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Triple clasp formulas for G_2

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Abstract. We use Kuperberg's diagrammatic description of homomorphisms between fundamental representations of G_2 to give explicit recursive formulas for the idempotent projecting to the highest weight irreducible summand in each tensor product of fundamental representations.

1. Introduction

1.1. History of clasp formulas

The discovery of the Jones polynomial in the early 1980s [11] triggered mathematical developments in areas including knot theory and quantum algebra. One way to define the Jones polynomial is by using the braiding in the Temperley–Lieb Category [12, Chapter 2]. Half a decade earlier, Rumer–Teller–Weyl found a description of morphisms between tensor products of the vector representation of $SL_2(\mathbb{C})$ in terms of cup and cap diagrams [24, equation (3)]. The q-analog of their result is that the Temperley–Lieb Category is monoidally equivalent to the full monoidal subcategory of $\mathbf{Rep}(U_q(\mathfrak{sl}_2))$ generated by the q-analogue of the vector representation.

Let $V \in \mathbf{Rep}(U_q(\mathfrak{sl}_2))$ denote the q-analogue of the vector representation of $SL_2(\mathbb{C})$. For each $n \in \mathbb{Z}_{\geq 0}$, there is an irreducible representation $V(n)$, which is a direct summand of $V^{\otimes n}$ and which is not a direct summand of $V^{\otimes m}$ for $m < n$. Note that

$$V(1) \cong V.$$

So, for each n, there is an idempotent in the Temperley–Lieb category which can be viewed as the idempotent in $\mathrm{End}_{U_q(\mathfrak{sl}_2)}(V^{\otimes n})$ with image $V(n)$. The condition that $V(n)$ is not a summand of $V^{\otimes m}$ for $m < n$ implies that composing a projector with any cap diagram will result in zero.

These idempotents are usually called Jones–Wenzl projectors, as they were first considered by Jones [10, Section 4.2], and the following explicit inductive formula

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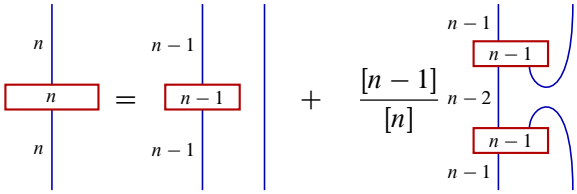
was first given by Wenzl [26]:



(1.1)

Here, we use the notation $[m]$ to denote the quantum integer

$$[m]_q := \frac{q^m - q^{-m}}{q - q^{-1}}$$

for each $m \in \mathbb{Z}$. Our convention is that a red box with label n is the morphism in the Temperley–Lieb category which corresponds to the idempotent with image $V(n)$.

The Jones–Wenzl projectors and the recursive formula in equation (1.1) describing them have been proven useful in link homology [6], Soergel bimodules [7], and the theory of subfactors and planar algebras [20]. The present work is concerned with generalizing equation (1.1) from $\mathfrak{sl}_2$ to the Lie algebra $\mathfrak{g}_2$. However, many things we say in the introduction make sense for all semisimple Lie algebras.

Fix a finite-dimensional semisimple Lie algebra $\mathfrak{g}$. There is an associated quantum enveloping algebra $U_q(\mathfrak{g})$, which is a $\mathbb{C}(q)$ algebra defined by generators and relations which "quantize" the Serre presentation of the usual enveloping algebra [9, Chapter 4]. The finite-dimensional irreducible type-**1** representations[1] are in bijection with the finite-dimensional irreducible representations of $\mathfrak{g}$; i.e., for each dominant integral weight λ, there is a finite-dimensional irreducible module of $U_q(\mathfrak{g})$, which we denote by $V(\lambda)$. We will abuse notation and write $\mathbf{Rep}(U_q(\mathfrak{g}))$ to refer to the category of finite-dimensional type-**1** representations of $U_q(\mathfrak{g})$. The algebra $U_q(\mathfrak{g})$ is a Hopf algebra [9, Section 4.8], and it turns out that $\mathbf{Rep}(U_q(\mathfrak{g}))$ is closed under taking tensor products. Furthermore, since we are working over $\mathbb{C}(q)$, where q is an indeterminant or a generic element of $\mathbb{C}$, the category $\mathbf{Rep}(U_q(\mathfrak{g}))$ is a semisimple tensor category, and the Grothendieck ring of $\mathbf{Rep}(U_q(\mathfrak{g}))$ is isomorphic to the Grothendieck ring of the category of finite-dimensional representations of $\mathfrak{g}$.

The recursion in equation (1.1) is expressed in terms of the Temperley–Lieb category, which describes the full monoidal subcategory of $\mathbf{Rep}(U_q(\mathfrak{sl}_2))$ generated by V. One way to generalize this subcategory to arbitrary quantum groups is proposed in the following definition.

[1] This means that, for all simple roots α, the element K_α acts on the μ weight space of any representation by $+q^{(\alpha,\mu)}$ [9, Section 5.2]. We will only consider type-**1** representations in this paper.

Definition 1.1. The category $\mathbf{Fund}(U_q(\mathfrak{g}))$ is the full subcategory of $\mathbf{Rep}(U_q(\mathfrak{g}))$, monoidally generated by the irreducible representations with highest weight a fundamental weight.

We denote the set of fundamental weights of $\mathfrak{g}$ by $\{\varpi_i\}$. Let λ be a dominant integral weight. Then, we can write

$$\lambda = \sum n_i \varpi_i,$$

where $n_i \in \mathbb{Z}_{\geq 0}$. There is a partial order on all weights, where $\mu \leq \lambda$ when $\lambda - \mu$ is a $\mathbb{Z}_{\geq 0}$-linear combination of positive roots. With respect to this partial order, the irreducible representation $V(\lambda)$ has highest weight λ. Also, $V(\lambda)$ is a direct summand of the tensor product $\bigotimes_i V(\varpi_i)^{\otimes n_i}$. Thus, there are projection and inclusion maps

$$\bigotimes_i V(\varpi_i)^{\otimes n_i} \to V(\lambda) \to \bigotimes_i V(\varpi_i)^{\otimes n_i}$$

such that the composition is an idempotent C_λ in $\mathbf{Fund}(U_q(\mathfrak{g}))$. We are interested in finding explicit descriptions of these idempotents, generalizing formula (1.1).

Unless λ is a fundamental weight or the zero weight, $V(\lambda)$ will not be an object in $\mathbf{Fund}(U_q(\mathfrak{g}))$. However, C_λ is a morphism in $\mathbf{Fund}(U_q(\mathfrak{g}))$, and we think of it as a replacement for $V(\lambda)$. Analogous to how $V(\lambda)$ is characterized as the irreducible representation with highest weight λ, the morphism C_λ is characterized as the non-zero idempotent endomorphism of $\bigotimes_i V(\varpi_i)^{\otimes n_i}$ such that if

$$f : \bigotimes_i V(\varpi_i)^{\otimes n_i} \to V(\varpi_{i_1}) \otimes \cdots \otimes V(\varpi_{i_r})$$

is a morphism in $\mathbf{Fund}(U_q(\mathfrak{g}))$ and $\sum_{k=1}^{r} \varpi_{i_k} < \lambda$, then $f \circ C_\lambda = 0$.

The Temperley–Lieb category provides a generators-and-relations description of $\mathbf{Fund}(U_q(\mathfrak{sl}_2))$. Kuperberg's paper [16] was the first attempt to generalize this to other $\mathfrak{g}$. In this work, Kuperberg gives generators and relations descriptions of the monoidal categories $\mathbf{Fund}(U_q(\mathfrak{g}))$ when $\mathfrak{g}$ is a rank two simple Lie algebra, i.e., $\mathfrak{g} = \mathfrak{sl}_3$, $\mathfrak{sp}_4$, or $\mathfrak{g}_2$. This work was later extended to types A [5, Theorem 3.3.1] and C [3, Theorem 1.4]. Following these authors, we refer to a diagrammatic category which is monoidally equivalent to $\mathbf{Fund}(U_q(\mathfrak{g}))$ as $\mathbf{Web}_q(\mathfrak{g})$. It remains an open problem to define web categories for all simple $\mathfrak{g}$.

Kuperberg also introduced the terminology *clasp* to refer to an idempotent projecting to the highest-weight irreducible summand of a tensor product of fundamental representations, viewed as a morphism in $\mathbf{Web}_q(\mathfrak{g})$. If the highest weight of this irreducible summand is λ, then we will call this idempotent a λ-clasp. To generalize the Jones–Wenzl recursion, a first step is to find recursive formulas of clasps in the rank two cases.

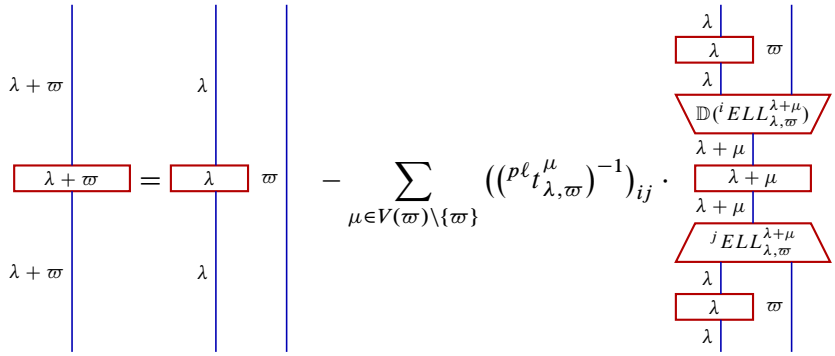
In the $\mathfrak{sl}_3$ case, a recursive formula was given by Ohtsuki and Yamada [18, Definition 2.4], where they called a clasp a "magic element". Later, Dongseok Kim found other recursive formulas for the $\mathfrak{sl}_3$ case as well [13, Theorem 3.3].

In [8, Conjecture 3.16], Elias made his type A clasp conjecture, which implies a recursive description of each $\mathfrak{sl}_n$ clasp using the language of $\mathfrak{sl}_n$ webs. Also, [8, Theorem 2.57] provides a basis for all homomorphism spaces between fundamental representations for $\mathfrak{sl}_n$. These bases have a particularly nice form which reduces the validity of the type A clasp conjecture to an explicit (but unwieldy) calculation. In [8, Sections 3.4 and 3.5], Elias proved the $\mathfrak{sl}_n$ clasp conjecture by hand for $n \leq 4$.

Since $\mathfrak{sp}_4$ is rank two, there are two simple roots: one short α_1 and one long α_2. We write ϖ_1 and ϖ_2 for the corresponding fundamental weights. In the $\mathfrak{sp}_4$ case, Kim gave recursive clasp formulas for the $a\varpi_1$-clasp [13, Corollary 4.3] and the $b\varpi_2$-clasp [13, Corollary 4.5]. However, an inductive formula for the $\mathfrak{sp}_4$ $a\varpi_1 + b\varpi_2$-clasp remained unknown until recently, when the first named author derived formulas generalizing Elias's type A clasp conjecture to type C_2 [1, Theorem 1.5].

In the $\mathfrak{g}_2$ case, little was known before the present work. Attempts at getting the $\mathfrak{g}_2$-clasp formulas have been made, including a few base-case calculations by Sakamoto and Yonezawa [25, Section 5]. In this paper, we give triple clasp expansions for the $\mathfrak{g}_2$ λ-clasps for all dominant integral weights λ. Our results are summarized in the following theorem.

Theorem 1.2. *Let λ be a dominant integral weight for $\mathfrak{g}_2$. Then, the $\lambda + \varpi$ clasp is given by the following recursive formula:*



$$- \sum_{\mu \in V(\varpi) \setminus \{\varpi\}} \left(\left({}^{p\ell}t^{\mu}_{\lambda,\varpi} \right)^{-1} \right)_{ij} \cdot$$

Here, a red box labeled by χ denotes the χ-clasp, i.e., the morphism in $\mathbf{Web}_q(\mathfrak{g}_2)$ which corresponds to the idempotent with image $V(\chi)$. The diagrams $\mathbb{D}({}^i ELL^{\lambda+\mu}_{\lambda,\varpi})$ and ${}^j ELL^{\lambda+\mu}_{\lambda,\varpi}$ are given explicitly in formula (3.1) and formula (3.2) in Section 3.1. The coefficients ${}^{p\ell}t^{\mu}_{\lambda,\varpi}$ are given explicitly by equations (3.3) to (3.25) in Section 3.1 and equations (A.1) to (A.3) in Appendix A.1.

1.2. Connection to the clasp conjecture

Let $\mathbb{F}$ be a field. Consider objects X and S in an additive $\mathbb{F}$-linear Karoubian category with duality $\mathbb{D}$, i.e., a contravariant endofunctor with $\mathbb{D}^2 \cong \mathrm{id}$ such that

$$\mathrm{End}(S) = \mathbb{F} \cdot \mathrm{id}_S, \quad \mathbb{D}(X) = X, \quad \text{and} \quad \mathbb{D}(S) = S.$$

Given

$$\pi : X \to S,$$

we obtain a map

$$\iota = \mathbb{D}(\pi) : S \to X$$

and

$$\pi \circ \iota = \kappa \, \mathrm{id}_S$$

for some $\kappa \in \mathbb{F}$. If $\kappa \neq 0$, then S is isomorphic to the image of the idempotent

$$e = \frac{1}{\kappa} \iota \circ \pi.$$

In [8, Definition 3.8], the coefficient κ, computed in the $\mathfrak{sl}_n$ web category, is called a local intersection form. We carry out analogous calculations in the $\mathfrak{g}_2$ web category and find that an analogue of Elias's clasp conjecture holds for G_2.

In fact, we expect that something in general will hold. Let $\mathfrak{g}$ be a simple Lie algebra, and let $U_q(\mathfrak{g})$ be the associated quantum group. Let W denote the Weyl group associated to $\mathfrak{g}$. For $V \in \mathbf{Rep}(U_q(\mathfrak{g}))$, we will write $\mathrm{wt}\, V$ to denote the set of all weights μ such that the μ weight space of V is non-zero. Fix a fundamental weight ϖ. For each $\mu \in \mathrm{wt}\, V(\varpi)$ such that μ is in the same W orbit as ϖ, there should be a clasped *elementary light ladder map*[2]

$$LL^{\lambda+\mu}_{\lambda,\varpi} : V(\lambda) \otimes V(\varpi) \to V(\lambda + \mu).$$

Definition 1.3. For an extremal weight μ in a fundamental representation (i.e., a weight in the W orbit of ϖ_i for some i), we write d_μ to denote the minimal length element $w \in W$ so that $w(\mu)$ is dominant. We also define Φ_μ to be the set of positive roots which are sent to negative roots by d_μ.

[2] In examples, this is most easily defined using web categories. But with some care should make sense in general, even without having a generators-and-relations presentation of $\mathbf{Fund}(\mathfrak{g})$. The main feature should be that the map is the composition of projectors and neutral maps with some fixed map from $V(\lambda_{min}) \otimes V(\varpi) \to V(\lambda_{min} + \mu)$, where λ_{min} is the smallest dominant weight so that $V(\lambda) \otimes V(\varpi)$ contains a copy of $V(\lambda + \mu)$.

Conjecture 1.4. *If we denote by* $\mathbb{D}$ *the duality*[3] *on* **Fund**$(\mathfrak{g})$ *and write*

$$LL^{\lambda+\mu}_{\lambda,\varpi} \circ \mathbb{D}(LL^{\lambda+\mu}_{\lambda,\varpi}) = \kappa^{\mu}_{\lambda,\varpi}\, \mathrm{id}_{V(\lambda+\mu)},$$

then

$$\kappa^{\mu}_{\lambda,\varpi} = \pm \prod_{\alpha\in\Phi_\mu} \frac{[(\alpha^\vee, \lambda+\rho)]_{q^{\ell(\alpha)}}}{[(\alpha^\vee, \lambda+\mu+\rho)]_{q^{\ell(\alpha)}}}.$$

Here, ρ *is the sum of the fundamental weights and*

$$l(\alpha) = (\alpha, \alpha)/2.$$

Remark 1.5. The conjecture is known to be true in types A_n for $n \leq 4$ [8, Section 3] and type C_2 [1, Corollary 3.7].

The following proposition is an elementary consequence of our main theorem.

Proposition 1.6. *The conjecture is true for* $\mathfrak{g}_2$.

Proof. See Corollary 3.1. ∎

Remark 1.7. We also expect a more general form of the conjecture which describes what happens for $\mu \in V(\varpi)$ which are not in the extremal Weyl orbit. The work in this paper and [1] could give enough data to guess the answer when $V(\varpi)_\mu$ is one dimensional, but we have not yet carried this out. We also hope the general form of the conjecture will give rise to a product formula which computes the elementary divisors of the matrix of local intersection forms when $\dim V(\varpi)_\mu > 1$.

1.3. Reshetikhin–Turaev invariants via skein theory

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$. In order to define an analogue of the Jones polynomial for $\mathfrak{g}$, Reshetikhin and Turaev defined a link invariant using the category **Rep**$(U_q(\mathfrak{g}))$ [22]. Their construction gives a knot invariant for every type-**1** representation of $U_q(\mathfrak{g})$. More generally, one can label each component of a link with an object in **Rep**$(U_q(\mathfrak{g}))$ and their construction gives the colored Reshetikhin–Turaev link invariant.

Kuperberg's first motivation for studying the category $\mathbf{Web}_q(\mathfrak{g}_2)$ was to compute the Reshetikhin–Turaev link invariant associated to $\mathfrak{g}_2$. Originally, he gave a diagrammatic method to compute the $\mathfrak{g}_2$ link invariant when each component is colored by

[3] Again, this is most easily defined in terms of webs, in which case it is just flipping the diagram upside down.

the first fundamental representation [15]. Soon after, using $\mathbf{Web}_{\mathrm{q}}(\mathfrak{g}_2)$, he gave diagrammatic tools for computing the $\mathfrak{g}_2$ link invariant colored by both fundamental representations [16, Section 4]. In this paper, we give explicit formulas for idempotents projecting to each irreducible $U_q(\mathfrak{g}_2)$ module. Combined with Kuperberg's earlier work, this gives a diagrammatic approach to computing the Reshetikhin–Turaev invariant of a link with components colored by any irreducible.

Reshitkhin–Turaev's paper about their link invariant was intended as a prequel to their work which gave an associated 3-manifold invariant [21]. The first step one takes to make sense of their 3-manifold invariant is to leave behind representation theory of $U_q(\mathfrak{g})$ for generic q and work instead with q specialized to a root of unity.

Let $U_{\mathbb{Z}[q,q^{-1}]}(\mathfrak{g}_2)$ be the $\mathbb{Z}[q,q^{-1}]$-subalgebra of $U_q(\mathfrak{g}_2)$ generated by $\frac{E_\alpha^{(a)}}{[a]_{q^{\ell(\alpha)}}!}$, $\frac{F_\alpha^{(a)}}{[a]_{q^{\ell(\alpha)}}!}$, and $K_\alpha^{\pm 1}$, for all simple roots α, and all $a \in \mathbb{Z}_{\geq 0}$. When ξ is a root of unity in $\mathbb{C}$, we can study the relation between $\mathbb{C} \otimes_{q=\xi} \mathbf{Web}_{\mathrm{q}}(\mathfrak{g}_2)$, and the category of tilting modules of $\mathbb{C} \otimes_{q=\xi} U_{\mathbb{Z}[q,q^{-1}]}(\mathfrak{g}_2)$. It is possible to adapt the approach from [2], which itself is based on [8], to prove that the Karoubi envelope of $\mathbb{C} \otimes_{q=\xi} \mathbf{Web}_{\mathrm{q}}(\mathfrak{g}_2)$ is equivalent to the category of tilting modules as long as $[2]_\xi, [3]_\xi \neq 0$. The same result is work in progress of Victor Ostrik and Noah Snyder, but they propose a slightly different approach.

When ξ is a root of unity of order greater than 5, the generators of the negligible ideal in the category of $U_\xi(\mathfrak{g}_2)$ tilting modules are (identity morphisms of) certain irreducible tilting modules with quantum dimension zero. Irreducible tilting modules are also Weyl modules, so these generating objects correspond to clasps in $\mathbf{Web}_{\mathrm{q}}(\mathfrak{g}_2)$. Moreover, the objects which survive in the negligible quotient are the irreducible Weyl modules with non-zero quantum dimension. Once the equivalence between the web category and the category of tilting modules is established, one can give a generators-and-relations presentation of the associated modular tensor category[4] using the $\mathfrak{g}_2$ triple clasp formulas for the negligible clasps. Combined with our description of the clasps corresponding to the irreducible Weyl modules with non-zero dimension, this gives an explicit way to compute the quantum $\mathfrak{g}_2$ 3-manifold invariant [21].

On the other hand, topologists have tried to understand the quantum 3-manifold invariants with graphical categories. A construction of the quantum $\mathfrak{sl}_2$ 3-manifold invariant using the Temperley–Lieb category was given by Lickorish [17]. This work was generalized to the $\mathfrak{sl}_3$ case by Ohtsuki and Yamada [18] with $\mathfrak{sl}_3$ webs. A self-contained proof of invariance under Kirby moves [14] using the graphical category was given in both cases. One can now give similar constructions and proofs in the $\mathfrak{g}_2$ case by using our clasp formulas.

[4] In the case that ξ is a root of unity, that actually gives rise to a modular category as the negligible quotient [23].

1.4. Structure of the paper

In Section 2, we recall the definition of $\mathfrak{g}_2$ webs and also the relation between $\mathfrak{g}_2$ webs and representations of the quantum group associated to $\mathfrak{g}_2$ [16]. Then, we give the definition of $\mathfrak{g}_2$ clasps and introduce the elementary light ladders as building blocks of the $\mathfrak{g}_2$ triple clasp expansions.

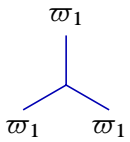
In Section 3, we give the triple clasp formulas for $\mathfrak{g}_2$ explicitly and then prove that a linear combination of webs given inductively by the $\mathfrak{g}_2$ triple clasp expansions satisfies the definition of a $\mathfrak{g}_2$ clasp.

In Appendix A.2, we give a complete list of recursive formulas which determine the coefficients in the $\mathfrak{g}_2$ triple clasp formulas. All the graphical calculations that lead to the recursive formulas can be found in Appendix A.4 in the arXiv version of this paper [4]. These calculations are a part of the proof of our main theorem. In Example 3.23, we give a detailed outline of how one such calculation goes. The SAGE code verifying that the explicit triple clasp coefficients satisfy the recursive formulas is included with the source file of the arXiv version of this paper [4].

2. Webs and clasps for $\mathfrak{g}_2$

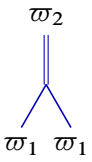
2.1. Definition of $\mathfrak{g}_2$ webs

Definition 2.1. The category $\mathbf{Web}_q(\mathfrak{g}_2)$ is the strict pivotal $\mathbb{C}(q)$-linear category, whose objects are generated by self-dual objects ϖ_1 and ϖ_2, and whose morphisms are generated by the following two trivalent vertices:



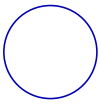
$\in \mathrm{Hom}_{\mathbf{Web}_q(\mathfrak{g}_2)}(\varpi_1 \otimes \varpi_1, \varpi_1)$

and

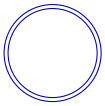


$\in \mathrm{Hom}_{\mathbf{Web}_q(\mathfrak{g}_2)}(\varpi_1 \otimes \varpi_1, \varpi_2)$

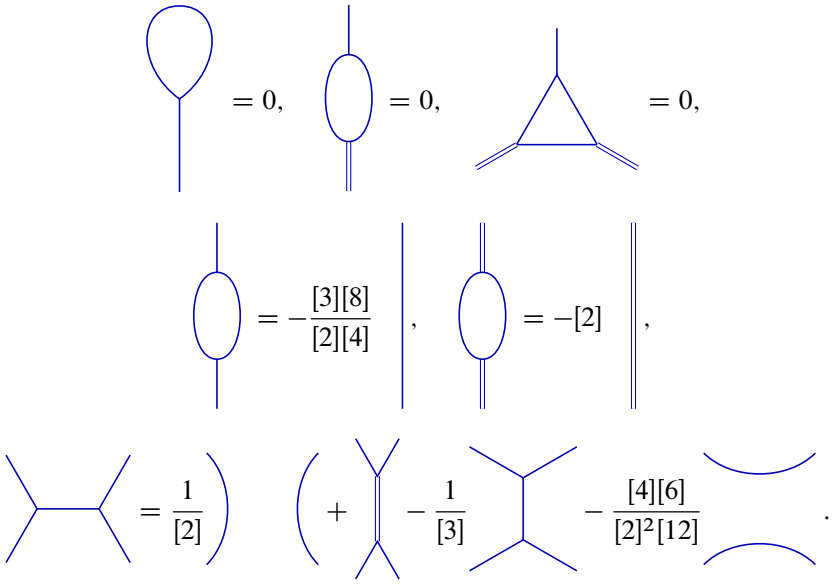
modulo the tensor-ideal generated by the following relations:



$= \dfrac{[2][7][12]}{[4][6]}$,



$= \dfrac{[7][8][15]}{[3][4][5]}$,



The tensor product of objects in $\mathbf{Web}_q(\mathfrak{g}_2)$ is concatenation of words. Tensor product of morphisms is horizontal concatenation. Composition of morphisms is vertical stacking.

Example 2.2. Let

$$f = \quad \in \mathrm{Hom}_{\mathbf{Web}_q(\mathfrak{g}_2)}(\varpi_1^{\otimes 3}, \varpi_1 \otimes \varpi_2),$$

$$g = \quad \in \mathrm{Hom}_{\mathbf{Web}_q(\mathfrak{g}_2)}(\varpi_1^{\otimes 2}, \varpi_1^{\otimes 3}).$$

Then,

$$f \otimes g = \quad \in \mathrm{Hom}_{\mathbf{Web}_q(\mathfrak{g}_2)}(\varpi_1^{\otimes 5}, \varpi_1 \otimes \varpi_2 \otimes \varpi_1^{\otimes 3})$$

and

$$f \circ g = \quad \in \mathrm{Hom}_{\mathbf{Web}_q(\mathfrak{g}_2)}(\varpi_1^{\otimes 2}, \varpi_1 \otimes \varpi_2).$$

Remark 2.3. Definition 2.1 is equivalent to the definition of $\mathfrak{g}_2$ spider given in [16, Section 4], with some rescaling of both types of trivalent vertices. See [4, Lemma A.1].

Notation 2.4. When $\underline{w} = \varpi_1^{\otimes a} \otimes \varpi_2^{\otimes b}$, we call $\underline{w}$ segregated. For example, $\varpi_1\varpi_1\varpi_2$ is segregated, while $\varpi_1\varpi_2\varpi_1$ is not.

We will write $\underline{s}_{(a,b)} := \varpi_1^{\otimes a} \otimes \varpi_2^{\otimes b}$, and we will denote

$$\mathrm{id}_{\underline{s}_{(a,b)}} = \left| \; \left| \cdots \right| \; \right\| \cdots \Big\| \in \mathrm{End}_{\mathbf{Web}_q(\mathfrak{g}_2)}(\underline{s}_{(a,b)})$$

as a labeled edge $\overset{(a,b)}{\left|\right.}$.

2.2. Kuperberg's results on equinumeration

We recall the results of [16] which describe the relation between $\mathfrak{g}_2$ webs and representations of the quantum group associated to $\mathfrak{g}_2$. We will only work over the field $\mathbb{C}(q)$, where q is either an indeterminant or a generic element of $\mathbb{C}^\times$.

Notation 2.5. Let Φ be the root system of type $\mathfrak{g}_2$ with Weyl group W and simple roots α_1 and α_2, where α_1 is the short root. It follows that the positive roots are

$$\Phi_+ = \{\alpha_1, 3\alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2, \alpha_1 + \alpha_2, \alpha_2\}.$$

Equip $\mathbb{Z}\Phi$ with the W invariant symmetric form determined by

$$(\alpha_1, \alpha_1) = 2, \quad (\alpha_1, \alpha_2) = -3 = (\alpha_2, \alpha_1), \quad \text{and} \quad (\alpha_2, \alpha_2) = 6.$$

We write X for the integral weight lattice and X_+ for the dominant integral weights. The fundamental weights are $\varpi_1 = 2\alpha_1 + \alpha_2$ and $\varpi_2 = 3\alpha_1 + 2\alpha_2$. We may use the notation (a, b) for $a\varpi_1 + b\varpi_2$; in particular,

$$X_+ = \{(a, b) \mid a, b \geq 0\}.$$

Definition 2.6. Let $\lambda, \mu \in X_+$. We define $\mu \leq \lambda$ if $\lambda - \mu$ is a non-negative linear combination of positive roots. We also write $\mu < \lambda$ if $\mu \leq \lambda$ and $\mu \neq \lambda$.

Definition 2.7 ([9, Section 4.3]). The algebra $U_q(\mathfrak{g}_2)$ is the $\mathbb{C}(q)$ algebra generated by elements F_1, F_2, $K_1^{\pm 1}$, $K_2^{\pm 1}$, E_1, and E_2 subject to the following relations:

$$K_1K_2 = K_2K_1,$$

$$K_2E_2 = q^6E_2K_2, \quad K_2E_1 = q^{-3}E_1K_2, \quad K_1E_1 = q^2E_1K_1, \quad K_1E_2 = q^{-3}K_1E_1,$$

$$K_2F_2 = q^{-6}F_2K_2, \quad K_2F_1 = q^3F_1K_2, \quad K_1F_1 = q^{-2}F_1K_1, \quad K_1F_2 = q^3F_2K_1,$$

$$E_1F_1 = F_1E_1 + \frac{K_1 - K_1^{-1}}{q - q^{-1}}, \quad E_2F_2 = F_2E_2 + \frac{K_2 - K_2^{-1}}{q^3 - q^{-3}},$$

$$E_2^2E_1 + E_1E_2^2 = \frac{[6]}{[3]}E_2E_1E_2, \quad F_2^2F_1 + F_1F_2^2 = \frac{[6]}{[3]}F_2F_1F_2,$$

$$E_1^4E_2 + \frac{[4][3]}{[2]}E_1^2E_2E_1^2 + E_2E_1^4 = [4]E_1^3E_2E_1 + [4]E_1E_2E_1^3,$$

$$F_1^4F_2 + \frac{[4][3]}{[2]}F_1^2F_2F_1^2 + F_2F_1^4 = [4]F_1^3F_2F_1 + [4]F_1F_2F_1^3.$$

The irreducible, finite dimensional, type-**1** representations of $U_q(\mathfrak{g}_2)$ are in bijection with the finite-dimensional irreducible representations of $\mathfrak{g}_2(\mathbb{C})$. For each $\lambda \in X_+$, we write $V(\lambda)$ for the $U_q(\mathfrak{g}_2)$ module which corresponds to the $\mathfrak{g}_2$ representation with highest weight λ.

The algebra $U_q(\mathfrak{g}_2)$ is a Hopf algebra, so its representation category is a monoidal category. We are only interested in type-**1** $U_q(\mathfrak{g}_2)$ modules, that is, modules such that K_{α_1} and K_{α_2} act diagonalizably with eigenvalues in $+q^m$ for $m \in \mathbb{Z}$. It is not hard to see that the condition of being type-**1** is closed under taking tensor product.

Notation 2.8. We write $\mathbf{Rep}(U_q(\mathfrak{g}_2))$ for the monoidal category of finite-dimensional type-**1** $U_q(\mathfrak{g}_2)$ modules.

The category $\mathbf{Rep}(U_q(\mathfrak{g}_2))$ is completely reducible [9, Theorem 5.17]. Moreover, we can determine how a module in $\mathbf{Rep}(U_q(\mathfrak{g}_2))$ decomposes by looking at its weight space decomposition.

The modules $V(\lambda)$ are type-**1**. Also, we have

$$V(\lambda) \otimes V(\mu) \cong \bigoplus_{\nu \in X_+} V(\nu)^{\oplus m_\nu^{\lambda,\mu}},$$

where the integers $m_\nu^{\lambda,\mu}$ are the same as those describing the tensor product decomposition of the analogous $\mathfrak{g}_2(\mathbb{C})$ modules. So, the tensor product of type-**1** modules are also type-**1**.

Definition 2.9 ([9, Section 5.1]). A module $W \in \mathbf{Rep}(U_q(\mathfrak{g}_2))$ decomposes as a direct sum

$$W = \bigoplus_{\mu \in X} W_\mu,$$

where

$$W_\mu = \{w \in W \mid K_1 w = q^{(\alpha_1,\mu)}w,\ K_2 w = q^{(\alpha_2,\mu)}w\}.$$

We will call this direct sum decomposition the *weight space decomposition* of W; say that W_μ is the μ *weight space* of W, and call $w \in W_\mu$ a *weight vector* of weight μ. We say that

$$\operatorname{wt} W := \{\mu \mid W_\mu \neq 0\}$$

is the set of *weights* of W.

Remark 2.10. It is well known that $\dim V(\lambda)_\mu$ is determined by the dimension of the μ weight space of the corresponding $\mathfrak{g}_2(\mathbb{C})$ module. In particular, $\dim V(\lambda)_\lambda = 1$ for all $\lambda \in X_+$.

Notation 2.11. Let W be a module in $\mathbf{Rep}(U_q(\mathfrak{g}_2))$. For each $\lambda \in X_+$, there are non-negative integers $m_\lambda(W)$ such that

$$W \cong \bigoplus_{\lambda \in X_+} V(\lambda)^{\oplus m_\lambda(W)}.$$

We write $[W : V(\lambda)] := m_\lambda(W)$ in this case.

Definition 2.12. The category $\mathbf{Fund}(U_q(\mathfrak{g}_2))$ is the full subcategory of $\mathbf{Rep}(U_q(\mathfrak{g}_2))$, monoidally generated by the fundamental representations $V(\varpi_1)$ and $V(\varpi_2)$.

Remark 2.13. The objects in the category $\mathbf{Fund}(U_q(\mathfrak{g}_2))$ are all isomorphic to iterated tensor products of fundamental representations. This includes the empty tensor product, which we take to be the trivial module, denoted by $\mathbf{1}$. The category is $\mathbb{C}(q)$-linear additive but is not closed under taking direct summands.

Definition 2.14. Let $\underline{\mathrm{w}}$ be an object in $\mathbf{Web}_{\mathrm{q}}(\mathfrak{g}_2)$. Then, $\underline{\mathrm{w}} = w_1 w_2 \cdots w_n$ for $w_i \in \{\varpi_1, \varpi_2\}$. We define

$$V(\underline{\mathrm{w}}) := V(w_1) \otimes V(w_2) \otimes \cdots \otimes V(w_n).$$

Remark 2.15. Note that

$$V(\underline{\mathrm{w}}) \cong \bigoplus_{\mu \in X_+} V(\mu)^{\oplus m_\mu^{\mathrm{w}}}.$$

The integers $m_\mu^{\mathrm{w}} := m_\mu(V(\underline{\mathrm{w}})) = [V(\underline{\mathrm{w}}) : V(\mu)]$ are the same as those describing the tensor product decomposition of the analogous $\mathfrak{g}_2(\mathbb{C})$ modules.

Notation 2.16. Given an object $\underline{\mathrm{w}} = w_1 w_2 \cdots w_n$, we write

$$\mathrm{wt}\,\underline{\mathrm{w}} = \sum_{i=1}^{n} \mathrm{wt}\, w_i.$$

Note that $\mathrm{wt}\,\underline{\mathrm{w}} \in X_+$ for all $\underline{\mathrm{w}}$.

Theorem 2.17 ([16, Theorem 5.1]). *There is an essentially surjective monoidal functor*

$$\Phi : \mathbf{Web}_{\mathrm{q}}(\mathfrak{g}_2) \to \mathbf{Fund}(U_q(\mathfrak{g}_2))$$

such that $\Phi(\varpi) = V(\varpi)$ *for* $\varpi \in \{\varpi_1, \varpi_2\}$.

Theorem 2.18 ([16, Theorem 6.10]). *Let* $\underline{\mathrm{w}}$ *and* $\underline{\mathrm{u}}$ *be objects in* $\mathbf{Web}_{\mathrm{q}}(\mathfrak{g}_2)$. *Then,*

$$\dim \mathrm{Hom}_{\mathbf{Web}_{\mathrm{q}}(\mathfrak{g}_2)}(\underline{\mathrm{w}}, \underline{\mathrm{u}}) = \dim \mathrm{Hom}_{\mathbf{Fund}(U_q(\mathfrak{g}_2))}(V(\underline{\mathrm{w}}), V(\underline{\mathrm{u}})),$$

and it follows that the functor Φ *is an equivalence of monoidal categories.*

Recall that, given a category $\mathcal{C}$, the Karoubi envelope of $\mathcal{C}$, is the category with objects: pairs (X, e), where X is an object in $\mathcal{C}$ and $e \in \mathrm{End}_{\mathcal{C}}(X)$ is an idempotent, and morphisms: triples $(e', f, e) : (X, e) \to (Y, e')$, where $f : X \to Y$ is a morphism in $\mathcal{C}$ so that $e' \circ f \circ e = f$. Given a $\mathbb{C}(q)$-linear category $\mathcal{C}$, the additive envelope of $\mathcal{C}$ is the category with objects formal direct sums of objects in $\mathcal{C}$ and morphisms matrices of morphisms in $\mathcal{C}$.

Definition 2.19. Let $\mathcal{C}$ be a $\mathbb{C}(q)$-linear category. Define the *Karoubi completion* of $\mathcal{C}$ to be the additive envelope of the Karoubi envelope of $\mathcal{C}$.

Corollary 2.20. *The functor* Φ *induces an equivalence of monoidal categories*

$$\mathrm{Kar}(\mathbf{Web}_{\mathrm{q}}(\mathfrak{g}_2)) \to \mathbf{Rep}(U_q(\mathfrak{g}_2))$$

such that $(\underline{\mathrm{w}}, e) \mapsto \mathrm{im}\, \Phi(e)$ *and* $(e', f, e) : (\underline{\mathrm{w}}, e) \to (\underline{\mathrm{w}}, e') \mapsto \Phi(e' \circ f \circ e)$.

Proof. Since every object in $\mathbf{Rep}(U_q(\mathfrak{g}_2))$ is a direct sum of direct summands of objects in $\mathbf{Fund}(U_q(\mathfrak{g}_2))$, this follows from Φ being an equivalence. ∎

Lemma 2.21. *Let* $D \in \mathrm{End}_{\mathbf{Web}_{\mathrm{q}}(\mathfrak{g}_2)}(\underline{\mathrm{w}})$ *such that* $\Phi(D)$ *acts as zero on* $V(\mathrm{wt}\, \underline{\mathrm{w}}) \overset{\oplus}{\subset} V(\underline{\mathrm{w}})$. *Then, we can write* D *as linear combination*

$$D = \sum_i A_i \circ B_i,$$

where $B_i \in \mathrm{Hom}_{\mathbf{Web}_{\mathrm{q}}(\mathfrak{g}_2)}(\underline{\mathrm{w}}, \underline{\mathrm{u}}_i)$ *and* $A_i \in \mathrm{Hom}_{\mathbf{Web}_{\mathrm{q}}(\mathfrak{g}_2)}(\underline{\mathrm{u}}_i, \underline{\mathrm{w}})$ *for some* $\underline{\mathrm{u}}_i$ *with* $\mathrm{wt}\, \underline{\mathrm{u}}_i < \mathrm{wt}\, \underline{\mathrm{w}}$.

Proof. Since

$$V(\underline{\mathrm{w}}) = V(\mathrm{wt}\, \underline{\mathrm{w}}) \bigoplus_{\mu < \mathrm{wt}\, \underline{\mathrm{w}}} V(\mu)^{\oplus m_{\mu}^{\underline{\mathrm{w}}}},$$

our hypothesis on $\Phi(D)$ implies that we can write $\Phi(D) = \sum_i \iota_i \circ \pi_i$, where for each i there is some $\mu_i < \mathrm{wt}\, \underline{\mathrm{w}}$ such that π_i is a projection $V(\underline{\mathrm{w}}) \to V(\mu_i)$ and ι_i is an inclusion $V(\mu_i) \to V(\underline{\mathrm{w}})$.

For each μ_i, fix an object $\underline{\mathrm{u}}_i$ in $\mathbf{Web}_{\mathrm{q}}(\mathfrak{g}_2)$ with $\mathrm{wt}\, \underline{\mathrm{u}}_i = \mu_i$. There is a projection $\gamma_i : V(\underline{\mathrm{u}}_i) \to V(\mu_i)$ and inclusion $\gamma^i : V(\mu_i) \to V(\underline{\mathrm{u}}_i)$ so that $\gamma_i \circ \gamma^i = \mathrm{id}_{V(\mu_i)}$. Then, we can write

$$\iota_i \circ \pi_i = \iota_i \circ \mathrm{id}_{V(\mu_i)} \circ \pi_i = \iota_i \circ \gamma_i \circ \gamma^i \circ \pi_i.$$

Thus, $\iota_i \circ \gamma_i \in \mathrm{Hom}_{U_q(\mathfrak{g}_2)}(V(\underline{\mathrm{u}}_i), V(\underline{\mathrm{w}}))$ and $\gamma^i \circ \pi_i \in \mathrm{Hom}_{U_q(\mathfrak{g}_2)}(V(\underline{\mathrm{w}}), V(\underline{\mathrm{u}}_i))$. The desired result now follows from Φ being an equivalence. ∎

Definition 2.22. The *neutral coefficient* of a diagram $D \in \mathrm{End}(\underline{\mathrm{w}})$ is the scalar by which $\Phi(D)$ acts on the one dimensional weight space $V(\underline{\mathrm{w}})_{\mathrm{wt}\,\underline{\mathrm{w}}}$. We write

$$\Phi(D)|_{V(\underline{\mathrm{w}})_{\mathrm{wt}\,\underline{\mathrm{w}}}} = N_D \cdot \mathrm{id}\,.$$

Lemma 2.23. *Let $D \in \mathrm{End}_{\mathbf{Web}_q(\mathfrak{g}_2)}(\underline{\mathrm{w}})$. Then, we can express D as a linear combination of diagrams*

$$D = N_D \cdot \mathrm{id}_{\underline{\mathrm{w}}} + \sum_i A_i \circ B_i\,,$$

where $B_i \in \mathrm{Hom}_{\mathbf{Web}_q(\mathfrak{g}_2)}(\underline{\mathrm{w}}, \underline{\mathrm{u}}_i)$ and $A_i \in \mathrm{Hom}_{\mathbf{Web}_q(\mathfrak{g}_2)}(\underline{\mathrm{u}}_i, \underline{\mathrm{w}})$ for some $\underline{\mathrm{u}}_i$ with $\mathrm{wt}\,\underline{\mathrm{u}}_i < \mathrm{wt}\,\underline{\mathrm{w}}$.

Proof. Consider $\Phi(D) - N_D\,\mathrm{id}_{V(\underline{\mathrm{w}})} \in \mathrm{End}(V(\underline{\mathrm{w}}))$. This endomorphism has $V(\underline{\mathrm{w}})_{\mathrm{wt}\,\underline{\mathrm{w}}}$ in its kernel and therefore also acts as zero on $V(\mathrm{wt}\,\underline{\mathrm{w}}) \overset{\oplus}{\subset} V(\underline{\mathrm{w}})$. The desired result now follows from Lemma 2.21. ∎

2.3. Definition of clasps

Definition 2.24. Let $\underline{\mathrm{w}} \in \mathbf{Web}_q(\mathfrak{g}_2)$. A diagrammatic $\underline{\mathrm{w}}$-**clasp** is a morphism $C_{\underline{\mathrm{w}}} \in \mathrm{End}_{\mathbf{Web}_q(\mathfrak{g}_2)}(\underline{\mathrm{w}})$ which satisfies the following conditions:

(1) $C_{\underline{\mathrm{w}}} \neq 0$;

(2) $C_{\underline{\mathrm{w}}} \circ C_{\underline{\mathrm{w}}} = C_{\underline{\mathrm{w}}}$;

(3) if $D \in \mathrm{Hom}_{\mathbf{Web}_q(\mathfrak{g}_2)}(\underline{\mathrm{w}}, \underline{\mathrm{u}})$ and $\mathrm{wt}\,\underline{\mathrm{u}} < \mathrm{wt}\,\underline{\mathrm{w}}$, then $D \circ C_{\underline{\mathrm{w}}} = 0$.

Remark 2.25. Note that we only use the terminology clasp to refer to idempotents. This is consistent with Kuperberg's original use of the term [16], but less general than Elias's [8, Definition 1.12]. In Section 2.4, we will define *generalized clasps*, which will agree with Elias's notion of clasp.

Lemma 2.26. *If the $\underline{\mathrm{w}}$ clasp exists, then it is unique and $N_{C_{\underline{\mathrm{w}}}} = 1$.*

Proof. Suppose that $C_{\underline{\mathrm{w}}}$ and $C'_{\underline{\mathrm{w}}}$ are both $\underline{\mathrm{w}}$-clasps. By Lemma 2.23, we can write $C_{\underline{\mathrm{w}}} = N_{C_{\underline{\mathrm{w}}}}\,\mathrm{id} + \sum_i A_i \circ B_i$ and $C'_{\underline{\mathrm{w}}} = N_{C'_{\underline{\mathrm{w}}}}\,\mathrm{id} + \sum_i A'_i \circ B'_i$. As a consequence of the definition of clasps, we find

$$C_{\underline{\mathrm{w}}} = C_{\underline{\mathrm{w}}} \circ C_{\underline{\mathrm{w}}} = \left(N_{C_{\underline{\mathrm{w}}}}\,\mathrm{id} + \sum_i A_i \circ B_i\right) \circ C_{\underline{\mathrm{w}}} = N_{C_{\underline{\mathrm{w}}}} C_{\underline{\mathrm{w}}}$$

and

$$C'_{\underline{\mathrm{w}}} = C'_{\underline{\mathrm{w}}} \circ C'_{\underline{\mathrm{w}}} = \left(N_{C'_{\underline{\mathrm{w}}}}\,\mathrm{id} + \sum_i A'_i \circ B'_i\right) \circ C'_{\underline{\mathrm{w}}} = N_{C'_{\underline{\mathrm{w}}}} C'_{\underline{\mathrm{w}}}.$$

Since $\underline{w}$-clasps are non-zero elements of the vector space $\mathrm{End}_{\mathbf{Web}_q(\mathfrak{g}_2)}(\underline{w})$, it follows that $N_{C_{\underline{w}}} = 1 = N_{C'_{\underline{w}}}$. Thus,

$$C'_{\underline{w}} = N_{C_{\underline{w}}} C'_{\underline{w}} = \left(N_{C_{\underline{w}}}\,\mathrm{id} + \sum_i A_i \circ B_i\right) \circ C'_{\underline{w}} = C_{\underline{w}} \circ C'_{\underline{w}}$$
$$= C_{\underline{w}} \circ \left(N_{C'_{\underline{w}}}\,\mathrm{id} + \sum_i A'_i \circ B'_i\right) = N_{C'_{\underline{w}}} C_{\underline{w}} = C_{\underline{w}}. \qquad \blacksquare$$

Definition 2.27. Let $\pi_{\underline{w}} \in \mathrm{End}_{U_q(\mathfrak{g}_2)}(V(\underline{w}))$ be the idempotent endomorphism with image $V(\mathrm{wt}\,\underline{w})$. The endomorphism $\Phi^{-1}(\pi_{\underline{w}})$ in $\mathrm{End}_{\mathbf{Web}_q(\mathfrak{g}_2)}(\underline{w})$ is the *algebraic* $\underline{w}$-*clasp*.

Lemma 2.28. *The algebraic clasp is a clasp, and* $\Phi^{-1}(\pi_{\underline{w}}) = C_{\underline{w}}$.

Proof. Since the algebraic clasp is non-zero and idempotent, we just need to argue that the algebraic clasp satisfies the third condition in the definition of clasp. Fix $\underline{u}$ such that $\mathrm{wt}\,\underline{u} < \mathrm{wt}\,\underline{w}$, and let $D \in \mathrm{Hom}_{\mathbf{Web}_q(\mathfrak{g}_2)}(\underline{w}, \underline{u})$. The module $V(\mathrm{wt}\,\underline{w})$ is not isomorphic to any summand of $V(\underline{u})$, so we know that $\mathrm{Hom}_{U_q(\mathfrak{g}_2)}(V(\mathrm{wt}\,\underline{w}), V(\underline{u})) = 0$. Therefore, $\Phi(D) \circ \pi_{\underline{w}} = 0$, and by Theorem 2.18, we may conclude that $D \circ (\Phi^{-1}(\pi_{\underline{w}})) = 0$. $\blacksquare$

Lemma 2.29. *Let* $\underline{w}$ *be an object in* $\mathbf{Web}_q(\mathfrak{g}_2)$*; then there is a unique diagrammatic* $\underline{w}$*-clasp* $C_{\underline{w}}$ *and* $\Phi(C_{\underline{w}})$ *is an idempotent endomorphism of* $V(\underline{w})$ *projecting to* $V(\mathrm{wt}\,\underline{w})$.

Proof. From Lemma 2.28, we see that clasps exist and map under Φ to the projector for $V(\mathrm{wt}\,\underline{w})$. Uniqueness follows from Lemma 2.26. $\blacksquare$

Lemma 2.30. *Let* $E \in \mathrm{End}_{\mathbf{Web}_q(\mathfrak{g}_2)}(\underline{w})$ *be a non-zero endomorphism so that* $E^2 = E$. *If* $C_{\underline{u}} \circ D \circ E = 0$ *for all* $D \in \mathrm{Hom}_{\mathbf{Web}_q(\mathfrak{g}_2)}(\underline{w}, \underline{u})$ *such that* $\mathrm{wt}\,\underline{u} < \mathrm{wt}\,\underline{w}$*, then* $E = C_{\underline{w}}$.

Proof. Since we assume that E is non-zero and idempotent, we just need to show that E satisfies the third condition in Definition 2.24. Fix $\underline{u}$ such that $\mathrm{wt}\,\underline{u} < \mathrm{wt}\,\underline{w}$. Suppose inductively that $B \circ E = 0$ for all $B \in \mathrm{Hom}_{\mathbf{Web}_q(\mathfrak{g}_2)}(\underline{w}, \underline{v})$, where $\mathrm{wt}\,\underline{v} < \mathrm{wt}\,\underline{u}$. By Lemma 2.23, we can write $C_{\underline{u}} = \mathrm{id} + \sum_i A_i \circ B_i$, where each A_i has domain $\underline{v}_i$ such that $\mathrm{wt}\,\underline{v}_i < \mathrm{wt}\,\underline{u}$. If $D \in \mathrm{Hom}_{\mathbf{Web}_q(\mathfrak{g}_2)}(\underline{w}, \underline{u})$, then

$$D \circ E = \mathrm{id} \circ D \circ E = C_{\underline{u}} \circ D \circ E - \sum_i A_i \circ B_i \circ D \circ E$$
$$= -\sum_i A_i \circ B_i \circ D \circ E. \tag{2.1}$$

Since $B_i \in \mathrm{Hom}_{\mathbf{Web}_q(\mathfrak{g}_2)}(\underline{u}, \underline{v}_i)$ and $D \in \mathrm{Hom}_{\mathbf{Web}_q(\mathfrak{g}_2)}(\underline{w}, \underline{u})$, we have

$$B_i \circ D \in \mathrm{Hom}_{\mathbf{Web}_q(\mathfrak{g}_2)}(\underline{w}, \underline{v}_i).$$

The induction hypothesis applies, so $(B_i \circ D) \circ E = 0$, and equation (2.1) implies that

$$D \circ E = 0. \qquad \blacksquare$$

Lemma 2.31 (Clasp Schur's lemma). *Let* $\underline{\mathrm{u}}$ *and* $\underline{\mathrm{v}}$ *be objects in* $\mathbf{Web}_{\mathrm{q}}(\mathfrak{g}_2)$ *and let* $D \in \mathrm{Hom}_{\mathbf{Web}_{\mathrm{q}}(\mathfrak{g}_2)}(\underline{\mathrm{u}}, \underline{\mathrm{v}})$. *Then, we have the following.*

(1) *If* $\mathrm{wt}\,\underline{\mathrm{u}} \neq \mathrm{wt}\,\underline{\mathrm{v}}$, *then* $C_{\underline{\mathrm{v}}} \circ D \circ C_{\underline{\mathrm{u}}} = 0$.

(2) *If* $\underline{\mathrm{u}} = \underline{\mathrm{v}}$, *then* $C_{\underline{\mathrm{v}}} \circ D \circ C_{\underline{\mathrm{u}}} = N_D \cdot C_{\underline{\mathrm{u}}}$.

Proof. By Corollary 2.20, we find that $\dim \mathrm{Hom}_{\mathrm{Kar}\,\mathbf{Web}_{\mathrm{q}}(\mathfrak{g}_2)}(C_{\underline{\mathrm{u}}}, C_{\underline{\mathrm{v}}}) = \delta_{\mathrm{wt}\,\underline{\mathrm{u}},\mathrm{wt}\,\underline{\mathrm{v}}}$. Thus, we can deduce the following.

(1) If $\mathrm{wt}\,\underline{\mathrm{u}} \neq \mathrm{wt}\,\underline{\mathrm{v}}$, then $\mathrm{Hom}_{\mathrm{Kar}\,\mathbf{Web}_{\mathrm{q}}(\mathfrak{g}_2)}(C_{\underline{\mathrm{u}}}, C_{\underline{\mathrm{v}}}) = 0$.

(2) If $\underline{\mathrm{u}} = \underline{\mathrm{v}}$, then $\mathrm{Hom}_{\mathrm{Kar}\,\mathbf{Web}_{\mathrm{q}}(\mathfrak{g}_2)}(C_{\underline{\mathrm{u}}}, C_{\underline{\mathrm{v}}}) = \mathbb{C}(q) \cdot C_{\underline{\mathrm{u}}}$. $\blacksquare$

Lemma 2.32. *Let* $\underline{\mathrm{w}}$, $\underline{\mathrm{u}}$, *and* $\underline{\mathrm{v}} \in \mathbf{Web}_{\mathrm{q}}(\mathfrak{g}_2)$. *If* $V(\mathrm{wt}\,\underline{\mathrm{v}})$ *is not a direct summand of* $V(\mathrm{wt}\,\underline{\mathrm{w}}) \otimes V(\mathrm{wt}\,\underline{\mathrm{u}})$, *then*

$$C_{\underline{\mathrm{v}}} \circ D \circ (C_{\underline{\mathrm{w}}} \otimes C_{\underline{\mathrm{u}}}) = 0$$

for all $D \in \mathrm{Hom}_{\mathbf{Web}_{\mathrm{q}}(\mathfrak{g}_2)}(\underline{\mathrm{w}} \otimes \underline{\mathrm{u}}, \underline{\mathrm{v}})$.

Proof. Corollary 2.20 implies that $\dim \mathrm{Hom}_{\mathrm{Kar}(\mathbf{Web}_{\mathrm{q}}(\mathfrak{g}_2))}(C_{\underline{\mathrm{w}}} \otimes C_{\underline{\mathrm{u}}}, C_{\underline{\mathrm{v}}}) = 0$, when $V(\mathrm{wt}\,\underline{\mathrm{v}})$ is not a direct summand of $V(\mathrm{wt}\,\underline{\mathrm{w}}) \otimes V(\mathrm{wt}\,\underline{\mathrm{u}})$. $\blacksquare$

Lemma 2.33 (Clasp absorption). *Let* $\underline{\mathrm{w}} = \underline{\mathrm{x}} \otimes \underline{\mathrm{y}} \otimes \underline{\mathrm{z}}$ *in* $\mathbf{Web}_{\mathrm{q}}(\mathfrak{g}_2)$, *then*

$$(\mathrm{id}_{\underline{\mathrm{x}}} \otimes C_{\underline{\mathrm{y}}} \otimes \mathrm{id}_{\underline{\mathrm{z}}}) \circ C_{\underline{\mathrm{w}}} = C_{\underline{\mathrm{w}}} = C_{\underline{\mathrm{w}}} \circ (\mathrm{id}_{\underline{\mathrm{x}}} \otimes C_{\underline{\mathrm{y}}} \otimes \mathrm{id}_{\underline{\mathrm{z}}}).$$

Proof. Since $V(\mathrm{wt}\,\underline{\mathrm{w}})$ appears with multiplicity one in $V(\underline{\mathrm{w}})$, it follows that $\pi_{\underline{\mathrm{w}}}$ is a central idempotent in $\mathrm{End}_{U_q(\mathfrak{g}_2)}(V(\underline{\mathrm{w}}))$. Therefore, $(\mathrm{id}_{\underline{\mathrm{x}}} \otimes C_{\underline{\mathrm{y}}} \otimes \mathrm{id}_{\underline{\mathrm{z}}}) \circ C_{\underline{\mathrm{w}}} = C_{\underline{\mathrm{w}}} \circ (\mathrm{id}_{\underline{\mathrm{x}}} \otimes C_{\underline{\mathrm{y}}} \otimes \mathrm{id}_{\underline{\mathrm{z}}})$ is also an idempotent and $C_{\underline{\mathrm{u}}} \circ D \circ (\mathrm{id}_{\underline{\mathrm{x}}} \otimes C_{\underline{\mathrm{y}}} \otimes \mathrm{id}_{\underline{\mathrm{z}}}) \circ C_{\underline{\mathrm{w}}} = 0$ for all $D \in \mathrm{Hom}_{\mathbf{Web}_{\mathrm{q}}(\mathfrak{g}_2)}(\underline{\mathrm{w}}, \underline{\mathrm{u}})$ such that $\mathrm{wt}\,\underline{\mathrm{u}} < \mathrm{wt}\,\underline{\mathrm{w}}$. Thus, by Lemma 2.30, it suffices to show that $(\mathrm{id}_{\underline{\mathrm{x}}} \otimes C_{\underline{\mathrm{y}}} \otimes \mathrm{id}_{\underline{\mathrm{z}}}) \circ C_{\underline{\mathrm{w}}} \neq 0$. This is deduced from observing that the morphism $\Phi((\mathrm{id}_{\underline{\mathrm{x}}} \otimes C_{\underline{\mathrm{y}}} \otimes \mathrm{id}_{\underline{\mathrm{z}}}) \circ C_{\underline{\mathrm{w}}})$ acts on $V(\underline{\mathrm{w}})_{\mathrm{wt}\,\underline{\mathrm{w}}}$ as multiplication by 1. $\blacksquare$

2.4. Neutral diagrams and generalized clasps

Definition 2.34. We will write $\mathsf{H}^{\varpi_2\varpi_1}_{\varpi_1\varpi_2} :=$  and $\mathsf{H}^{\varpi_1\varpi_2}_{\varpi_2\varpi_1} :=$  . These are the *basic neutral diagrams*.

Lemma 2.35 (Neutral absorption). *If* $\underline{\mathrm{w}} = \underline{\mathrm{w}}_1\, \varpi_1 \varpi_2\, \underline{\mathrm{w}}_2$ *and* $\underline{\mathrm{w}}' = \underline{\mathrm{w}}_1\, \varpi_2 \varpi_1\, \underline{\mathrm{w}}_2$, *then*

$$(\mathrm{id}_{\underline{\mathrm{w}}_1} \otimes \mathsf{H}^{\varpi_2\varpi_1}_{\varpi_1\varpi_2} \otimes \mathrm{id}_{\underline{\mathrm{w}}_2}) \circ C_{\underline{\mathrm{w}}} \circ (\mathrm{id}_{\underline{\mathrm{w}}_1} \otimes \mathsf{H}^{\varpi_1\varpi_2}_{\varpi_2\varpi_1} \otimes \mathrm{id}_{\underline{\mathrm{w}}_2}) = C_{\underline{\mathrm{w}}'}$$

in $\mathrm{End}_{\mathbf{Web}_{\mathrm{q}}(\mathfrak{g}_2)}(\underline{\mathrm{w}}')$.

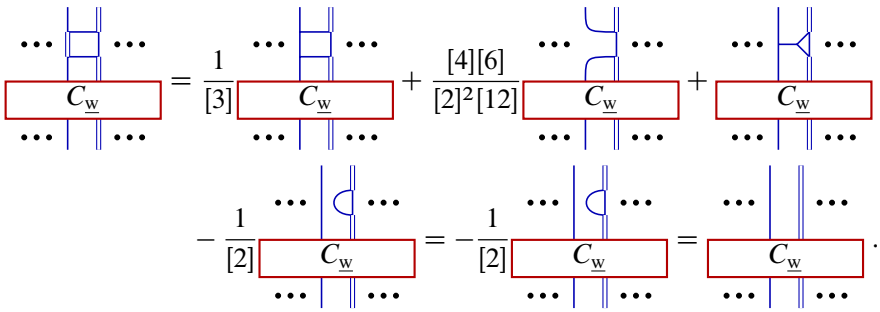
Proof. Write $\mathsf{H}^{\underline{w}'}_{\underline{w}} := \mathrm{id}_{\underline{w}_1} \otimes \mathsf{H}^{\varpi_2\varpi_1}_{\varpi_1\varpi_2} \otimes \mathrm{id}_{\underline{w}_2}$ and $\mathsf{H}^{\underline{w}}_{\underline{w}'} := \mathrm{id}_{\underline{w}_1} \otimes \mathsf{H}^{\varpi_1\varpi_2}_{\varpi_2\varpi_1} \otimes \mathrm{id}_{\underline{w}_2}$. By Lemma 2.26, we only need to show that $(\mathsf{H}^{\underline{w}'}_{\underline{w}}) \circ C_{\underline{w}} \circ (\mathsf{H}^{\underline{w}}_{\underline{w}'})$ satisfies the defining properties of a clasp.

Let $D \in \mathrm{Hom}_{\mathbf{Web}_q(\mathfrak{g}_2)}(\underline{w}', \underline{u})$, where $\mathrm{wt}\,\underline{u} < \mathrm{wt}\,\underline{w}'$. Then,

$$D \circ \mathsf{H}^{\underline{w}'}_{\underline{w}} \in \mathrm{Hom}_{\mathbf{Web}_q(\mathfrak{g}_2)}(\underline{w}, \underline{u})$$

and $\mathrm{wt}\,\underline{w} = \mathrm{wt}\,\underline{w}' > \mathrm{wt}\,\underline{u}$, so $D \circ \mathsf{H}^{\underline{w}'}_{\underline{w}} \circ C_{\underline{w}} = 0$. So, $\mathsf{H}^{\underline{w}'}_{\underline{w}} \circ C_{\underline{w}} \circ \mathsf{H}^{\underline{w}}_{\underline{w}'}$ satisfies the third condition in the definition of clasps.

The following calculation shows that $\mathsf{H}^{\underline{w}}_{\underline{w}'} \circ \mathsf{H}^{\underline{w}'}_{\underline{w}} \circ C_{\underline{w}} = C_{\underline{w}}$:



So, we have

$$\begin{aligned}(\mathsf{H}^{\underline{w}'}_{\underline{w}} \circ C_{\underline{w}} \circ \mathsf{H}^{\underline{w}}_{\underline{w}'}) \circ (\mathsf{H}^{\underline{w}'}_{\underline{w}} \circ C_{\underline{w}} \circ \mathsf{H}^{\underline{w}}_{\underline{w}'}) &= \mathsf{H}^{\underline{w}'}_{\underline{w}} \circ C_{\underline{w}} \circ (\mathsf{H}^{\underline{w}}_{\underline{w}'} \circ \mathsf{H}^{\underline{w}'}_{\underline{w}} \circ C_{\underline{w}}) \circ \mathsf{H}^{\underline{w}}_{\underline{w}'} \\ &= \mathsf{H}^{\underline{w}'}_{\underline{w}} \circ C_{\underline{w}} \circ C_{\underline{w}} \circ \mathsf{H}^{\underline{w}}_{\underline{w}'} = \mathsf{H}^{\underline{w}'}_{\underline{w}} \circ C_{\underline{w}} \circ \mathsf{H}^{\underline{w}}_{\underline{w}'}.\end{aligned}$$

This tells us that $\mathsf{H}^{\underline{w}'}_{\underline{w}} \circ C_{\underline{w}} \circ \mathsf{H}^{\underline{w}}_{\underline{w}'}$ satisfies the second condition in the definition of clasps.

What is more, $\mathsf{H}^{\underline{w}'}_{\underline{w}} \circ C_{\underline{w}} \circ \mathsf{H}^{\underline{w}}_{\underline{w}'} \neq 0$. Otherwise,

$$C_{\underline{w}} = \mathsf{H}^{\underline{w}}_{\underline{w}'} \circ (\mathsf{H}^{\underline{w}'}_{\underline{w}} \circ C_{\underline{w}} \circ \mathsf{H}^{\underline{w}}_{\underline{w}'}) \circ \mathsf{H}^{\underline{w}'}_{\underline{w}} = 0,$$

which is a contradiction. ∎

Definition 2.36. A *neutral diagram* $\mathsf{N}^{\underline{w}'}_{\underline{w}} \in \mathrm{Hom}_{\mathbf{Web}_q(\mathfrak{g}_2)}(\underline{w}, \underline{w}')$ is a composition of tensor products of identity diagrams and basic neutral diagrams. A *reduced neutral diagram* is a neutral diagram such that $\mathsf{H}^{\varpi_1\varpi_2}_{\varpi_2\varpi_1} \circ \mathsf{H}^{\varpi_2\varpi_1}_{\varpi_1\varpi_2}$ or $\mathsf{H}^{\varpi_2\varpi_1}_{\varpi_1\varpi_2} \circ \mathsf{H}^{\varpi_1\varpi_2}_{\varpi_2\varpi_1}$ do not occur as subdiagrams of $\mathsf{N}^{\underline{w}'}_{\underline{w}}$.

Lemma 2.37. *Fix* $\underline{w}$ *and* $\underline{w}'$. *We obtain the following.*

(1) *There is a neutral diagram* $\mathsf{N}^{\underline{w}'}_{\underline{w}}$ *if and only if* $\mathrm{wt}\,\underline{w} = \mathrm{wt}\,\underline{w}'$.

(2) *If* $\mathrm{wt}\,\underline{w} = \mathrm{wt}\,\underline{w}'$, *then there is a reduced neutral diagram in* $\mathrm{Hom}_{\mathbf{Web}_q(\mathfrak{g}_2)}(\underline{w}, \underline{w}')$.

(3) *Reduced neutral diagrams are unique.*

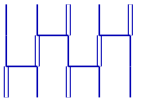
(4) *Suppose that* ${}^1\mathsf{N}_{\underline{w}}^{\underline{w}'}$ *and* ${}^2\mathsf{N}_{\underline{w}}^{\underline{w}'}$ *are two neutral diagrams. Then,*

$$ {}^1\mathsf{N}_{\underline{w}}^{\underline{w}'} \circ C_{\underline{w}} = {}^2\mathsf{N}_{\underline{w}}^{\underline{w}'} \circ C_{\underline{w}}. $$

Proof. It is omitted. ∎

Notation 2.38. Suppose that $\operatorname{wt}\underline{w} = \operatorname{wt}\underline{w}'$; then we will write $\mathsf{H}_{\underline{w}}^{\underline{w}'}$ for the reduced neutral diagram in $\mathrm{Hom}_{\mathbf{Web}_q(\mathfrak{g}_2)}(\underline{w}, \underline{w}')$.

Example 2.39. Consider $\underline{w} = \varpi_2\varpi_1\varpi_2\varpi_1\varpi_1$, $\underline{w}' = \varpi_1\varpi_1\varpi_2\varpi_1\varpi_2$. We know that $\operatorname{wt}\underline{w} = \operatorname{wt}\underline{w}' = (3,2)$. The reduced neutral diagram is

$\mathsf{H}_{\underline{w}}^{\underline{w}'} =$  .

Definition 2.40. Given a diagram D in $\mathbf{Web}_q(\mathfrak{g}_2)$, we will write $\mathbb{D}(D)$ for the diagram obtained by flipping D upside down. Note that $\mathbb{D}(\mathsf{H}_{\underline{w}}^{\underline{w}'}) = \mathsf{H}_{\underline{w}'}^{\underline{w}}$.

Definition 2.41. Given $\underline{x}, \underline{y}$ so that $\operatorname{wt}\underline{x} = \operatorname{wt}\underline{w} = \operatorname{wt}\underline{y}$, we define the *generalized clasp* $C_{\underline{x}}^{\underline{y}} := \mathsf{H}_{\underline{w}}^{\underline{y}} \circ C_{\underline{w}} \circ \mathsf{H}_{\underline{x}}^{\underline{w}}$. From Lemma 2.37, it follows that if $\mathsf{N}_{\underline{w}}^{\underline{y}}$ and $\mathsf{N}_{\underline{x}}^{\underline{w}}$ are any neutral diagrams, then $C_{\underline{x}}^{\underline{y}} = \mathsf{N}_{\underline{w}}^{\underline{y}} \circ C_{\underline{w}} \circ \mathsf{N}_{\underline{x}}^{\underline{w}}$.

Proposition 2.42. *The generalized clasps satisfy the following properties:*

(1) $C_{\underline{x}}^{\underline{x}} = C_{\underline{x}}$,

(2) $C_{\underline{x}}^{\underline{y}} \circ \mathsf{H}_{\underline{z}}^{\underline{x}} = C_{\underline{z}}^{\underline{y}}$,

(3) $\mathsf{H}_{\underline{y}}^{\underline{z}} \circ C_{\underline{x}}^{\underline{y}} = C_{\underline{x}}^{\underline{z}}$,

(4) $C_{\underline{y}}^{\underline{z}} \circ C_{\underline{x}}^{\underline{y}} = C_{\underline{x}}^{\underline{z}}$,

(5) $\mathbb{D}(C_{\underline{x}}^{\underline{y}}) = C_{\underline{y}}^{\underline{x}}$.

Proof. It is left as an exercise for the reader. For hints, see [8, Proposition 3.2]. ∎

2.5. Elementary light ladders

Notation 2.43. We write $f_1^{(k)} := \frac{f_1^k}{[k]_q!}$ and $f_2^{(k)} := \frac{f_2^k}{[k]_{q^3}!}$, where

$$ [k]_q! := [k][k-1]\cdots[2][1] $$

and

$$ [k]_{q^3}! := [k]_{q^3}[k-1]_{q^3}\cdots[2]_{q^3}[1]_{q^3}. $$

Note that $[k]_{q^3} = \frac{[3k]}{[3]}$.

For each fundamental weight

$$\varpi \in \{\varpi_1, \varpi_2\},$$

we choose a basis $\{{}^i v_{\mu,\varpi}\}_{i=1,\dots,\dim V(\varpi)_\mu}$ for all weight spaces $V(\varpi)_\mu$. Our convention will be to not record the superscript i in ${}^i v_{\mu,\varpi}$ when the weight space is multiplicity one. Explicitly, we choose the following basis of $V(\varpi_1)$:

$$\begin{aligned}
v_{(1,0),\varpi_1} &= v_1, & v_{(-1,1),\varpi_1} &= f_1 v_1,\\
v_{(2,-1),\varpi_1} &= f_2 f_1 v_1, & v_{(0,0),\varpi_1} &= f_1 f_2 f_1 v_1,\\
v_{(-2,1),\varpi_1} &= f_1^{(2)} f_2 f_1 v_1, & v_{(1,-1),\varpi_1} &= f_2 f_1^{(2)} f_2 f_1 v_1,\\
v_{(-1,0),\varpi_1} &= f_1 f_2 f_1^{(2)} f_2 f_1 v_1,
\end{aligned}$$

and the following basis of $V(\varpi_2)$:

$$\begin{aligned}
v_{(0,1),\varpi_2} &= v_2, & v_{(3,-1),\varpi_2} &= f_2 v_2,\\
v_{(1,0),\varpi_2} &= f_1 f_2 v_2, & v_{(-1,1),\varpi_2} &= f_1^{(2)} f_2 f_1 v_2,\\
v_{(2,-1),\varpi_2} &= f_2 f_1^{(2)} f_2 v_2, & v_{(-3,2),\varpi_2} &= f_1^{(3)} f_2 v_2,\\
{}^1 v_{(0,0),\varpi_2} &= f_1 f_2 f_1^{(2)} f_2 v_2, & {}^2 v_{(0,0),\varpi_2} &= f_2 f_1^{(3)} f_2 v_2,\\
v_{(3,-2),\varpi_2} &= f_2^{(2)} f_1^{(3)} f_2 v_2, & v_{(-2,1),\varpi_2} &= f_1^{(2)} f_2 f_1^{(2)} f_2 v_2,\\
v_{(1,-1),\varpi_2} &= f_1 f_2^{(2)} f_1^{(3)} f_2 v_2, & v_{(-1,0),\varpi_2} &= f_1^{(2)} f_2^{(2)} f_1^{(3)} f_2 v_2,\\
v_{(-3,1),\varpi_2} &= f_1^{(3)} f_2^{(2)} f_1^{(3)} f_2 v_2, & v_{(0,-1),\varpi_2} &= f_2 f_1^{(3)} f_2^{(2)} f_1^{(3)} f_2 v_2.
\end{aligned}$$

Remark 2.44. The following relation holds in $V(\varpi_2)$:

$$f_1 f_2^{(2)} f_1^{(3)} f_2 v_2 = f_2 f_1^{(2)} f_2 f_1^{(2)} f_2 v_2.$$

Thus, there are two ways to present the vector $v_{(1,-1),\varpi_2}$.

Definition 2.45. For each vector ${}^i v_{\mu,\varpi} \in V(\varpi)$ we associate a diagram in $\mathbf{Web}_q(\mathfrak{g}_2)$ denoted ${}^i L_{\mu,\varpi}$. Our convention will be to not record the superscript i in ${}^i L_{\mu,\varpi}$ when the weight space is multiplicity one

$L_{(1,0),\varpi_1} :=$  ,
$L_{(-1,1),\varpi_1} :=$  ,
$L_{(2,-1),\varpi_1} :=$  ,

$L_{(0,0),\varpi_1} :=$  ,
$L_{(-2,1),\varpi_1} :=$  ,
$L_{(1,-1),\varpi_1} :=$  ,

$L_{(-1,0),\varpi_1} :=$  ,
$L_{(0,1),\varpi_2} :=$  ,
$L_{(3,-1),\varpi_2} :=$  ,

$L_{(1,0),\varpi_2} :=$  ,
$L_{(-1,1),\varpi_2} :=$  ,
$L_{(2,-1),\varpi_2} :=$  ,

$L_{(-3,2),\varpi_2} :=$  , ${}^1L_{(0,0),\varpi_2} :=$  , ${}^2L_{(0,0),\varpi_2} :=$  ,

$L_{(3,-2),\varpi_2} :=$  , $L_{(-2,1),\varpi_2} :=$  , $L_{(1,-1),\varpi_2} :=$  ,

$L_{(-1,0),\varpi_2} :=$  , $L_{(-3,1),\varpi_2} :=$  , $L_{(0,-1),\varpi_2} :=$  .

The diagram ${}^iL_{\mu,\varpi}$ is a morphism from ${}^i\underline{\mathrm{x}}_{\mu,\varpi} \otimes \varpi \to {}^i\underline{\mathrm{y}}_{\mu,\varpi}$. We will refer to ${}^i\underline{\mathrm{x}}_{\mu,\varpi}$ as the *in strand* of ${}^iL_{\mu,\varpi}$ and ${}^i\underline{\mathrm{y}}_{\mu,\varpi}$ as the *out strand* of ${}^iL_{\mu,\varpi}$. Note that

$$\mu = \mathrm{wt}({}^i\underline{\mathrm{y}}_{\mu,\varpi}) - \mathrm{wt}({}^i\underline{\mathrm{x}}_{\mu,\varpi}).$$

Example 2.46. For $L_{(-2,1),\varpi_2} :=$  , we have $\underline{\mathrm{x}}_{(-2,1),\varpi_2} = \varpi_1\varpi_1$, $\underline{\mathrm{y}}_{(-2,1),\varpi_2} = \varpi_2$, and $(-2,1) = \mathrm{wt}(\underline{\mathrm{y}}_{(-2,1),\varpi_2}) - \mathrm{wt}(\underline{\mathrm{x}}_{(-2,1),\varpi_2})$.

Notation 2.47. If W is a subspace of a $U_q(\mathfrak{g}_2)$ module, we will write

$$\mathrm{Ker}_{(a,b)}(W) := \{w \in W : e_1^{a+1}w = 0 = e_2^{b+1}w\}.$$

Lemma 2.48. *Let $a, b \in \mathbb{Z}_{\geq 0}$. Fix a fundamental weight ϖ and let $\mu \in \mathrm{wt}\, V(\varpi)$. Then,*

$$[V(a,b) \otimes V(\varpi) : V((a,b)+\mu)] = \dim \mathrm{Ker}_{(a,b)}(V(\varpi)_\mu). \tag{2.2}$$

Proof. It follows from [19, Theorem 2.1]. ∎

Lemma 2.49. *The following are equivalent.*

(1) ${}^iv_{\mu,\varpi} \in \mathrm{Ker}_{(a,b)}(V(\varpi)_\mu)$.

(2) *There is $(c,d) \in \mathbb{N} \times \mathbb{N}$ such that* $\mathrm{wt}({}^i\underline{\mathrm{x}}_{\mu,\varpi}) + (c,d) = (a,b)$.

Proof. The lemma can be deduced from the following claim: the weight of the in strand for ${}^iL_{\mu,\varpi}$, $\mathrm{wt}({}^i\underline{\mathrm{x}}_{\mu,\varpi})$ is equal to the minimal (a,b) so that

$${}^iv_{\mu,\varpi} \in \mathrm{Ker}_{(a,b)}(V(\varpi)_\mu).$$

The claim is verified from the vector to diagram correspondence ${}^iv_{\mu,\varpi} \mapsto {}^iL_{\mu,\varpi}$, along with equation (2.2), and the description of action of e_1 and e_2 on the vectors in each fundamental representation. Computing $e_k \cdot {}^iv_{\mu,\varpi}$ is left as an exercise; the most interesting case is the zero weight space for the second fundamental representation. ∎

Example 2.50. When $a \geq 3$ and $b \geq 2$, $[V(a,b) \otimes V(\varpi_2) : V((a,b)+\mu)] = 1$ when $\mu \neq (0,0)$, and $[V(a,b) \otimes V(\varpi_2) : V(a,b)] = 2$. The reader should compare this with the observation that for each ${}^iL_{\mu,\varpi_2}$ the number of ϖ_1 colored in strands is less than or equal to 3 and the number of ϖ_2 colored in strands is less than or equal to 2.

For each dominant integral weight $\lambda = a\varpi_1 + b\varpi_2 \in X_+$, we choose a distinguished object $\underline{\mathrm{u}}_\lambda \in \mathbf{Web}_{\mathrm{q}}(\mathfrak{g}_2)$ such that $\mathrm{wt}\,\underline{\mathrm{u}}_\lambda = \lambda$.

Example 2.51. We must have $\underline{\mathrm{u}}_{(2,0)} = \varpi_1\varpi_1$, and for $\underline{\mathrm{u}}_{(1,1)}$, we choose one of $\varpi_1\varpi_2$ or $\varpi_2\varpi_1$.

Definition 2.52. Let $\underline{\mathrm{w}}$ be an object in $\mathbf{Web}_{\mathrm{q}}(\mathfrak{g}_2)$, and let $\lambda = \mathrm{wt}\,\underline{\mathrm{w}}$. Suppose that ${}^i v_{\mu,\varpi} \in \mathrm{Ker}_\lambda(V(\varpi)_\mu)$, so in particular $\lambda + \mu \in X_+$. Let

$$(c,d) := \lambda - \mathrm{wt}({}^i\underline{\mathrm{x}}_{\mu,\varpi});$$

then, by Lemma 2.49, we have $(c,d) \in \mathbb{N} \times \mathbb{N}$, so there is a reduced neutral diagram

$$\mathsf{H}^{\underline{s}_{(c,d)} \otimes ({}^i\underline{\mathrm{x}}_{\mu,\varpi})}_{\underline{\mathrm{w}}} : \underline{\mathrm{w}} \to \underline{s}_{(c,d)} \otimes ({}^i\underline{\mathrm{x}}_{\mu,\varpi}).$$

Since $(c,d) + \mathrm{wt}({}^i\underline{\mathrm{y}}_{\mu,\varpi}) = \lambda + \mu$, there is also a reduced neutral diagram

$$\mathsf{H}^{\underline{\mathrm{u}}_{\lambda+\mu}}_{\underline{s}_{(c,d)} \otimes ({}^i\underline{\mathrm{y}}_{\mu,\varpi})} : \underline{s}_{(c,d)} \otimes ({}^i\underline{\mathrm{y}}_{\mu,\varpi}) \to \underline{\mathrm{u}}_{\lambda+\mu}.$$

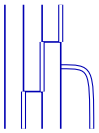
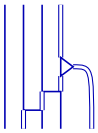
We define the *elementary light ladder diagram* to be

$${}^i ELL^{\underline{\mathrm{u}}_{\lambda+\mu}}_{\underline{\mathrm{w}},\varpi} := \mathsf{H}^{\underline{\mathrm{u}}_{\lambda+\mu}}_{\underline{s}_{(c,d)} \otimes ({}^i\underline{\mathrm{y}}_{\mu,\varpi})} \circ \left(\mathrm{id}_{\underline{s}_{(c,d)}} \otimes {}^i L_{\mu,\varpi}\right) \circ \left(\mathsf{H}^{\underline{s}_{(c,d)} \otimes ({}^i\underline{\mathrm{x}}_{\mu,\varpi})}_{\underline{\mathrm{w}}} \otimes \mathrm{id}_\varpi\right).$$

Example 2.53. Consider $\underline{\mathrm{w}} = \varpi_1\varpi_2\varpi_1\varpi_1$ and $\varpi = \varpi_2$. We know that $\lambda = \mathrm{wt}\,\underline{\mathrm{w}} = (3,1)$. When $\mu = (-2,1)$, so $\lambda + \mu = (1,2)$, choose $\underline{\mathrm{u}}_{\lambda+\mu} = \varpi_1\varpi_2\varpi_2$. Then,

$${}^1 ELL^{\underline{\mathrm{u}}_{\lambda+\mu}}_{\underline{\mathrm{w}},\varpi} =$$  .

When $\mu = (0,0)$, $\lambda + \mu = (3,1)$, choose $\underline{\mathrm{u}}_{\lambda+\mu} = \varpi_1\varpi_1\varpi_1\varpi_2$. Then,

$${}^1 ELL^{\underline{\mathrm{u}}_{\lambda+\mu}}_{\underline{\mathrm{w}},\varpi} =$$  and $${}^2 ELL^{\underline{\mathrm{u}}_{\lambda+\mu}}_{\underline{\mathrm{w}},\varpi} =$$  .

Definition 2.54. Let $\underline{\mathrm{w}} \in \mathbf{Web}_{\mathrm{q}}(\mathfrak{g}_2)$. Write $\lambda = \mathrm{wt}\,\underline{\mathrm{w}}$, and suppose that ${}^i v_{\mu,\varpi} \in \mathrm{Ker}_\lambda(V(\varpi)_\mu)$. We define the *(clasped) light ladder diagram* to be the following diagram:

$${}^i LL^{\underline{\mathrm{u}}_{\lambda+\mu}}_{\underline{\mathrm{w}},\varpi} := C_{\underline{\mathrm{u}}_{\lambda+\mu}} \circ ({}^i ELL^{\underline{\mathrm{u}}_{\lambda+\mu}}_{\underline{\mathrm{w}},\varpi}) \circ (C_{\underline{\mathrm{w}}} \otimes \mathrm{id}_\varpi).$$

Lemma 2.55. *Let* $\mathsf{N}^{\underline{\mathrm{w}}'}_{\underline{\mathrm{w}}} : \underline{\mathrm{w}} \to \underline{\mathrm{w}}'$ *be a neutral diagram. Then,*

$${}^i LL^{\underline{\mathrm{u}}_{\lambda+\mu}}_{\underline{\mathrm{w}}',\varpi} \circ (N^{\underline{\mathrm{w}}'}_{\underline{\mathrm{w}}} \otimes \mathrm{id}_\varpi) \circ C_{\underline{\mathrm{w}}} \otimes \mathrm{id}_\varpi = {}^i LL^{\underline{\mathrm{u}}_{\lambda+\mu}}_{\underline{\mathrm{w}},\varpi}.$$

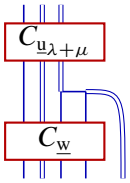
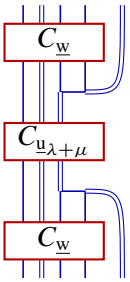
Proof. It follows from Lemma 2.37. ∎

Definition 2.56. Suppose that ${}^{i}v_{\mu,\varpi}, {}^{j}v_{\mu,\varpi} \in \mathrm{Ker}_{\mathrm{wt}\,\underline{\mathrm{w}}}(V(\varpi)_\mu)$ (we allow for $i = j$). We define the *(clasped) double ladder diagram* to be the following diagram in $\mathrm{End}_{\mathbf{Web}_q(\mathfrak{g}_2)}(\underline{\mathrm{w}} \otimes \varpi)$:

$$ {}^{ij}\mathbb{LL}^{\underline{\mathrm{u}}_{\lambda+\mu}}_{\underline{\mathrm{w}},\varpi} := (\mathbb{D}({}^{i}LL^{\underline{\mathrm{u}}_{\lambda+\mu}}_{\underline{\mathrm{w}},\varpi})) \circ ({}^{j}LL^{\underline{\mathrm{u}}_{\lambda+\mu}}_{\underline{\mathrm{w}},\varpi}). $$

In the case that $V(\varpi)_\mu$ is one dimensional, we will drop the superscripts ij in double ladders and drop superscript i in (clasped) light ladders.

Example 2.57. Consider $\underline{\mathrm{w}} = \varpi_1\varpi_2\varpi_1\varpi_1$, $\varpi = \varpi_2$. When $\mu = (-2, 1)$, choose $\underline{\mathrm{u}}_{\lambda+\mu} = \varpi_1\varpi_2\varpi_2$; then

$LL^{\underline{\mathrm{u}}_{\lambda+\mu}}_{\underline{\mathrm{w}},\varpi} =$  and $\mathbb{LL}^{\underline{\mathrm{u}}_{\lambda+\mu}}_{\underline{\mathrm{w}},\varpi} =$  .

Remark 2.58. Using the definition of the elementary light ladder, and basic properties of clasps, we can expand the clasped light ladder

$$ \begin{aligned} {}^{i}LL^{\underline{\mathrm{u}}_{\lambda+\mu}}_{\underline{\mathrm{w}},\varpi} &:= C_{\underline{\mathrm{u}}_{\lambda+\mu}} \circ ({}^{i}ELL^{\underline{\mathrm{u}}_{\lambda+\mu}}_{\underline{\mathrm{w}},\varpi}) \circ (C_{\underline{\mathrm{w}}} \otimes \mathrm{id}_\varpi) \\ &= C^{\underline{\mathrm{u}}_{\lambda+\mu}}_{\underline{s}_{(c,d)} \otimes ({}^{i}\underline{\mathrm{y}}_{\mu,\varpi})} \circ \left((\mathrm{id}_{\underline{s}_{(c,d)}}) \otimes {}^{i}L_{\mu,\varpi}\right) \circ \left(C^{\underline{s}_{(c,d)} \otimes ({}^{i}\underline{\mathrm{x}}_{\mu,\varpi})}_{\underline{\mathrm{w}}} \otimes \mathrm{id}_\varpi\right). \end{aligned} $$

We can similarly expand the clasped double ladder

$$ \begin{aligned} {}^{ij}\mathbb{LL}^{\underline{\mathrm{u}}_{\lambda+\mu}}_{\underline{\mathrm{w}},\varpi} &:= (\mathbb{D}({}^{i}LL^{\underline{\mathrm{u}}_{\lambda+\mu}}_{\underline{\mathrm{w}},\varpi})) \circ ({}^{j}LL^{\underline{\mathrm{u}}_{\lambda+\mu}}_{\underline{\mathrm{w}},\varpi}) \\ &= (C_{\underline{\mathrm{w}}} \otimes \mathrm{id}_\varpi) \circ (\mathbb{D}({}^{i}ELL^{\underline{\mathrm{u}}_{\lambda+\mu}}_{\underline{\mathrm{w}},\varpi})) \circ C_{\underline{\mathrm{u}}_{\lambda+\mu}} \circ ({}^{j}ELL^{\underline{\mathrm{u}}_{\lambda+\mu}}_{\underline{\mathrm{w}},\varpi}) \circ (C_{\underline{\mathrm{w}}} \otimes \mathrm{id}_\varpi) \\ &= \left(C^{\underline{\mathrm{w}}}_{\underline{s}_{(c,d)} \otimes ({}^{i}\underline{\mathrm{x}}_{\mu,\varpi})} \otimes \mathrm{id}_\varpi\right) \circ \left((\mathrm{id}_{\underline{s}_{(c,d)}}) \otimes \mathbb{D}({}^{i}L_{\mu,\varpi})\right) \circ C^{\underline{s}_{(c,d)} \otimes ({}^{i}\underline{\mathrm{y}}_{\mu,\varpi})}_{\underline{s}_{(e,f)} \otimes ({}^{j}\underline{\mathrm{y}}_{\mu,\varpi})} \\ &\quad \circ \left((\mathrm{id}_{\underline{s}_{(e,f)}}) \otimes {}^{j}L_{\mu,\varpi}\right) \circ \left(C^{\underline{s}_{(e,f)} \otimes ({}^{j}\underline{\mathrm{x}}_{\mu,\varpi})}_{\underline{\mathrm{w}}} \otimes \mathrm{id}_\varpi\right). \end{aligned} $$

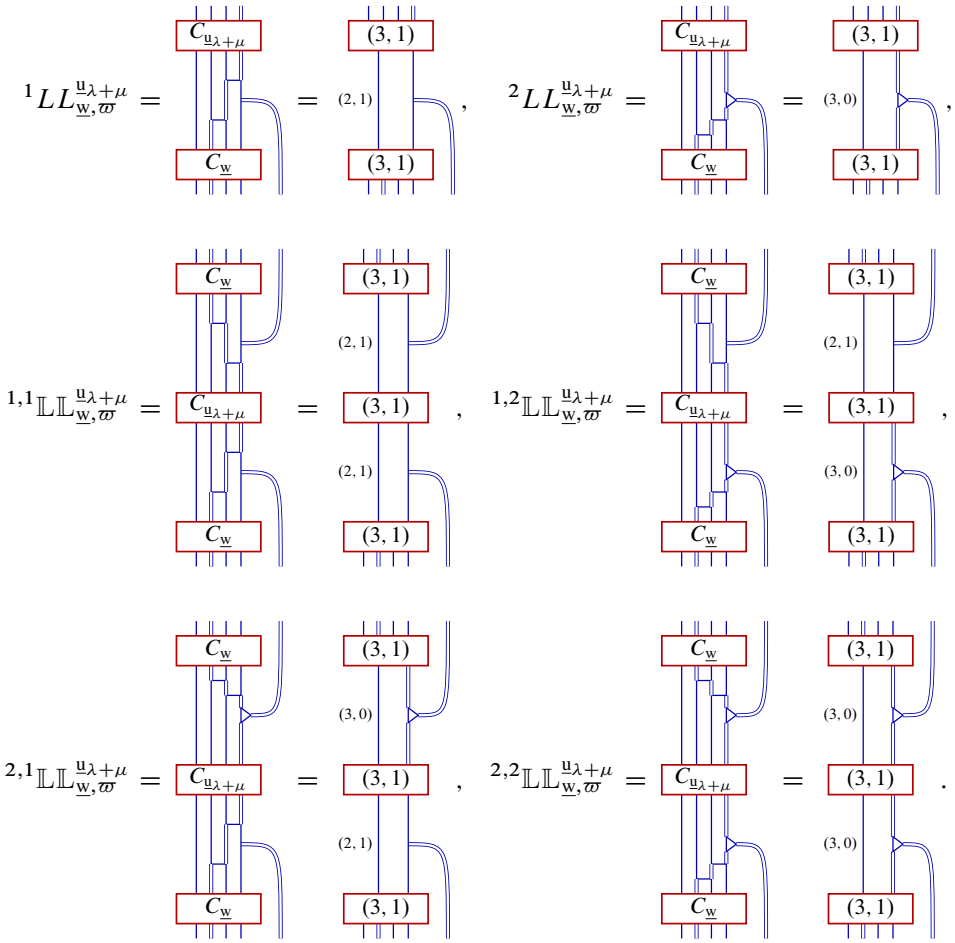
This more complicated-looking expanded formula is actually simpler when viewed in terms of the graphical calculus, as we illustrate in Example 2.60.

Notation 2.59. When $\mathrm{wt}\,\underline{\mathrm{x}} = \mathrm{wt}\,\underline{\mathrm{y}} = (a, b)$, we will use an (a, b)-labeled box in $\mathrm{Hom}_{\mathbf{Web}_q(\mathfrak{g}_2)}(\underline{\mathrm{x}}, \underline{\mathrm{y}})$ to denote $C^{\underline{\mathrm{y}}}_{\underline{\mathrm{x}}}$.

Example 2.60. Let $\underline{w} = \varpi_1\varpi_2\varpi_1\varpi_1$, $\varpi = \varpi_2$. When $\mu = (0,0)$, choose

$$\underline{u}_{\lambda+\mu} = \varpi_1\varpi_1\varpi_1\varpi_2;$$

then (using Lemma 2.35), we have



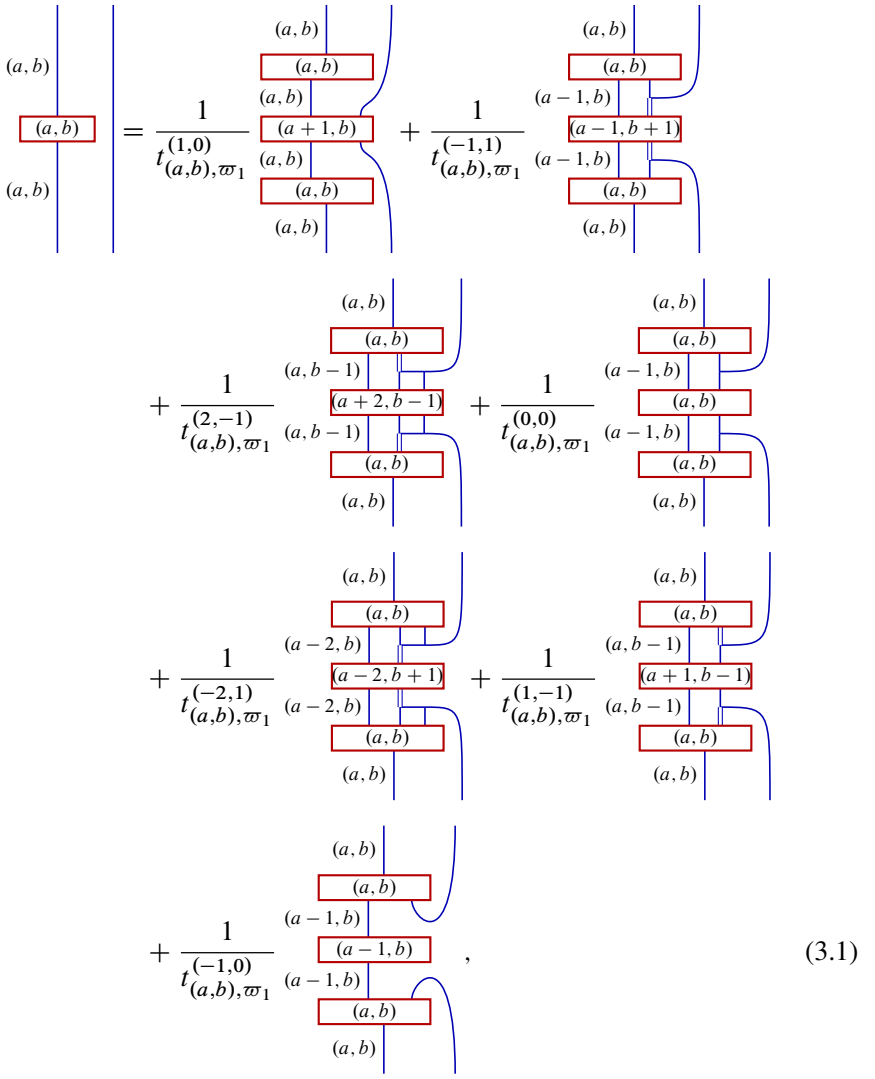
Remark 2.61. As (a,b) varies, so does $\mathrm{Ker}_{(a,b)}(V(\varpi))$. However, the vector $v_\varpi = v_{\varpi,1}$ is always contained in $\mathrm{Ker}_{(a,b)}(V(\varpi))$. Moreover, the associated elementary light ladder is just a composition of neutral diagrams, so the associated (clasped) double ladder can be simplified to

$$\mathbb{L}\mathbb{L}^{\underline{u}_{\lambda+\varpi}}_{\underline{w},\varpi} = C_{\underline{w}\otimes\varpi}.$$

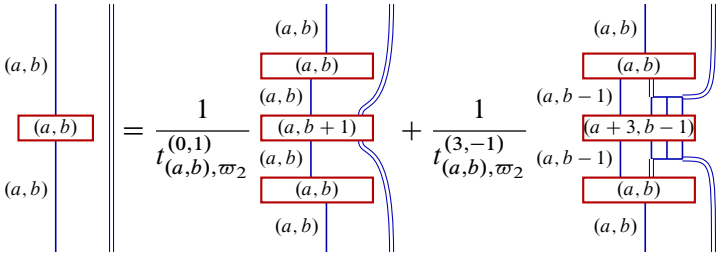
Thus, the (clasped) double ladder associated to the highest weight vector in $V(\varpi)$ is itself a clasp.

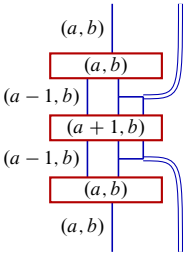
3. Triple clasp formula

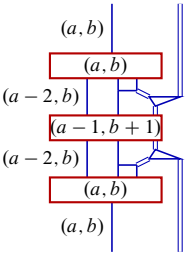
3.1. Formulas

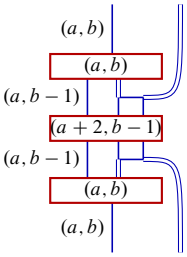


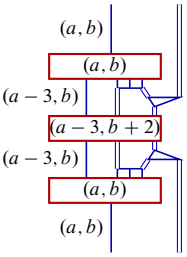
(3.1)

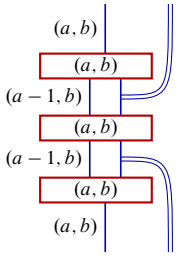


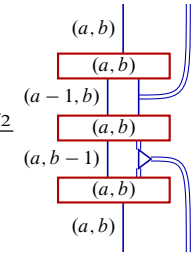
$$+\frac{1}{t^{(1,0)}_{(a,b),\varpi_2}}$$


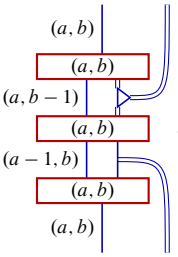
$$+\frac{1}{t^{(-1,1)}_{(a,b),\varpi_2}}$$


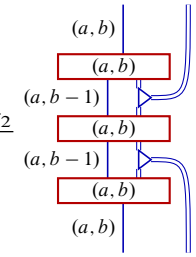
$$+\frac{1}{t^{(2,-1)}_{(a,b),\varpi_2}}$$


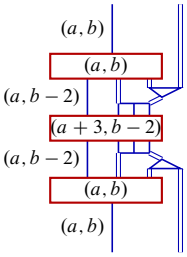
$$+\frac{1}{t^{(-3,2)}_{(a,b),\varpi_2}}$$


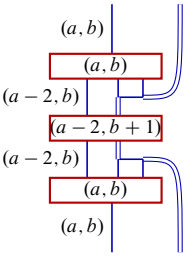
$$+\frac{{}^{2,2}t^{(0,0)}_{(a,b),\varpi_2}}{D_{(a,b)}}$$


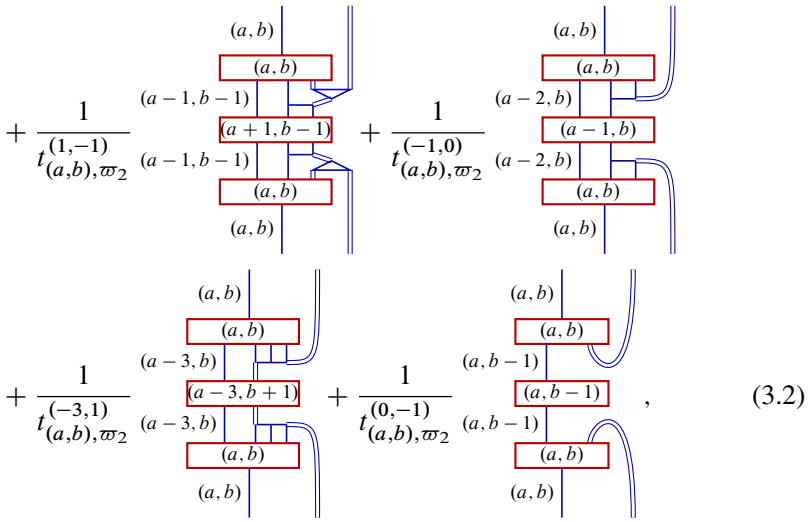
$$-\frac{{}^{1,2}t^{(0,0)}_{(a,b),\varpi_2}}{D_{(a,b)}}$$


$$-\frac{{}^{2,1}t^{(0,0)}_{(a,b),\varpi_2}}{D_{(a,b)}}$$


$$+\frac{{}^{1,1}t^{(0,0)}_{(a,b),\varpi_2}}{D_{(a,b)}}$$


$$+\frac{1}{t^{(3,-2)}_{(a,b),\varpi_2}}$$


$$+\frac{1}{t^{(-2,1)}_{(a,b),\varpi_2}}$$




(3.2)

$$t^{(1,0)}_{(a,b),\varpi_1} = 1, \tag{3.3}$$

$$t^{(-1,1)}_{(a,b),\varpi_1} = -\frac{[a+1]}{[a]}, \tag{3.4}$$

$$t^{(2,-1)}_{(a,b),\varpi_1} = \frac{[3b+3][a+3b+4]}{[3b][a+3b+3]}, \tag{3.5}$$

$$t^{(0,0)}_{(a,b),\varpi_1} = -\frac{[a+2][a+3b+5][2a+3b+6]}{[2][a][a+3b+3][2a+3b+4]}, \tag{3.6}$$

$$t^{(-2,1)}_{(a,b),\varpi_1} = \frac{[a+1][2a+3b+5][3a+3b+6]}{[a-1][2a+3b+4][3a+3b+3]}, \tag{3.7}$$

$$t^{(1,-1)}_{(a,b),\varpi_1} = -\frac{[3b+3][a+3b+4][2a+3b+5][3a+6b+9]}{[3b][a+3b+2][2a+3b+4][3a+6b+6]}, \tag{3.8}$$

$$t^{(-1,0)}_{(a,b),\varpi_1} = \frac{[a+1][a+3b+4][2a+3b+5][3a+3b+6][3a+6b+9]}{[a][a+3b+3][2a+3b+3][3a+3b+3][3a+6b+6]}, \tag{3.9}$$

$$t^{(0,1)}_{(a,b),\varpi_2} = 1, \tag{3.10}$$

$$t^{(3,-1)}_{(a,b),\varpi_2} = -\frac{[3b+3]}{[3b]}, \tag{3.11}$$

$$t^{(1,0)}_{(a,b),\varpi_2} = \frac{[a+3][a+3b+6]}{[3][a][a+3b+3]}, \tag{3.12}$$

$$t^{(-1,1)}_{(a,b),\varpi_2} = -\frac{[a+1][a+2][2a+3b+7]}{[3][a-1][a][2a+3b+4]}, \tag{3.13}$$

$$t^{(2,-1)}_{(a,b),\varpi_2} = \frac{[3b+3][a+3b+4][a+3b+5][2a+3b+7]}{[3][3b][a+3b+2][a+3b+3][2a+3b+4]}, \tag{3.14}$$

$$t^{(-3,2)}_{(a,b),\varpi_2} = \frac{[a+1][3a+3b+6]}{[a-2][3a+3b+3]}, \tag{3.15}$$

$$t^{(3,-2)}_{(a,b),\varpi_2} = \frac{[3b+3][a+3b+4][3a+6b+9]}{[3b-3][a+3b+1][3a+6b+6]}, \tag{3.16}$$

$$t^{(-2,1)}_{(a,b),\varpi_2} = \frac{[a+1][a+3b+6][2a+3b+5][2a+3b+6][3a+3b+6]}{[3][a-1][a+3b+3][2a+3b+3][2a+3b+4][3a+3b+3]}, \tag{3.17}$$

$$t^{(1,-1)}_{(a,b),\varpi_2} = -\frac{[a+3][3b+3][a+3b+4][2a+3b+5][2a+3b+6][3a+6b+9]}{[3][a][3b][a+3b+2][2a+3b+3][2a+3b+4][3a+6b+6]}, \tag{3.18}$$

$$t^{(-1,0)}_{(a,b),\varpi_2} = \frac{[a+1][a+2][a+3b+4][a+3b+5][2a+3b+5][3a+3b+6][3a+6b+9]}{[3][a-1][a][a+3b+2][a+3b+3][2a+3b+3][3a+3b+3][3a+6b+6]}, \tag{3.19}$$

$$t^{(-3,1)}_{(a,b),\varpi_2} = -\frac{[a+1][2a+3b+5][3a+3b+6][3a+6b+9]}{[a-2][2a+3b+2][3a+3b][3a+6b+6]}, \tag{3.20}$$

$$t^{(0,-1)}_{(a,b),\varpi_2} = \frac{[3b+3][a+3b+4][2a+3b+5][3a+3b+6][3a+6b+9]}{[3b][a+3b+1][2a+3b+2][3a+3b+3][3a+6b+3]}, \tag{3.21}$$

$$\left({}^{p,\ell}t^{(0,0)}_{(a,b),\varpi_2}\right) = \begin{pmatrix} {}^{1,1}t^{(0,0)}_{(a,b),\varpi_2} & {}^{1,2}t^{(0,0)}_{(a,b),\varpi_2} \\ {}^{2,1}t^{(0,0)}_{(a,b),\varpi_2} & {}^{2,2}t^{(0,0)}_{(a,b),\varpi_2} \end{pmatrix}, \tag{3.22}$$

$$\mathcal{D}_{(a,b)} := \det\left({}^{p,\ell}t^{(0,0)}_{(a,b),\varpi_2}\right). \tag{3.23}$$

Moreover,

$${}^{1,2}t^{(0,0)}_{(a,b),\varpi_2} = {}^{2,1}t^{(0,0)}_{(a,b),\varpi_2}, \tag{3.24}$$

$$\mathcal{D}_{(a,b)} = \frac{[4][6][a+2][3b+6][a+3b+5][2a+3b+6][3a+3b+9][3a+6b+12]}{[2][3][12][a][3b][a+3b+3][2a+3b+4][3a+3b+3][3a+6b+6]}, \tag{3.25}$$

and the entries of the matrix can be computed from the relations in Appendix A.1.

3.2. Verifying the clasp conjecture

Before proving our main theorem, we will prove the following, which implies the clasp conjecture in type G_2.

Corollary 3.1. *Fix $\lambda = a\varpi_1 + b\varpi_2$ with $a, b \in \mathbb{Z}_{\geq 0}$. Let $\varpi \in \{\varpi_1, \varpi_2\}$ be a fundamental weight, and let $\mu \in W \cdot \varpi$ be a weight in the Weyl group orbit of ϖ. Then,*

$$t^{\mu}_{\lambda,\varpi} = \pm \prod_{\alpha \in \Phi_\mu} \frac{[(\alpha^\vee, \lambda + \rho)]_{\ell^{\ell(\alpha)}}}{[(\alpha^\vee, \lambda + \varpi + \rho)]_{\ell^{\ell(\alpha)}}}.$$

Proof. In type G_2, the W invariant bilinear pairing on $\mathbb{Z}\Phi$ is determined by

$$(\alpha_1, \alpha_1) = 2 \quad \text{and} \quad (\alpha_2, \alpha_2) = 6.$$

In particular, $l(\alpha_1) = 1$ and $l(\alpha_2) = 3$. We set $\alpha^\vee = 2\alpha/(\alpha, \alpha)$. The positive roots are

$$\alpha_1, 3\alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2, \alpha_1 + \alpha_2, \alpha_2;$$

the corresponding coroots are

$$\alpha_1^\vee, \alpha_1^\vee + \alpha_2^\vee, 2\alpha_1^\vee + 3\alpha_2^\vee, \alpha_1^\vee + 2\alpha_2^\vee, \alpha_1^\vee + 3\alpha_2^\vee, \alpha_2^\vee.$$

To simplify notation, we will write $s_i := s_{\alpha_i}$. It is not hard to see that

$$\begin{aligned}
&d_{(1,0)} = 1, && d_{(0,1)} = 1, && d_{(-2,1)} = s_1 s_2 s_1, && d_{(3,-2)} = s_2 s_1 s_2,\\
&d_{(-1,1)} = s_1, && d_{(3,-1)} = s_2, && d_{(1,-1)} = s_1 s_2 s_1 s_2, && d_{(-3,1)} = s_2 s_1 s_2 s_1,\\
&d_{(2,-1)} = s_1 s_2, && d_{(-3,2)} = s_2 s_1, && d_{(-1,0)} = s_1 s_2 s_1 s_2 s_1, && d_{(0,-1)} = s_2 s_1 s_2 s_1 s_2.
\end{aligned}$$

The claim then follows from the formulas for $t^{\mu}_{\lambda,\varpi}$ in Section 3.1. One verifies this by using the fact that if $w = s_{\beta_1} s_{\beta_2} \cdots s_{\beta_n}$, then

$$\{\alpha \in \Phi_+ : w\alpha \in \Phi_-\} = \{\beta_n, s_{\beta_n}(\beta_{n-1}), s_{\beta_n} s_{\beta_{n-1}}(\beta_{n-2}), \ldots, s_{\beta_n} s_{\beta_{n-1}} \cdots s_{\beta_2}(\beta_1)\},$$

along with the quantum number identity $[n]_{q^3} = [3n]/[3]$. ∎

3.3. Proof of triple clasp formula

Suppose that $V(\operatorname{wt} \underline{\mathrm{u}})$ is a summand of $V(\operatorname{wt} \underline{\mathrm{w}}) \otimes V(\varpi)$ and that $(a, b) = \lambda = \operatorname{wt} \underline{\mathrm{w}}$ and $(m, n) = \mu = \operatorname{wt} \underline{\mathrm{u}} - \operatorname{wt} \underline{\mathrm{w}}$. Then, we will write

$${}^{p\ell} t^{\underline{\mathrm{u}}}_{\underline{\mathrm{w}},\varpi} := {}^{p\ell} t^{(m,n)}_{(a,b),\varpi}.$$

Our convention is that the $p\ell$ superscript is neglected when $\dim V(\varpi)_\mu = 1$. By definition, the elements ${}^{p\ell} t^{\underline{\mathrm{u}}}_{\underline{\mathrm{w}},\varpi}$ only depend on the weights $\operatorname{wt} \underline{\mathrm{w}}$ and $\operatorname{wt} \underline{\mathrm{u}}$, not the words $\underline{\mathrm{w}}$ and $\underline{\mathrm{u}}$.

We will write $({}^{p\ell} t^{\underline{\mathrm{u}}}_{\underline{\mathrm{w}},\varpi})$ to denote the matrix of scalars ${}^{p\ell} t^{\underline{\mathrm{u}}}_{\underline{\mathrm{w}},\varpi}$ for

$$v_p, v_\ell \in \operatorname{Ker}_{\operatorname{wt} \underline{\mathrm{w}}}(V(\varpi)_\mu).$$

Lemma 3.2. *The matrix* $({}^{p\ell} t^{\underline{\mathrm{u}}}_{\underline{\mathrm{w}},\varpi})$ *is invertible over* $\mathbb{C}(q)$.

Proof. From Section 3.1, using equations (3.3) to (3.21), and (3.25), one can check that the determinant of this matrix is invertible in $\mathbb{C}(q)$. ∎

Definition 3.3. Let $\underline{w} \in \mathbf{Web}_q(\mathfrak{g}_2)$, write $\lambda = \operatorname{wt}\underline{w}$, and let $\varpi \in \{\varpi_1, \varpi_2\}$. We define the *triple clasp* to be the following inductively defined diagram:

$$T_{\underline{w}\otimes\varpi} := T_{\underline{w}} \otimes \mathrm{id}_\varpi - \sum_{\substack{\mu\in \operatorname{wt} V(\varpi)\setminus\{\varpi\} \\ v_i, v_j \in \operatorname{Ker}_{\operatorname{wt}\underline{w}}(V(\varpi)_\mu)}} ((p^\ell t^{\underline{u}_{\lambda+\mu}}_{\underline{w},\varpi})^{-1})_{ij} \cdot {}^{ij}\mathbb{T}\,\mathbb{T}^{\underline{u}_{\lambda+\mu}}_{\underline{w},\varpi},$$

where

$$\begin{aligned} {}^{ij}\mathbb{T}\,\mathbb{T}^{\underline{u}_{\lambda+\mu}}_{\underline{w},\varpi} := (T_{\underline{w}} \otimes \mathrm{id}_\varpi) \circ (\mathbb{D}({}^i ELL^{\underline{u}_{\lambda+\mu}}_{\underline{w},\varpi})) \circ T_{\underline{u}_{\lambda+\mu}} \\ \circ ({}^j ELL^{\underline{u}_{\lambda+\mu}}_{\underline{w},\varpi}) \circ (T_{\underline{w}} \otimes \mathrm{id}_\varpi). \end{aligned}$$

Definition 3.4. By Lemma 2.31, there is a scalar ${}^{p\ell}\kappa^{\underline{u}_{\lambda+\mu}}_{\underline{w},\varpi}$ such that

$${}^{p}LL^{\underline{u}_{\lambda+\mu}}_{\underline{w},\varpi} \circ \mathbb{D}({}^{\ell}LL^{\underline{u}_{\lambda+\mu}}_{\underline{w},\varpi}) = {}^{p\ell}\kappa^{\underline{u}_{\lambda+\mu}}_{\underline{w},\varpi} \cdot C_{\underline{u}_{\lambda+\mu}}.$$

We call the matrix

$$({}^{p\ell}\kappa^{\underline{u}_{\lambda+\mu}}_{\underline{w},\varpi})$$

such that $v_p, v_\ell \in \operatorname{Ker}_{\operatorname{wt}\underline{w}}(V(\varpi)_\mu)$, a *local intersection form matrix*.

Lemma 3.5. *If* $\operatorname{wt}\underline{w} = \operatorname{wt}\underline{w}'$, *then*

$${}^{p\ell}\kappa^{\underline{u}_{\lambda+\mu}}_{\underline{w},\varpi} = {}^{p\ell}\kappa^{\underline{u}_{\lambda+\mu}}_{\underline{w}',\varpi}.$$

Proof. First, observe that

$$\begin{aligned} {}^{p\ell}\kappa^{\underline{u}_{\lambda+\mu}}_{\underline{w},\varpi} \cdot C_{\underline{u}_{\lambda+\mu}} &= {}^{p}LL^{\underline{u}_{\lambda+\mu}}_{\underline{w},\varpi} \circ \mathbb{D}({}^{\ell}LL^{\underline{u}_{\lambda+\mu}}_{\underline{w},\varpi}) \\ &= {}^{p}LL^{\underline{u}_{\lambda+\mu}}_{\underline{w},\varpi} \circ (C_{\underline{w}} \otimes \mathrm{id}_\varpi) \circ \mathbb{D}({}^{\ell}LL^{\underline{u}_{\lambda+\mu}}_{\underline{w},\varpi}) \\ &= {}^{p}LL^{\underline{u}_{\lambda+\mu}}_{\underline{w},\varpi} \circ (\mathsf{H}^{\underline{w}}_{\underline{w}'} \otimes \mathrm{id}_\varpi) \\ &\quad \circ (C_{\underline{w}'} \otimes \mathrm{id}_\varpi) \circ (\mathsf{H}^{\underline{w}'}_{\underline{w}} \otimes \mathrm{id}_\varpi) \circ \mathbb{D}({}^{\ell}LL^{\underline{u}_{\lambda+\mu}}_{\underline{w},\varpi}) \\ &= {}^{p}LL^{\underline{u}_{\lambda+\mu}}_{\underline{w}',\varpi} \circ (C_{\underline{w}'} \otimes \mathrm{id}_\varpi) \circ \mathbb{D}({}^{\ell}LL^{\underline{u}_{\lambda+\mu}}_{\underline{w}',\varpi}) \\ &= {}^{p}LL^{\underline{u}_{\lambda+\mu}}_{\underline{w}',\varpi} \circ \mathbb{D}({}^{\ell}LL^{\underline{u}_{\lambda+\mu}}_{\underline{w}',\varpi}) \\ &= {}^{p\ell}\kappa^{\underline{u}_{\lambda+\mu}}_{\underline{w}',\varpi} \cdot C_{\underline{u}_{\lambda+\mu}}. \end{aligned}$$

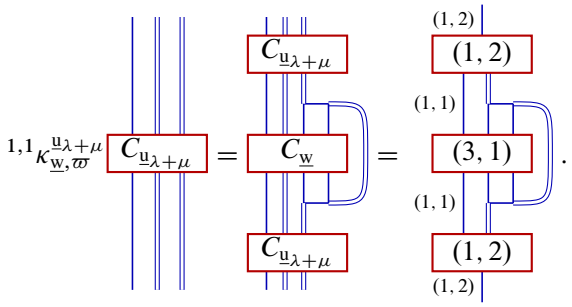
The claim follows from comparing neutral coefficients. ∎

Notation 3.6. We will write

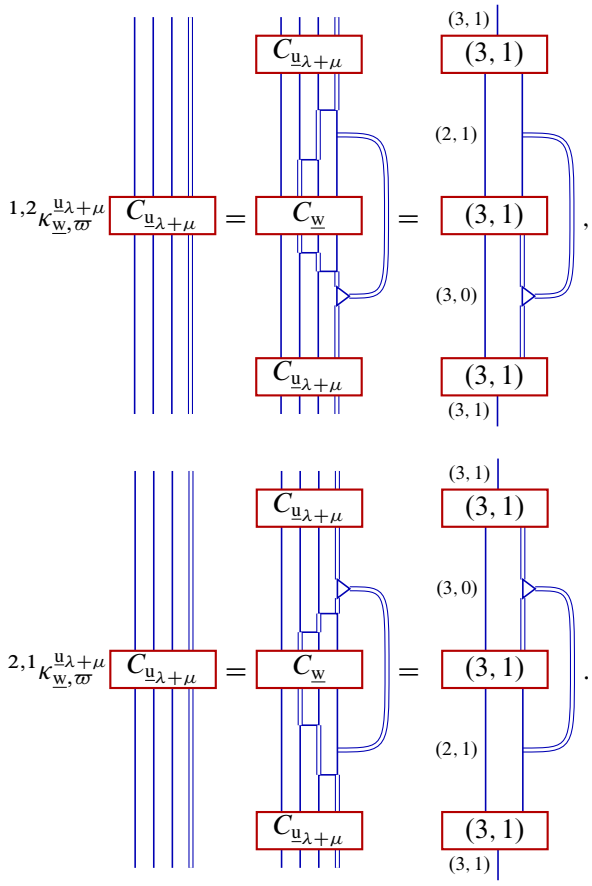
$${}^{p\ell}\kappa^{\mu}_{\lambda,\varpi} := {}^{p\ell}\kappa^{\underline{u}_{\lambda+\mu}}_{\underline{w},\varpi},$$

where $\operatorname{wt}\underline{w} = (a, b)$.

Example 3.7. Consider $\underline{w} = \varpi_1\varpi_2\varpi_1\varpi_1$ and $\varpi = \varpi_2$. When $\mu = (-2, 1)$, choose $\underline{u}_{\lambda+\mu} = \varpi_1\varpi_2\varpi_2$. Then,



When $\mu = (0, 0)$, choose $\underline{u}_{\lambda+\mu} = \varpi_1\varpi_1\varpi_1\varpi_2$. Then, we have the following:



Remark 3.8. When $\mu = (0, 0)$, by taking the quantum trace, we know that $^{1,2}\kappa^{\underline{u}_{\lambda+\mu}}_{\underline{w},\varpi} = {}^{2,1}\kappa^{\underline{u}_{\lambda+\mu}}_{\underline{w},\varpi}$. So, the local intersection form matrix $({}^{p\ell}\kappa^{\underline{u}_{\lambda+\mu}}_{\underline{w},\varpi})$ is symmetric.

Notation 3.9. Fix the following set of formal variables:

$$\mathcal{X} := \{x^{(c,d)}_{(a,b),\varpi} \mid a,b,c,d \in \mathbb{Z}_{\geq 0} \text{ and } \varpi \in \{\varpi_1, \varpi_2\}\}.$$

We will consider elements in the ring $\mathcal{A} := \mathbb{C}(q)[x^{\pm 1} \mid x \in \mathcal{X}]$.

Suppose that $(a,b) = \lambda = \operatorname{wt} \underline{w}$ and $(m,n) = \mu = \operatorname{wt} \underline{u}_{\lambda+\mu} - \operatorname{wt} \underline{w}$; then we will write

$${}^{p\ell}\rho^{\underline{u}_{\lambda+\mu}}_{\underline{w},\varpi} := {}^{p\ell}\rho^{(m,n)}_{(a,b),\varpi} \in \mathcal{A},$$

where ${}^{p\ell}\rho^{(m,n)}_{(a,b),\varpi}$ is the recursive relation described by equations (A.4) to (A.25) in Appendix A.2.

We also write ${}^{p\ell}\rho^{(m,n)}_{(a,b),\varpi}(\kappa)$ to denote the right-hand side of the recursive relation with each ${}^{p\ell}x^{\mu}_{\lambda,\varpi}$ replaced with ${}^{p\ell}\kappa^{\mu}_{\lambda,\varpi}$. Similarly, we write ${}^{p\ell}\rho^{(m,n)}_{(a,b),\varpi}(t)$ to denote the right-hand side of the recursive relation with each ${}^{p\ell}x^{\mu}_{\lambda,\varpi}$ replaced by ${}^{p\ell}t^{\mu}_{\lambda,\varpi}$.

Our convention is that the $p\ell$ superscript is neglected when

$$\dim V(\varpi)_{\mu} = 1.$$

Example 3.10. Consider $\underline{w}$ and $\underline{u}_{\lambda+\mu}$ in $\mathbf{Web}_q(\mathfrak{g}_2)$ such that $\operatorname{wt} \underline{w} = (a,b)$ and $\operatorname{wt} \underline{u}_{\lambda+\mu} = (a+1,b)$. Also, let $\varpi = \varpi_2$. By equation (A.13),

$$\rho^{\underline{u}_{\lambda+\mu}}_{\underline{w},\varpi_2} = \rho^{(1,0)}_{(a,b),\varpi_2} = \frac{[7]}{[3]} - \frac{1}{x^{(-1,1)}_{(a-1,b),\varpi_1}} x^{(3,-1)}_{(a-2,b+1),\varpi_2} - \frac{1}{x^{(2,-1)}_{(a-1,b),\varpi_1}}.$$

Remark 3.11. Since the recursive relations in Appendix A.2 are elements of $\mathcal{A}$, there is no question whether a particular element in $\mathcal{X}$ appearing in a relation is invertible or not. Thanks to Lemma 3.2, the elements ${}^{p\ell}\rho^{(m,n)}_{(a,b),\varpi}(t)$ are also always well defined. This is not obviously true for ${}^{p\ell}\rho^{(m,n)}_{(a,b),\varpi}(\kappa)$. However, we will prove that $\kappa^{\underline{u}_{\lambda+\mu}}_{\underline{w},\varpi} = t^{\underline{u}_{\lambda+\mu}}_{\underline{w},\varpi}$.

Theorem 3.12. *If* $\underline{w} \in \mathbf{Web}_q(\mathfrak{g}_2)$, $\lambda = \operatorname{wt} \underline{w}$, *and* $\varpi \in \{1,2\}$, *then*

$$T_{\underline{w}} = C_{\underline{w}} \quad \text{and} \quad \kappa^{\underline{u}_{\lambda+\mu}}_{\underline{w},\varpi} = t^{\underline{u}_{\lambda+\mu}}_{\underline{w},\varpi}.$$

We will prove Theorem 3.12 by induction. To simplify the arguments, we will break the various steps of the proof into smaller lemmas about the following statements. In what follows, we write $\lambda = \operatorname{wt} \underline{w}$:

$$\begin{aligned} S_1(\underline{w}) &:= (T_{\underline{w}} = C_{\underline{w}}), \\ S_1'(\underline{w},\varpi) &:= \big({}^{p\ell}\kappa^{\underline{u}_{\lambda+\mu}}_{\underline{w},\varpi} = {}^{p\ell}t^{\underline{u}_{\lambda+\mu}}_{\underline{w},\varpi} \text{ for all } \mu \in \operatorname{wt} V(\varpi) \\ &\qquad \text{and for all } v_{\mu,p}, v_{\mu,\ell} \in \operatorname{Ker}_{\lambda}(V(\varpi)_{\mu})\big), \end{aligned}$$

$$S_2(\underline{w},\varpi) := \Big(C_{\underline{w}} \otimes \mathrm{id}_\varpi \in \mathrm{span} \bigcup_{\substack{\mu\in \mathrm{wt}\, V(\varpi)\\ v_p,v_\ell\in \mathrm{Ker}_\lambda(V(\varpi)_\mu)}} \{{}^{p\ell}\mathbb{L}\mathbb{L}^{\underline{u}_{\lambda+\mu}}_{\underline{w},\varpi}\}\Big),$$

$$S_2'(\underline{w},\varpi) := \big(\{{}^{p}LL^{\underline{u}_{\lambda+\mu}}_{\underline{w},\varpi}\}_{v_p\in \mathrm{Ker}_\lambda(V(\varpi)_\mu)} \text{ is a linearly independent set for all } \mu\in \mathrm{wt}\, V(\varpi)\big),$$

$$S_3(\underline{w},\varpi) := \big(\{{}^{p}LL^{\underline{u}_{\lambda+\mu}}_{\underline{w},\varpi}\}_{v_p\in \mathrm{Ker}_\lambda(V(\varpi)_\mu)} \text{ is a basis for } \mathrm{Hom}_{\mathrm{Kar}\,\mathbf{Web}_q(\mathfrak{g}_2)}(C_{\underline{w}}\otimes \mathrm{id}_\varpi, C_{\underline{u}_{\lambda+\mu}}) \text{ for all } \mu\in \mathrm{wt}\, V(\varpi)\big),$$

$$S_4(\underline{w},\varpi) := \big({}^{p\ell}\kappa^{\underline{u}_{\lambda+\mu}}_{\underline{w},\varpi} = {}^{p\ell}\rho^{\underline{u}_{\lambda+\mu}}_{\underline{w},\varpi}(\kappa) \text{ for all } \mu\in \mathrm{wt}\, V(\varpi)\big),$$

$$S_5(\underline{w},\varpi) := \Big(T_{\underline{w}\otimes\varpi} = C_{\underline{w}}\otimes \mathrm{id}_\varpi - \sum_{\substack{\mu\in \mathrm{wt}\, V(\varpi)\setminus\{\varpi\}\\ v_i,v_j\in \mathrm{Ker}_\lambda(V(\varpi)_\mu)}} ({}^{p\ell}\kappa^{\underline{u}_{\lambda+\mu}}_{\underline{w},\varpi})^{-1}_{ij}\cdot {}^{ij}\mathbb{L}\mathbb{L}^{\underline{u}_{\lambda+\mu}}_{\underline{w},\varpi}\Big),$$

$$S_6(\underline{w}) := \big(T^2_{\underline{w}} = T_{\underline{w}}\big),$$

$$S_6'(\underline{w},\varpi) := \big(C_{\underline{u}}\circ D\circ T_{\underline{w}\otimes\varpi} = 0 \text{ for all } \underline{u} \text{ such that } (\mathrm{wt}\,\underline{u}-\lambda)\in \mathrm{wt}\, V(\varpi)\setminus\{\varpi\} \text{ and for all possible diagrams } D\big).$$

Lemma 3.13. *If $V(\lambda+\mu)$ is a summand of $V(\lambda)\otimes V(\varpi)$, then $\lambda+\mu\le\lambda+\varpi$.*

Proof. Suppose that $V(\lambda+\mu)$ is a summand of $V(\lambda)\otimes V(\varpi)$. Then, $V(\varpi)_\mu\neq 0$. It follows that $\mu\in\varpi+\mathbb{Z}_{\le 0}\Phi_+$, so $\varpi-\mu\ge 0$. ∎

Lemma 3.14. *If $S_1(\underline{x})$ for all $\underline{x}$ such that $\mathrm{wt}\,\underline{x}\le \mathrm{wt}(\underline{w}\otimes\varpi)$, then $S_2(\underline{w},\varpi)$.*

Proof. Write $\lambda=\mathrm{wt}\,\underline{w}$. Since $T_{\underline{w}\otimes\varpi}=C_{\underline{w}\otimes\varpi}$,

$$C_{\underline{w}\otimes\varpi} = T_{\underline{w}\otimes\varpi} = T_{\underline{w}}\otimes \mathrm{id}_\varpi - \sum_{\substack{\mu\in \mathrm{wt}\, V(\varpi)\setminus\{\varpi\}\\ v_i,v_j\in \mathrm{Ker}_\lambda(V(\varpi)_\mu)}} ({}^{p\ell}t^{\underline{u}_{\lambda+\mu}}_{\underline{w},\varpi})^{-1}_{ij}\cdot {}^{ij}\mathbb{T}\,\mathbb{T}^{\underline{u}_{\lambda+\mu}}_{\underline{w},\varpi}.$$

Also, we have

$$\mathrm{wt}\,\underline{w} < \mathrm{wt}(\underline{w}\otimes\varpi) \quad\text{and}\quad \mathrm{wt}\,\underline{u}_{\lambda+\mu}\le \mathrm{wt}(\underline{w}\otimes\varpi)$$

for all μ such that $V(\lambda+\mu)$ is a summand of $V(\lambda)\otimes V(\varpi)$, so

$$T_{\underline{w}} = C_{\underline{w}} \quad\text{and}\quad T_{\underline{u}_{\lambda+\mu}} = C_{\underline{u}_{\lambda+\mu}}.$$

Therefore,

$$\begin{aligned}
{}^{ij}\mathbb{T}\,\mathbb{T}^{\underline{u}_{\lambda+\mu}}_{\underline{w},\varpi} &:= (T_{\underline{w}}\otimes\mathrm{id}_\varpi)\circ(\mathbb{D}({}^{i}ELL^{\underline{u}_{\lambda+\mu}}_{\underline{w},\varpi}))\circ T_{\underline{u}_{\lambda+\mu}}\circ({}^{j}ELL^{\underline{u}_{\lambda+\mu}}_{\underline{w},\varpi})\circ(T_{\underline{w}}\otimes\mathrm{id}_\varpi)\\
&= (C_{\underline{w}}\otimes\mathrm{id}_\varpi)\circ(\mathbb{D}({}^{i}ELL^{\underline{u}_{\lambda+\mu}}_{\underline{w},\varpi}))\circ C_{\underline{u}_{\lambda+\mu}}\circ({}^{j}ELL^{\underline{u}_{\lambda+\mu}}_{\underline{w},\varpi})\circ(C_{\underline{w}}\otimes\mathrm{id}_\varpi)\\
&= {}^{ij}\mathbb{L}\mathbb{L}^{\underline{u}_{\lambda+\mu}}_{\underline{w},\varpi}.
\end{aligned}$$

The claim follows from observing that $C_{\underline{w}\otimes\varpi} = \mathbb{L}\mathbb{L}^{\underline{u}_{\lambda+\varpi}}_{\underline{w},\varpi}$, which is a consequence of Lemmas 2.33 and 2.35 (i.e., clasp absorption and neutral absorption). ∎

Lemma 3.15. *If* $S_1'(\underline{w},\varpi)$*, then* $S_2'(\underline{w},\varpi)$.

Proof. Write $\lambda = \operatorname{wt}\underline{w}$. For each μ such that $V(\lambda+\mu)$ is a summand of $V(\lambda)\otimes V(\varpi)$, consider the linear relation

$$\sum_p \xi_p \cdot {}^p LL^{\underline{u}_{\lambda+\mu}}_{\underline{w},\varpi} = 0.$$

We can precompose the relation with $\mathbb{D}({}^\ell LL^{\underline{u}_{\lambda+\mu}}_{\underline{w},\varpi})$ for all $v_\ell \in \operatorname{Ker}_\lambda(V(\varpi)_\mu)$ to obtain a family of relations

$$\sum_p \xi_p \cdot {}^{p\ell}\kappa^{\underline{u}_{\lambda+\mu}}_{\underline{w},\varpi} = 0.$$

By our hypothesis, we obtain

$$\sum_p \xi_p \cdot {}^{p\ell}t^{\underline{u}_{\lambda+\mu}}_{\underline{w},\varpi} = 0,$$

and it follows from Lemma 3.2 that each $\xi_p = 0$. ∎

Lemma 3.16. *If* $S_2'(\underline{w},\varpi)$*, then* $S_3(\underline{w},\varpi)$.

Proof. Let $\mu = \operatorname{wt}\underline{u} - \operatorname{wt}\underline{w}$. By combining Corollary 2.20 with equation (2.2), we may deduce the following:

$$\begin{aligned}\dim \operatorname{Hom}_{\operatorname{Kar}\mathbf{Web}_q(\mathfrak{g}_2)}(C_{\underline{w}}\otimes \mathrm{id}_\varpi, C_{\underline{u}}) &= \dim \operatorname{Hom}_{U_q(\mathfrak{g}_2)}(V(\operatorname{wt}\underline{w})\otimes V(\varpi), V(\operatorname{wt}\underline{u}))\\ &= \dim \operatorname{Ker}_{\operatorname{wt}\underline{w}}(V(\varpi)_\mu).\end{aligned}$$

The claim follows by observing that a linearly independent set with cardinality equal to the dimension of the vector space must be a spanning set. ∎

Lemma 3.17. *If* $S_4(\underline{w},\varpi)$ *and* $S_1'(\underline{x},\psi)$ *whenever* $\operatorname{wt}(\underline{x}\otimes\psi) \le \operatorname{wt}\underline{w}$*, then* $S_1'(\underline{w},\varpi)$.

Proof. The right-hand side of the equation ${}^{p\ell}\kappa^{\underline{u}}_{\underline{w},\varpi} = {}^{p\ell}\rho^{\underline{u}}_{\underline{w},\varpi}(\kappa)$ only involves terms ${}^{ij}\kappa^{\underline{y}}_{\underline{x},\psi}$ such that $\operatorname{wt}(\underline{x}\otimes\psi) \le \operatorname{wt}\underline{w}$. If we write ${}^{p\ell}\rho^{\underline{u}}_{\underline{w},\varpi}(t)$ to denote the same formula with each ${}^{ij}\kappa^{\underline{y}}_{\underline{x},\psi}$ replaced by ${}^{ij}t^{\underline{y}}_{\underline{x},\psi}$, then our hypotheses imply that

$${}^{p\ell}\kappa^{\underline{u}}_{\underline{w},\varpi} = {}^{p\ell}\rho^{\underline{u}}_{\underline{w},\varpi}(t).$$

Thus, to show that $S_1'(\underline{w},\varpi)$ holds, we must verify the following equality of rational functions in $\mathbb{C}(q)$:

$${}^{p\ell}t^{\underline{u}}_{\underline{w},\varpi} = {}^{p\ell}\rho^{\underline{u}}_{\underline{w},\varpi}(t),$$

which we verified using the SAGE code included with the source file of [4]. ∎

Example 3.18. We take verification of equation (A.13) as an example. In order to verify that

$$t^{(1,0)}_{(a,b),\varpi_2} = \rho^{(1,0)}_{(a,b),\varpi_2}(t) = \frac{[7]}{[3]} - \frac{1}{t^{(-1,1)}_{(a-1,b),\varpi_1}} t^{(3,-1)}_{(a-2,b+1),\varpi_2} - \frac{1}{t^{(2,-1)}_{(a-1,b),\varpi_1}},$$

we first write the $t^{(s,t)}_{(x,y),\varpi}$'s explicitly using equations (3.12), (3.4), (3.11), and (3.5) to obtain

$$\frac{[a+3][a+3b+6]}{[3][a][a+3b+3]} = \frac{[7]}{[3]} - \frac{1}{\left(-\frac{[a]}{[a-1]}\right)}\left(-\frac{[3b+6]}{[3b+3]}\right) - \frac{1}{\frac{[3b+3][a+3b+3]}{[3b][a+3b+2]}}.$$

We can rewrite this as

$$\begin{aligned}
&\frac{(q^{a+3}-q^{-a-3})(q^{a+3b+6}-q^{-a-3b-6})(q-q^{-1})}{(q^3-q^{-3})(q^a-q^{-a})(q^{a+3b+3}-q^{-a-3b-3})} \\
&\quad = \frac{q^7-q^{-7}}{q^3-q^{-3}} - \frac{(q^{a-1}-q^{-a+1})(q^{3b+6}-q^{-3b-6})}{(q^a-q^{-a})(q^{(3b+3)}-q^{-3b-3})} \\
&\qquad - \frac{(q^{3b}-q^{-3b})(q^{a+3b+2}-q^{-a-3b-2})}{(q^{3b+3}-q^{-3b-3})(q^{a+3b+3}-q^{-a-3b-3})}.
\end{aligned}$$

Making the substitutions $A = q^a$ and $B = q^b$, we obtain

$$\begin{aligned}
&\frac{(Aq^3-A^{-1}q^{-3})(AB^3q^6-A^{-1}B^{-3}q^{-6})(q-q^{-1})}{(q^3-q^{-3})(A-A^{-1})(AB^3q^3-A^{-1}B^{-3}q^{-3})} \\
&\quad = \frac{q^7-q^{-7}}{q^3-q^{-3}} - \frac{(Aq^{-1}-A^{-1}q)(B^3q^6-B^{-3}q^{-6})}{(A-A^{-1})(B^3q^3-B^{-3}q^{-3})} \\
&\qquad - \frac{(B^3-B^{-3})(AB^3q^2-A^{-1}B^{-3}q^{-2})}{(B^3q^3-B^{-3}q^{-3})(AB^3q^3-A^{-1}B^{-3}q^{-3})}.
\end{aligned}$$

Then, we can use .simplify_full() in SAGE to simplify the rational function of A, B, and q, which is given by the difference of the left-hand side and right-hand side of the above equation. The result computed by SAGE is equal to 0, which tells us that equation (A.13) holds.

Lemma 3.19. *If $S_1(\underline{x})$ for all $\underline{x}$ such that $\operatorname{wt}\underline{x} < \operatorname{wt}(\underline{w}\otimes\varpi)$ and $S_1'(\underline{w},\varpi)$, then $S_5(\underline{w},\varpi)$.*

Proof. By the definition of $T_{\underline{w}\otimes\varpi}$, we find that

$$T_{\underline{w}\otimes\varpi} := T_{\underline{w}}\otimes \mathrm{id}_\varpi - \sum_{\substack{\mu\in \operatorname{wt} V(\varpi)\setminus\{\varpi\} \\ v_i,v_j\in \operatorname{Ker}_{\operatorname{wt}\underline{w}}(V(\varpi)_\mu)}} (p^{\ell_{\underline{u}_{\operatorname{wt}\underline{w}}+\mu}} t^{\underline{u}_{\operatorname{wt}\underline{w}}+\mu}_{\underline{w},\varpi})^{-1}_{ij} \cdot {}^{ij}\mathbb{T}\,\mathbb{T}^{\underline{u}_{\operatorname{wt}\underline{w}}+\mu}_{\underline{w},\varpi}.$$

Then, by our hypotheses, we deduce that

$$T_{\underline{\mathrm{w}}\otimes\varpi} = C_{\underline{\mathrm{w}}}\otimes \mathrm{id}_{\varpi} - \sum_{\substack{\mu\in \mathrm{wt}\, V(\varpi)\setminus\{\varpi\}\\ v_i,v_j\in \mathrm{Ker}_{\mathrm{wt}\,\underline{\mathrm{w}}}(V(\varpi)_\mu)}} ({}^{p\ell}\kappa^{\underline{\mathrm{u}}_{\mathrm{wt}\,\underline{\mathrm{w}}+\mu}}_{\underline{\mathrm{w}},\varpi})^{-1}_{ij}\cdot {}^{ij}\mathbb{LL}^{\underline{\mathrm{u}}_{\mathrm{wt}\,\underline{\mathrm{w}}+\mu}}_{\underline{\mathrm{w}},\varpi}. \qquad \blacksquare$$

Lemma 3.20. *If $S_5(\underline{\mathrm{w}},\varpi)$, then $S_6(\underline{\mathrm{w}}\otimes\varpi)$.*

Proof. By Lemma 2.31, we deduce the following multiplication formula for double ladders:

$${}^{p\ell}\mathbb{LL}^{\underline{\mathrm{u}}}_{\underline{\mathrm{w}},\varpi}\circ {}^{rs}\mathbb{LL}^{\underline{\mathrm{v}}}_{\underline{\mathrm{w}},\varpi} = \delta_{\mathrm{wt}\,\underline{\mathrm{u}},\mathrm{wt}\,\underline{\mathrm{v}}}\,{}^{\ell r}\kappa^{\underline{\mathrm{u}}}_{\underline{\mathrm{w}},\varpi}\cdot {}^{ps}\mathbb{LL}^{\underline{\mathrm{u}}}_{\underline{\mathrm{w}},\varpi}.$$

Using the expression for $T_{\underline{\mathrm{w}}\otimes\varpi}$ from $S_5(\underline{\mathrm{w}},\varpi)$ and the above formula, one can explicitly compute to verify that $T_{\underline{\mathrm{w}},\varpi}$ is idempotent. $\blacksquare$

Lemma 3.21. *If $S_5(\underline{\mathrm{w}},\varpi)$ and $S_3(\underline{\mathrm{w}},\varpi)$, then $S_6'(\underline{\mathrm{w}},\varpi)$.*

Proof. Write $\lambda = \mathrm{wt}\,\underline{\mathrm{w}}$. Let $\mu\in \mathrm{wt}\, V(\varpi)\setminus\{\varpi\}$, and let $\underline{\mathrm{u}}\in \mathbf{Web}_{\mathrm{q}}(\mathfrak{g}_2)$ such that $\mathrm{wt}\,\underline{\mathrm{u}} = \lambda+\mu$. Let $D\in \mathrm{Hom}_{\mathbf{Web}_{\mathrm{q}}(\mathfrak{g}_2)}(\underline{\mathrm{w}}\otimes\varpi,\underline{\mathrm{u}})$. Consider the neutral diagram $\mathsf{H}^{\underline{\mathrm{u}}_{\lambda+\mu}}_{\underline{\mathrm{u}}}:\underline{\mathrm{u}}\to \underline{\mathrm{u}}_{\lambda+\mu}$, and write $D' = \mathsf{H}^{\underline{\mathrm{u}}_{\lambda+\mu}}_{\underline{\mathrm{u}}}\circ C_{\underline{\mathrm{u}}}\circ D$.

Combining the fact that clasps are idempotent with Lemma 2.31 we find, that

$$C_{\underline{\mathrm{u}}_{\lambda+\mu}}\circ D'\circ {}^{ij}\mathbb{LL}^{\underline{\mathrm{u}}_{\lambda+\nu}}_{\underline{\mathrm{w}},\varpi} = \delta_{\mu,\nu}\cdot C_{\underline{\mathrm{u}}_{\lambda+\mu}}\circ D'\circ (C_{\underline{\mathrm{w}}}\otimes \mathrm{id}_{\varpi})\circ {}^{ij}\mathbb{LL}^{\underline{\mathrm{u}}_{\lambda+\mu}}_{\underline{\mathrm{w}},\varpi}.$$

By $S_3(\underline{\mathrm{w}},\varpi)$, there are scalars ξ_k such that

$$C_{\underline{\mathrm{u}}_{\lambda+\mu}}\circ D'\circ (C_{\underline{\mathrm{w}}}\otimes \mathrm{id}_{\varpi}) = \sum_{v_k\in \mathrm{Ker}_{\mathrm{wt}\,\underline{\mathrm{w}}}(V(\varpi)_\mu)} \xi_k\cdot {}^{k}LL^{\underline{\mathrm{u}}_{\lambda+\mu}}_{\underline{\mathrm{w}},\varpi}.$$

Thus, using $S_5(\underline{\mathrm{w}},\varpi)$, we can rewrite $C_{\underline{\mathrm{u}}_{\lambda+\mu}}\circ D'\circ T_{\underline{\mathrm{w}}\otimes\varpi}$ as follows:

$$\begin{aligned}
&C_{\underline{\mathrm{u}}_{\lambda+\mu}}\circ D'\circ (C_{\underline{\mathrm{w}}}\otimes \mathrm{id}_{\varpi})\\
&- \sum_{v_i,v_j\in \mathrm{Ker}_{\lambda}(V(\varpi)_\mu)} ({}^{p\ell}\kappa^{\underline{\mathrm{u}}_{\lambda+\mu}}_{\underline{\mathrm{w}},\varpi})^{-1}_{ij}\cdot C_{\underline{\mathrm{u}}_{\lambda+\mu}}\circ D'\circ (C_{\underline{\mathrm{w}}}\otimes \mathrm{id}_{\varpi})\circ {}^{ij}\mathbb{LL}^{\underline{\mathrm{u}}_{\lambda+\mu}}_{\underline{\mathrm{w}},\varpi}\\
&= C_{\underline{\mathrm{u}}_{\lambda+\mu}}\circ D'\circ (C_{\underline{\mathrm{w}}}\otimes \mathrm{id}_{\varpi})\\
&\quad - \sum_{v_i,v_j\in \mathrm{Ker}_{\mathrm{wt}\,\underline{\mathrm{w}}}(V(\varpi)_\mu)}\sum_{k} ({}^{p\ell}\kappa^{\underline{\mathrm{u}}_{\lambda+\mu}}_{\underline{\mathrm{w}},\varpi})^{-1}_{ij}\xi_k\cdot {}^{k}LL^{\underline{\mathrm{u}}_{\lambda+\mu}}_{\underline{\mathrm{w}},\varpi}\circ {}^{ij}\mathbb{LL}^{\underline{\mathrm{u}}_{\lambda+\mu}}_{\underline{\mathrm{w}},\varpi}\\
&= C_{\underline{\mathrm{u}}_{\lambda+\mu}}\circ D'\circ (C_{\underline{\mathrm{w}}}\otimes \mathrm{id}_{\varpi})\\
&\quad - \sum_{v_i,v_j\in \mathrm{Ker}_{\mathrm{wt}\,\underline{\mathrm{w}}}(V(\varpi)_\mu)}\sum_{k} ({}^{p\ell}\kappa^{\underline{\mathrm{u}}_{\lambda+\mu}}_{\underline{\mathrm{w}},\varpi})^{-1}_{ij}\xi_k\,{}^{ki}\kappa^{\underline{\mathrm{u}}_{\lambda+\mu}}_{\underline{\mathrm{w}},\varpi}\cdot {}^{j}LL^{\underline{\mathrm{u}}_{\lambda+\mu}}_{\underline{\mathrm{w}},\varpi}\\
&= C_{\underline{\mathrm{u}}_{\lambda+\mu}}\circ D'\circ (C_{\underline{\mathrm{w}}}\otimes \mathrm{id}_{\varpi}) - \sum_{v_j\in \mathrm{Ker}_{\mathrm{wt}\,\underline{\mathrm{w}}}(V(\varpi)_\mu)}\sum_{k}\xi_k\delta_{k,j}\cdot {}^{j}LL^{\underline{\mathrm{u}}_{\lambda+\mu}}_{\underline{\mathrm{w}},\varpi}\\
&= C_{\underline{\mathrm{u}}_{\lambda+\mu}}\circ D'\circ (C_{\underline{\mathrm{w}}}\otimes \mathrm{id}_{\varpi}) - \sum_{v_j\in \mathrm{Ker}_{\mathrm{wt}\,\underline{\mathrm{w}}}(V(\varpi)_\mu)}\xi_j\cdot {}^{j}LL^{\underline{\mathrm{u}}_{\lambda+\mu}}_{\underline{\mathrm{w}},\varpi} = 0.
\end{aligned}$$

Using Lemmas 2.33 and 2.35, it is not hard to see that

$$C_{\underline{u}} \circ \mathsf{H}^{\underline{u}}_{\underline{u}_{\lambda+\mu}} \circ C_{\underline{u}_{\lambda+\mu}} \circ D' = C_{\underline{u}} \circ D,$$

and it follows that $C_{\underline{u}} \circ D \circ T_{\underline{w} \otimes \varpi} = 0$. ∎

Lemma 3.22. *Let* $\underline{w} \in \mathbf{Web}_q(\mathfrak{g}_2)$*, and let* ϖ *be a fundamental weight. If* $S_2(\underline{x}, \psi)$ *and* $S_2'(\underline{x}, \psi)$ *whenever* $\operatorname{wt}(\underline{x} \otimes \psi) \leq \operatorname{wt} \underline{w}$*, then* $S_4(\underline{w}, \varpi)$*.*

Proof. Consider $\underline{x}$, ψ such that $\operatorname{wt}(\underline{x} \otimes \psi) \leq \operatorname{wt} \underline{w}$. By $S_2(\underline{x}, \psi)$, we obtain the following:

$$C_{\underline{x}} \otimes \mathrm{id}_\psi = \sum_{\substack{\mu \in \operatorname{wt} V(\psi) \\ v_i, v_j \in \operatorname{Ker}_{\operatorname{wt}\underline{x}}(V(\psi)_\mu)}} {}^{ij}\xi^{\underline{u}_{\operatorname{wt}\underline{x}}+\mu}_{\underline{x},\psi} \cdot {}^{ij}\mathbb{LL}^{\underline{u}_{\operatorname{wt}\underline{x}}+\mu}_{\underline{x},\psi}.$$

Postcomposing with ${}^{p}LL^{\underline{u}_{\operatorname{wt}\underline{x}}+\mu}_{\underline{x},\psi}$ and using Lemma 2.31 results in the sequence of equalities:

$$\begin{aligned} {}^{p}LL^{\underline{u}_{\operatorname{wt}\underline{x}}+\mu}_{\underline{x},\psi} &= \sum_{v_i, v_j \in \operatorname{Ker}_{\operatorname{wt}\underline{x}}(V(\psi)_\mu)} {}^{ij}\xi^{\underline{u}_{\operatorname{wt}\underline{x}}+\mu}_{\underline{x},\psi} \cdot {}^{p}LL^{\underline{u}_{\operatorname{wt}\underline{x}}+\mu}_{\underline{x},\psi} \circ {}^{ij}\mathbb{LL}^{\underline{u}_{\operatorname{wt}\underline{x}}+\mu}_{\underline{x},\psi} \\ &= \sum_{v_i, v_j \in \operatorname{Ker}_{\operatorname{wt}\underline{x}}(V(\psi)_\mu)} {}^{ij}\xi^{\underline{u}_{\operatorname{wt}\underline{x}}+\mu}_{\underline{x},\psi} \cdot {}^{pi}\kappa^{\underline{u}_{\operatorname{wt}\underline{x}}+\mu}_{\underline{x},\psi} \cdot {}^{j}LL^{\underline{u}_{\operatorname{wt}\underline{x}}+\mu}_{\underline{x},\psi}. \end{aligned}$$

By $S_2'(\underline{x}, \psi)$, it follows that

$$\sum_{v_i \in \operatorname{Ker}_{\operatorname{wt}\underline{x}}(V(\psi)_\mu)} {}^{ij}\xi^{\underline{u}_{\operatorname{wt}\underline{x}}+\mu}_{\underline{x},\psi} \cdot {}^{pi}\kappa^{\underline{u}_{\operatorname{wt}\underline{x}}+\mu}_{\underline{x},\psi} = \delta_{jp}.$$

From Definition 2.54, we can use the fact that clasps are idempotent to write

$$\begin{aligned} LL^{\underline{u}_{\operatorname{wt}\underline{x}}+\psi}_{\underline{x},\psi} \circ \mathbb{D}(LL^{\underline{u}_{\operatorname{wt}\underline{x}}+\psi}_{\underline{x},\psi}) = C_{\underline{u}_{\operatorname{wt}\underline{x}}+\psi} \circ ELL^{\underline{u}_{\operatorname{wt}\underline{x}}+\psi}_{\underline{x},\psi} \circ (C_{\underline{x}} \otimes \mathrm{id}_\psi) \\ \circ \mathbb{D}(ELL^{\underline{u}_{\operatorname{wt}\underline{x}}+\psi}_{\underline{x},\psi}) \circ C_{\underline{u}_{\operatorname{wt}\underline{x}}+\psi}. \end{aligned}$$

By Definition 2.52, $ELL^{\underline{u}_{\operatorname{wt}\underline{x}}+\psi}_{\underline{x},\psi}$ is a neutral map. Therefore, Lemmas 2.37, 2.35, and 2.33, along with clasps being idempotent, imply that

$$LL^{\underline{u}_{\operatorname{wt}\underline{x}}+\psi}_{\underline{x},\psi} \circ \mathbb{D}(LL^{\underline{u}_{\operatorname{wt}\underline{x}}+\psi}_{\underline{x},\psi}) = C_{\underline{u}_{\operatorname{wt}\underline{x}}+\psi}.$$

It follows that $\kappa^{\underline{u}_{\operatorname{wt}\underline{x}}+\psi}_{\underline{x},\psi} = 1$. Thus,

$$\xi^{\underline{u}_{\operatorname{wt}\underline{x}}+\psi}_{\underline{x},\psi} = 1$$

and

$$C_{\underline{x}\otimes\psi} = C_{\underline{x}} \otimes \mathrm{id}_\psi - \sum_{\substack{\mu \in \operatorname{wt} V(\psi)\setminus\{\psi\} \\ v_i, v_j \in \operatorname{Ker}_{\operatorname{wt}\underline{x}}(V(\psi)_\mu)}} ({}^{p\ell}\kappa^{\underline{u}_{\operatorname{wt}\underline{x}}+\mu}_{\underline{x},\psi})^{-1}_{ij} \cdot {}^{ij}\mathbb{LL}^{\underline{u}_{\operatorname{wt}\underline{x}}+\mu}_{\underline{x},\psi}. \tag{3.26}$$

Observe that

$$\begin{aligned} {}^{p\ell}\kappa^{\underline{u}}_{\underline{w},\varpi} C_{\underline{u}} &= {}^{p}LL^{\underline{u}}_{\underline{w},\varpi} \circ \mathbb{D}({}^{\ell}LL^{\underline{u}}_{\underline{w},\varpi}) \\ &= C_{\underline{u}} \circ {}^{p}ELL^{\underline{u}}_{\underline{w},\varpi} \circ (C_{\underline{w}} \otimes \mathrm{id}_{\varpi}) \circ \mathbb{D}({}^{\ell}ELL^{\underline{u}}_{\underline{w},\varpi}) \circ C_{\underline{u}}. \end{aligned}$$

Then, use equation (3.26) for $\underline{w} = \underline{x} \otimes \psi$ to rewrite the $C_{\underline{w}}$ term on the right-hand side. This new sum will reduce to a scalar multiple of $C_{\underline{u}}$ by repeatedly applying graphical reductions or by replacing another clasp, necessarily of the form $C_{\underline{y} \otimes \varpi}$ for some $\underline{y}$, ϖ such that $\mathrm{wt}(\underline{y} \otimes \varpi) \leq \mathrm{wt}\,\underline{w}$, using equation (3.26). The exact form of the coefficient is determined via the calculations in [4, Appendix A.4], where it is shown to be equal to ${}^{p\ell}\rho^{\underline{u}}_{\underline{w},\varpi}(\kappa)$. Therefore, ${}^{p\ell}\kappa^{\underline{u}}_{\underline{w},\varpi} C_{\underline{u}} = {}^{p\ell}\rho^{\underline{u}}_{\underline{w},\varpi}(\kappa) C_{\underline{u}}$, and the desired result follows from looking at the neutral coefficient of each map. ∎

Example 3.23. The above argument is best illustrated by an example. Consider $\underline{w}$ with $\mathrm{wt}\,\underline{w} = (a, b)$, and assume that $S_2(\underline{x}, \psi)$ and $S_2'(\underline{x}, \psi)$ whenever $\mathrm{wt}(\underline{x} \otimes \psi) \leq \mathrm{wt}\,\underline{w}$. Note that

$$\rho^{(-1,1)}_{(a,b),\varpi_1} := -[2] - \frac{1}{x^{(-1,1)}_{(a-1,b),\varpi_1}}.$$

We will show that $\kappa^{\underline{u}_{(a-1,b+1)}}_{\underline{w},\varpi_1} = \rho^{(-1,1)}_{(a,b),\varpi_1}(\kappa)$.

Let $\mathrm{wt}(\underline{v} \otimes \varpi_1) = \mathrm{wt}\,\underline{w}$. It follows from Lemma 3.5 that

$$\kappa^{\underline{u}_{(a-1,b+1)}}_{\underline{w},\varpi_1} = \kappa^{\underline{u}_{(a-1,b+1)}}_{\underline{v} \otimes \varpi_1,\varpi_1}.$$

By definition, we have that $\kappa^{\underline{u}_{(a-1,b+1)}}_{\underline{v} \otimes \varpi_1,\varpi_1} C_{\underline{u}_{(a-1,b+1)}}$ is equal to

$$C_{\underline{u}_{(a-1,b+1)}} \circ ELL^{\underline{u}_{(a-1,b+1)}}_{\underline{v} \otimes \varpi_1,\varpi_1} \circ (C_{\underline{v} \otimes \varpi_1} \otimes \mathrm{id}_{\varpi_1}) \circ \mathbb{D}(ELL^{\underline{u}_{(a-1,b+1)}}_{\underline{v} \otimes \varpi_1,\varpi_1}) \circ C_{\underline{u}_{(a-1,b+1)}}.$$

As in the first half of the proof of Lemma 3.22, our hypotheses allow us to write

$$C_{\underline{v} \otimes \varpi_1} = C_{\underline{v}} \otimes \mathrm{id}_{\varpi_1} - \sum_{\substack{\mu \in \mathrm{wt}\,V(\varpi_1) \setminus \{\varpi_1\} \\ v_i, v_j \in \mathrm{Ker}_{\mathrm{wt}\,\underline{v}}(V(\varpi_1)_\mu)}} ({}^{p\ell}\kappa^{\underline{u}_{\mathrm{wt}\,\underline{v}+\mu}}_{\underline{v},\varpi_1})^{-1}_{ij} \cdot {}^{ij}\mathbb{L}\mathbb{L}^{\underline{u}_{\mathrm{wt}\,\underline{v}+\mu}}_{\underline{v},\varpi_1}.$$

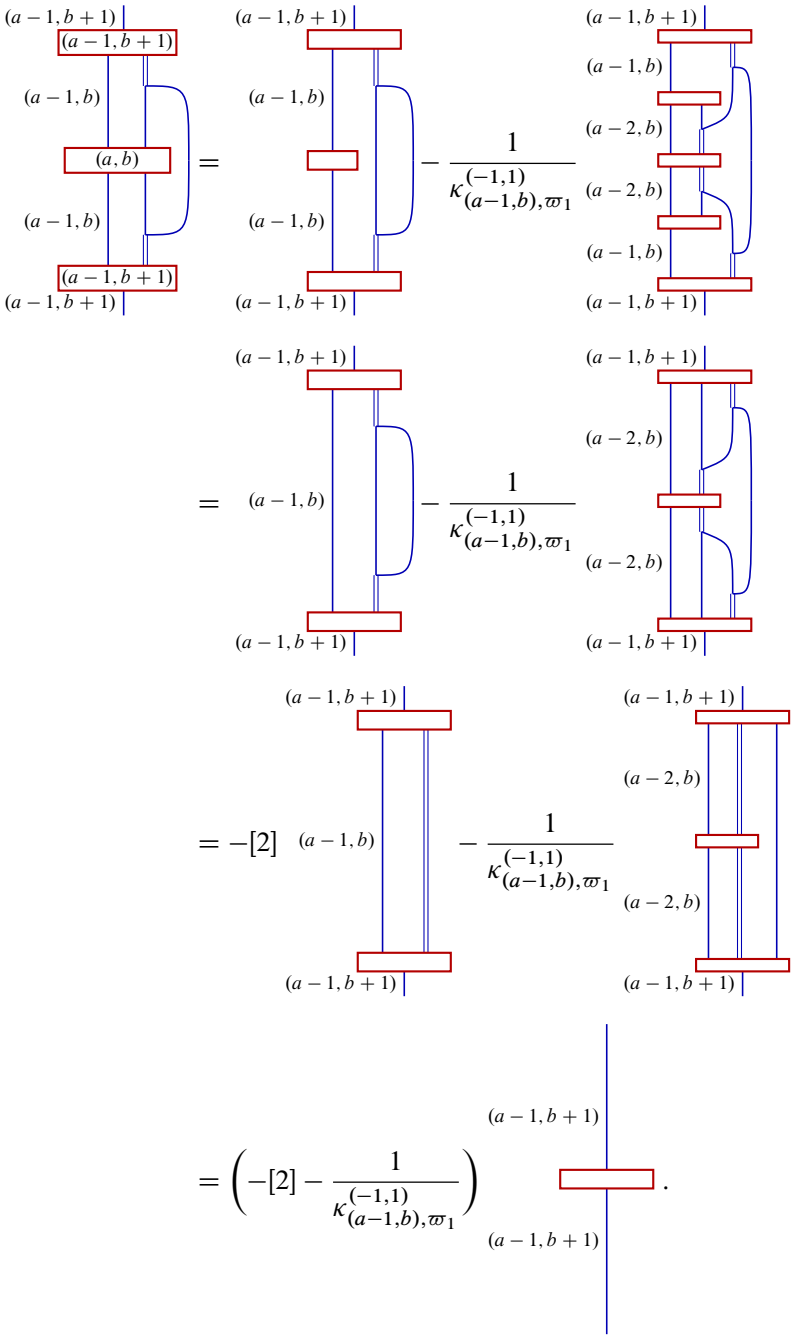
Using Lemma 2.32, we observe that if $\mu \neq (-1, 1)$, then

$$\begin{aligned} C_{\underline{u}_{(a-1,b+1)}} \circ ELL^{\underline{u}_{(a-1,b+1)}}_{\underline{v} \otimes \varpi_1,\varpi_1} \circ \Big(({}^{p\ell}\kappa^{\underline{u}_{\mathrm{wt}\,\underline{v}+\mu}}_{\underline{v},\varpi_1})^{-1}_{ij} \cdot {}^{ij}\mathbb{L}\mathbb{L}^{\underline{u}_{\mathrm{wt}\,\underline{v}+\mu}}_{\underline{v},\varpi_1} \otimes \mathrm{id}_{\varpi_1} \Big) \\ \circ \mathbb{D}(ELL^{\underline{u}_{(a-1,b+1)}}_{\underline{v} \otimes \varpi_1,\varpi_1}) \circ C_{\underline{u}_{(a-1,b+1)}} \end{aligned}$$

is zero. Finally, applying web relations (and properties of clasps), we find that

$$\kappa^{(-1,1)}_{(a,b),\varpi_1} C_{\underline{u}_{(a-1,b+1)}} = \left(-[2] - \frac{1}{\kappa^{(-1,1)}_{(a-1,b),\varpi_1}} \right) C_{\underline{u}_{(a-1,b+1)}}.$$

We conclude with a schematic of the graphical calculations involved as follows:



Finally, we combine the previous lemmas to deduce the result of our main theorem.

Proof of Theorem 3.12. We will prove the result by induction on $\operatorname{wt}\underline{w}$ with respect to $\leq$. The base case follows from observing that $T_\emptyset$ is 1 times the empty diagram, which agrees with $C_\emptyset$. Assume that $S_1(\underline{x})$ holds for all $\underline{x}$ such that $\operatorname{wt}\underline{x} < \operatorname{wt}\underline{w}$, and assume that $S_1'(\underline{y}, \psi)$ holds for all $\underline{y}$, ψ such that $\operatorname{wt}(\underline{y} \otimes \psi) < \operatorname{wt}\underline{w}$. We will show $S_1(\underline{w})$ and $S_1'(\underline{w}', \varpi)$, where

$$\underline{w} = \underline{w}' \otimes \varpi.$$

Consider $\underline{y}$, ψ such that $\operatorname{wt}(\underline{y} \otimes \psi) < \operatorname{wt}\underline{w}$. Then, $S_1(\underline{x})$ holds whenever

$$\operatorname{wt}\underline{x} \leq \operatorname{wt}(\underline{y} \otimes \psi),$$

and by Lemma 3.14, we deduce $S_2(\underline{y}, \psi)$. Thus, $S_2(\underline{y}, \psi)$ holds for all $\underline{y}$, ψ such that $\operatorname{wt}(\underline{y} \otimes \psi) < \operatorname{wt}\underline{w}$.

If $\underline{y}$, ψ is such that $\operatorname{wt}(\underline{y} \otimes \psi) < \operatorname{wt}\underline{w}$, then our inductive hypothesis also says that $S_1'(\underline{y}, \psi)$ holds. Along with Lemmas 3.15 and 3.16, this implies $S_2'(\underline{y}, \psi)$ and $S_3(\underline{y}, \psi)$. Hence, $S_2'(\underline{y}, \psi)$ and $S_3(\underline{y}, \psi)$ holds for all $\underline{y}, \psi$ such that $\operatorname{wt}(\underline{y} \otimes \psi) < \operatorname{wt}\underline{w}$.

For $\underline{x}$ such that $\operatorname{wt}\underline{x} < \operatorname{wt}\underline{w}$, we have $S_2(\underline{y}, \psi)$ and $S_2'(\underline{y}, \psi)$ whenever

$$\operatorname{wt}(\underline{y} \otimes \psi) \leq \operatorname{wt}\underline{x}\,.$$

So, from Lemma 3.22, we deduce $S_4(\underline{x}, \varpi)$ for all $\underline{x}$ such that $\operatorname{wt}\underline{x} < \operatorname{wt}\underline{w}$ and for arbitrary ϖ.

If $\underline{w} = \underline{w}' \otimes \varpi$, then $\operatorname{wt}\underline{w}' < \operatorname{wt}\underline{w}$, so $S_4(\underline{w}', \varpi)$. Also, if $\operatorname{wt}(\underline{y} \otimes \psi) \leq \operatorname{wt}\underline{w}'$, then

$$\operatorname{wt}(\underline{y} \otimes \psi) < \operatorname{wt}\underline{w},$$

so $S_1'(\underline{y}, \psi)$ holds whenever $\operatorname{wt}(\underline{y} \otimes \psi) \leq \operatorname{wt}\underline{w}'$. Thus, Lemma 3.17 implies $S_1'(\underline{w}', \varpi)$.

At this point, we know that $S_1(\underline{x})$ whenever $\operatorname{wt}\underline{x} < \operatorname{wt}(\underline{w}' \otimes \varpi)$ and that $S_1'(\underline{w}', \varpi)$ holds. Therefore, Lemma 3.19 implies that $S_5(\underline{w}', \varpi)$ holds too. Then, from Lemma 3.20, we deduce that $S_6(\underline{w}' \otimes \varpi)$ is true.

Moreover, since $S_1'(\underline{w}', \varpi)$ is true, Lemma 3.15 and Lemma 3.16 together imply that $S_3(\underline{w}', \varpi)$. Therefore, we can use Lemma 3.21 to deduce $S_6'(\underline{w}', \varpi)$.

If we show that $T_{\underline{w}' \otimes \varpi} \neq 0$, then Definition 2.24 and Lemma 2.26 will tell us that $S_6(\underline{w}' \otimes \varpi)$ and $S_6'(\underline{w}', \varpi)$ imply $S_1(\underline{w}' \otimes \varpi)$, so we are then done by induction. To see that $T_{\underline{w}' \otimes \varpi}$ is not 0, we apply Φ from Theorem 2.17 and evaluate on a weight vector in $V(\underline{w}' \otimes \varpi)_{\operatorname{wt}\underline{w}' + \varpi}$. Using $S_5(\underline{w}', \varpi)$, along with the observations that for all $\underline{u}$ such that

$$\operatorname{wt}\underline{u} - \operatorname{wt}\underline{w}' = \mu \in \operatorname{wt} V(\varpi) \backslash \{\varpi\},$$

the map $\Phi({}^{ij}\mathbb{L}\mathbb{L}^{\underline{u}}_{\underline{w}', \varpi})$ acts as zero on $V(\underline{w}' \otimes \varpi)_{\operatorname{wt}\underline{w}' + \varpi}$ (since these maps factor through representations which do not have $\operatorname{wt}\underline{w}' + \varpi$ as a weight) and $\Phi\big(C_{\underline{w}'} \otimes \mathrm{id}_\varpi\big)$ acts on $V(\underline{w}' \otimes \varpi)_{\operatorname{wt}\underline{w}' + \varpi}$ as multiplication by 1, we deduce that $T_{\underline{w}' \otimes \varpi}$ is non-zero. ∎

A. Appendix

A.1. Relations for computing the ${}^{p\ell}t^{(0,0)}_{(a,b),\varpi_2}$ coefficients

$$\begin{aligned}
&\frac{{}^{2,2}t^{(0,0)}_{(a,b),\varpi_2}}{[2]} + {}^{1,2}t^{(0,0)}_{(a,b),\varpi_2} \\
&= \frac{-[3][a+2][3a+6b+9]}{[2][3b][a+3b+3][2a+3b+4][3a+3b+3][3a+6b+6]} \\
&\quad \cdot \Bigg([3a+3b+6][2a+3b+5] + [a+3b+4][3b+3] \\
&\qquad + \frac{[3a+3b+6][3b+3][2]^2}{[3]} + [a+4] - [a-2] \Bigg),
\end{aligned} \tag{A.1}$$

$$\begin{aligned}
{}^{1,1}t^{(0,0)}_{(a,b),\varpi_2} &= -\frac{[6][8][15]}{[3][5][12]} + \frac{[2][a+2][a+3b+5][2a+3b+2]}{[3][a+1][a+3b+4][2a+3b+4]} \\
&\quad + \frac{[a-2][3b+6][2a+3b+2][3a+3b]}{[a][3b+3][2a+3b+3][3a+3b+3]} \\
&\quad + \frac{[a-1][3b+6][a+3b+5][a+3b+6][2a+3b+6]}{[3][a][3b+3][a+3b+3][a+3b+4][2a+3b+3]} \\
&\quad + \frac{[3b][a+3b+1][2a+3b+2][3a+6b+3]}{[3b+3][a+3b+3][2a+3b+3][3a+6b+6]} \\
&\quad + \frac{[a+2][a+3][3b][a+3b+2][2a+3b+6]}{[3][a][a+1][3b+3][a+3b+3][2a+3b+3]},
\end{aligned} \tag{A.2}$$

$$\begin{aligned}
&\frac{{}^{2,2}t^{(0,0)}_{(a,b),\varpi_2}}{\mathcal{D}_{(a,b)}} \left(\frac{[4][6]^2}{[2][3]^2[12]} \right)^2 + 2 \cdot \frac{{}^{1,2}t^{(0,0)}_{(a,b),\varpi_2}}{\mathcal{D}_{(a,b)}} \frac{[4]^2[6]^3}{[2]^2[3]^2[12]^2} + \frac{{}^{1,1}t^{(0,0)}_{(a,b),\varpi_2}}{\mathcal{D}_{(a,b)}} \left(\frac{[4][6]}{[2][12]} \right)^2 \\
&= -\frac{[6][8][15]}{[3][5][12]} + \frac{[a+3][3b][a+3b+2][a+3b+3][2a+3b+4]}{[3][a+2][3b+3][a+3b+4][a+3b+5][2a+3b+7]} \\
&\quad + \frac{[a-1][a][3b+6][a+3b+6][2a+3b+4]}{[3][a+1][a+2][3b+3][a+3b+5][2a+3b+7]} \\
&\quad + \frac{[2][a][a+3b+3][2a+3b+8]}{[3][a+1][a+3b+4][2a+3b+6]} \\
&\quad + \frac{[a+4][3b][2a+3b+8][3a+3b+12]}{[a+2][3b+3][2a+3b+7][3a+3b+9]} \\
&\quad + \frac{[3b+6][a+3b+7][2a+3b+8][3a+6b+15]}{[3b+3][a+3b+5][2a+3b+7][3a+6b+12]}.
\end{aligned} \tag{A.3}$$

A.2. Recursions for the coefficients

We write $x^{(c,d)}_{(a,b)} := x^{(c,d)}_{(a,b),\varpi_1}$ and $y^{(c,d)}_{(a,b)} := x^{(c,d)}_{(a,b),\varpi_2}$:

$$\rho^{(1,0)}_{(a,b),\varpi_1} = 1, \tag{A.4}$$

$$\rho^{(-1,1)}_{(a,b),\varpi_1} = -[2] - \frac{1}{x^{(-1,1)}_{(a-1,b)}}, \tag{A.5}$$

$$\rho^{(2,-1)}_{(a,b),\varpi_1} = \frac{[7]}{[3]} - \frac{1}{y^{(3,-1)}_{(a,b-1)}} x^{(-1,1)}_{(a+3,b-2)} - \frac{1}{y^{(1,0)}_{(a,b-1)}} \frac{1}{[3]^2}, \tag{A.6}$$

$$\rho^{(0,0)}_{(a,b),\varpi_1} = -\frac{[3][8]}{[2][4]} - \frac{x^{(2,-1)}_{(a-2,b+1)}}{x^{(-1,1)}_{(a-1,b)}} - \frac{x^{(-1,1)}_{(a+1,b-1)}}{x^{(2,-1)}_{(a-1,b)}} - \frac{1}{[2]^2 x^{(0,0)}_{(a-1,b)}}, \tag{A.7}$$

$$\begin{aligned}\rho^{(-2,1)}_{(a,b),\varpi_1} = {} & -\frac{[3][8]}{[2][4]} x^{(-1,1)}_{(a-1,b)} - \frac{1}{x^{(-1,1)}_{(a-1,b)}} \left(\frac{1}{x^{(-1,1)}_{(a-2,b)}} \right)^2 x^{(0,0)}_{(a-2,b+1)} \\ & - \frac{1}{x^{(0,0)}_{(a-1,b)}} \left(\frac{[3]}{[2]} + \frac{1}{x^{(-1,1)}_{(a-2,b)}} \right)^2 x^{(-1,1)}_{(a-1,b)} \\ & - \frac{1}{x^{(-2,1)}_{(a-1,b)}} \left(-\frac{[3]}{[2]} + \frac{1}{x^{(-1,1)}_{(a-2,b)}} \frac{1}{x^{(-1,1)}_{(a-3,b)}} \frac{1}{[2]} \right)^2,\end{aligned} \tag{A.8}$$

$$\begin{aligned}\rho^{(1,-1)}_{(a,b),\varpi_1} = {} & -\frac{[6][8][15]}{[3][5][12]} - \frac{1}{y^{(3,-1)}_{(a,b-1)}} x^{(-2,1)}_{(a+3,b-2)} - \frac{1}{y^{(1,0)}_{(a,b-1)}} \left(\frac{[2]}{[3]} \right)^2 x^{(0,0)}_{(a+1,b-1)} \\ & - \frac{1}{y^{(2,-1)}_{(a,b-1)}} \left(\frac{1}{[3]} \right)^2 x^{(-1,1)}_{(a+2,b-2)} - \frac{1}{y^{(-1,1)}_{(a,b-1)}} \left(\frac{1}{[3]} \right)^2 x^{(2,-1)}_{(a-1,b)} \\ & - \frac{1}{{}_{1,1}y^{(0,0)}_{(a,b-1)}} \left(\frac{[4][6]^2}{[2][3]^2[12]} \right)^2 - \frac{1}{{}_{2,2}y^{(0,0)}_{(a,b-1)}} \left(\frac{[4][6]}{[2][12]} \right)^2 \\ & + \left(\frac{1}{{}_{1,2}y^{(0,0)}_{(a,b-1)}} + \frac{1}{{}_{2,1}y^{(0,0)}_{(a,b-1)}} \right) \frac{[4][6]^2}{[2][3]^2[12]} \frac{[4][6]}{[2][12]},\end{aligned} \tag{A.9}$$

$$\begin{aligned}\rho^{(-1,0)}_{(a,b),\varpi_1} = {} & \frac{[2][7][12]}{[4][6]} - \frac{1}{x^{(-1,1)}_{(a-1,b)}} x^{(1,-1)}_{(a-2,b+1)} - \frac{1}{x^{(2,-1)}_{(a-1,b)}} x^{(-2,1)}_{(a+1,b-1)} - 1 \\ & - \frac{1}{x^{(-2,1)}_{(a-1,b)}} x^{(2,-1)}_{(a-3,b+1)} - \frac{1}{x^{(1,-1)}_{(a-1,b)}} x^{(-1,1)}_{(a,b-1)} - \frac{1}{x^{(-1,0)}_{(a-1,b)}},\end{aligned} \tag{A.10}$$

$$\rho^{(0,1)}_{(a,b),\varpi_2} = 1, \tag{A.11}$$

$$\rho^{(3,-1)}_{(a,b),\varpi_2} = -\frac{[6]}{[3]} - \frac{1}{y^{(3,-1)}_{(a,b-1)}}, \tag{A.12}$$

$$\rho^{(1,0)}_{(a,b),\varpi_2} = \frac{[7]}{[3]} - \frac{1}{x^{(-1,1)}_{(a-1,b)}} y^{(3,-1)}_{(a-2,b+1)} - \frac{1}{x^{(2,-1)}_{(a-1,b)}}, \tag{A.13}$$

$$\begin{aligned}\rho^{(-1,1)}_{(a,b),\varpi_2} &= -\frac{[7]}{[2]} - \frac{1}{x^{(-1,1)}_{(a-2,b)}}\frac{[7]}{[3]} - \frac{1}{x^{(-1,1)}_{(a-1,b)}}\left(\frac{1}{x^{(-1,1)}_{(a-2,b)}}\right)^2 y^{(1,0)}_{(a-2,b+1)} \\ &\quad - \frac{1}{x^{(0,0)}_{(a-1,b)}}\left(\frac{[3]}{[2]} + \frac{1}{x^{(-1,1)}_{(a-2,b)}}\right)^2,\end{aligned} \tag{A.14}$$

$$\begin{aligned}\rho^{(2,-1)}_{(a,b),\varpi_2} &= \frac{[8][10]}{[3]^2[5]} - \frac{1}{y^{(3,-1)}_{(a,b-1)}} y^{(-1,1)}_{(a+3,b-2)} - \frac{1}{y^{(1,0)}_{(a,b-1)}}\frac{[2]^2}{[3]^2} y^{(1,0)}_{(a+1,b-1)} \\ &\quad - \frac{1}{y^{(-1,1)}_{(a,b-1)}}\frac{1}{[3]^2} y^{(3,-1)}_{(a-1,b)} - \frac{1}{y^{(2,-1)}_{(a,b-1)}}\frac{1}{[3]^2},\end{aligned} \tag{A.15}$$

$$\begin{aligned}\rho^{(-3,2)}_{(a,b),\varpi_2} &= \frac{[7]}{[3]}\left(-\frac{[3]}{[2]}x^{(-1,1)}_{(a-2,b)} + \frac{1}{x^{(-1,1)}_{(a-2,b)}\left(x^{(-1,1)}_{(a-3,b)}\right)^2}\left(\frac{[3]}{[2]} + \frac{1}{x^{(-1,1)}_{(a-4,b+1)}}\right)\right) \\ &\quad - \frac{y^{(-1,1)}_{(a-2,b+1)}}{x^{(-1,1)}_{(a-1,b)}\left(x^{(-1,1)}_{(a-2,b)}x^{(-1,1)}_{(a-3,b)}\right)^2} - \frac{1}{x^{(-2,1)}_{(a-1,b)}}\left(-\frac{[3]}{[2]}x^{(-1,1)}_{(a-2,b)}\right. \\ &\quad \left. + \frac{1}{x^{(-1,1)}_{(a-2,b)}}\left(\frac{1}{x^{(-1,1)}_{(a-3,b)}}\right)^2\left(\frac{[3]}{[2]} + \frac{1}{x^{(-1,1)}_{(a-4,b+1)}}\right)\right)^2,\end{aligned} \tag{A.16}$$

$$\begin{aligned}{}^{1,1}\rho^{(0,0)}_{(a,b),\varpi_2} &= -\frac{[6][8][15]}{[3][5][12]} - \frac{1}{x^{(-1,1)}_{(a-1,b)}} y^{(2,-1)}_{(a-2,b+1)} - \frac{1}{x^{(2,-1)}_{(a-1,b)}} y^{(-1,1)}_{(a+1,b-1)} \\ &\quad - \frac{1}{x^{(0,0)}_{(a-1,b)}} y^{(1,0)}_{(a-1,b)} - \frac{1}{x^{(-2,1)}_{(a-1,b)}} y^{(3,-1)}_{(a-3,b+1)} - \frac{1}{x^{(1,-1)}_{(a-1,b)}},\end{aligned} \tag{A.17}$$

$$\begin{aligned}{}^{2,2}\rho^{(0,0)}_{(a,b),\varpi_2} &= -\frac{[4][6]^2[18]}{[3][9][12]} - \frac{y^{(-3,2)}_{(a+3,b-2)}}{y^{(3,-1)}_{(a,b-1)}} - \frac{y^{(-1,1)}_{(a+1,b-1)}}{y^{(1,0)}_{(a,b-1)}} - \frac{y^{(1,0)}_{(a-1,b)}}{y^{(-1,1)}_{(a,b-1)}} - \frac{y^{(3,-1)}_{(a-3,b+1)}}{y^{(-3,2)}_{(a,b-1)}} \\ &\quad - \frac{{}^{2,2}y^{(0,0)}_{(a,b-1)}}{\mathcal{D}_{(a,b-1)}}\left(\frac{[4][6]}{[2][12]}\right)^2 - \left(\frac{{}^{1,2}y^{(0,0)}_{(a,b-1)}}{\mathcal{D}_{(a,b-1)}} + \frac{{}^{2,1}y^{(0,0)}_{(a,b-1)}}{\mathcal{D}_{(a,b-1)}}\right)\frac{[4]^2[6]^2}{[2][12]^2} \\ &\quad - \frac{{}^{1,1}y^{(0,0)}_{(a,b-1)}}{\mathcal{D}_{(a,b-1)}}\left(\frac{[4][6]}{[12]}\right)^2,\end{aligned} \tag{A.18}$$

$$\begin{aligned}{}^{1,2}\rho^{(0,0)}_{(a,b),\varpi_2} &= \frac{[4][6]^2[18]}{[2][3][9][12]} + \frac{y^{(-1,1)}_{(a+1,b-1)}}{y^{(1,0)}_{(a,b-1)}}\left(\frac{[4]}{[3]} + \frac{1}{x^{(-1,1)}_{(a-1,b-1)}}\right) \\ &\quad + \frac{y^{(1,0)}_{(a-1,b)}}{y^{(-1,1)}_{(a,b-1)}}\left(\frac{[4]}{[3]} - \frac{[2]}{[3]x^{(-1,1)}_{(a-1,b-1)}x^{(-1,1)}_{(a-2,b-1)}}\right)\end{aligned}$$

$$
\begin{aligned}
&+\frac{y^{(3,-1)}_{(a-3,b+1)}}{y^{(-3,2)}_{(a,b-1)}}\left(\frac{[4]}{[3]}+\frac{1}{[3]x^{(-1,1)}_{(a-1,b-1)}x^{(-1,1)}_{(a-2,b-1)}x^{(-1,1)}_{(a-3,b-1)}}\right)\\
&+\frac{{}^{2,1}y^{(0,0)}_{(a,b-1)}}{\mathcal{D}_{(a,b-1)}}\left(\frac{[4][6]}{[2][12]}\right)^2+\frac{{}^{1,1}y^{(0,0)}_{(a,b-1)}}{\mathcal{D}_{(a,b-1)}}\frac{[4]^2[6]^2}{[2][12]^2}\\
&-\frac{{}^{2,2}y^{(0,0)}_{(a,b-1)}}{\mathcal{D}_{(a,b-1)}}\frac{[4][6]}{[2][12]}\left(\frac{1}{[2]}+\frac{1}{[3]x^{(-1,1)}_{(a-1,b-1)}}-\frac{[4][6]}{[2]^2[12]}\right)\\
&-\frac{{}^{1,2}y^{(0,0)}_{(a,b-1)}}{\mathcal{D}_{(a,b-1)}}\frac{[4][6]}{[12]}\left(\frac{1}{[2]}+\frac{1}{[3]x^{(-1,1)}_{(a-1,b-1)}}-\frac{[4][6]}{[2]^2[12]}\right),
\end{aligned}
\tag{A.19}
$$

$$
\begin{aligned}
\rho^{(3,-2)}_{(a,b),\varpi_2}=&-\frac{[4][6]^2[18]}{[3][9][12]}y^{(3,-1)}_{(a,b-1)}\\
&-\frac{1}{y^{(3,-1)}_{(a,b-1)}}\left([3]^2{}^{1,1}y^{(0,0)}_{(a+3,b-2)}-\frac{2\cdot[3]}{y^{(3,-1)}_{(a,b-2)}}{}^{1,2}y^{(0,0)}_{(a+3,b-2)}+\frac{{}^{2,2}y^{(0,0)}_{(a+3,b-2)}}{\left(y^{(3,-1)}_{(a,b-2)}\right)^2}\right)\\
&-\frac{y^{(2,-1)}_{(a+1,b-1)}}{y^{(1,0)}_{(a,b-1)}}-\frac{1}{y^{(2,-1)}_{(a,b-1)}}\left(y^{(3,-1)}_{(a,b-1)}\right)^2y^{(1,0)}_{(a+2,b-2)}\\
&-\frac{{}^{1,1}y^{(0,0)}_{(a,b-1)}}{\mathcal{D}_{(a,b-1)}}\left(\frac{[4][6]}{[12]}+y^{(3,-1)}_{(a,b-1)}\right)^2y^{(3,-1)}_{(a,b-1)}\\
&-2\frac{{}^{1,2}y^{(0,0)}_{(a,b-1)}}{\mathcal{D}_{(a,b-1)}}\frac{[4][6]}{[2][12]}\left(\frac{[4][6]}{[12]}+y^{(3,-1)}_{(a,b-1)}\right)y^{(3,-1)}_{(a,b-1)}\\
&-\frac{{}^{2,2}y^{(0,0)}_{(a,b-1)}}{\mathcal{D}_{(a,b-1)}}\left(\frac{[4][6]}{[2][12]}\right)^2y^{(3,-1)}_{(a,b-1)}\\
&-\frac{1}{y^{(3,-2)}_{(a,b-1)}}\left(\frac{[4][6]}{[12]}-\frac{[6]}{[3]}+\frac{1}{y^{(3,-1)}_{(a,b-2)}}\left(\frac{[4][6]^2}{[3][12]}-1+\frac{1}{y^{(3,-1)}_{(a,b-3)}}\frac{[4][6]}{[12]}\right)\right)^2,
\end{aligned}
\tag{A.20}
$$

$$
\begin{aligned}
\rho^{(-2,1)}_{(a,b),\varpi_2}=&-\frac{[6][8][15]}{[3][5][12]}x^{(-1,1)}_{(a-1,b)}\\
&-\frac{1}{x^{(-1,1)}_{(a-1,b)}}\left(\frac{{}^{1,1}y^{(0,0)}_{(a-2,b+1)}}{\left(x^{(-1,1)}_{(a-2,b)}\right)^2}-\frac{2\cdot\left({}^{1,2}y^{(0,0)}_{(a-2,b+1)}\right)}{x^{(-1,1)}_{(a-2,b)}}+{}^{2,2}y^{(0,0)}_{(a-2,b+1)}\right)\\
&-\frac{1}{x^{(2,-1)}_{(a-1,b)}}y^{(-3,2)}_{(a+1,b-1)}-\frac{1}{x^{(0,0)}_{(a-1,b)}}\frac{y^{(-1,1)}_{(a-1,b)}}{\left(x^{(-1,1)}_{(a-2,b)}\right)^2}\\
&-\frac{1}{x^{(-2,1)}_{(a-1,b)}}\frac{y^{(1,0)}_{(a-3,b+1)}}{\left(x^{(-1,1)}_{(a-2,b)}x^{(-1,1)}_{(a-3,b)}\right)^2}-\frac{1}{x^{(-1,0)}_{(a-1,b)}}\left(x^{(-1,1)}_{(a-1,b)}\right)^2,
\end{aligned}
\tag{A.21}
$$

$$\rho^{(1,-1)}_{(a,b),\varpi_2} = -\frac{[6][8][15]}{[3][5][12]} x^{(2,-1)}_{(a-1,b)} - \frac{1}{x^{(-1,1)}_{(a-1,b)}} y^{(3,-2)}_{(a-2,b+1)}$$
$$- \frac{{}^{1,1}y^{(0,0)}_{(a+1,b-1)}}{x^{(2,-1)}_{(a-1,b)}} \left(\left(\frac{[2]^2}{[3]} - \frac{1}{y^{(1,0)}_{(a-1,b-1)}} \frac{1}{[3]^2} + \frac{1}{y^{(3,-1)}_{(a-1,b-1)}} \frac{1}{x^{(-1,1)}_{(a+1,b-2)}} \right)^2 \right.$$
$$+ \frac{{}^{2,2}y^{(0,0)}_{(a+1,b-1)}}{\left(y^{(3,-1)}_{(a-1,b-1)}\right)^2} - 2 \cdot \left(\frac{[2]^2}{[3]} - \frac{1}{y^{(1,0)}_{(a-1,b-1)}} \frac{1}{[3]^2} \right.$$
$$\left. + \frac{1}{y^{(3,-1)}_{(a-1,b-1)}} \frac{1}{x^{(-1,1)}_{(a+1,b-2)}} \right) \frac{1}{y^{(3,-1)}_{(a-1,b-1)}} {}^{1,2}y^{(0,0)}_{(a+1,b-1)} \Bigg)$$
$$- \frac{1}{x^{(0,0)}_{(a-1,b)}} \left(\frac{[2]}{[3]} + \frac{1}{y^{(3,-1)}_{(a-1,b-1)}} \frac{1}{x^{(-1,1)}_{(a+1,b-2)}} \right.$$
$$\cdot \left(\frac{1}{[2]} + \frac{1}{x^{(-1,1)}_{(a,b-2)}} \frac{1}{x^{(-1,1)}_{(a-1,b-2)}} \left(- \frac{[3]}{[2]} - \frac{1}{x^{(-1,1)}_{(a-2,b-1)}} \right) \right)$$
$$\left. - \frac{1}{y^{(1,0)}_{(a-1,b-1)}} \frac{1}{[3]} \left(\frac{1}{[2][3]} - \frac{[3]}{[2]} - \frac{1}{x^{(-1,1)}_{(a-2,b-1)}} \right) \right)^2 y^{(2,-1)}_{(a-1,b)}$$
$$- \frac{1}{x^{(1,-1)}_{(a-1,b)}} \left(- \frac{[4]}{[3]} - \frac{1}{y^{(3,-1)}_{(a-1,b-1)}} \frac{1}{x^{(-1,1)}_{(a+1,b-2)}} \frac{1}{x^{(-1,1)}_{(a,b-2)}} - \frac{1}{y^{(1,0)}_{(a-1,b-1)}} \frac{[2]}{[3]^2} \right)^2 y^{(1,0)}_{(a,b-1)}$$
$$- \frac{1}{x^{(-1,0)}_{(a-1,b)}} \left(- \frac{[4]}{[3]} - \frac{1}{y^{(3,-1)}_{(a-1,b-1)}} \left(\frac{1}{[3]} \right.\right.$$
$$+ \frac{1}{x^{(-1,1)}_{(a+1,b-2)}} \frac{1}{x^{(-1,1)}_{(a,b-2)}} \frac{1}{x^{(-1,1)}_{(a-1,b-2)}} \left(\frac{[4]}{[3]} + \frac{1}{x^{(-1,1)}_{(a-2,b-1)}} \right) \Big)$$
$$\left. + \frac{1}{y^{(1,0)}_{(a-1,b-1)}} \frac{1}{[3]} \left(\frac{[4]}{[3]} + \frac{1}{x^{(-1,1)}_{(a-2,b-1)}} \right) \right)^2 y^{(3,-1)}_{(a-2,b)}, \tag{A.22}$$
$$\rho^{(-1,0)}_{(a,b),\varpi_2} = -\frac{[6][8][15]}{[3][5][12]} x^{(0,0)}_{(a-1,b)} - \frac{1}{x^{(-1,1)}_{(a-1,b)}} \frac{1}{\left(x^{(-1,1)}_{(a-2,b)}\right)^2} y^{(1,-1)}_{(a-2,b+1)}$$
$$- \frac{y^{(-2,1)}_{(a+1,b-1)}}{x^{(2,-1)}_{(a-1,b)}} \left(\frac{1}{[2]} + \frac{1}{x^{(2,-1)}_{(a-2,b)}} \left(\frac{[5]}{[2][3]} + \frac{1}{y^{(3,-1)}_{(a-2,b-1)}} \left(\frac{1}{[2]^2} \right.\right.\right.$$
$$\left.\left.\left. + \frac{1}{[2]^2 x^{(-1,1)}_{(a,b-2)} x^{(-1,1)}_{(a-1,b-2)}} \right) + \frac{1}{[2][3]^2 y^{(1,0)}_{(a-2,b-1)}} \right) \right)^2$$
$$- \frac{1}{x^{(0,0)}_{(a-1,b)}} \left\{ \left(- \frac{[3]}{[2]} - \frac{1}{x^{(-1,1)}_{(a-1,b-1)}} \left(\frac{1}{x^{(-1,1)}_{(a-2,b)} y^{(3,-1)}_{(a-3,b)}} - \frac{1}{x^{(2,-1)}_{(a-2,b)}} \right) - \frac{1}{[2]^2 x^{(0,0)}_{(a-2,b)}} \right.\right.$$
$$\left.\left. - \frac{1}{x^{(-1,1)}_{(a-2,b)}} \left(\frac{[2]^2}{[3]} - \frac{1}{[3]^2 y^{(1,0)}_{(a-3,b)}} \right) \right)^2 {}^{1,1}y^{(0,0)}_{(a-1,b)} \right.$$

$$
\begin{aligned}
&+2\cdot\Bigg(-\frac{[3]}{[2]}-\frac{1}{[2]^2x^{(0,0)}_{(a-2,b)}}-\frac{1}{x^{(-1,1)}_{(a-2,b)}}\Bigg(\frac{[2]^2}{[3]}-\frac{1}{[3]^2y^{(1,0)}_{(a-3,b)}}\Bigg)\\
&-\frac{1}{x^{(-1,1)}_{(a-1,b-1)}}\Bigg(\frac{1}{x^{(-1,1)}_{(a-2,b)}y^{(3,-1)}_{(a-3,b)}}-\frac{1}{x^{(2,-1)}_{(a-2,b)}}\Bigg)\Bigg)\Bigg(\frac{1}{x^{(-1,1)}_{(a-2,b)}y^{(3,-1)}_{(a-3,b)}}\\
&-\frac{1}{x^{(2,-1)}_{(a-2,b)}}\Bigg)^{1,2}y^{(0,0)}_{(a-1,b)}+\Bigg(\frac{1}{x^{(-1,1)}_{(a-2,b)}y^{(3,-1)}_{(a-3,b)}}-\frac{1}{x^{(2,-1)}_{(a-2,b)}}\Bigg)^{2,2}y^{(0,0)}_{(a-1,b)}\Bigg\}\\
&-\frac{1}{x^{(-2,1)}_{(a-1,b)}}\Bigg(\frac{[3]}{[2]}-\frac{1}{x^{(-1,1)}_{(a-2,b)}}\Bigg(\frac{[2]}{[3]x^{(-1,1)}_{(a-3,b)}}+\frac{1}{[3]y^{(1,0)}_{(a-3,b)}x^{(-1,1)}_{(a-3,b)}}\Bigg(\frac{[4]}{[3]}+\frac{1}{x^{(-1,1)}_{(a-4,b)}}\Bigg)\Bigg)\\
&-\frac{1}{[2]x^{(0,0)}_{(a-2,b)}}\Bigg(\frac{[3]}{[2]}+\frac{1}{x^{(-1,1)}_{(a-3,b)}}\Bigg)\Bigg)^2y^{(2,-1)}_{(a-3,b+1)}-\frac{1}{x^{(1,-1)}_{(a-1,b)}}\Bigg(-\frac{[3]}{[2]}\\
&+\frac{1}{x^{(2,-1)}_{(a-2,b)}}\Bigg(\frac{[2]}{[3]^2y^{(1,0)}_{(a-2,b-1)}}+\frac{[5]}{[2][3]}+\frac{1}{y^{(3,-1)}_{(a-2,b-1)}x^{(-1,1)}_{(a,b-2)}x^{(-1,1)}_{(a-1,b-2)}}\Bigg)\\
&+\Bigg(\frac{1}{[2]x^{(0,0)}_{(a-2,b)}}+\frac{1}{x^{(2,-1)}_{(a-2,b)}x^{(-1,1)}_{(a-1,b-1)}x^{(-1,1)}_{(a-2,b-1)}}\Bigg)\\
&\cdot\Bigg(\frac{[4]}{[2]^2}-\frac{1}{x^{(2,-1)}_{(a-3,b)}}\Bigg(\frac{[5]}{[2][3]}+\frac{1}{y^{(3,-1)}_{(a-3,b-1)}}\Bigg(\frac{1}{[2]^2}+\frac{1}{[2]^2x^{(-1,1)}_{(a-1,b-2)}x^{(-1,1)}_{(a-2,b-2)}}\Bigg)\\
&+\frac{1}{[2][3]^2y^{(1,0)}_{(a-3,b-1)}}\Bigg)\Bigg)\Bigg)^2y^{(-1,1)}_{(a,b-1)}-\frac{1}{x^{(-1,0)}_{(a-1,b)}}\Bigg(\frac{1}{x^{(-1,1)}_{(a-2,b)}}\Bigg(\frac{[4]}{[3]}+\frac{[2]}{[3]^2y^{(1,0)}_{(a-3,b)}}\Bigg)\\
&+\Bigg(-\frac{1}{x^{(2,-1)}_{(a-2,b)}}+\frac{1}{x^{(-1,1)}_{(a-2,b)}y^{(3,-1)}_{(a-3,b)}}\Bigg)\frac{1}{x^{(-1,1)}_{(a-1,b-1)}x^{(-1,1)}_{(a-2,b-1)}}-\frac{1}{[2]x^{(0,0)}_{(a-2,b)}}\Bigg)^2y^{(1,0)}_{(a-2,b)},
\end{aligned}
\tag{A.23}
$$

$$
\begin{aligned}
\rho^{(-3,1)}_{(a,b),\varpi_2}&=-\frac{[6][8][15]}{[3][5][12]}x^{(-2,1)}_{(a-1,b)}-\frac{1}{x^{(-1,1)}_{(a-1,b)}}\frac{y^{(-1,0)}_{(a-2,b+1)}}{\big(x^{(-1,1)}_{(a-2,b)}x^{(-1,1)}_{(a-3,b)}\big)^2}\\
&-\frac{1}{x^{(0,0)}_{(a-1,b)}}\Bigg(-\frac{[3]}{[2]}+\frac{1}{x^{(-1,1)}_{(a-2,b)}x^{(-1,1)}_{(a-3,b)}}\Bigg(\frac{[2]}{[3]}+\frac{1}{[3]y^{(1,0)}_{(a-3,b)}}\Bigg(\frac{[4]}{[3]}+\frac{1}{x^{(-1,1)}_{(a-4,b)}}\Bigg)\Bigg)\\
&+\frac{1}{[2]x^{(0,0)}_{(a-2,b)}}\Bigg(\frac{[3]}{[2]}+\frac{1}{x^{(-1,1)}_{(a-3,b)}}\Bigg)\Bigg)^2y^{(-2,1)}_{(a-1,b)}\\
&-\frac{1}{x^{(-2,1)}_{(a-1,b)}}\Bigg\{\Bigg(\frac{[3]}{[2]x^{(-1,1)}_{(a-3,b)}}-\frac{1}{x^{(-1,1)}_{(a-2,b)}\big(x^{(-1,1)}_{(a-3,b)}\big)^2}\Bigg(-\frac{[3]}{[2]}-\frac{1}{x^{(-1,1)}_{(a-4,b+1)}}\Bigg(\frac{[2]^2}{[3]}\\
&+\frac{1}{y^{(3,-1)}_{(a-5,b+1)}x^{(-1,1)}_{(a-3,b)}}-\frac{1}{[3]^2y^{(1,0)}_{(a-5,b+1)}}\Bigg)+\frac{1}{x^{(2,-1)}_{(a-4,b+1)}x^{(-1,1)}_{(a-3,b)}}-\frac{1}{[2]^2x^{(0,0)}_{(a-4,b+1)}}\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{x^{(0,0)}_{(a-2,b)}}\left(\frac{[3]}{[2]}+\frac{1}{x^{(-1,1)}_{(a-3,b)}}\right)^2\frac{1}{x^{(-1,1)}_{(a-3,b)}}\\
&-\frac{1}{x^{(-2,1)}_{(a-2,b)}}\left(\frac{[3]}{[2]}-\frac{1}{[2]x^{(-1,1)}_{(a-3,b)}x^{(-1,1)}_{(a-4,b)}}\right)^2\Bigg)^2{}_{1,1}y^{(0,0)}_{(a-3,b+1)}\\
&+2\Bigg(-\frac{1}{x^{(-1,1)}_{(a-2,b)}\left(x^{(-1,1)}_{(a-3,b)}\right)^2}\left(\frac{1}{x^{(-1,1)}_{(a-4,b+1)}y^{(3,-1)}_{(a-5,b+1)}}-\frac{1}{x^{(2,-1)}_{(a-4,b+1)}}\right)\\
&-\frac{1}{x^{(0,0)}_{(a-2,b)}}\left(\frac{[3]}{[2]}+\frac{1}{x^{(-1,1)}_{(a-3,b)}}\right)^2-\frac{[3]}{[2]}\Bigg)\Bigg(\frac{[3]}{[2]x^{(-1,1)}_{(a-3,b)}}-\frac{1}{x^{(-1,1)}_{(a-2,b)}\left(x^{(-1,1)}_{(a-3,b)}\right)^2}\Bigg(-\frac{[3]}{[2]}\\
&-\frac{1}{x^{(-1,1)}_{(a-4,b+1)}}\left(\frac{[2]^2}{[3]}+\frac{1}{y^{(3,-1)}_{(a-5,b+1)}x^{(-1,1)}_{(a-3,b)}}-\frac{1}{[3]^2y^{(1,0)}_{(a-5,b+1)}}\right)+\frac{1}{x^{(2,-1)}_{(a-4,b+1)}x^{(-1,1)}_{(a-3,b)}}\\
&-\frac{1}{[2]^2x^{(0,0)}_{(a-4,b+1)}}\Bigg)+\frac{1}{x^{(0,0)}_{(a-2,b)}}\left(\frac{[3]}{[2]}+\frac{1}{x^{(-1,1)}_{(a-3,b)}}\right)^2\frac{1}{x^{(-1,1)}_{(a-3,b)}}\\
&-\frac{1}{x^{(-2,1)}_{(a-2,b)}}\left(\frac{[3]}{[2]}-\frac{1}{[2]x^{(-1,1)}_{(a-3,b)}x^{(-1,1)}_{(a-4,b)}}\right)^2\Bigg)^{1,2}y^{(0,0)}_{(a-3,b+1)}\\
&+\Bigg(\frac{-1}{x^{(-1,1)}_{(a-2,b)}\left(x^{(-1,1)}_{(a-3,b)}\right)^2}\left(\frac{1}{x^{(-1,1)}_{(a-4,b+1)}y^{(3,-1)}_{(a-5,b+1)}}-\frac{1}{x^{(2,-1)}_{(a-4,b+1)}}\right)-\frac{[3]}{[2]}\\
&-\frac{1}{x^{(0,0)}_{(a-2,b)}}\left(\frac{[3]}{[2]}+\frac{1}{x^{(-1,1)}_{(a-3,b)}}\right)^2\Bigg)^2{}_{2,2}y^{(0,0)}_{(a-3,b+1)}\Bigg\}\\
&-\frac{1}{x^{(1,-1)}_{(a-1,b)}}\left(\frac{[3]}{[2]}+\frac{1}{x^{(0,0)}_{(a-2,b)}}\left(\frac{[3]}{[2]}+\frac{1}{x^{(-1,1)}_{(a-3,b)}}\right)\left(\frac{-1}{x^{(2,-1)}_{(a-3,b)}}\left(\frac{[5]}{[2][3]}\right.\right.\right.\\
&\left.\left.\left.-\frac{1}{[2]y^{(3,-1)}_{(a-3,b-1)}x^{(-1,1)}_{(a-1,b-2)}}+\frac{1}{[2][3]^2y^{(1,0)}_{(a-3,b-1)}}\right)+\frac{[4]}{[2]^2}\right)\Bigg)^2y^{(-3,2)}_{(a,b-1)}\\
&-\frac{1}{x^{(-1,0)}_{(a-1,b)}}\Bigg(\frac{1}{x^{(-1,1)}_{(a-2,b)}}\frac{1}{x^{(-1,1)}_{(a-3,b)}}\Bigg(-\frac{[3]}{[2]}+\frac{1}{x^{(2,-1)}_{(a-4,b+1)}}\\
&\cdot\frac{1}{x^{(-1,1)}_{(a-3,b)}}\Bigg(-\frac{[5]}{[3]}-\frac{1}{[3]^2y^{(1,0)}_{(a-4,b)}}+\frac{1}{y^{(3,-1)}_{(a-4,b)}x^{(-1,1)}_{(a-2,b-1)}}\Bigg)\\
&+\frac{1}{[2]x^{(0,0)}_{(a-4,b+1)}}\left(\frac{[4]}{[2]^2}-\frac{1}{x^{(2,-1)}_{(a-5,b+1)}}\left(\frac{[5]}{[2][3]}-\frac{1}{[2]y^{(3,-1)}_{(a-5,b)}x^{(-1,1)}_{(a-3,b-1)}}\right.\right.\\
&\left.\left.+\frac{1}{[2][3]^2y^{(1,0)}_{(a-5,b)}}\right)\right)\Bigg)-\frac{1}{x^{(0,0)}_{(a-2,b)}}\left(\frac{[3]}{[2]}+\frac{1}{x^{(-1,1)}_{(a-3,b)}}\right)\frac{1}{x^{(-1,1)}_{(a-3,b)}}\\
&+\frac{1}{x^{(-2,1)}_{(a-2,b)}}\left(\frac{[3]}{[2]}-\frac{1}{[2]x^{(-1,1)}_{(a-3,b)}x^{(-1,1)}_{(a-4,b)}}\right)\Bigg)^2y^{(-1,1)}_{(a-2,b)},
\end{aligned}
\tag{A.24}
$$

$$\begin{aligned}\rho^{(0,-1)}_{(a,b),\varpi_2} = {} & \frac{[7][8][15]}{[3][4][5]} - \frac{1}{y^{(3,-1)}_{(a,b-1)}} y^{(-3,1)}_{(a+3,b-2)} - \frac{1}{y^{(1,0)}_{(a,b-1)}} y^{(-1,0)}_{(a+1,b-1)} \\ & - \frac{1}{y^{(-1,1)}_{(a,b-1)}} y^{(1,-1)}_{(a-1,b)} - \frac{1}{y^{(2,-1)}_{(a,b-1)}} y^{(-2,1)}_{(a+2,b-2)} - \frac{1}{y^{(-3,2)}_{(a,b-1)}} y^{(3,-2)}_{(a-3,b+1)} \\ & - \frac{{}^{2,2}y^{(0,0)}_{(a,b-1)}}{\mathcal{D}_{(a,b-1)}} {}^{1,1}y^{(0,0)}_{(a,b-1)} + \frac{{}^{1,2}y^{(0,0)}_{(a,b-1)}}{\mathcal{D}_{(a,b-1)}} {}^{1,2}y^{(0,0)}_{(a,b-1)} \\ & + \frac{{}^{2,1}y^{(0,0)}_{(a,b-1)}}{\mathcal{D}_{(a,b-1)}} {}^{2,1}y^{(0,0)}_{(a,b-1)} - \frac{{}^{1,1}y^{(0,0)}_{(a,b-1)}}{\mathcal{D}_{(a,b-1)}} {}^{2,2}y^{(0,0)}_{(a,b-1)} \\ & - \frac{1}{y^{(3,-2)}_{(a,b-1)}} y^{(-3,2)}_{(a+3,b-3)} - \frac{1}{y^{(-2,1)}_{(a,b-1)}} y^{(2,-1)}_{(a-2,b)} - \frac{1}{y^{(1,-1)}_{(a,b-1)}} y^{(-1,1)}_{(a+1,b-2)} \\ & - \frac{1}{y^{(-1,0)}_{(a,b-1)}} y^{(1,0)}_{(a-1,b-1)} - \frac{1}{y^{(-3,1)}_{(a,b-1)}} y^{(3,-1)}_{(a-3,b)} - \frac{1}{y^{(0,-1)}_{(a,b-1)}} y^{(0,1)}_{(a,b-2)}. \end{aligned} \tag{A.25}$$

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