_1$, we have the equivalence

$$K_t(f, Y_0, Y_1) \sim K_t(f, X_0, X_1),$$

for any $t > 0$, where the implicit constants do not depend on t or f. Trivially, we have that

$$K_t(f, Y_0, Y_1) \ge K_t(f, X_0, X_1),$$

and hence, in order to verify the K-closedness of the subcouple (Y_0, Y_1), it suffices to check the inequality

$$K_t(f, Y_0, Y_1) \lesssim K_t(f, X_0, X_1).$$

In other words, it suffices to verify that for any fixed $t > 0$ and any decomposition $f = f_0 + f_1$ (depending on t), where $f_0 \in X_0$, $f_1 \in X_1$, with

$$\|f_0\|_{X_0} + t\|f_1\|_{X_1} \le 1,$$

there exist $g_0 \in Y_0$, $g_1 \in Y_1$ (depending on t) such that $f = g_0 + g_1$ and

$$\|g_0\|_{Y_0} + t\|g_1\|_{Y_1} \lesssim 1,$$

where the implicit constant does not depend on t or f.

The notion of K-closedness was introduced by Pisier in [33] in order to give a short proof of Bourgain's result that the disc algebra has the Grothendieck property. Other authors like Bourgain [6] or Kislyakov and Kruglyak [21] further developed the theory of K-closed subcouples deriving interpolation properties for the Hardy and the Sobolev spaces. Here, we only use the notion of K-closedness in a simple situation. We need the following lemma on quotient spaces that follows from the work of Pisier and Janson.

Lemma 3.20. *Let (X_0, X_1) be a compatible couple of Banach spaces such that $X_0 \cap X_1$ is dense in X_0 and X_1. Suppose M is a finite dimensional subspace of $X_0 \cap X_1$.*

(i) *We have that*

$$(X_0/M) \cap (X_1/M) = (X_0 \cap X_1)/M.$$

(ii) *For any $\theta \in (0,1)$ and any $q \in (1,\infty)$, we have*

$$(X_0/M, X_1/M)_{\theta,q} = (X_0, X_1)_{\theta,q}/M.$$

Proof. Part (i) follows from the work of Pisier (see, for instance, Theorem 4.1 (i) and (iv) in [17]), and (ii) is a consequence of Theorem 4.2 in [17]. First, let us recall that, since M is finite dimensional, any two norms are equivalent on M. It is easy to see that (M, M) is a normal subcouple of (X_0, X_1) (see Definition on p. 317 of [17]). Note that M is a finite dimensional subspace of $X_0 + X_1$, and consequently M is complemented in $X_0 + X_1$. Therefore, there exists an onto bounded projection $P_M \colon X_0 + X_1 \to M$. We get

$$\|P_M f\|_M \lesssim \|f\|_{X_0+X_1} \le \min(\|f\|_{X_0}, \|f\|_{X_1}),$$

for any $f \in X_0 \cap X_1$. Since

$$\|P_M f\|_M \sim \|P_M f\|_{X_0} \sim \|P_M f\|_{X_1},$$

the operator $P_M \colon X_j \to X_j$ is bounded for any $j = 0, 1$.

One can see now that (M, M) is a K-closed subcouple of (X_0, X_1). Indeed, fix some $t > 0$. Consider some $f \in M$ and a decomposition $f = f_0 + f_1$, with $f_0 \in X_0$, $f_1 \in X_1$, such that

$$\|f_0\|_{X_0} + t\|f_1\|_{X_1} \le 1. \tag{3.31}$$

Observe that

$$f = P_M f = P_M f_0 + P_M f_1,$$

and by the boundedness of P_M on X_j, together with (3.31), we get

$$\|P_M f_0\|_{X_0} + t\|P_M f_1\|_{X_1} \le \|P_M\| < \infty.$$

Now, we can apply Theorem 4.1 (i) and (iv) in [17] and Theorem 4.2 in [17] to the subcouple (M, M) in order to obtain (i) and (ii), respectively. ∎

Proof of Proposition 3.19. It suffices to prove Proposition 3.19 in the case $t = p \in (1,\infty)$ (or $t = 2$), $\tau = q$, with nonequality instead of nonembedding. One can then prove the general case as we deduced Proposition 3.18 from Corollary 3.17. Suppose by contradiction that

$$(\dot{W}^{s,p}(\mathbb{R}^d), \dot{W}^{r,\infty}(\mathbb{R}^d))_{\theta,q} = \dot{B}^{\sigma,q}_q(\mathbb{R}^d). \tag{3.32}$$

Using Lemma 2.4, this is equivalent to

$$(\dot{W}^{s,p}(\mathbb{R}^d), \dot{C}^r_0(\mathbb{R}^d))_{\theta,q} = \dot{B}^{\sigma,q}_q(\mathbb{R}^d). \tag{3.33}$$

Let l be the smallest integer with $l \geq s$ and consider $k := \max\{l, r\}$. By the Poincaré inequality (see, for instance, (2.6)), we have

$$\|f\|_{L^p(Q)/\mathcal{P}^l} \lesssim \|f\|_{\dot{W}^{s,p}(\mathbb{R}^d)},$$

for any Schwartz function f, where $\mathcal{P}^l$ is the space of polynomials of degree at most $l-1$. Now, using this and the trivial inequality

$$\|f\|_{\dot{W}^{s,p}(Q)/\mathcal{P}^l} \leq \|f\|_{\dot{W}^{s,p}(Q)} \leq \|f\|_{\dot{W}^{s,p}(\mathbb{R}^d)},$$

together with Lemma 3.20 (i) (for $M = \mathcal{P}^l$) and the fact that $W^{s,p} = L^p \cap \dot{W}^{s,p}$, we get

$$\|f\|_{W^{s,p}(Q)/\mathcal{P}^l} \lesssim \|f\|_{\dot{W}^{s,p}(\mathbb{R}^d)},$$

for any Schwartz function f. Hence, since $k \geq l$,

$$\|f\|_{W^{s,p}(Q)/\mathcal{P}^k} \lesssim \|f\|_{\dot{W}^{s,p}(\mathbb{R}^d)}, \tag{3.34}$$

for any Schwartz function f. By (3.34), the restriction operator R_Q ($R_Q f := f|_Q$ for any Schwartz f) is bounded from $\dot{W}^{s,p}(\mathbb{R}^d)$ to $W^{s,p}(Q)/\mathcal{P}^k$. Also, by using the mean value theorem (see also (2.6)), R_Q is bounded from $\dot{C}^r_0(\mathbb{R}^d)$ to $C^r(Q)/\mathcal{P}^r$. Since $k \geq r$, R_Q is bounded from $\dot{C}^r_0(\mathbb{R}^d)$ to $C^r(Q)/\mathcal{P}^k$. By interpolation, we get

$$R_Q(\dot{W}^{s,p}(\mathbb{R}^d), \dot{C}^r_0(\mathbb{R}^d))_{\theta,q} \hookrightarrow (W^{s,p}(Q)/\mathcal{P}^k, C^r(Q)/\mathcal{P}^k)_{\theta,q}.$$

Using (3.33), we can rewrite this as

$$R_Q(\dot{B}^{\sigma,q}_q(\mathbb{R}^d)) \hookrightarrow (W^{s,p}(Q)/\mathcal{P}^k, C^r(Q)/\mathcal{P}^k)_{\theta,q}. \tag{3.35}$$

For any element of $B^{\sigma,q}_q(Q)/\mathcal{P}^k$, there exists a representative $f \in B^{\sigma,q}_q(Q)$ and there exists an extension $\tilde{f} \in B^{\sigma,q}_q(\mathbb{R}^d) \hookrightarrow \dot{B}^{\sigma,q}_q(\mathbb{R}^d)$ such that $R_Q\tilde{f} = f$. Hence,

$$B^{\sigma,q}_q(Q)/\mathcal{P}^k \hookrightarrow R_Q(\dot{B}^{\sigma,q}_q(\mathbb{R}^d)),$$

and combining this with (3.35), we get

$$B^{\sigma,q}_q(Q)/\mathcal{P}^k \hookrightarrow (W^{s,p}(Q)/\mathcal{P}^k, C^r(Q)/\mathcal{P}^k)_{\theta,q}.$$

Using this and Lemma 3.20 (ii) (for $M = \mathcal{P}^k$), we get

$$B_q^{\sigma,q}(Q)/\mathcal{P}^k \hookrightarrow (W^{s,p}(Q), C^r(Q))_{\theta,q}/\mathcal{P}^k. \tag{3.36}$$

Now, by using Lemma 2.6 (and by a standard application of the retraction method), we have

$$(W^{s,p}(Q), C^r(Q))_{\theta,q} \hookrightarrow (W^{s,p}(Q), F_2^{r,\infty}(Q))_{\theta,q} = B_q^{\sigma,q}(Q),$$

and hence, by (3.30), the space $(W^{s,p}(Q), C^r(Q))_{\theta,q}$ is strictly smaller that $B_q^{\sigma,q}(Q)$. In other words, re-denoting the spaces as $A_1 := B_q^{\sigma,q}(Q)$ and $A_2 := (B_p^{s,p}(Q), C^r(Q))_{\theta,q}$, there exists a sequence $(f_n)_{n\geq 1}$ of Schwartz functions on $\mathbb{R}^d$ such that

$$\|f_n\|_{A_2} = 1 \quad \text{and} \quad \|f_n\|_{A_1} \to 0, \tag{3.37}$$

when $n \to \infty$. Consider the sets $U_1 := (-2, 1/2) \times (-2, 2)^{d-1}$, $U_2 := (-1/2, 2) \times (-2, 2)^{d-1}$ that form an open covering of Q. For these open sets, one can find two smooth functions $\chi_j \in C_c^\infty(U_j)$, $j = 0, 1$, with $\chi_1 + \chi_2 = 1$ on Q. We can observe that, for any Schwartz function f on $\mathbb{R}^d$, we have

$$\|\chi_j f\|_A \lesssim \|f\|_A, \tag{3.38}$$

when $A = A_1$ or $A = A_2$. This is a standard fact when $A = A_1$. To obtain the inequality for the case $A = A_2$, we observe that (3.38) holds for $A = B_p^{s,p}(Q)$ and $A = C^r(Q)$, and then apply the standard real interpolation method. Now we can write

$$\|f\|_{A_j} \leq \|\chi_1 f\|_{A_j} + \|\chi_2 f\|_{A_j} \lesssim \|f\|_{A_j},$$

and combining this with (3.37),

$$\|\chi_1 f_n\|_{A_2} + \|\chi_2 f_n\|_{A_2} \sim 1,$$

uniformly in n and

$$\|\chi_1 f_n\|_{A_1} + \|\chi_2 f_n\|_{A_1} \to 0,$$

when $n \to \infty$. Considering a subsequence (which for simplicity will be also denoted by $(f_n)_{n\geq 1}$) we can suppose without loss of generality that

$$\|\chi_1 f_n\|_{A_2} \sim 1 \quad \text{and} \quad \|\chi_1 f_n\|_{A_1} \to 0,$$

when $n \to \infty$. In other words, introducing the functions $g_n := \chi_1 f_n$, we have

$$\|g_n\|_{A_2} \sim 1 \quad \text{and} \quad \|g_n\|_{A_1} \to 0, \tag{3.39}$$

when $n \to \infty$. There exist two sequences $(p_n^1)_{n\geq 1}, (p_n^2)_{n\geq 1}$ of polynomials[9] in $\mathcal{P}^k$ such that, for any $j = 1, 2$,

$$\|g_n - p_n^j\|_{A_j} \sim \|g_n\|_{A_j/\mathcal{P}^k}.$$

[9]These polynomials are not the same as the polynomials p_k^l introduced in Section 2.1.

Hence, by (3.36),

$$\|g_n - p_n^2\|_{A_2} \lesssim \|g_n - p_n^1\|_{A_1} \leq \|g_n\|_{A_1}, \tag{3.40}$$

uniformly in n. Consider some $\psi \in C_c^\infty(\Omega)$, not identically 0, where $\Omega := [2/3, 1] \times [-1, 1]^{d-1}$. As in (3.38), we get

$$\|\psi p_n^2\|_{A_2} = \|\psi(g_n - p_n^2)\|_{A_2} \lesssim \|g_n - p_n^2\|_{A_2},$$

where we have used the fact that, since $\operatorname{supp} g_n \subset U_1$, the sets $\operatorname{supp} g_n$ and Ω are disjoint. This inequality implies that all the coefficients of the polynomial p_n^2 are uniformly bounded by $\|g_n - p_n^2\|_{A_2}$. Hence, by (3.39) and (3.40), all the coefficients of the polynomial p_n^2 are converging to 0 uniformly. Since $\mathcal{P}^k$ is finite dimensional, any two norms are equivalent on $\mathcal{P}^k$, and consequently $\|p_n^2\|_{A_2}$ is equivalent to the sum of the absolute values of the coefficients of p_n^2. By the previous discussion, we get $\|p_n^2\|_{A_2} \to 0$, when $n \to \infty$. By this, (3.39) and (3.40), we have

$$\|g_n\|_{A_2} \leq \|g_n - p_n^2\|_{A_2} + \|p_n^2\|_{A_2} \lesssim \|g_n\|_{A_1} + \|p_n^2\|_{A_2} \to 0,$$

when $n \to \infty$. However, this contradicts (3.39). This proves that (3.32) does not hold. ∎

By duality, Proposition 3.18 (or Proposition 3.19 for the homogeneous case) immediately implies a partial converse to Proposition 1.3 in the case where $l \leq 0$.

Corollary 3.21. *Consider some parameters $l \in \mathbb{Z}$, $s \in \mathbb{R}$, $p, t \in (1, \infty)$. If $l \leq 0$ and $l - d/p' \leq s < l$, then, for any $\theta \in (0, 1)$, we have*

$$(W^{l,1}(\mathbb{R}^d), F_t^{s,p}(\mathbb{R}^d))_{\theta,q} \neq B_q^{\sigma,q}(\mathbb{R}^d),$$

where $\sigma = (1-\theta)l + \theta s$ and $1/q = 1 - \theta + \theta/p$.

Indeed, if for some $r = -l \geq 0$ and some $s \in [-r - d/p', -r)$, we have

$$(W^{-r,1}, F_t^{s,p})_{\theta,q} = B_q^{\sigma,q},$$

then, by duality, we get

$$(W^{r,\infty}, F_{t'}^{-s,p'})_{\theta,q'} = B_{q'}^{-\sigma,q'}. \tag{3.41}$$

However, since $r < -s \leq r + d/p'$, by Corollary 3.17 the equality (3.41) cannot be true.

Remark 3.22. A direct consequence of Corollary 3.21 and the proof of Proposition 3.11 is the fact that the range of the parameter γ in Theorem 3.9 cannot be improved when $l \leq 0$. In other words, for any $\gamma \in [0, 1]$ and any integer $l \leq 0$, we have the noninequality

$$\|(f_I)_I\|_{w\ell_1^\gamma(\omega_l)} \not\lesssim \|f\|_{W^{l,1}},$$

for Schwartz functions f. This is in contrast with the case $l \geq 1$, where a larger range of γ is available (see Remark 3.5).

Since we have proved the nonpathological case of Theorem 1.4, Proposition 3.18 and Proposition 3.19, we have proved Theorem 1.4.

One can observe that Theorem 1.4 implies, via the reiteration theorem, the following sharpening of the conclusion in the pathological case.

Proposition 3.23. *Consider the parameters $s \in \mathbb{R}$, $r \in \mathbb{N}$, with $s \neq r$, and let $p, q \in (1,\infty)$, $\theta \in (0,1)$, $t \in [1,\infty]$, $\sigma \in \mathbb{R}$ be as in the statement of Theorem 1.4. Let X be a space in the class $\mathcal{C}(\theta, F_t^{s,p}, C^r)$. Then, if $s \in (r, r + d/p]$, no space $F_\tau^{\sigma,q}$, with $\tau \in [1,\infty]$, embedds in X. A similar fact holds in the case of the homogeneous spaces.*

Proof. Indeed, suppose by contradiction that $F_\tau^{\sigma,q} \hookrightarrow X$. Then, as in the proof of Proposition 3.18, we get, by the reiteration theorem (and Lemma 2.6), that

$$B_\rho^{\sigma_1,\rho} = (F_t^{s,p}, F_\tau^{\sigma,q})_{1/2,\rho} \hookrightarrow (F_t^{s,p}, X)_{1/2,\rho} = (F_t^{s,p}, W^{r,\infty})_{\phi,\rho},$$

where $\phi = \theta/2$, $\sigma_1 = s/2 + \sigma/2$ and $1/\rho = 1/(2p) + 1/(2q)$. However, this contradicts Theorem 1.4. ∎

Proof of Proposition 1.6. Suppose by contradiction that $F_\tau^{\sigma,q}$ is an interpolation space of exponent θ with respect to $(F_t^{s,p}, W^{r,\infty})$. By a simple regularization argument, we get that $F_\tau^{\sigma,q}$ is an interpolation space of exponent θ with respect to $(F_t^{s,p}, C^r)$. Then, by the Aronzajn–Gagliardo theorem (see [3] or Exercise 2.8.4 on p. 33 of [5]), there exists an interpolation method (functor) G_θ of exponent θ such that

$$F_\tau^{\sigma,q} = G_\theta(F_t^{s,p}, C^r).$$

However, since $F_t^{s,p} \cap C^r$ is dense in $F_t^{s,p}$ and C^r, by the extremal property of the real interpolation method (see Theorem 3.9.1 on p. 58 of [5]), we get

$$(F_t^{s,p}, C^r)_{\theta,1} \hookrightarrow G_\theta(F_t^{s,p}, C^r) \hookrightarrow (F_t^{s,p}, C^r)_{\theta,\infty}.$$

In particular, according to Theorem 3.5.2 on p. 49 of [5], $G_\theta(F_t^{s,p}, C^r)$ is a space of class $\mathcal{C}(\theta, F_t^{s,p}, C^r)$. Hence, by the reiteration theorem (as in the proof of Proposition 3.18 and Proposition 3.23),

$$B_\rho^{\sigma_1,\rho} = (F_t^{s,p}, F_\tau^{\sigma,q})_{1/2,\rho} = (F_t^{s,p}, G_\theta(F_t^{s,p}, C^r))_{1/2,\rho} = (F_t^{s,p}, W^{r,\infty})_{\phi,\rho},$$

where $\phi = \theta/2$, $\sigma_1 = s/2 + \sigma/2$ and $1/\rho = 1/(2p) + 1/(2q)$. This contradicts Theorem 1.4. ∎

At least in the case of homogeneous spaces Proposition 1.6 can be strengthened as follows.

Corollary 3.24. *Consider the parameters $s \in \mathbb{R}$, $r \in \mathbb{N}$, with $s \neq r$, and let $p, q \in (1,\infty)$, $\theta \in (0,1)$, $t \in [1,\infty]$, $\sigma \in \mathbb{R}$ be as in the statement of Theorem 1.4. Moreover, suppose that $s \in (r, r + d/p)$ and fix some $\tau \in [1,\infty]$. Then there exists a linear operator*

$$T : \dot{F}_t^{s,p}(\mathbb{R}^d) + \dot{W}^{r,\infty}(\mathbb{R}^d) \to \dot{F}_t^{s,p}(\mathbb{R}^d) + \dot{W}^{r,\infty}(\mathbb{R}^d),$$

such that

(i) *T is bounded on $\dot{F}_t^{s,p}(\mathbb{R}^d)$,*
(ii) *T is bounded on $\dot{W}^{r,\infty}(\mathbb{R}^d)$,*
(iii) *T is not bounded on $\dot{F}_\tau^{\sigma,q}(\mathbb{R}^d)$.*

Proof. Suppose by contradiction that any operator

$$T : \dot{F}_t^{s,p} + \dot{W}^{r,\infty} \to \dot{F}_t^{s,p} + \dot{W}^{r,\infty},$$

that is bounded on $\dot{F}^{s,p}_t$ and on $\dot{W}^{r,\infty}$, has to be bounded on $\dot{F}^{\sigma,q}_\tau$. By Theorem 2.4.2 in [5], we have

$$\|T\|_{\dot{F}^{\sigma,q}_\tau \to \dot{F}^{\sigma,q}_\tau} \lesssim \|T\|_{\dot{F}^{s,p}_t \to \dot{F}^{s,p}_t} + \|T\|_{\dot{W}^{r,\infty} \to \dot{W}^{r,\infty}}, \tag{3.42}$$

where the implicit constant does not depend on T. Fix such an operator T. For any $\lambda > 0$, we introduce the new operator T_λ formally defined by

$$T^\lambda f = (Tf)^\lambda,$$

where $f^\lambda(\cdot) = f(\lambda \cdot)$ (see Section 2.1, (2.3)). We can easily verify that

$$\|T^\lambda\|_{\dot{F}^{\sigma,q}_\tau \to \dot{F}^{\sigma,q}_\tau} \sim \lambda^{\sigma - d/q} \|T\|_{\dot{F}^{\sigma,q}_\tau \to \dot{F}^{\sigma,q}_\tau},$$

and get similar relations corresponding to the spaces $\dot{F}^{s,p}_t$, $\dot{W}^{r,\infty}$. From (3.42), we get

$$\|T\|_{\dot{F}^{\sigma,q}_\tau \to \dot{F}^{\sigma,q}_\tau} \lesssim \lambda^{\theta(s-r-1/p)} \|T\|_{\dot{F}^{s,p}_t \to \dot{F}^{s,p}_t} + \lambda^{-(1-\theta)(s-r-1/p)} \|T\|_{\dot{W}^{r,\infty} \to \dot{W}^{r,\infty}}$$

and, by setting

$$\lambda = \left(\frac{\|T\|_{\dot{W}^{r,\infty} \to \dot{W}^{r,\infty}}}{\|T\|_{\dot{F}^{s,p}_t \to \dot{F}^{s,p}_t}} \right)^{1/(s-r-1/p)},$$

we obtain

$$\|T\|_{\dot{F}^{\sigma,q}_\tau \to \dot{F}^{\sigma,q}_\tau} \lesssim \|T\|^{1-\theta}_{\dot{F}^{s,p}_t \to \dot{F}^{s,p}_t} \|T\|^{\theta}_{\dot{W}^{r,\infty} \to \dot{W}^{r,\infty}}.$$

However, this implies that $\dot{F}^{\sigma,q}_\tau$ is an interpolation space of exponent η with respect to $(\dot{F}^{s,p}_t, \dot{W}^{r,\infty})$, which by Proposition 1.6 cannot be true. ∎

3.3. Pathological sums of Sobolev spaces on $\mathbb{R}^d$

As we have seen in the previous section, the statement of Proposition 3.16 gives more than the fact that the space $B^{\sigma,q}_q$ is not equal to a space obtained by interpolating $B^{l,1}_1$ and C^r, when $l \in (r, r+d]$. Namely, $B^{\sigma,q}_q$ cannot be even embedded in the sum $B^{l,1}_1 + C^r$. It is natural to ask if the same phenomenon occurs in the pathological cases described by Proposition 3.18 or Proposition 3.19. The homogeneous spaces that appear in the statement of Proposition 3.19 are in most cases convenient for converting the result of nonequality into a stronger result of nonembeddability into the sum. More precisely, we have the following partial result.

Proposition 3.25. *Let r be a nonnegative integer and let $p, q \in [1, \infty)$, $\theta \in (0,1)$, $t, \tau \in [1, \infty]$, $s, \sigma \in \mathbb{R}$ be some parameters such that $1/q = (1-\theta)/p$ and $\sigma = (1-\theta)s + \theta r$. If $r < s < r + d/p$, then*

$$\dot{F}^{\sigma,q}_\tau(\mathbb{R}^d) \not\hookrightarrow \dot{F}^{s,p}_t(\mathbb{R}^d) + \dot{W}^{r,\infty}(\mathbb{R}^d).$$

Proof. For simplicity, let us denote the spaces as follows: $A_0 := \dot{F}^{s,p}_t$, $A_1 := \dot{W}^{r,\infty}$ and $Y := \dot{F}^{\sigma,q}_\tau$. Suppose for contradiction that we have

$$Y \hookrightarrow A_0 + A_1.$$

This implies that, for any $f \in Y$,

$$\|f^\lambda\|_{A_0 + A_1} \lesssim \|f^\lambda\|_Y, \tag{3.43}$$

for any $\lambda > 0$, where $f^\lambda(\cdot) := f(\lambda\cdot)$. Introducing the number $t := \lambda^{r-s+d/p}$ (recall that $r - s + d/p \neq 0$), one can see by a direct computation that (see (2.2))

$$\|f^\lambda\|_{A_0+A_1} = \lambda^{s-d/p} K_t(f, A_0, A_1). \tag{3.44}$$

Since, $\|f^\lambda\|_Y \sim \lambda^{\sigma-d/q}\|f\|_Y$, by (3.43) and (3.44), we obtain

$$\begin{aligned}\|f\|_Y &\gtrsim \lambda^{(s-d/p)-(\sigma-d/q)} K_t(f, A_0, A_1)\\ &= \lambda^{\eta(s-d/p-r)} K_t(f, A_0, A_1) = t^{-\theta} K_t(f, A_0, A_1),\end{aligned}$$

which means that

$$Y \hookrightarrow (A_0, A_1)_{\theta,\infty}.$$

Let us consider the spaces $X_0 = A_0$ and $X_1 = (A_0, A_1)_{\theta,\infty}$. Note that X_0 is in the class $\mathcal{C}(0, A_0, A_1)$ and X_1 is in the class $\mathcal{C}(\theta, A_0, A_1)$ (again see Definition 3.51 and Theorem 3.5.2 in pp. 48–49 of [5]). We can apply the reiteration theorem (see Theorem 3.5.3 on p. 50 of [5]) and (3.44) to conclude that, for any $\rho \in [1, \infty]$,

$$(A_0, Y)_{1/2,\rho} \hookrightarrow (X_0, X_1)_{1/2,\rho} = (A_0, A_1)_{\phi,\sigma}, \tag{3.45}$$

where $\phi = \theta/2$. Fix $\rho \in (0, 1)$ such that $1/\rho = 1/(2p) + 1/(2q)$. In this case, by Lemma 2.6, we have that

$$(A_0, Y)_{1/2,\rho} = (\dot{F}^{s,p}_t, \dot{F}^{\sigma,q}_\tau)_{1/2,\rho} = \dot{B}^{\beta,\rho}_\rho,$$

where $\beta := s/2 + \sigma/2$, which together with (3.45) gives

$$\dot{B}^{\beta,\rho}_\rho \hookrightarrow (\dot{F}^{s,p}_t, \dot{W}^{r,\infty})_{\phi,\rho}.$$

By the fact that $L^\infty \hookrightarrow \mathrm{BMO} = \dot{F}^{0,\infty}_2$, we have

$$\begin{aligned}(\dot{F}^{s,p}_t, \dot{W}^{r,\infty})_{\phi,\rho} \hookrightarrow (\dot{F}^{s,p}_t, \dot{F}^{r,\infty}_2)_{\phi,\rho} &= (\dot{F}^{-s,p'}_{t'}, \dot{F}^{-r,1}_2)^*_{\phi,\rho'}\\ &= (\dot{B}^{-\beta,\rho'}_{\rho'})^*_{\phi,\rho'} = \dot{B}^{\beta,\rho}_\rho,\end{aligned} \tag{3.46}$$

where we also have used Lemma 2.6. Hence,

$$\dot{B}^{\beta,\rho}_\rho = (\dot{F}^{s,p}_t, \dot{W}^{r,\infty})_{\phi,\rho}. \tag{3.47}$$

As we can easily check, $\beta = (1-\phi)s + \phi r$ and $1/\rho = \phi/p$. This shows, via Proposition 3.18, that (3.47) does not hold. ■

Proof of Proposition 1.5. Suppose by contradiction that

$$W^{\sigma,q} \hookrightarrow W^{s,p} + W^{r,\infty}.$$

Since $s > 0$, from this we get that

$$W^{\sigma,q} \hookrightarrow \dot{W}^{s,p} + \dot{W}^{r,\infty}.$$

Combining this with Lemma 2.3 (i), we get the estimate

$$\|L_l f\|_{\dot{W}^{s,p}+\dot{W}^{r,\infty}} \lesssim \|L_l f\|_{W^{\sigma,q}} \lesssim \|f\|_{\dot{W}^{\sigma,q}}, \tag{3.48}$$

for any Schwartz function f, where l is the smallest integer with $l \geq \sigma$. By Lemma 2.3 (ii), we also have

$$\|\tilde{L}_l f\|_{\dot{W}^{s,p}+\dot{W}^{r,\infty}} \lesssim \|\tilde{L}_l f\|_{W^{r,\infty}} \lesssim \|f\|_{\dot{W}^{\sigma,q}}. \tag{3.49}$$

Now, by the triangle inequality, (3.48) and (3.49) imply

$$\|f\|_{\dot{W}^{s,p}+\dot{W}^{r,\infty}} \leq \|L_l f\|_{\dot{W}^{s,p}+\dot{W}^{r,\infty}} + \|\tilde{L}_l f\|_{\dot{W}^{s,p}+\dot{W}^{r,\infty}} \lesssim \|f\|_{\dot{W}^{\sigma,q}},$$

for any Schwartz function f. However, this implies the embedding

$$\dot{W}^{\sigma,q} \hookrightarrow \dot{W}^{s,p} + \dot{W}^{r,\infty},$$

which, by Proposition 3.25, cannot hold. ∎

Note that we deduced Proposition 1.5 from Proposition 3.25, which is a result of a global nature. It will be interesting to see if the conclusion of Proposition 1.5 remains true for spaces defined on compact domains. This can be compared with the situation when the parameters s, r, p are in the nonpathological case. More precisely, suppose Q is the cube $[-1,1]^d$ and the parameters $s > 0$, $r \in \mathbb{N}$, $p \in (1,\infty)$ satisfy $s \notin [r, r + d/p]$ (recall that we always consider $s \neq r$). Then

$$W^{\sigma,q}(Q) \hookrightarrow W^{s,p}(Q) + W^{r,\infty}(Q),$$

where the parameters σ, q are as in the statement of Proposition 1.5. In fact, we have the more general embedding

$$F^{\sigma,q}_{\tau}(Q) \hookrightarrow F^{s,p}_{t}(Q) + W^{r,\infty}(Q),$$

where $t, \tau \in [1,\infty)$. Indeed, if $s < r$, then we can write

$$\Big(\sigma - \frac{d}{q}\Big) - \Big(s - \frac{d}{p}\Big) = \theta\Big(r + \frac{d}{p} - s\Big) > 0.$$

We also have that $\sigma = (1-\theta)s + \theta r > s$ (since $s > r$), and we get, from the classical embedding Theorem 1(3) on p. 82 of [35], that

$$F^{\sigma,q}_{\tau}(Q) \hookrightarrow F^{s,p}_{t}(Q) \hookrightarrow F^{s,p}_{t}(Q) + W^{r,\infty}(Q).$$

If $s > r + d/p$, then

$$\sigma - r - \frac{d}{q} = (1-\theta)\Big(s - r - \frac{d}{p}\Big) > 0,$$

and this implies (using the classical embedding $F^{\sigma-r,q}_{\tau} \hookrightarrow L^\infty$, when $\sigma > r + d/q$) that

$$F^{\sigma,q}_{\tau}(Q) \hookrightarrow W^{r,\infty}(Q) \hookrightarrow F^{s,p}_{t}(Q) + W^{r,\infty}(Q).$$

3.4. The unboundedness of the Riesz transforms

Let us turn the attention to the behavior of some operators given by singular integrals that act on some "pathological" interpolation spaces. Here, we restrict ourselves to a class of pathologies that are subject of Proposition 3.19. Let us recall that when $\theta \in (0,1)$, $p \in (1,\infty)$, $t \in [1,\infty]$, $s \in \mathbb{R}$ are some parameters such that $0 < s \le d/p$, by Proposition 3.19 (setting $r = 0$), we have

$$(\dot{F}^{s,p}_t(\mathbb{R}^d), L^\infty(\mathbb{R}^d))_{\theta,q} \neq \dot{B}^{\sigma,q}_q(\mathbb{R}^d), \tag{3.50}$$

where $\sigma = (1-\theta)s$ and $1/q = (1-\theta)/p$. It turns out that much more is true. Namely, at least when $s \neq d/p$, none of the Riesz transforms is bounded on the space defined by the left-hand side of (3.50). This can be deduced from (3.50) and the following result.

Proposition 3.26. *Consider the function space*

$$X := (\dot{F}^{s,p}_t(\mathbb{R}^d), L^\infty(\mathbb{R}^d))_{\theta,q},$$

where $\theta \in (0,1)$, $p \in (1,\infty)$, $t \in [1,\infty)$, $1/q = (1-\theta)/p$ and $s > 0$, with $s \neq d/p$. If $X \neq \dot{B}^{\sigma,q}_q$, then none of the Riesz transforms is bounded on X. The same fact holds for the inhomogeneous spaces.

Remark 3.27. Let us note that, by Proposition 3.26, one can construct in a natural way function spaces with some special properties. Suppose that X is the space defined in Proposition 3.26 above and $0 < s < d/p$. Suppose also that $t \in (1,\infty)$. Then $\dot{F}^{s,p}_t$ is uniformly convex and, by a theorem of Beauzamy (see, for instance, Theorem 2.g.21 on p. 229 of [26]), the space X is uniformly convex. One can immediately check that X is translation and rotation invariant. Now, by Proposition 3.26 and (3.50), the space X is a uniformly convex function space on $\mathbb{R}^d$, translation and rotation invariant, and no R_j is bounded on X. The same observation applies in the inhomogeneous case.

Remark 3.28. Proposition 3.26 implies, in particular, that X is never a Besov space. Otherwise, we would get that $X = B^{\sigma,q}_\tau$ for some $\tau \in [1,\infty]$. However, the Riesz transforms are bounded on $B^{\sigma,q}_\tau$ and not on X.

In order to prove Proposition 3.26, we need the following homogeneous variant of a result of Adams and Frazier (Theorem 2 in [1]).[10]

Lemma 3.29. *Suppose $s > 0$ and $p \in (1,\infty)$. For any $f \in \dot{W}^{s,p} \cap \mathrm{BMO}$, there exist $f_0, f_1, \dots, f_d \in \dot{W}^{s,p} \cap L^\infty$ such that*

$$f = f_0 + R_1 f_1 + \dots + R_d f_d$$

and

$$\sum_{j=0}^{d} \|f_j\|_{\dot{W}^{s,p}\cap L^\infty} \lesssim \|f\|_{\dot{W}^{s,p}\cap \mathrm{BMO}}.$$

[10] Lemma 3.29 (that follows) can be proven on the same lines as Theorem 2 in [1]. However, since the proof in Theorem 2 in [1] is too involved to be outlined here, we prefer to deduce Lemma 3.29 from Theorem 2 in [1].

Proof. Let l be the smallest integer with $l \geq s$ and fix some $f \in \dot{W}^{s,p} \cap \mathrm{BMO}$. By Lemma 2.3 (i), we have

$$\|L_l f\|_{W^{s,p}} \lesssim \|f\|_{\dot{W}^{s,p}} \lesssim \|f\|_{\dot{W}^{s,p} \cap \mathrm{BMO}},$$

and, by Lemma 2.3 (ii),

$$\|L_l f\|_{\mathrm{BMO}} \lesssim \|f\|_{\mathrm{BMO}} + \|\tilde{L}_l f\|_{\mathrm{BMO}} \lesssim \|f\|_{\mathrm{BMO}} + \|\tilde{L}_l f\|_{L^\infty} \lesssim \|f\|_{\dot{W}^{s,p} \cap \mathrm{BMO}}.$$

Hence, we can write

$$\|L_l f\|_{W^{s,p} \cap \mathrm{BMO}} \lesssim \|f\|_{\dot{W}^{s,p} \cap \mathrm{BMO}}. \tag{3.51}$$

Using the Adams–Frazier theorem, Theorem 2 in [1], we have a decomposition of the form

$$L_l f = f_0' + R_1 f_1' + \cdots + R_d f_d'$$

such that

$$\sum_{j=0}^{d} \|f_j'\|_{\dot{W}^{s,p} \cap L^\infty} \lesssim \sum_{j=0}^{d} \|f_j'\|_{W^{s,p} \cap L^\infty} \lesssim \|L_l f\|_{W^{s,p} \cap \mathrm{BMO}}. \tag{3.52}$$

Now we can set $f_0 := f_0' + \tilde{L}_l f$ and $f_j := f_j'$, for any $1 \leq j \leq d$. Thanks to the decomposition

$$f = L_l f + \tilde{L}_l f,$$

we have

$$f = f_0 + R_1 f_1 + \cdots + R_d f_d.$$

Also, by (3.51) and (3.52),

$$\sum_{j=1}^{d} \|f_j\|_{\dot{W}^{s,p} \cap L^\infty} \lesssim \|f\|_{\dot{W}^{s,p} \cap \mathrm{BMO}}$$

and

$$\|f_0'\|_{\dot{W}^{s,p} \cap L^\infty} \lesssim \|f\|_{\dot{W}^{s,p} \cap \mathrm{BMO}}.$$

We combine this with the estimate

$$\|\tilde{L}_l f\|_{\dot{W}^{s,p} \cap L^\infty} \lesssim \|f\|_{\dot{W}^{s,p}} \lesssim \|f\|_{\dot{W}^{s,p} \cap \mathrm{BMO}}$$

(see Lemma 2.3 (ii)), and we get

$$\|f_0\|_{\dot{W}^{s,p} \cap L^\infty} \leq \|f_0'\|_{\dot{W}^{s,p} \cap L^\infty} + \|\tilde{L}_l f\|_{\dot{W}^{s,p} \cap L^\infty} \lesssim \|f\|_{\dot{W}^{s,p} \cap \mathrm{BMO}},$$

concluding the proof of Lemma 3.29. ∎

Proof of Proposition 3.26. It suffices to prove Proposition 3.26 in the case $t = p$ or $t = 2$. One can then prove the general case by reiteration as in the proof of Proposition 3.18. By Lemma 3.29, for any $g \in \dot{W}^{s,p} \cap \mathrm{BMO}$, there exist $g_0, g_1, \ldots, g_d \in \dot{W}^{s,p} \cap L^\infty$ such that

$$g = g_0 + R_1 g_1 + \cdots + R_d g_d \tag{3.53}$$

and

$$\sum_{j=0}^{d} \|g_j\|_{\dot{W}^{s,p} \cap L^\infty} \lesssim \|g\|_{\dot{W}^{s,p} \cap \mathrm{BMO}}. \tag{3.54}$$

Let $f \in \dot{W}^{-s,p'} + H^1$ and consider some $g \in \dot{W}^{s,p} \cap \mathrm{BMO}$, with $\|g\|_{\dot{W}^{s,p}\cap \mathrm{BMO}} = 1$, such that

$$\|f\|_{\dot{W}^{-s,p'}+H^1} \le 2\langle f, g\rangle. \tag{3.55}$$

Using the decomposition (3.53) for this g, and the estimate (3.54), we get

$$\begin{aligned}\langle f, g\rangle &= \langle f, g_0\rangle + \langle R_1 f, g_1\rangle + \cdots + \langle R_d f, g_d\rangle \\ &\lesssim \|f\|_{L^1+\dot{W}^{-s,p'}} + \|R_1 f\|_{L^1+\dot{W}^{-s,p'}} + \cdots + \|R_d f\|_{L^1+\dot{W}^{-s,p'}},\end{aligned}$$

which, together with (3.55), gives

$$\|f\|_{\dot{W}^{-s,p'}+H^1} \lesssim \|\tilde{R} f\|_{\dot{W}^{-s,p'}+L^1},$$

where $\tilde{R} f := (f, R_1 f, \ldots, R_d f)$. Thanks to the fact that the Riesz transforms are bounded from H^1 to L^1 (and hence from $\dot{W}^{-s,p'} + H^1$ to $\dot{W}^{-s,p'} + L^1$), we have

$$\|f\|_{\dot{W}^{-s,p'}+H^1} \sim \|\tilde{R} f\|_{\dot{W}^{-s,p'}+L^1}. \tag{3.56}$$

Now consider some $\lambda > 0$ and define $f^\lambda(\cdot) := f(\lambda\cdot)$. It is easy to see that for $t := \lambda^{-s+d/p}$ (see (2.2)),

$$\|f^\lambda\|_{\dot{W}^{-s,p'}+H^1} = \lambda^{-d} K_t(f, \dot{W}^{-s,p'}, H^1),$$

and in a similar way,

$$\|\tilde{R} f^\lambda\|_{\dot{W}^{-s,p'}+L^1} = \|(\tilde{R} f)^\lambda\|_{\dot{W}^{-s,p'}+L^1} = \lambda^{-d} K_t(\tilde{R} f, \dot{W}^{-s,p'}, L^1).$$

Now, we get from (3.56) that

$$K_t(f, \dot{W}^{-s,p'}, H^1) \sim K_t(\tilde{R} f, \dot{W}^{-s,p'}, L^1),$$

for any $t > 0$, and consequently

$$\|f\|_{(\dot{W}^{-s,p'},H^1)_{\theta,q'}} \sim \|\tilde{R} f\|_{(\dot{W}^{-s,p'},L^1)_{\theta,q'}}. \tag{3.57}$$

Notice that the dual of the closed subspace

$$V := \{\tilde{R} f \mid f \in (\dot{W}^{-s,p'}, L^1)_{\theta,q'}\},$$

of $((\dot{W}^{-s,p'}, L^1)_{\theta,q'})^{1+d}$ is the space of all functions g that can be decomposed as in (3.53) with the norm given by

$$\|g\|_{V^*} = \inf \sum_{j=0}^{d} \|g_j\|_X,$$

where the infimum is taken over all decompositions of the form given in (3.53). Schematically, we can write this as

$$V^* = X + R_1 X + \cdots + R_d X. \tag{3.58}$$

The dual of $(\dot{W}^{-s,p'}, H^1)_{\theta,q'}$ is (see, for instance, (3.46))

$$(\dot{W}^{s,p}, \mathrm{BMO})_{\theta,q} = (\dot{W}^{s,p}, \dot{F}_2^{0,\infty})_{\theta,q} = \dot{B}_q^{\sigma,q}. \tag{3.59}$$

By (3.57), (3.58) and (3.59), we get that

$$\dot{B}_q^{\sigma,q} = X + R_1 X + \cdots + R_d X. \tag{3.60}$$

Observe that $X \hookrightarrow \dot{B}_q^{\sigma,q}$. If all the Riesz transforms are bounded on X, then by (3.60) we obtain $X = \dot{B}_q^{\sigma,q}$, which is not the case. Hence, at least one of the Riesz transforms, suppose R_1, is not bounded on X. Any R_j can be obtained from R_1 by a rotation of the coordinates and we can immediately see that X is invariant to rotations. It follows that any R_j is unbounded on X. We have proved Proposition 3.26 in the homogeneous case.

Now, let us prove Proposition 3.26 in the case of the inhomogeneous spaces. Consider some parameters $\lambda > 0$, $n \in \mathbb{N}^*$ and the operator T_λ^n defined by

$$T_\lambda^n f := \Big(\sum_{-n \le k \le n} P_k f \Big)^\lambda$$

for any Schwartz function f, where $P_k f$ $(k \in \mathbb{Z})$ are the Littlewood–Paley pieces of f. Let $\Phi \in C_c^\infty(B(0,2))$, with $\Phi \equiv 1$ on $B(0,1)$, be the function such that (see Section 2.1)

$$\widehat{P_k f}(\xi) = (\Phi(\xi/2^k) - \Phi(\xi/2^{k-1}))\, \hat{f}(\xi).$$

We get

$$\sum_{-n \le k \le n} \widehat{P_k f}(\xi) = (\Phi(\xi/2^n) - \Phi(\xi/2^{-n-1}))\, \hat{f}(\xi).$$

Hence, we have $T_\lambda^n f := (f * \varsigma_n)^\lambda$, where

$$\varsigma_n := (\check{\Phi})_{2^{-n}} - (\check{\Phi})_{2^{n+1}}.$$

(Here, $(\check{\Phi})_\lambda(\cdot) = \lambda^{-d} \check{\Phi}(\lambda \cdot)$.)

Using the Littlewood–Paley square function theorem for L^p, it is easy to see that

$$\begin{aligned} \|T_\lambda^n f\|_{F_t^{s,p}} &= \lambda^{-d/p} \|f * \varsigma_n\|_{L^p} + \lambda^{s-d/p} \|f * \varsigma_n\|_{\dot{F}_t^{s,p}} \\ &\lesssim \lambda^{-d/p} C_n \|f\|_{\dot{F}_t^{s,p}} + \lambda^{s-d/p} \|f * \varsigma_n\|_{\dot{F}_t^{s,p}} \\ &\lesssim \lambda^{s-d/p} (1 + \lambda^{-s} C_n) \|f\|_{\dot{F}_t^{s,p}}, \end{aligned} \tag{3.61}$$

where C_n is a constant depending on n, s, p and t. We also have the estimate

$$\|T_\lambda^n f\|_{L^\infty} = \|f * \varsigma_n\|_{L^\infty} \le \|\varsigma_n\|_{L^1} \|f\|_{L^\infty} \le 2 \|\check{\Phi}\|_{L^1} \|f\|_{L^\infty}. \tag{3.62}$$

From (3.61), (3.62), we get by interpolation that

$$\|T_\lambda^n f\|_{(F_t^{s,p}, L^\infty)_{\eta,q}} \lesssim (1 + \lambda^{-s} C_n)^{1-\eta} \lambda^{\sigma - d/q} \|f\|_{(\dot{F}_t^{s,p}, L^\infty)_{\eta,q}}, \tag{3.63}$$

for any Schwartz function f, where the implicit constant does not depend on n or λ. Suppose now that

$$R_j : (F_t^{s,p}, L^\infty)_{\theta,q} \to (F_t^{s,p}, L^\infty)_{\theta,q}$$

is bounded. Since $(F_t^{s,p}, L^\infty)_{\theta,q} \hookrightarrow (\dot{F}_t^{s,p}, L^\infty)_{\theta,q}$, we get

$$\lambda^{-\sigma+d/q} \|R_j T_\lambda^n f\|_{(\dot{F}_t^{s,p},L^\infty)_{\theta,q}} \lesssim \lambda^{-\sigma+d/q} \|T_\lambda^n f\|_{(F_t^{s,p},L^\infty)_{\theta,q}}, \tag{3.64}$$

for any Schwartz function f. It is easy to verify that

$$\lambda^{-\sigma+d/q} \|R_j T_\lambda^n f\|_{(\dot{F}_t^{s,p},L^\infty)_{\theta,q}} \sim \|R_j T_1^n f\|_{(\dot{F}_t^{s,p},L^\infty)_{\theta,q}}.$$

Also, by (3.63), when $\lambda \to \infty$, we have

$$\liminf_{\lambda\to\infty} \lambda^{-\sigma+d/q} \|T_\lambda^n f\|_{(F_t^{s,p},L^\infty)_{\theta,q}} \lesssim \|f\|_{(\dot{F}_t^{s,p},L^\infty)_{\theta,q}},$$

and now (3.64) implies

$$\|R_j T_1^n f\|_{(\dot{F}_t^{s,p},L^\infty)_{\theta,q}} \lesssim \|f\|_{(\dot{F}_t^{s,p},L^\infty)_{\theta,q}}, \tag{3.65}$$

where the implicit constant does not depend on n. Since $R_j T_1^n f \to R_j f$ in the sense of distributions (when $n \to \infty$), it remains to observe that

$$\|R_j f\|_{(\dot{F}_t^{s,p},L^\infty)_{\theta,q}} \le \liminf_{n\to\infty} \|R_j T_1^n f\|_{(\dot{F}_t^{s,p},L^\infty)_{\theta,q}},$$

and by (3.65) we have reduced the boundedness of R_j on $(F_t^{s,p}, L^\infty)_{\theta,q}$ to the boundedness on $(\dot{F}_t^{s,p}, L^\infty)_{\theta,q}$. Now, we can apply the homogeneous case of Proposition 3.26 in order to complete the proof. ■

Recall that we have the embeddings

$$(F_t^{s,p}, L^\infty)_{\theta,1} \hookrightarrow (F_t^{s,p}, L^\infty)_{\theta,q} \hookrightarrow (F_t^{s,p}, L^\infty)_{\theta,1},$$

where all the parameters are as in the statement of Proposition 3.26 above. In view of this embedding, we can strengthen Proposition 3.26 by observing that R_j is not bounded from $(F_t^{s,p}, L^\infty)_{\theta,1}$ to $(F_t^{s,p}, L^\infty)_{\theta,\infty}$. Indeed, if

$$R_j : (F_t^{s,p}, L^\infty)_{\theta,1} \to (F_t^{s,p}, L^\infty)_{\theta,\infty}$$

were bounded, then, since

$$R_j : F_t^{s,p} \to F_t^{s,p}$$

is bounded, we would get by interpolation that

$$R_j : (F_t^{s,p}, (F_t^{s,p}, L^\infty)_{\theta,1})_{1/2,\rho} \to (F_t^{s,p}, (F_t^{s,p}, L^\infty)_{\theta,\infty})_{1/2,\rho}$$

is bounded, where $1/\rho = (1-\theta/2)/p$. Since $F_t^{s,p}$ is a space of class $\mathcal{C}(0, F_t^{s,p}, L^\infty)$ and $(F_t^{s,p}, L^\infty)_{\theta,1}$, $(F_t^{s,p}, L^\infty)_{\theta,\infty}$ are spaces of class $\mathcal{C}(\theta, F_t^{s,p}, L^\infty)$, we get by reiteration that

$$(F_t^{s,p}, (F_t^{s,p}, L^\infty)_{\theta,1})_{1/2,\rho} = (F_t^{s,p}, (F_t^{s,p}, L^\infty)_{\theta,\infty})_{1/2,\rho} = (F_t^{s,p}, L^\infty)_{\theta/2,\rho}.$$

This combined with (3.65) gives the boundedness of the operator

$$R_j : (F_t^{s,p}, L^\infty)_{\theta/2,\rho} \to (F_t^{s,p}, L^\infty)_{\theta/2,\rho},$$

which, as Proposition 3.26 shows, cannot hold.

To summarize, we have obtained the following result.

Proposition 3.30. *Let $\theta \in (0,1)$, $p \in (1,\infty)$, $t \in [1,\infty]$, $s \in \mathbb{R}$ be such that $0 < s < d/p$. Then, none of the Riesz transforms is bounded from $(F_t^{s,p}, L^\infty)_{\theta,1}$ to $(F_t^{s,p}, L^\infty)_{\theta,\infty}$.*

4. The complex method

4.1. A nonembedding property

We give now a proof of Proposition 1.7 based on standard trace theory and basic embedding properties of Triebel–Lizorkin spaces.

Proof of Proposition 1.7. By Lemma 2.4 (see (2.10)), it suffices to prove that

$$F_t^{\sigma,\rho}(\mathbb{R}^d) \not\hookrightarrow (F_q^{s,p}(\mathbb{R}^d), C^l(\mathbb{R}^d))_\theta.$$

Suppose by contradiction that

$$F_t^{\sigma,\rho}(\mathbb{R}^d) \hookrightarrow (F_q^{s,p}(\mathbb{R}^d), C^l(\mathbb{R}^d))_\theta. \tag{4.1}$$

By standard trace theory, we have

$$\operatorname{Tr} F_t^{\sigma,\rho}(\mathbb{R}^d) = B_r^{\sigma-1/\rho,\rho}(\mathbb{R}^{d-1}).$$

Also, since the trace operator $\operatorname{Tr}: F_q^{s,p}(\mathbb{R}^d) \to B_p^{s-1/p,p}(\mathbb{R}^{d-1})$, $\operatorname{Tr}: C^l(\mathbb{R}^d) \to C^l(\mathbb{R}^{d-1})$ is bounded, (4.1) implies

$$\begin{aligned} B_\rho^{\sigma-1/\rho,\rho}(\mathbb{R}^{d-1}) = \operatorname{Tr}(F_t^{\sigma,\rho}(\mathbb{R}^d)) &\hookrightarrow \operatorname{Tr}(F_q^{s,p}(\mathbb{R}^d), C^l(\mathbb{R}^d))_\theta \\ &\hookrightarrow (\operatorname{Tr} F_q^{s,p}(\mathbb{R}^d), \operatorname{Tr} C^l(\mathbb{R}^d))_\theta \\ &= (B_p^{s-1/p,p}(\mathbb{R}^{d-1}), C^l(\mathbb{R}^{d-1}))_\theta. \end{aligned} \tag{4.2}$$

Thanks to the fact that $C^l(\mathbb{R}^{d-1}) \hookrightarrow F_2^{l,\infty}(\mathbb{R}^{d-1})$ (recall that $C(\mathbb{R}^{d-1}) \hookrightarrow \operatorname{bmo}(\mathbb{R}^{d-1}) = F_2^{0,\infty}(\mathbb{R}^{d-1})$), we have

$$\begin{aligned} (B_p^{s-1/p,p}(\mathbb{R}^{d-1}), C^l(\mathbb{R}^{d-1}))_\theta &\hookrightarrow (B_p^{s-1/p,p}(\mathbb{R}^{d-1}), F_2^{l,\infty}(\mathbb{R}^{d-1}))_\theta \\ &= F_{\rho_1}^{\sigma-1/\rho,\rho}(\mathbb{R}^{d-1}), \end{aligned}$$

and from (4.2), one gets

$$B_\rho^{\sigma-1/\rho,\rho}(\mathbb{R}^{d-1}) \hookrightarrow F_{\rho_1}^{\sigma-1/\rho,\rho}(\mathbb{R}^{d-1}), \tag{4.3}$$

where $1/\rho_1 = (1-\theta)/p + \theta/2$.

However, since $\rho_1 < \rho$, the embedding (4.3) is false (see, e.g., Theorem 3.1.1 (i) in [37]). ∎

Remark 4.1. Since, as long as $1 < p < \infty$, the spaces $W^{s,p}$ and $W^{\sigma,\rho}$ are Triebel–Lizorkin spaces, Proposition 1.7 gives, in particular, that

$$(W^{s,p}(\mathbb{R}^d), W^{l,\infty}(\mathbb{R}^d))_\theta \neq W^{\sigma,\rho}(\mathbb{R}^d).$$

Remark 4.2. Note that we get a similar statement if we consider the corresponding spaces on $\mathbb{T}^d$. Namely, as long as $l \geq 0$ is an integer and $1 < p, q < \infty$, if $s > 1/p$, then (keeping the notation from the statement of Proposition 1.7)

$$F_t^{\sigma,r}(\mathbb{T}^d) \not\hookrightarrow (F_q^{s,p}(\mathbb{T}^d), W^{l,\infty}(\mathbb{T}^d))_\theta.$$

In particular, we have

$$(W^{s,p}(\mathbb{T}^d), W^{l,\infty}(\mathbb{T}^d))_\theta \neq W^{\sigma,\rho}(\mathbb{T}^d),$$

which easily implies that

$$(W_\sharp^{s,p}(\mathbb{T}^d), W_\sharp^{l,\infty}(\mathbb{T}^d))_\theta \neq W_\sharp^{\sigma,\rho}(\mathbb{T}^d).$$

Corollary 4.3. *Let $l \geq 1$ be an integer. Fix some $\theta \in (0,1)$ and define $p := 1/(1-\theta)$. We have*

$$(W^{l,1}(\mathbb{R}^d), W^{l,\infty}(\mathbb{R}^d))_\theta \neq W^{l,p}(\mathbb{R}^d).$$

Proof. Suppose by contradiction that

$$(W^{l,1}, W^{l,\infty})_\theta = W^{l,p}. \tag{4.4}$$

Since $1 < p < \infty$, by Milman's result (see Theorem B in [29]), we have

$$(W^{l,1}, W^{l,p})_\eta = W^{l,p_1}, \tag{4.5}$$

for some $\eta \in (0,1)$, where $1/p_1 := 1 - \eta + \eta/p$. By (4.4) and reiteration, one can rewrite (4.5) as

$$(W^{l,1}, W^{l,\infty})_{\eta_1} = (W^{l,1}, (W^{l,1}, W^{l,\infty})_\theta)_\eta = W^{l,p_1}, \tag{4.6}$$

where $\eta_1 := (1-\theta)\eta + \theta$. Using now (4.6) and the reiteration theorem (see Theorem 4.6.1 on p. 101 of [5]), we have

$$(W^{l,p_1}, W^{l,\infty})_{\eta_2} = ((W^{l,1}, W^{l,\infty})_{\eta_1}, W^{l,\infty})_{\eta_2} = (W^{l,1}, W^{l,\infty})_\theta = W^{l,p}, \tag{4.7}$$

where $\eta_1 := \theta/\eta_1$. However, since $1 < p_1 < \infty$, according to Proposition 1.7 (see also Remark 4.1), we cannot have (4.7). This concludes the proof of Corollary 4.3. ∎

4.2. Some noncomplemented subspaces of $(C(\mathbb{T}^d))^N$

Consider some function $m = (m_1, \ldots, m_N): \mathbb{Z}^d \to \mathbb{R}^N$ and let us denote by $\|m\|$ the function

$$|m| := (m_1^2 + \cdots + m_N^2)^{1/2}.$$

By $m(\nabla)$ we mean the Fourier multiplier whose symbol is m. In other words, for any trigonometric polynomial f, we have

$$m(\nabla)f := (m_1(\nabla)f, \ldots, m_N(\nabla)f),$$

where

$$m_j(\nabla)(e^{2\pi i\langle n,\cdot\rangle})(x) = m_j(n)e^{2\pi i\langle n,x\rangle},$$

for any $n \in \mathbb{Z}^d$ and any $1 \le j \le N$.

Fix some $s > 0$. We say that m is s-admissible if:

(i) for any $\rho \in (1,\infty)$ and any trigonometric polynomial f on $\mathbb{T}^d$, we have

$$\|m(\nabla)f\|_{L^\rho(\mathbb{T}^d)} \sim_\rho \||\nabla|^s f\|_{L^\rho(\mathbb{T}^d)};$$

(ii) there exists $N_\flat < N$ such that $m_\flat := (m_1, \ldots, m_{N_\flat})$ depends only on the first $d_\flat$ coordinates $(n_1, \ldots, n_{d_\flat})$ for some $d_\flat < d$ and, for any trigonometric polynomial f on $\mathbb{T}^{d_\flat}$,

$$\|m(\nabla)f\|_{\mathrm{bmo}(\mathbb{T}^{d_\flat})} \sim \||\nabla|^s f\|_{\mathrm{bmo}(\mathbb{T}^{d_\flat})}.$$

Given a vector space X of distributions on $\mathbb{T}^d$, we denote by $G_m(X)$ the vector space

$$G_m(X) := \{m(\nabla)g \mid g \in \mathcal{D}'(\mathbb{T}^d) \text{ such that } m(\nabla)g \in X^N\} \subseteq X^N.$$

When X is a Banach function space, we endow $G_m(X)$ with the norm induced by X^N.

The spaces $C^m(\mathbb{T}^d)$, $W^mL^\rho(\mathbb{T}^d)$ and $\mathrm{bmo}^{m_\flat}(\mathbb{T}^{d_\flat})$ are spaces of distributions f on $\mathbb{T}^d$ (or $\mathbb{T}^{d_\flat}$), for which the following norms are finite:

$$\|f\|_{C^m(\mathbb{T}^d)} := \|f\|_{C(\mathbb{T}^d)} + \|m(\nabla)f\|_{C(\mathbb{T}^d)},$$
$$\|f\|_{W^mL^\rho(\mathbb{T}^d)} := \|f\|_{L^\rho(\mathbb{T}^d)} + \|m(\nabla)f\|_{L^\rho(\mathbb{T}^d)}$$

and

$$\|f\|_{\mathrm{bmo}^{m_\flat}(\mathbb{T}^{d_\flat})} := \|f\|_{\mathrm{bmo}(\mathbb{T}^{d_\flat})} + \|m(\nabla)f\|_{\mathrm{bmo}(\mathbb{T}^{d_\flat})},$$

respectively.

Note that property (i) implies

$$W^mL^\rho(\mathbb{T}^d) = F_2^{s,\rho}(\mathbb{T}^d), \tag{4.8}$$

for any $\rho \in (1,\infty)$. Also, by (ii), we can see that

$$\mathrm{bmo}^{m_\flat}(\mathbb{T}^{d_\flat}) = F_2^{s,\infty}(\mathbb{T}^{d_\flat}). \tag{4.9}$$

Lemma 4.4. *For m as above and $\theta \in (0,1)$, we have*

$$(C^m(\mathbb{T}^d), W^mL^p(\mathbb{T}^d))_\theta \neq W^mL^q(\mathbb{T}^d),$$

for any $p \in (d_\flat/s, \infty)$ and $1/q = \theta/p$. We also have

$$(C^m_\sharp(\mathbb{T}^d), W^mL^p_\sharp(\mathbb{T}^d))_\theta \neq W^mL^q_\sharp(\mathbb{T}^d).$$

Proof. The argument is similar to the one used in the proof of Proposition 1.7. Suppose by contradiction that we have

$$W^m L^\rho(\mathbb{T}^d) = (C^m(\mathbb{T}^d), W^m L^p(\mathbb{T}^d))_\theta. \tag{4.10}$$

Since $s > d_\flat/p > d_\flat/q$, we can write (using (4.8)),

$$B_p^{s-(d-d_\flat)/p,p}(\mathbb{T}^{d'}) = \mathrm{Tr}_{d_\flat} F_2^{s,p}(\mathbb{T}^d) = \mathrm{Tr}_{d_\flat} W^m L^p(\mathbb{T}^d)$$

and, similarly,

$$B_q^{s-(d-d_\flat)/q,q}(\mathbb{T}^{d_\flat}) = \mathrm{Tr}_{d_\flat} W^m L^q(\mathbb{T}^d).$$

Also, we see directly that

$$\mathrm{Tr}_{d_\flat} C^m(\mathbb{T}^d) \hookrightarrow C^{m_\flat}(\mathbb{T}^{d_\flat}).$$

Using these considerations and (4.10), (4.9),

$$\begin{aligned} B_p^{s-(d-d_\flat)/q,q}(\mathbb{T}^{d_\flat}) &= \mathrm{Tr}_{d_\flat}(C^m(\mathbb{T}^d), F_2^{s,p}(\mathbb{T}^d))_\theta \\ &\hookrightarrow (\mathrm{Tr}_{d_\flat} C^m(\mathbb{T}^d), \mathrm{Tr}_{d_\flat} F_2^{s,p}(\mathbb{T}^d))_\theta \\ &\hookrightarrow (C^{m_\flat}(\mathbb{T}^{d_\flat}),\ F_p^{s-(d-d_\flat)/p,p}(\mathbb{T}^{d_\flat}))_\theta \\ &\hookrightarrow (\mathrm{bmo}^{m_\flat}(\mathbb{T}^{d_\flat}),\ F_p^{s-(d-d_\flat)/p,p}(\mathbb{T}^{d_\flat}))_\theta \\ &= (F_2^{s,\infty}(\mathbb{T}^{d_\flat}),\ F_p^{s-(d-d_\flat)/p,p}(\mathbb{T}^{d_\flat}))_\theta \\ &= F_t^{s-(d-d_\flat)/q,q}(\mathbb{T}^{d_\flat}), \end{aligned}$$

where $1/t = (1-\theta)/2 + \theta/p > 1/q$.

As in the proof of Proposition 1.7, we conclude that this embedding is false, and hence (4.10) must be false. The first nonequality of Lemma 4.4 is proved. The second nonequality follows immediately from the first one or by slightly modifying the proof of the first nonequality. ∎

Using this, we can prove the following.

Theorem 4.5. *Suppose $m = (m_1, \dots, m_N)$ is s-admissible for some $s > 0$. Then the space $G_m(C)$ is not complemented in $(C(\mathbb{T}^d))^N$.*

Proof. We will prove it by contradiction. Suppose that there exists a bounded onto projection $P: (C(\mathbb{T}^d))^N \to G_m(C)$. Define $\tilde{P}: (C(\mathbb{T}^d))^N \to (C(\mathbb{T}^d))^N$ by

$$\tilde{P} f := \int_{\mathbb{T}^d} \tau_{-y} P \tau_y f \, dy, \tag{4.11}$$

for any $f \in (C(\mathbb{T}^d))^N$, where τ_y is the translation operator of the vector y ($\tau_y f(x) = f(x+y)$, for any $x \in \mathbb{T}^d$) and dy is the normalized Haar measure on $\mathbb{T}^d$. It is easy to verify that $\tilde{P}$ is indeed bounded on $(C(\mathbb{T}^d))^N$. One can also verify that $\tilde{P}: (C(\mathbb{T}^d))^N \to G_m(C)$ is an onto projection. Indeed, if $f \in G_m(C)$, then $\tau_y f \in G_m(C)$, and hence $P\tau_y f = \tau_y f$, for all $y \in \mathbb{T}^d$. This and (4.11) give that $\tilde{P} f = f$, for every $f \in G_m(C)$. Conversely, if $f \in (C(\mathbb{T}^d))^N$, using the boundedness of P, we have $\tilde{P} f \in G_m(C)$.

Now, one can observe that by (4.11) the projection $\tilde{P}$ is invariant to translations, and hence it is a Fourier multiplier. In other words, for each $n \in \mathbb{Z}^d$, there exists a matrix $M(n) = (M_{ij}(n))_{i,j=1,\dots,N}$ such that

$$\widehat{\tilde{P} f}(n) = M(n)\hat{f}(n),$$

for all n. Thanks to the boundedness of $\tilde{P}$ on $(C(\mathbb{T}^d))^N$, one gets

$$\Big\| \sum_{n\in\mathbb{Z}^d} M(n)\hat{f}(n)e^{2\pi i\langle n,\cdot\rangle} \Big\|_{L^\infty} \lesssim \|f\|_{L^\infty}, \tag{4.12}$$

for any $f \in (C(\mathbb{T}^d))^N$. Let $h \in C(\mathbb{T}^d)$ and fix some indices $i, j \in \{1,\dots,N\}$. By setting $f = (0,\dots,0,h,0,\dots,0)$, with h on the j-th position, the bound (4.12) implies

$$\begin{aligned}\Big\| \sum_{n\in\mathbb{Z}^d} M_{ij}(n)\hat{h}(n)\, e^{2\pi i\langle n,\cdot\rangle} \Big\|_{L^\infty} &\le \sum_{k=1}^{N} \Big\| \sum_{n\in\mathbb{Z}^d} M_{kj}(n)\,\hat{h}(n)\, e^{2\pi i\langle n,\cdot\rangle} \Big\|_{L^\infty} \\ &= \Big\| \sum_{n\in\mathbb{Z}^d} M(n)\hat{f}(n)\, e^{2\pi i\langle n,\cdot\rangle} \Big\|_{L^\infty} \lesssim \|f\|_{L^\infty} = \|h\|_{L^\infty}.\end{aligned}$$

Consequently, each operator $M_{ij}(\nabla)$ is bounded on $C(\mathbb{T}^d)$, and by duality, also on $L^1(\mathbb{T}^d)$. Hence, by interpolation, each operator $M_{ij}(\nabla)$ is bounded on $L^p(\mathbb{T}^d)$ for any $p \in (1,\infty)$. For such p, this gives that $\tilde{P}$ is bounded on $(L^p(\mathbb{T}^d))^N$. Using some standard density arguments, it is also easy to verify that $\tilde{P}: (L^p(\mathbb{T}^d))^N \to G_m(L^p)$ is onto. Fix some $p \in (1/s,\infty)$.

Note that each $f \in G_m(\mathcal{D})$ can be written as $f = \nabla^r g$ for some $g \in \mathcal{D}(\mathbb{T}^d)$. Consider the operator $\Psi: G_m(\mathcal{D}) \to \mathcal{D}(\mathbb{T}^d)$ defined by

$$\Psi(m(\nabla)g) = g - \hat{g}(0).$$

We have that Ψ maps $G_k(C)$ to $C^m_\sharp(\mathbb{T}^d)$, respectively, $G_m(L^p)$ to $W^m L^p_\sharp(\mathbb{T}^d)$, isometrically. Hence, $\tilde{P}\Psi: (C(\mathbb{T}^d))^N \to C^m_\sharp(\mathbb{T}^d)$ and $\tilde{P}\Psi: (L^p(\mathbb{T}^d))^N \to W^m L^p_\sharp(\mathbb{T}^d)$ are bounded operators. Note that operator $E = \Psi^{-1} \circ \iota$ (where ι is the canonical embedding $\iota: G_m(C) \to (C(\mathbb{T}^d))^N$ and $\iota: G_m(L^p) \to (L^p(\mathbb{T}^d))^N$) is bounded from $C^m_\sharp(\mathbb{T}^d)$ to $(C(\mathbb{T}^d))^N$ and from $W^m L^p_\sharp(\mathbb{T}^d)$ to $(L^p(\mathbb{T}^d))^N$. Also, E is an extension for $\tilde{P}\Psi$, i.e., $\tilde{P}\Psi \circ E = \mathrm{id}$ on $C^m_\sharp(\mathbb{T}^d) + W^m L^p_\sharp(\mathbb{T}^d)$.

Using this, by the retraction method, we get that, for any $\theta \in (0,1)$,

$$(C^m_\sharp(\mathbb{T}^d), W^m L^p_\sharp(\mathbb{T}^d))_\theta = \tilde{P}\Psi((C(\mathbb{T}^d))^N, (L^p(\mathbb{T}^d))^N)_\theta = \tilde{P}\Psi(L^q(\mathbb{T}^d))^N,$$

where $1/q = \theta/p$.

Note that, by complex interpolation, $\tilde{P}$ is bounded on

$$((C(\mathbb{T}^d))\ ^N, (L^p(\mathbb{T}^d))^N)_\theta = (L^q(\mathbb{T}^d))^N,$$

and E is bounded from $W^m L^q_\sharp(\mathbb{T}^d)$ to $(L^q(\mathbb{T}^d))^N$. Also, as above, one can verify that $\tilde{P}: (L^q(\mathbb{T}^d))^N \to G_m(L^q)$ is onto. From this, we get that $\tilde{P}\Psi(L^q(\mathbb{T}^d))^N = W^m L^q_\sharp(\mathbb{T}^d)$, which combined with (4.12) gives

$$(C^m_\sharp(\mathbb{T}^d), W^m L^p_\sharp(\mathbb{T}^d))_\theta = W^m L^q_\sharp(\mathbb{T}^d).$$

However, by Lemma 4.4 (since $s > 1/p$), this identity is false. ■

A. Appendix

We give below some results concerning the existence and the nonexistence of traces of Besov spaces on some particular subsets of $\mathbb{R}^d$. All the results we give below are known. We only state them here in a form that is convenient for us. Before proceeding to these results, we make some conventions in order to simplify the presentation.

Let $\Gamma \subset \mathbb{R}^d$ be a Borel set, and denote by Tr_Γ the corresponding trace operator. Let $V(\mathbb{R}^d)$ be a Banach function space on $\mathbb{R}^d$ and let $V_1(\mathbb{R}^d)$ be the normed space of all smooth compactly supported functions that are in $V(\mathbb{R}^d)$. The norm on $V_1(\mathbb{R}^d)$ is induced by the norm of $V(\mathbb{R}^d)$. We say that $V(\mathbb{R}^d)$ has no trace on Γ if for any Banach function space $Y(\Gamma)$ on Γ, the trace operator Tr_Γ is not bounded from $V_1(\mathbb{R}^d)$ to $Y(\Gamma)$.

In the case where $\Gamma = \mathbb{R}^l \times \{0\}^{d-l} \simeq \mathbb{R}^l$, we write Tr_l instead of Tr_Γ and even Tr when $l = d-1$.

We are now interested in some critical situations. Proposition A.1 below is essentially known (see, for instance, p. 220 in Section 4.4.3 of [38]). In our applications, one can use instead Theorem A.2; however, the proof we give below of Proposition A.1 is much easier, and in some applications, Proposition A.1 suffices (see, for instance, the proof of Theorem 1.1).

Proposition A.1. *Let $1 < p, q < \infty$ be some parameters and let l be an integer with $0 < l < d$. Then the space $B_q^{l/p,p}(\mathbb{R}^d)$ has no trace on $\mathbb{R}^{d-l}$.*

Proof. Note that if $l \geq 2$, by the standard theory of traces, we have that

$$\mathrm{Tr}_{d-l+1} B_q^{l/p,p}(\mathbb{R}^d) = B_q^{1/p,p}(\mathbb{R}^{d-l+1}).$$

Hence, it suffices to prove Proposition A.1 in the case $l = d-1$.

As long as $0 < s < 1$, we have the following equivalent of the norm of $B_q^{s,p}(\mathbb{R}^d)$ (see, for instance, Theorem 7.47 on p. 242 of [2]):

$$\|v\|_{B_q^{s,p}(\mathbb{R}^d)} \sim \|v\|_{L^p(\mathbb{R}^d)} + \left(\int_{B(0,1)} |h|^{-sq} \, \|\Delta_h v\|_{L^p(\mathbb{R}^d)}^q \, \frac{dh}{|h|^d} \right)^{1/q}, \tag{A.1}$$

for any Schwartz function v, where $B(0,1)$ is the open unit ball in $\mathbb{R}^d$ and

$$\Delta_h v(x) := v(x+h) - v(x).$$

Consider some Banach function space $Y(\mathbb{R}^{d-1})$ and let $F \in C_c^\infty(\mathbb{R}^{d-1})$ be a nontrivial function in $Y(\mathbb{R}^{d-1})$. Consider also some $\psi \in C_c^\infty(\mathbb{R})$. Setting $u := F \otimes \psi$, we see that

$$\|u\|_{B_q^{1/p,p}(\mathbb{R}^d)} \lesssim \|F\|_{B_q^{1/p,p}(\mathbb{R}^{d-1})} \, \|\psi\|_{B_q^{1/p,p}(\mathbb{R})}. \tag{A.2}$$

In order to prove (A.2), we apply (A.1) for $s = 1/p$. We have

$$\|u\|_{L^p(\mathbb{R}^d)} = \|F\|_{L^p(\mathbb{R}^{d-1})} \, \|\psi\|_{L^p(\mathbb{R})}. \tag{A.3}$$

Also, since

$$\begin{aligned}\Delta_h u(x', x_d) &= (F(x' + h') - F(x'))\psi(x_d + h_d) + F(x')(\psi(x_d + h_d) - \psi(x_d)) \\ &= (\Delta_{h'} F(x'))\psi(x_d + h_d) + F(x')(\Delta_{h_d}\psi(x_d)),\end{aligned}$$

the triangle inequality allows us to bound the term

$$\Big(\int_{B(0,1)} |h|^{-q/p} \|\Delta_h u\|^q_{L^p(\mathbb{R}^d)} \frac{dh}{|h|^d}\Big)^{1/q}$$

by

$$\tag{A.4} \begin{aligned}&\Big(\int_{B(0,1)} |h|^{-q/p} \|\Delta_{h'} F\|^q_{L^p(\mathbb{R}^{d-1})} \frac{dh}{|h|^d}\Big)^{1/q} \|\psi\|_{L^p(\mathbb{R})} \\ &\quad + \Big(\int_{B(0,1)} |h|^{-q/p} \|\Delta_{h_d}\psi\|^q_{L^p(\mathbb{R}^{d-1})} \frac{dh}{|h|^d}\Big)^{1/q} \|F\|_{L^p(\mathbb{R}^d)}.\end{aligned}$$

Since

$$\int_0^1 |h|^{-q/p-d}\, dh_d \sim |h'|^{-q/p-d+1},$$

we have

$$\tag{A.5} \begin{aligned}&\Big(\int_{B(0,1)} |h|^{-q/p}\, \|\Delta_{h'} F\|^q_{L^p(\mathbb{R}^{d-1})} \frac{dh}{|h|^d}\Big)^{1/q} \\ &\lesssim \Big(\int_{B'(0,1)} \Big(\int_0^1 |h|^{-q/p-d}\, dh_d\Big) \|\Delta_{h'} F\|^q_{L^p(\mathbb{R}^{d-1})}\, dh'\Big)^{1/q} \\ &\sim \Big(\int_{B'(0,1)} |h'|^{-q/p-d+1}\, \|\Delta_{h'} F\|^q_{L^p(\mathbb{R}^{d-1})}\, dh'\Big)^{1/q} \lesssim \|F\|_{B^{1/p,p}_q(\mathbb{R}^{d-1})},\end{aligned}$$

where $B'(0, 1)$ is the open unit ball in $\mathbb{R}^{d-1}$.

In a similar way, using the fact that

$$\int_{B'(0,1)} |h|^{-q/p-d}\, dh' \sim |h_d|^{-q/p-1},$$

we obtain

$$\tag{A.6} \Big(\int_{B(0,1)} |h|^{-q/p}\, \|\Delta_{h_d}\psi\|^q_{L^p(\mathbb{R}^{d-1})} \frac{dh}{|h|^d}\Big)^{1/q} \lesssim \|\psi\|_{B^{1/p,p}_q(\mathbb{R})}.$$

From (A.4), (A.5) and (A.6), we get

$$\Big(\int_{B(0,1)} |h|^{-q/p}\, \|\Delta_h u\|^q_{L^p(\mathbb{R}^d)} \frac{dh}{|h|^d}\Big)^{1/q} \lesssim \|F\|_{B^{1/p,p}_q(\mathbb{R}^{d-1})} \|\psi\|_{B^{1/p,p}_q(\mathbb{R})},$$

which, together with (A.3), gives (A.2).

Now suppose that the trace operator is bounded from $V(\mathbb{R}^d)$ to $Y(\mathbb{R}^{d-1})$. Since, $\operatorname{Tr} u(x') = F(x')\psi(0)$, we can write

$$\|F\|_Y |\psi(0)| \lesssim \|u\|_{B_q^{1/p,p}(\mathbb{R}^d)}.$$

Using (A.2) and the fact that $\|F\|_Y > 0$ (note that F is not trivial), we get

$$|\psi(0)| \lesssim \|\psi\|_{B_q^{1/p,p}(\mathbb{R})},$$

for any $\psi \in C_c^\infty(\mathbb{R})$, where the implicit constant does not depend on ψ. However, by a translation argument, this implies the false embedding $B_q^{1/p,p}(\mathbb{R}) \hookrightarrow C(\mathbb{R})$. ■

Consider some Borel set $\Gamma \subset \mathbb{R}^d$ and some number $\delta \in (0,d)$. We say that Γ is δ-full if it is of Hausdorff dimension δ and there exist two constants $c_1, c_2 > 0$ such that

$$c_1 R^\delta \le \mathcal{H}^\delta(B(x,R)\cap\Gamma) \le c_2 R^\delta,$$

for any $x \in \mathbb{R}^d$ and any $R > 0$, where $\mathcal{H}^\delta$ is the δ-Hausdorff measure. It is easy to see that for any $\delta \in (0,d)$, there exist full Borel subsets of $\mathbb{R}^d$ of Hausdorff dimension δ.

For such subsets we have the following result borrowed from Theorem 3.3.1 (ii) in [7] and Proposition 2.9 in [8].

Theorem A.2. *Let $\Gamma \subset \mathbb{R}^d$ be a Borel set that is δ-full for some $\delta \in (0,d)$. Then,*

(i) *for any $0 < p < \infty$ and $0 < q \le \min(p,1)$, we have that*

$$\operatorname{Tr}_\Gamma B_q^{(d-\delta)/p,p}(\mathbb{R}^d) = L^p(\Gamma),$$

where $L^p(\Gamma)$ is considered with respect to the δ-Hausdorff measure.

(ii) *If $0 < p < \infty$ and $1 < q < \infty$, then $C_c^\infty(\mathbb{R}^d \setminus \Gamma)$ is dense in $B_q^{(d-\delta)/p,p}(\mathbb{R}^d)$. It follows that $B_q^{(d-\delta)/p,p}(\mathbb{R}^d)$ has no trace on Γ.*

Let us justify the last assertion of (ii). Suppose $Y(\Gamma)$ is Banach space of functions on Γ and pick some $F \in C_c^\infty(\mathbb{R}^d)$ such that $\operatorname{Tr}_\Gamma F \in Y(\Gamma)$ and $\operatorname{Tr}_\Gamma F$ is not identically 0. Then we have that the operator Tr_Γ cannot be bounded from $B_q^{(d-\delta)/p,p}(\mathbb{R}^d)$ to $Y(\Gamma)$. Indeed, since $C_c^\infty(\mathbb{R}^d \setminus \Gamma)$ is dense in $B_q^{(d-\delta)/p,p}(\mathbb{R}^d)$ and $F \in B_q^{(d-\delta)/p,p}(\mathbb{R}^d)$, there exists a sequence $(F_n)_{n\ge1}$ of functions in $C_c^\infty(\mathbb{R}^d \setminus \Gamma)$ such that

$$\|F - F_n\|_{B_q^{(d-\delta)/p,p}(\mathbb{R}^d)} \to 0 \quad \text{when } n \to \infty. \tag{A.7}$$

If the operator Tr_Γ is bounded from $B_q^{(d-\delta)/p,p}(\mathbb{R}^d)$ to $Y(\Gamma)$, then we must have

$$\|\operatorname{Tr}_\Gamma F\|_{Y(\Gamma)} = \|\operatorname{Tr}_\Gamma (F - F_n)\|_{Y(\Gamma)} \lesssim \|F - F_n\|_{B_q^{(d-\delta)/p,p}(\mathbb{R}^d)},$$

for any $n \ge 1$, and by (A.7), we get that $\operatorname{Tr}_\Gamma F \equiv 0$, which contradicts the choice of F.

The part (i) of Theorem A.2 is a direct consequence of Theorem 5.9 in [7]. The part (ii) of Theorem A.2 is a direct consequence of Proposition 3.16 in [8]. We mention that both parts (i) and (ii) hold in a more general context, namely, when Γ is assumed to be an h-set satisfying the porosity condition (see, for instance, Definition 2.8 in [8] for a definition of porosity and Proposition 2.9 in [8] for a characterization that shows, in particular, that any δ-full Borel set with $\delta \in (0,d)$ satisfies the porosity condition).

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References

[1] Adams, D. R. and Frazier, M.: BMO and smooth truncation in Sobolev spaces. *Studia Math.* **89** (1988), no. 3, 241–260.

[2] Adams, R. A. and Fournier, J. J. F.: *Sobolev spaces*. Second edition. Pure and Applied Mathematics (Amsterdam) 140, Elsevier/Academic Press, Amsterdam, 2003.

[3] Aronszajn, N. and Gagliardo, E.: Interpolation spaces and interpolation methods. *Ann. Mat. Pura Appl. (4)* **68** (1965), 51–117.

[4] Bennett, C. and Sharpley, R.: *Interpolation of operators*. Pure and Applied Mathematics 129, Academic Press, Boston, MA, 1988.

[5] Bergh, J. and Löfström, J.: *Interpolation spaces. An introduction*. Grundlehren der Mathematischen Wissenschaften 223, Springer, Berlin-New York, 1976.

[6] Bourgain, J.: Some consequences of Pisier's approach to interpolation. *Israel J. Math.* **77** (1992), no. 1-2, 165–185.

[7] Bricchi, M.: Tailored Besov spaces and h-sets. *Math. Nachr.* **263/264** (2004), 36–52.

[8] Caetano, A. M. and Haroske, D. D.: Traces of Besov spaces on fractal h-sets and dichotomy results. *Studia Math.* **231** (2015), no. 2, 117–147.

[9] Calderón, A.-P.: Intermediate spaces and interpolation, the complex method. *Studia Math.* **24** (1964), 113–190.

[10] Cohen, A.: Ondelettes, espaces d'interpolation et applications. In *Séminaire: équations aux dérivées partielles, 1999–2000*, Exp. no. I, 14 pp. Sémin. Équ. Dériv. Partielles, École Polytech., Palaiseau, 2000.

[11] Cohen, A., Dahmen, W., Daubechies, I. and DeVore, R.: Harmonic analysis of the space BV. *Rev. Mat. Iberoamericana* **19** (2003), no. 1, 235–263.

[12] Cohen, A., DeVore, R., Petrushev, P. and Xu, H.: Nonlinear approximation and the space $\mathrm{BV}(\mathbb{R}^2)$. *Amer. J. Math.* **121** (1999), no. 3, 587–628.

[13] Costabel, M. and McIntosh, A.: On Bogovskiĭ and regularized Poincaré integral operators for de Rham complexes on Lipschitz domains. *Math. Z.* **265** (2010), no. 2, 297–320.

[14] DeVore, R. and Scherer, K.: Interpolation of linear operators on Sobolev spaces. *Ann. of Math. (2)* **109** (1979), no. 3, 583–599.

[15] Grafakos, L.: *Classical Fourier analysis*. Second edition. Graduate Texts in Mathematics 249, Springer, New York, 2008.

[16] Henkin, G. M.: Impossibility of a uniform homeomorphism between spaces of smooth functions of one and of n variables ($n \geq 2$). *Mat. Sb. (N. S.)* **74 (116)** (1967), 595–607. Translation in *Math. USSR-Sb.* **3** (1967), no. 4, 551–561.

[17] Janson, S.: Interpolation of subcouples and quotient couples. *Ark. Mat.* **31** (1993), no. 2, 307–338.

[18] Jones, P. W.: *Complex interpolation between Sobolev spaces*. In *Linear and complex analysis problem book. 199 research problems*. Lecture Notes in Mathematics 1043, Springer, Berlin-Heidelberg, 1984.

[19] Kislyakov, S. V.: Sobolev imbedding operators, and the nonisomorphism of certain Banach spaces. *Funkcional. Anal. i Priložen.* **9** (1975), no. 4, 22–27.

[20] Kislyakov, S. V., Kruglyak, N. Y.: Stability of approximation under singular integrals, and Calderón–Zygmund type decompositions. PDMI Preprint no. 7, Petersburg Department of Steklov Institute of Mathematics, 2005.

[21] Kislyakov, S. V. and Kruglyak, N. Y.: *Extremal problems in interpolation theory, Whitney–Besicovitch coverings, and singular integrals*. Instytut Matematyczny Polskiej Akademii Nauk. Monografie Matematyczne (New Series) 74, Birkhäuser/Springer, Basel, 2013.

[22] Kislyakov, S. V., Maksimov, D. V. and Stolyarov, D. M.: Differential expressions with mixed homogeneity and spaces of smooth functions they generate in arbitrary dimension. *J. Funct. Anal.* **269** (2015), no. 10, 3220–3263.

[23] Kreyszig, E.: *Introductory functional analysis with applications*. Wiley, New York-London-Sydney, 1978.

[24] Kruglyak, N. Y.: *Investigations in the theory of real interpolation*. PhD Thesis, Steklov Mathematical Institute, St. Petersburg, 1996.

[25] Kruglyak, N. Y.: Smooth analogues of the Calderón–Zygmund decomposition, quantitative covering theorems and the K-functional for the couple $(L_q, \dot{W}_p^k)$. *Algebra i Analiz* **8** (1996), no. 4, 110–160; *St. Petersburg Math. J.* **8** (1997), no. 4, 617–649.

[26] Lindenstrauss, J. and Tzafriri, L.: *Classical Banach spaces. II*. Ergebnisse der Mathematik und ihrer Grenzgebiete 97, Springer, Berlin-New York, 1979.

[27] Liu, F.-C. and Tai, W.-S.: Lusin properties and interpolation of Sobolev spaces. *Topol. Methods Nonlinear Anal.* **9** (1997), no. 1, 163–177.

[28] Meyer, Y.: Wavelets and operators. In *Analysis at Urbana, Vol. I (Urbana, IL, 1986–1987)*, pp. 256–365. London Math. Soc. Lecture Note Ser. 137, Cambridge University Press, Cambridge, 1989.

[29] Milman, M.: Fourier type and complex interpolation. *Proc. Amer. Math. Soc.* **89** (1983), no. 2, 246–248.

[30] Mironescu, P.: Sum-intersection property of Sobolev spaces. In *Current research in nonlinear analysis*, pp. 203–228. Springer Optim. Appl. 135, Springer, Cham, 2018.

[31] Mironescu, P. and Russ, E.: Traces of weighted Sobolev spaces. Old and new. *Nonlinear Anal.* **119** (2015), 354–381.

[32] Peetre, J.: *New thoughts on Besov spaces*. Duke University Mathematics Series 1, Duke University, Mathematics Department, Durham, NC, 1976.

[33] Pisier, G.: A simple proof of a theorem of Jean Bourgain. *Michigan Math. J.* **39** (1992), no. 3, 475–484.

[34] Pisier, G.: *Martingales in Banach spaces*. Cambridge Studies in Advanced Mathematics 155, Cambridge University Press, Cambridge, 2016.

[35] Runst, T. and Sickel, W.: *Sobolev spaces of fractional order, Nemytskij operators, and nonlinear partial differential equations*. De Gruyter Series in Nonlinear Analysis and Applications 3, De Gruyter, Berlin, 1996.

[36] Schmeisser, H.-J. and Triebel, H.: *Topics in Fourier analysis and function spaces*. A Wiley-Interscience Publication, Wiley, Chichester, 1987.

[37] Sickel, W. and Triebel, H.: Hölder inequalities and sharp embeddings in function spaces of B^s_{pq} and F^s_{pq} type. *Z. Anal. Anwendungen* **14** (1995), no. 1, 105–140.

[38] Triebel, H.: *Theory of function spaces. II*. Monographs in Mathematics 84, Birkhäuser, Basel, 1992.

[39] Uspenskiĭ, S. V.: Imbedding theorems for classes with weights. *Trudy Mat. Inst. Steklov.* **60** (1961), 282–303.

[40] Wolff, T. H.: A note on interpolation spaces. In *Harmonic analysis (Minneapolis, Minn., 1981)*, pp. 199–204. Lecture Notes in Math. 908, Springer, Berlin-New York, 1982.

[41] Ziemer, W. P.: *Weakly differentiable functions*. Graduate Texts in Mathematics 120, Springer, New York, 1989.

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