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Nijenhuis geometry III: gl-regular Nijenhuis operators

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Abstract. We study Nijenhuis operators, that is, $(1,1)$-tensors with vanishing Nijenhuis torsion under the additional assumption that they are gl-regular, i.e., every eigenvalue has geometric multiplicity one. We prove the existence of a coordinate system in which the operator takes first or second companion form, and give a local description of such operators. We apply this local description to study singular points. In particular, we obtain normal forms of gl-regular Nijenhuis operators near singular points in dimension two and discover topological restrictions for the existence of gl-regular Nijenhuis operators on closed surfaces.

1. Basic definitions and main results

Given a $(1,1)$ tensor field L on a manifold M^n, one defines the *Nijenhuis torsion* of L as

$$\mathcal{N}_L(\xi,\eta) = L^2[\xi,\eta] - L[L\xi,\eta] - L[\xi,L\eta] + [L\xi,L\eta], \tag{1.1}$$

where ξ and η are arbitrary vector fields. If $\mathcal{N}_L$ identically vanishes, then L is said to be a *Nijenhuis operator*.

Nijenhuis geometry studies Nijenhuis operators and their properties, both local and global (see e.g. the videocourse [24]). A research programme and a general strategy for studying such operators were suggested in [8]. This paper is devoted to the next item of our agenda (after [8] and [16], see also [6, 7, 9, 10, 12]), and is focused on Nijenhuis operators satisfying gl-regularity condition.

We start with the following equivalent definitions of gl-*r*egular operators $L\colon \mathbb{R}^n \to \mathbb{R}^n$, see e.g. the Wikipedia pages [27, 28] (the same notation L will be used for the matrix corresponding to this operator, with appropriate amendments under coordinate transformations if necessary):

- L is a regular element of the Lie algebra $\mathrm{gl}(n,\mathbb{R})$ in the sense that the adjoint orbit $\mathcal{O}(L) = \{PLP^{-1} \mid P \in \mathrm{GL}(n,\mathbb{R})\} \subset \mathrm{gl}(n,\mathbb{R})$ has maximal dimension.
- The operators $\mathrm{Id}, L, \dots, L^{n-1}$ are linearly independent.

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- For each eigenvalue of L there is exactly one Jordan block in its Jordan normal form (this includes complex eigenvalues).
- The minimal polynomial of L coincides with the characteristic polynomial

$$\chi_L(\lambda) = \det(\lambda \cdot \mathrm{Id} - L) = \lambda^n - c_1\lambda^{n-1} - \dots - c_n.$$

- L is similar to the *first companion form*

$$\begin{pmatrix} c_1 & 1 & 0 & \dots & 0 \\ c_2 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ c_{n-1} & 0 & \dots & 0 & 1 \\ c_n & 0 & \dots & 0 & 0 \end{pmatrix},$$

  where c_i are the coefficients of the characteristic polynomial $\chi_L(\lambda)$.
- L is similar to the *second companion form*

$$\begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \dots & 0 & 1 \\ c_n & c_{n-1} & \dots & c_2 & c_1 \end{pmatrix},$$

  where c_i are the coefficients of the characteristic polynomial $\chi_L(\lambda)$.

We say that a Nijenhuis operator L defined on a smooth manifold M is gl-*regular* if it is gl-regular at every point $\mathsf{p} \in \mathsf{M}$, see Definition 2.9 in [8]. Many results in our paper are local, and in this case M is an open domain in $\mathbb{R}^n$.

Note that the eigenvalues of gl-regular operators are not necessarily smooth, as the following example shows. Consider the gl-regular Nijenhuis operator

$$L = \begin{pmatrix} x & 1 \\ y & 0 \end{pmatrix}$$

on $\mathbb{R}^2(x, y)$. Its eigenvalues are

$$\lambda_{1,2} = \frac{x \pm \sqrt{x^2 + 4y}}{2}.$$

On the curve $x^2 + 4y = 0$, L is similar to a single Jordan block with eigenvalue $x/2$. If $x^2 + 4y > 0$, then L is semisimple with distinct real eigenvalues (thus, $\mathbb{R}$-diagonalisable) whereas for $x^2 + 4y < 0$ this operator has two complex conjugate eigenvalues. In particular, this shows that gl-regular operators may admit singular points (cf. Definition 2.8 in [8]) at which the algebraic structure of L changes.

All the objects we are dealing with are supposed to be real analytic. The first result of the paper is the following theorem, which gives a local characterisation of gl-regular Nijenhuis operators of any algebraic type.

Theorem 1.1. *Consider a real analytic* gl*-regular operator L with characteristic polynomial*

$$\chi_L(\lambda) = \det(\lambda \cdot \mathrm{Id} - L) = \lambda^n - f_1 \lambda^{n-1} - \cdots - f_n$$

for $n \geq 2$ in a sufficiently small neighbourhood of a point $\mathsf{p} \in \mathsf{M}$*. Then the following are equivalent.*

(i) *L is Nijenhuis.*

(ii) *There exists a local coordinate system $x = (x^1, \dots, x^n)$ in which L takes the following form*:

$$L_{\mathsf{comp1}}(x) = \begin{pmatrix} f_1 & 1 & 0 & \cdots & 0 \\ f_2 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ f_{n-1} & 0 & \cdots & 0 & 1 \\ f_n & 0 & \cdots & 0 & 0 \end{pmatrix}, \tag{1.2}$$

where $f_i = f_i(x)$ are coefficients of the characteristic polynomial in this coordinate system. These coefficients satisfy the following system of PDEs:

$$\begin{aligned} \frac{\partial f_i}{\partial x^j} &= f_i \frac{\partial f_1}{\partial x^{j+1}} + \frac{\partial f_{i+1}}{\partial x^{j+1}}, \\ \frac{\partial f_n}{\partial x^j} &= f_n \frac{\partial f_1}{\partial x^{j+1}}, \end{aligned} \tag{1.3}$$

for $1 \leq i, j \leq n-1$.

(iii) *There exists a local coordinate system $x = (x^1, \dots, x^n)$ in which L takes the following form*:

$$L_{\mathsf{comp2}}(x) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ f_n & f_{n-1} & \cdots & f_2 & f_1 \end{pmatrix}, \tag{1.4}$$

where $f_i = f_i(x)$ are coefficients of the characteristic polynomial in this coordinate system. These coefficients satisfy a system of PDEs that can be written in the form

$$\mathrm{d}\omega = 0, \quad \mathrm{d}(L^* \omega) = 0, \tag{1.5}$$

where $\omega = f_n \, \mathrm{d}x^1 + \cdots + f_1 \, \mathrm{d}x^n$.

Following terminology from linear algebra, we will refer to (1.2) and (1.4) as the *first* and *second companion forms* of L.

Remark 1.2. If a Nijenhuis operator L is differentially non-degenerate at a point $\mathsf{p} \in \mathsf{M}$ (see[1] [8], Definition 2.10), then there are two distinguished coordinate systems in which L takes the first and second companion form. Namely, if we take the coefficients of the cha-

[1] Recall that this condition means that the differentials $\mathrm{d} f_1(\mathsf{p}), \dots, \mathrm{d} f_n(\mathsf{p})$ are linearly independent.

racteristic polynomial of L as local coordinates, i.e., set $x^i = f_i$, then in these coordinates L takes the form

$$L_{\mathsf{comp1}}(x) = \begin{pmatrix} x^1 & 1 & 0 & \cdots & 0 \\ x^2 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ x^{n-1} & 0 & \cdots & 0 & 1 \\ x^n & 0 & \cdots & 0 & 0 \end{pmatrix}. \tag{1.6}$$

Similarly, if we set $x^1 = \operatorname{tr} L, x^2 = \frac{1}{2}\operatorname{tr} L^2, \dots, x^n = \frac{1}{n}\operatorname{tr} L^n$ then, in these coordinates, we have

$$L_{\mathsf{comp2}}(x) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ f_n(x) & f_{n-1}(x) & \cdots & f_2(x) & f_1(x) \end{pmatrix},$$

where $f_i(x)$ are the so-called Newton–Girard polynomials that express the coefficients of the characteristic polynomial in terms of the traces of powers of L appropriately rescaled, see Appendix B in [9] for details.

The point of Theorem 1.1, however, is that such a nice companion form exists for any gl-regular Nijenhuis operator so that in the real analytic category the differential non-degeneracy condition is not actually important.

Remark 1.3. The existence of the first companion form for an operator L is equivalent to the existence of a vector field ξ such that $\xi, L\xi, L^2\xi, \dots, L^{n-1}\xi$ pairwise commute and are linearly independent (for L_{comp1}, this vector field is $\xi = \partial_{x^n}$). Similarly, the existence of the second companion form for L is equivalent to the existence of a closed 1-form α such that the forms $\alpha, L^*\alpha, (L^*)^2\alpha, \dots, (L^*)^{n-1}\alpha$ are all closed and linearly independent (for L_{comp2}, we can take $\alpha = \mathrm{d}x^1$).

Remark 1.4. The reducibility of an operator to a companion form by a coordinate transformation is a non-trivial condition. Indeed, companion forms (1.2) and (1.4) are parametrised by n functions (in n variables). The coordinate change is also parametrized by n functions. At the same time, an operator field L (not necessarily Nijenhuis) is parametrised by n^2 functions. For $n > 2$ one has $n^2 > 2n$, and thus, almost no operator field L can be brought to companion form.

As a specific example, consider L such that the coefficients f_i of its characteristic polynomial $\chi_L(\lambda)$ are all constant. The companion form for L will then be a constant matrix. Hence, if L is reducible to companion form by a suitable coordinate transformation, then its Nijenhuis torsion $\mathcal{N}_L$ necessarily vanishes, which is not always the case. Indeed, take

$$L = \begin{pmatrix} 0 & 1 & 0 \\ -(y^2+1) & 0 & 1 \\ 0 & (y^2+1) & 0 \end{pmatrix}.$$

This operator is nilpotent, but $\mathcal{N}_L \neq 0$. Thus, L cannot be brought to companion form.

Remark 1.5. The set of coordinate systems in which a gl-regular Nijenhuis operator L is in first or second companion form is parametrised by n functions of one variable, which is the maximal number of possible parameters. In more precise terms, the equations defining the corresponding coordinate transformations (see (3.3) and (3.6) below) are in involution for a gl-regular operator L if and only if L is Nijenhuis (see Propositions 3.3 and 3.4).

Theorem 1.1 characterises gl-regular Nijenhuis operators but, in fact, should not be interpreted as their local description. To get such a description, one needs another important step. Namely, one needs to solve the PDE system (1.3) in order to find functions f_i from the first column of L_{comp1}. The second result of our paper is an algebraic method for solving this system for arbitrary initial conditions.

Theorem 1.6. *For n arbitrary real analytic functions $v_1(t), \dots, v_n(t)$ defined in a neighbourhood of zero, consider the function*

$$r(\lambda, t) = \lambda^n - v_1(t)\lambda^{n-1} - v_2(t)\lambda^{n-2} - \dots - v_{n-1}(t)\lambda - v_n(t)$$

and the matrix relation

$$r(L, M) = 0,$$

where $M = x^1 L^{n-1} + x^2 L^{n-2} + \dots + x^{n-1} L + x^n \operatorname{Id}$, and L is a gl-regular $n \times n$ matrix. Then

- *from this matrix relation, the coefficients $f_1, \dots, f_n$ of the characteristic polynomial of L can be uniquely expressed in a neighbourhood of $x = 0$ as real analytic functions in $x^1, \dots, x^n$ (by the implicit function theorem).*
- *The functions $f_1(x), \dots, f_n(x)$ so obtained are solutions of (1.3) satisfying the initial condition*

$$\begin{aligned} f_1(0, \dots, 0, x^n) &= v_1(x^n), \\ f_2(0, \dots, 0, x^n) &= v_2(x^n), \\ &\vdots \\ f_n(0, \dots, 0, x^n) &= v_n(x^n). \end{aligned} \tag{1.7}$$

This theorem gives local description for all gl-regular Nijenhuis operators and therefore provides a "list" of all possible singularities that can occur for gl-regular operators (Example 4.7 demonstrates how it works in practice). One should, however, remember that the first companion form for a Nijenhuis operator L is not unique. In other words, different companion forms can be equivalent. Speaking in rigorous terms, on the space of all (Nijenhuis) companion forms L_{comp1} given by (1.2), we can introduce a natural action of the groupoid that consists of coordinate transformations sending one companion form into another. Local classification of gl-regular operators in proper sense amounts to the orbit classification for this action. For $n \geq 3$, we hope to address this problem elsewhere.

In the two-dimensional case, which is somehow rather special, the local classification of gl-regular Nijenhuis operators is obtained in Section 5, see Theorem 5.1. In addition to three (algebraically) generic types of gl-regular operators, this theorem describes five types[2] of singular points (series L_{nc}, M, O, P and S) for gl-regular operators in dimen-

[2]The other two series L_{nil} and N from Theorem 5.1 are not singular as the algebraic type of these operators does not change, at each point the operator is a 2×2 Jordan block.

sion 2. It appears that locally every gl-regular Nijenhuis operator can be reduced to an explicit polynomial canonical form, which is quite different from the companion form. Our choice is explained by the following natural reason. The functions f_1 and f_2 involved in L_{comp1} are solutions of (1.3), and Theorem 1.6 suggests that they can be found explicitly only in exceptional cases. Despite its elegance and convenience for various theoretical purposes, the companion form L_{comp1} does not provide description in elementary functions. However, such a description can be achieved by an appropriate change of variables, and that is what Theorem 5.1 does.

Based on this theorem, we obtain the following global description of Nijenhuis operators on closed two-dimensional manifolds.

Theorem 1.7. *Let* (M^2, L) *be a closed connected* gl-*regular Nijenhuis* 2-*manifold. Then one of the following holds*:

(1) M^2 *is orientable and* $L = \alpha \operatorname{Id} + \beta A$, *where* A *is a complex structure on* M^2 *and* $\alpha, \beta \in \mathbb{R}$ *are constants,* $\beta \neq 0$.

(2) M^2 *is homeomorphic to either a torus or a Klein bottle, and* L *has two distinct real eigenvalues on* M^2 *at each point.*

(3) M^2 *is homeomorphic to a torus, and* L *is similar to a Jordan block at each point of* M^2.

(4) M^2 *is homeomorphic to either a torus or a Klein bottle, and one of the eigenvalues of* L *is constant.*

In the first three cases, the algebraic type of L remains the same at each point of the surface. In other words, the set of singular points is empty. In the fourth case, the eigenvalues of L may collide, and we show in Proposition 5.7 that the corresponding singular point necessarily belongs to the M-series, one of five series from Theorem 5.1. In particular, the other types of singular points cannot occur on compact surfaces.

Theorem 1.7 provides topological obstructions for existence of (non-trivial) gl-regular Nijenhuis operators in dimension 2.

Corollary 1.8. *Let* M^2 *be either a sphere or a closed Riemann surface of genus* ≥ 2. *Then* M^2 *cannot carry any* gl-*regular Nijenhuis operator* L *except for* $L = \alpha \operatorname{Id} + \beta A$, *where* A *is a complex structure on* M^2 *and* $\alpha, \beta \in \mathbb{R}$, $\beta \neq 0$.

Corollary 1.9. *A non-orientable closed* 2-*manifold different from a Klein bottle cannot carry any* gl-*regular Nijenhuis operator.*

Another result of our paper is description of various scenarios for Nijenhuis perturbations of a Jordan block. Assume that at a given point p, all the coefficients $f_1, \dots, f_n$ of the characteristic polynomial of a Nijenhuis operator L vanish so that $L(\mathsf{p})$ is similar to a Jordan block with zero eigenvalues. What can we say about the algebraic type of L at a generic point q from an open neighbourhood $U(\mathsf{p})$ of p? Formula (1.6) gives an example when $L(\mathsf{q})$ typically becomes semisimple, moreover for any prescribed collection of eigenvalues $\lambda_1, \dots, \lambda_n$ (with arbitrary multiplicities and including complex conjugate pairs) there exists exactly one point q that realises this spectrum of L. This scenario coincides with the versal deformation of a Jordan block in terms of V. Arnold [2]. But can L split into two Jordan blocks? Or, more generally, does there exist a Nijenhuis perturbation of a Jordan block $J_0 = L(\mathsf{p})$ such that at a generic point $\mathsf{q} \in U(\mathsf{p})$ the operator $L(\mathsf{q})$ has a prescribed algebraic type?

We use Theorem 1.6 to show that the answer is positive: all scenarios are possible. To state this result in a rigorous way, recall that in the space of all $n \times n$ matrices, which we interpret as the Lie algebra $\mathrm{gl}(n, \mathbb{R})$, we can introduce a natural partition $\mathrm{gl}(n, \mathbb{R}) = \sqcup_\alpha W_\alpha$ into families of adjoint orbits having the same algebraic type (Segre characteristic). Such families are sometimes called *layers*. For regular orbits, their algebraic type is defined by multiplicities $k_1, \dots, k_s$ of eigenvalues[3], so that we can write

$$\mathrm{gl}(n, \mathbb{R})^{\mathrm{reg}} = \bigsqcup_{\sum k_s = n} W_{k_1,\dots,k_s}, \quad \text{for } k_1 \le \dots \le k_s,\ s \in \mathbb{N},\ k_i \in \mathbb{N},$$

where $W_{k_1,\dots,k_s} \subset \mathrm{gl}(n, \mathbb{R})$ is the subset of gl-regular operators having s distinct eigenvalues with multiplicities $k_1, \dots, k_s$ (regularity will automatically imply that each eigenvalue contributes exactly one Jordan block into the Jordan normal form of the operator). Notice that the Jordan block J_0 belongs to the closure of each regular layer.

Theorem 1.10. *For any regular layer $W_{k_1,\dots,k_s} \subset \mathrm{gl}(n, \mathbb{R})$, there exists a Nijenhuis operator L defined in a small neighbourhood of $0 \in \mathbb{R}^n$ such that $L(0) = J_0$ and $L(x) \in \overline{W}_{k_1,\dots,k_s}$ for all $x \in U(0)$, where $\overline{W}_{k_1,\dots,k_s}$ is the closure of $W_{k_1,\dots,k_s}$ (in the usual or in the Zariski topology).*

The structure of the paper is as follows. The proof of Theorem 1.1 is given in Section 3. Section 4 is devoted to Theorems 1.6 and 1.10. In Section 5, we obtain local classification of all gl-regular Nijenhuis operators in dimension 2 and prove Theorem 1.7. These sections are mainly independent on each other and contain no cross references.

2. Outlook and motivation

Our motivation for studying gl-regular Nijenhuis operators was based on a very naive question: "What is the most natural genericity assumption for $(1, 1)$-tensor fields similar to non-degeneracy of bilinear forms, symmetric or skew-symmetric?". In a general algebraic context, the latter condition simply means that a bilinear form belongs to the "largest" orbit of the natural $\mathrm{GL}(n)$-action, and hence is the most typical. As a matter of fact, such an orbit, in this case, is open. For operators, there are no open orbits, but we may still consider $\mathrm{GL}(n)$-orbits of maximal dimension, which is exactly the gl-regularity assumption[4]. In this view, gl-regular operators can be thought of as natural analogs of symplectic forms and (pseudo)-Riemannian metrics.

Another naive way to look at $(1, 1)$-tensor fields is to think of them as families of matrices depending on parameters (coordinates on the manifold). Then the next natural question would obviously be: "Which bifurcations are typical in such families?". The most typical bifurcation is a collision of two (or several) eigenvalues resulting in appearance

[3] Though we deal with real matrices, we make no difference between complex and real roots.

[4] The non-degeneracy assumption $\det L \neq 0$ is much less relevant in Nijenhuis geometry as many problems one has to deal with are invariant with respect to shifts $L \mapsto L + \mathrm{const} \cdot \mathrm{Id}$.

of a Jordan block. That is exactly a singularity which we may observe in the case of gl-regular Nijenhuis operators. One could, of course, avoid collision of eigenvalues by requiring that L has no multiple eigenvalues, but would make the definition too rigid and exclude many important examples and interesting phenomena. It is worth mentioning that the complement to the set of matrices with no multiple roots has codimension one, whereas the complement to the set of gl-regular matrices is much smaller and has codimension 3.

The "converse" question, naturally appearing in applications, can be stated as follows: "What happens to a Jordan block under a perturbation?". The answer depends on the number of parameters involved in perturbation and additional assumptions imposed on it. We refer to the famous paper [2] by Arnold devoted to this subject, which contains, in particular, an elegant solution in terms of versal deformations. In the context of Nijenhuis geometry, it is quite natural to ask: "What are *Nijenhuis* perturbations of a Jordan block? Can we describe *all* of them? Which of them are generic (*versal* in the sense of Arnold)?". This is again a question on gl-regular Nijenhuis operators. It is amazing that the answer turns out to be very similar to that given by Arnold: there is a very simple generic Nijenhuis perturbation of a Jordan block (see formula (1.6) and Proposition 4.5), which is unique and coincides exactly with the one given in [2]. All the others can be derived from this canonical one by solving a system of integrable PDEs. We give a purely algebraic algorithm (see Theorem 1.6) to do it for arbitrary initial conditions, i.e., for finding *all* the solutions.

Notice that gl-regular operators may have different types but still possess many common properties. There are many facts well known for diagonalisable operators with simple spectrum that still hold true for gl-regular operators. If an operator is diagonalisable almost everywhere and has no multiple eigenvalues, then some (but not all!) of these results can be transferred to the gl-regular case by continuity. However, even this procedure is often non-trivial, as one needs to show that "transferring objects", e.g., conservation laws or commuting flows, remain smooth and independent (linearly or functionally or otherwise), i.e., they neither explode nor blow up. Moreover, there are many occasions when a given Nijenhuis operator is not diagonalisable at all, but gl-regularity still guarantees good properties.

For this reason, we are trying to use "invariant language" in our proof. This makes things technically a bit more complicated (for Nijenhuis operators written in diagonal form some of our proofs would be just one line) but, as a reward, we manage to cover many different cases by using one universal approach suitable for all Nijenhuis operators satisfying just one additional condition, namely gl-regularity.

We are confident that our results can and will have many applications. Indeed, Nijenhuis operators naturally appear in many unrelated topics in differential geometry and mathematical physics. A possible explanation for this "experimentally observed phenomenon" is as follows. For many geometric systems of partial differential equations, their coefficients are constructed from a certain operator, i.e., a $(1,1)$-tensor field $L = (L^i_j(u))$. If such a system is invariant with respect to diffeomorphisms, then the compatibility and involutivity conditions can be invariantly written in terms of L. The point is that vanishing of the Nijenhuis torsion of L is, in a certain sense (see e.g. discussion in the introduction of [8]), the simplest non-trivial condition of this kind.

This "experimental observation" suggests that any progress in Nijenhuis geometry might and should be applied in different areas where Nijenhuis operators have appeared, by combining the questions/methods from those topics with new results on Nijenhuis operators.

Until very recently, the list of known results in Nijenhuis geometry was very limited: Haantjes' theorem [15], the Newlander–Nirenberg theorem [21] and Thompson's theorem [25]. These results have been extensively used as a simplifying ansatz in those situations where Nijenhuis operators appear: customary, one works with those coordinates in which the operator takes the "best" possible form provided by these theorems (e.g., in the case of the Haantjes theorem, L reduces to diagonal form with diagonal elements $\lambda_i = \lambda_i(u_i)$, and in the case of the Thompson and Newlander–Nirenberg theorems, one works in a coordinate system where L has constant entries).

The assumptions of the theorems of Haantjes, Newlander–Nirenberg and Thompson essentially limit their applications. They all require that L is algebraically stable, i.e., has the same Segre characteristic at every point. Moreover, they have strong conditions on the Segre characteristic: in the Haantjes and Newlander–Nirenberg theorems, the operator L is semi-simple (diagonalisable over complex numbers). The Thompson and Newlander–Nirenberg theorems assume that the eigenvalues of L are constant.

This paper, as well as its predecessors [8,9,16], aim to repair this situation. An important ingredient of our strategy described in [8] is to develop tools to study and describe Nijenhuis operators near those points where the Segre characteristic changes (singular points, in the terminology of [8]) and also on closed manifolds. Any such tool can be applied wherever Nijenhuis operators naturally appear.

Obviously, there are many different types of singularities for Nijenhuis operators. We started our research with two opposite cases: the paper [16] (see also [8], §5) studies the so-called singular points of scalar type, i.e., those where the operator L vanishes (we may think of them as the *most* singular points). In the present paper, we come from the other side and consider singular points at which the operator L remains gl-regular, see Definition 2.9 in [8] (the *least* singular points). Our first main result, Theorem 1.1, provides a common framework for studying such singularities: it allows one to assume without loss of generality that L locally takes the first or second companion form (see (1.2) and (1.4)). Similar to the diagonal form from the Haantjes theorem, the companion forms (1.2) and (1.4) depend on an arbitrary choice of n functions of one variable. In contrast to the Haantjes theorem, they allow bifurcations of the eigenvalues, and in Section 4 we discuss the freedom in such bifurcations.

A demonstration that our strategy works is Theorem 5.1, that describes all possible singularities for gl-regular Nijenhuis operators in dimension 2. As a corollary we have Theorem 1.7 on topological obstructions for the existence of regular Nijenhuis operators on closed two-dimensional surfaces.

We expect many applications of our results. For example, with the help of Theorem 1.7 one can easily reprove most results of the paper [19] devoted to geodesically equivalent metrics on two dimensional semi-Riemannian manifolds. By [11], a pair of such metrics allows one to construct a Nijenhuis operator. One can easily show, applying the trick from Section 3.3 in [20], that on a closed surface this operator is always gl-regular provided the metrics are semi-Riemannian. Case (1) of Theorem 1.7 corresponds to a trivial geodesic equivalence, and cases (2), (3) and (4), translated to the language of geodesically equivalent metrics, imply most results in [19], and in particular allow to prove the natural generalisation of the projective Obata conjecture for the 2-torus.

We expect that our results may be effectively used in the theory of (infinite-dimensional) integrable systems of hydrodynamic type. They are partial differential equation systems of the form

$$u^i_t = \sum_j A^i_j(u)\, u^j_x, \tag{2.1}$$

where $u(t,x) = (u^1(t,x), \dots, u^n(t,x))$ is an unknown vector-function. In this case, the matrix $A = A(u)$ can be seen as an operator on an n-dimensional manifold with local coordinates $(u^1, \dots, u^n)$. The integrability of this system amounts to a certain condition on the operator A (more general than vanishing of the Nijenhuis torsion, see [26]).

One of the standard methods to work with systems (2.1) is based on the so-called Riemann invariants, which are closely related to finding a polynomial p with coefficients depending on u such that $p(A)$ is a Nijenhuis operator (the eigenvalues of the operator $p(A)$ are precisely the Riemann invariants).

The overwhelming majority of results on integrable systems of hydrodynamic type assume that the operator A is simple (i.e., has n different eigenvalues). Our results allow one to avoid this assumption. In particular, they can be applied to study stability of solutions of (2.1) near the points where the eigenvalues collide. The "proof of concept" is, in fact, the Appendix of the arXiv version [5] of this paper, where we demonstrate how it works in the simplest case, when the operator A is itself a Nijenhuis operator.

Notice that not diagonalisable but still gl-regular operators naturally appear in differential geometry and mathematical physics in the context of integrable PDEs of type (2.1), see e.g. [1, 3, 4, 13, 26]. Moreover, they often resemble the companion form discussed in Theorem 1.1.

3. Proof of Theorem 1.1

First of all, we observe that an operator L_{comp1} given by (1.2) is Nijenhuis if and only if relations (1.3) hold. And, similarly, an operator L_{comp2} given by (1.4) is Nijenhuis if and only if (1.5) holds. The verification of this fact is straightforward and we omit it. In terms of Theorem 1.1 this means, in particular, that (ii) $\Rightarrow$ (i) and (iii) $\Rightarrow$ (i).

It remains to show that every gl-regular Nijenhuis operator L can be (locally) reduced to either of the companion forms L_{comp1} and L_{comp2}. Since the proofs for L_{comp1} and L_{comp2} are rather similar, we will do reduction simultaneously for both of them following the same scheme.

Consider a gl-regular Nijenhuis operator L in a neghbourhood $U(\mathsf{p})$ of a point $\mathsf{p} \in \mathsf{M}$, and choose local coordinates $u = (u^1, \dots, u^n)$ in this neighbourhood. Our goal is to find coordinate transformations bringing L to the first companion form (1.2) and second companion form (1.4).

For the first companion form, such a coordinate transformation $u = u(x)$, where $x = (x^1, \dots, x^n)$ is a new coordinate system, satisfies the following system of PDEs:

$$\left(\frac{\partial u}{\partial x}\right)^{-1} L(u) \left(\frac{\partial u}{\partial x}\right) = L_{\mathsf{comp1}}(x), \tag{3.1}$$

where $\left(\frac{\partial u}{\partial x}\right)$ denotes the Jacobi matrix of the transformation $u = u(x)$:

$$\left(\frac{\partial u}{\partial x}\right) = \begin{pmatrix} u^1_{x^1} & u^1_{x^2} & \cdots & u^1_{x^n} \\ u^2_{x^1} & u^2_{x^2} & \cdots & u^2_{x^n} \\ \vdots & \vdots & \ddots & \vdots \\ u^n_{x^1} & u^n_{x^2} & \cdots & u^n_{x^n} \end{pmatrix}.$$

Here and throughout the paper, when doing matrix computation, we consider u and x as column-vectors; also, we use u_{x^i} or $u^j_{x^i}$ for partial derivatives.

Rewriting (3.1) as

$$\left(\frac{\partial u}{\partial x}\right) L_{\mathsf{comp1}} = L\left(\frac{\partial u}{\partial x}\right), \tag{3.2}$$

we see that the columns u_{x^i} of $\left(\frac{\partial u}{\partial x}\right)$ satisfy the equations $Lu_{x^i} = u_{x^{i-1}}$, or equivalently,

$$u_{x^{n-k}} = L^k u_{x^n}, \quad \text{where } L^k = \underbrace{L \cdot L \cdots L}_{k \text{ times}}, \ k = 1, \dots, n-1. \tag{3.3}$$

Lemma 3.1. *Systems* (3.2) *and* (3.3) *are equivalent. In particular,* (3.1) *is equivalent to* (3.3) *provided the Jacobi matrix* $(\partial u / \partial x)$ *is invertible.*

Proof. By construction, (3.3) simply means that all the columns of the matrices in the left and right-hand sides of (3.2) coincide except for the first column. In other words, system (3.2), as compared to (3.3), contains one additional vector relation for the first columns of the left-hand and the right-hand side of (3.3), namely

$$f_1 u_{x^1} + f_2 u_{x^2} + \cdots + f_n u_{x^n} = L u_{x^1} \tag{3.4}$$

We need to show that this relation follows from (3.3). This is an easy corollary of the Cayley–Hamilton theorem. Indeed, substituting $u_{x^{n-k}} = L^k u_{x^n}$ into (3.4) gives

$$f_1 L^{n-1} u_{x^n} + f_2 L^{n-2} u_{x^n} + \cdots + f_n u_{x^n} = L^n u_{x^n},$$

or equivalently,

$$(L^n - f_1 L^{n-1} - f_2 L^{n-2} - \cdots - f_n \,\mathrm{Id})\, u_{x^n} = \chi_L(L) u_{x^n} = 0,$$

which holds true automatically by the Cayley–Hamilton theorem. ∎

Similarly, to bring L to the second companion form, we need to find an invertible transformation $u = u(x)$ such that

$$\left(\frac{\partial u}{\partial x}\right)^{-1} L(u) \left(\frac{\partial u}{\partial x}\right) = L_{\mathsf{comp2}}(x), \tag{3.5}$$

where L_{comp2} is given by (1.4). Proceeding in a similar way as above, we get $L\left(\frac{\partial u}{\partial x}\right) = \left(\frac{\partial u}{\partial x}\right) L_{\mathsf{comp2}}$. This gives the following relation on the columns of the Jacobi matrix: $Lu_{x^i} = u_{x^{i-1}} - f_{n-i} u_{x^n}$. For $i = n$, we get $Lu_{x^n} = u_{x^{n-1}} + f_1 u_{x^{n-1}}$, which yields

$$u_{x^{n-1}} = M_1 u_{x^n} \quad \text{with } M_1 = L - f_1 \cdot \mathrm{Id}\,.$$

Next, for $i = n-1$, we get $Lu_{x^{n-1}} = u_{x^{n-2}} + f_2 u_{x^n}$, yielding

$$u_{x^{n-2}} = M_2 u_{x^n}, \quad \text{with } M_2 = LM_1 - f_2 \cdot \mathrm{Id},$$

and so on. Finally, we come to the following system of PDEs:

$$u_{x^{n-k}} = M_k u_{x^n}, \quad \text{where } \begin{cases} M_1 = L - f_1 \cdot \mathrm{Id}, \\ M_k = LM_{k-1} - f_k \cdot \mathrm{Id}, \quad 2 \le k \le n, \end{cases} \tag{3.6}$$

and where $f_1, \dots, f_n$ are the coefficients of the characteristic polynomial of L. Equivalently,

$$M_k = L^k - f_1 L^{k-1} - f_2 L^{k-2} - \dots - f_{k-1} L - f_k \cdot \mathrm{Id}, \quad k = 1, \dots, n-1. \tag{3.7}$$

This system is equivalent to (3.5), cf. Lemma 3.1.

Thus, we see that reducing L to both the first and second companion forms amounts to solving a quasilinear system of PDEs of the form

$$u_{x^{n-k}} = A_k(u) u_{x^n}, \quad 1 \le k \le n-1, \tag{3.8}$$

where for the first companion form we set $A_k = L^k$, while for the second companion form, $A_k = M_k$ with M_k given by (3.6) or (3.7).

Notice that (3.8) is overdetermined and, in general, not necessarily consistent. However, the conditions under which local solutions exist for all initial data (in other words, the system is in involution) are well known.

Proposition 3.2. *The following properties of* (3.8) *are equivalent.*

(A) *For any real analytic initial condition*

$$\begin{matrix} u^1(0, \dots, 0, x^n) = h^1(x^n), \\ u^2(0, \dots, 0, x^n) = h^2(x^n), \\ \dots \\ u^n(0, \dots, 0, x^n) = h^n(x^n), \end{matrix} \qquad \textit{or shortly,} \quad u(0, \dots, 0, x^n) = h(x^n),$$

where h is a real analytic vector-function of one variable, there exists a unique real analytic solution $u = u(x)$ of system (3.8).

(B) *The operators A_k pairwise commute (i.e., $A_k A_j = A_j A_k$), and*

$$\langle A_k, A_j \rangle(\xi, \xi) \stackrel{\text{def}}{=} [A_k \xi, A_j \xi] - A_j[A_k \xi, \xi] - A_k[\xi, A_j \xi] = 0 \tag{3.9}$$

for any vector field ξ and $k, j = 1, \dots, n-1$.

Proof. The existence of solutions of (3.8) for all initial conditions in a more general case is discussed in [9] (and, in fact, can be derived from the Cartan–Kähler theorem [14]). The necessary and sufficient condition is $D_{x^{n-i}}(A_j u_{x^n}) = D_{x^{n-j}}(A_i u_{x^n})$ on $U(\mathsf{p})$, where D_{x^k} stands for the derivative in virtue of (3.8). For quasilinear systems, this calculation is well known (see [23], [17]) and leads to (B). ∎

We now apply this proposition in our special case.

Proposition 3.3. *Both systems* (3.3) *and* (3.6) *satisfy Property* (B)*, and therefore, Property* (A) *from Proposition* 3.2*.*

Proof. Although the verification of (B) for operators $A_k = L^k$ and $A_k = M_k$ is a nice exercise in tensor calculus, we prefer to make use of an elegant theory of bidifferential ideals introduced by F. Magri in [18], and then developed by F. Magri and P. Lorenzoni in [17], in particular, to construct hierarchies of commuting flows of hydrodynamic type. They are defined recursively by setting (cf. (3.6))

$$A_0 = \mathrm{Id}, \quad A_k = A_{k-1}L - a_k\,\mathrm{Id}, \quad k = 1, 2, \dots$$

for any chain of functions $a_1, a_2, \dots$ satisfying the relations

$$\mathrm{d}a_{k+1} = L^* \,\mathrm{d}a_k - a_k\,\mathrm{d}a_1. \tag{3.10}$$

Under these conditions, the operators A_k generate commuting flows (see Proposition 2 in [17]), i.e., satisfy (B).

Our situation is just a particular case of this construction. Indeed, setting $a_k = 0$, we obtain the sequence of operators $A_k = L^k$. Hence Property (B) holds for (3.3). Of course, this fact is easy to check independently.

In the case of system (3.6), we only need to check that the coefficients f_k of the characteristic polynomial of L satisfy (3.10) (we may formally set $f_k = 0$ for $k > n$), but these are exactly relations from Proposition 2.2 in [8]. Hence Property (B) holds for (3.6). It is worth noticing that (3.6) can also be understood as an ε-system in the sense of M. Pavlov [22] for $\varepsilon = -1$. ∎

We have just shown that the PDE systems (3.3) and (3.6) are both in involution and their (local) solutions $u(x)$ are parametrised by n functions of one variable (initial conditions $h^1(x^n), \dots, h^n(x^n)$). To make sure that such a solution $u(x)$ defines a desired coordinate transformation, we need to check that the Jacobi matrix $\left(\frac{\partial u}{\partial x}\right)$ is non-degenerate at least at the initial point. Almost all solutions satisfy this property due to gl-regularity of L (moreover this condition is necessary).

Indeed, for system (3.3), choose the initial condition $u(0, \dots, 0, x^n) = h(x^n)$ in such a way that the vector $\xi = u_{x^n}(0) = h_{x^n}(0)$ is such that $L^{n-1}\xi, \dots, L\xi, \xi$ are linearly independent. Since L is gl-regular, almost all vectors ξ satisfy this condition. Due to (3.3), they form the columns of the Jacobi matrix $\left(\frac{\partial u}{\partial x}\right)$ at the initial point $x = (0, \dots, 0, 0)$. Hence, at this point, $\det\left(\frac{\partial u}{\partial x}\right) \neq 0$ as required.

The same conclusion for solutions of system (3.6) immediately follows from the fact that $\mathrm{Span}(M_{n-1}\xi, \dots, M_1\xi, \xi) = \mathrm{Span}(L^{n-1}\xi, \dots, L\xi, \xi)$. This completes the proof of Theorem 1.1.

We see from this proof that reducibility of L to companion forms (1.2) and (1.4) follows from the involutivity (Property (B) from Proposition 3.2) of the PDE systems (3.3) and (3.6), respectively. This property, in turn, follows from the fact that L is Nijenhuis. It is natural to ask if the latter condition is also necessary for (3.3) and (3.6) to be in involution. The answer is positive under the additional assumption that L is gl-regular.

Proposition 3.4. *Let $n = \dim \mathsf{M} > 2$ and let L be* gl*-regular.*

(1) *If $\langle L^i, L^j \rangle = 0$ for $1 \le i < j \le n-1$, i.e., if (3.3) is in involution, then L is a Nijenhuis operator.*

(2) *If $\langle M_i, M_j \rangle = 0$ for $1 \le i < j \le n-1$, where M_i is defined as in (3.6), i.e., if (3.6) is in involution, then L is a Nijenhuis operator.*

The proof of this proposition is rather technical, and can be seen in Appendix A.

Remark 3.5. The gl-regularity assumption in Proposition 3.4 is essential. Indeed, consider an operator L such that $L^2 = \mathrm{Id}$ or $L^2 = 0$. Then the involutivity conditions $\langle L^i, L^j \rangle = 0$ and $\langle M_i, M_j \rangle = 0$ obviously hold. However, L does not need to be Nijenhuis.

4. Proof of Theorems 1.6 and 1.10

The goal of this section is to solve the PDE system:

$$\begin{aligned} \frac{\partial f_i}{\partial x^j} &= f_i \frac{\partial f_1}{\partial x^{j+1}} + \frac{\partial f_{i+1}}{\partial x^{j+1}}, \\ \frac{\partial f_n}{\partial x^j} &= f_n \frac{\partial f_1}{\partial x^{j+1}}, \end{aligned} \tag{4.1}$$

$1 \le i, j \le n-1$. According to Theorem 1.1, every collection of functions f_i satisfying this system defines a gl-regular Nijenhuis operator of the form

$$L(x) = L_{\mathsf{comp1}}(x) = \begin{pmatrix} f_1 & 1 & 0 & \cdots & 0 \\ f_2 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ f_{n-1} & 0 & \cdots & 0 & 1 \\ f_n & 0 & \cdots & 0 & 0 \end{pmatrix}, \tag{4.2}$$

and vice versa, if this operator is Nijenhuis, then these functions satisfy (4.1).

The proof of Theorem 1.6 presented below is based on an analysis of the evolution of the eigenvalues of $L(x)$, and differs from that in the first version [5] of this paper, available on arXiv. The alternative proof from [5] is perhaps a little more complicated in terms of the formulas involved, but has a certain advantage of not using “nasty” functions like eigenvalues which behave badly at singular points.

Consider the eigenvalues of the Nijenhuis operator (4.2). At singular points where some of them collide, the eigenvalues are not necessarily smooth in local coordinates (see an example just before Theorem 1.1). However, at those points $x = (x^1, \dots, x^n)$ where their multiplicities are locally constant, they are smooth (perhaps complex-valued) functions of x. In the case of gl-regular operators, such points are exactly those which were called *algebraically generic* in Definition 2.7 of [8]. They form an open dense subset. It appears that in companion coordinates $(x^1, \dots, x^n)$, the eigenvalues of Nijenhuis operators satisfy a rather simple system of PDEs.

Lemma 4.1. *Assume that $x_0 = (x_0^1, \dots, x_0^n)$ is algebraically generic. Then* (4.1) *implies that every eigenvalue λ of L, in a neighbourhood of this point, satisfies the following system of PDEs*:

$$\frac{\partial \lambda}{\partial x^{n-k}} = \lambda^k \frac{\partial \lambda}{\partial x^n}, \quad k = 1, \dots, n-1. \tag{4.3}$$

Conversely, if we have n functions $\lambda_1, \dots, \lambda_n$ each of which satisfies (4.3)*, then the coefficients f_i of the polynomial*

$$\prod_{i=1}^{n} (\lambda - \lambda_i) = \lambda^n - \sum_{i=1}^{n} f_i \, \lambda^{n-i}$$

satisfy (4.1).

Remark 4.2. Equation (4.3) makes sense both for real and complex eigenvalues. In the latter case, $\lambda(x) = u(x) + i v(x)$ should be understood as complex-valued smooth function in n real variables $x^1, \dots, x^n$.

Proof. Let $L_{\mathsf{comp1}}(F)$ be as in (4.2). Notice that our (quasilinear) PDE system (4.1) can be written in the following matrix form:

$$F_{x^i} = L_{\mathsf{comp1}}(F) \, F_{x^{i+1}}, \quad F = (f_1, \dots, f_n)^{\mathsf{T}}. \tag{4.4}$$

Let $\Lambda = (\lambda_1, \dots, \lambda_n)^{\mathsf{T}}$ be the roots of the polynomial $\lambda^n - f_1 \lambda^{n-1} - \dots - f_n$ so that we have standard polynomial expressions for f_i in terms of Λ. Then (4.4) can be rewritten as

$$\Big(\frac{\partial F}{\partial \Lambda}\Big) \, \Lambda_{x^i} = L_{\mathsf{comp1}}(F) \Big(\frac{\partial F}{\partial \Lambda}\Big) \, \Lambda_{x^{i+1}}, \tag{4.5}$$

where $\big(\frac{\partial F}{\partial \Lambda}\big)$ denotes the Jacobi matrix. We now use the following algebraic identity:

$$\Big(\frac{\partial F}{\partial \Lambda}\Big) \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} = L_{\mathsf{comp1}}(F) \Big(\frac{\partial F}{\partial \Lambda}\Big). \tag{4.6}$$

If λ_i's are pairwise distinct, then $\big(\frac{\partial F}{\partial \Lambda}\big)$ is invertible and (4.5) is equivalent to

$$\Lambda_{x^i} = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \Lambda_{x^{i+1}}, \tag{4.7}$$

which coincides with (4.3), as required.

However, if some of λ_i's coincide, this method does not work directly. In this case, we consider (4.5) as a system of linear equations on Λ_{x^i} and notice that (4.7) is a particular solution of this system in view of (4.6). Of course, there are many other solutions. However, we are only interested in those that satisfy the property $\partial \lambda_s / \partial x^i = \partial \lambda_t / \partial x^i$ whenever $\lambda_s = \lambda_t$. It is an easy exercise in linear algebra to show that such a solution is unique. ∎

The next lemma describes both the real and complex solutions of (4.3).

Lemma 4.3. (1) *Let $\mu(t)$ be an arbitrary real analytic function. Then the solution $\lambda(x)$ of* (4.3) *with initial condition $\lambda(0,0,\dots,0,x^n) = \mu(x^n)$ can be found by resolving the following algebraic relation*:

$$\lambda = \mu\,(x^1\lambda^{n-1} + x^2\lambda^{n-2} + \cdots + x^{n-1}\lambda + x^n). \tag{4.8}$$

Every local real analytic solution of (4.3) *can be obtained in this way.* (*Here we can replace "real analytic" by "smooth".*)

(2) *Let $\mu(z)$ be an arbitrary complex analytic function. Then the complex-valued solution $\lambda(x) = u(x) + i\,v(x)$ of* (4.3) *with the initial condition*

$$\lambda(0,0,\dots,0,x^n) = u(0,0,\dots,0,x^n) + i\,v(0,0,\dots,0,x^n) = \mu(x^n)$$

can be found by resolving the following (*complex*) *algebraic relation*:

$$\lambda = \mu\,(x^1\lambda^{n-1} + x^2\lambda^{n-2} + \cdots + x^{n-1}\lambda + x^n). \tag{4.9}$$

Every local complex analytic solution $\lambda(x) = u(x) + i\,v(x)$ of (4.3) *can be obtained in this way.*

Proof. We first check that the implicit solution $\lambda(x)$ of relation (4.8) satisfies the PDE system (4.3) (for the complex relation (4.9), which is "formally" the same as (4.8), the proof is similar). Indeed, differentiating (4.8) with respect to x^i, we get

$$\lambda_{x^i} = \mu'\cdot(c\,\lambda_{x^i} + \lambda^{n-i}), \quad \text{where } c = \sum_{\alpha=1}^{n-1} x^\alpha\,(n-\alpha)\,\lambda^{n-\alpha-1}.$$

Hence,

$$\lambda_{x^i}\,(1-\mu'\,c) = \mu'\,\lambda^{n-i}, \tag{4.10}$$

and similarly (replacing i with $i+1$),

$$\lambda_{x^{i+1}}\,(1-\mu'\,c) = \mu'\,\lambda^{n-i-1}. \tag{4.11}$$

Now multiplying (4.11) by λ and subtracting from (4.10) we get that $(\lambda_{x^i} - \lambda\,\lambda_{x^{i+1}})$ $(1-\mu'\,c) = 0$. It remains to notice that $c = 0$ on the initial line $x^1 = x^2 = \cdots = x^{n-1} = 0$ and (4.3) follows.

The fulfilment of the initial condition is straightforward. The uniqueness of the solution with given initial condition follows from the Cauchy–Kovalevskaya theorem. Moreover, in the complex case, for any real analytic initial condition $u(0,0,\dots,0,x^n) + i\,v(0,0,\dots,0,x^n) = u_0(x^n) + i\,v_0(x^n)$, there exists a unique complex analytic function $\mu(z)$, $z = x + iy$ such that on the real line we have $\mu(x) = u_0(x) + i\,v_0(x)$. ∎

To prove Theorem 1.6, we need to show that the solution $f(x) = (f_1(x),\dots,f_n(x))$ of (4.1) with prescribed initial conditions (1.7) can be obtained by resolving the relation

$$L^n - v_1(M)L^{n-1} - v_2(M)L^{n-2} - \cdots - v_{n-1}(M)L - v_n(M) = 0, \tag{4.12}$$

where $M = \sum_{i=1}^n x^i L^{n-i}$, with respect to the coefficients of the characteristic polynomial of L. We first notice that this relation is invariant in algebraic sense so that we may consider the matrices L and M in any basis we like. We will assume that L is written in companion form (4.2), with $f_1, \dots, f_n$ being the entries of the first column of $L = L_{\mathsf{comp1}}$.

The matrix $\sum v_i(M) L^{n-i}$ commutes with L and its entries are analytic functions in x and f. This matrix can be uniquely presented as linear combination

$$\sum v_i(M) L^{n-i} = g_1 L^{n-1} + \dots + g_{n-1} L + g_n \,\mathrm{Id},$$

where $g_i = g_i(x, f)$ are nothing else but the entries of the last column of $\sum v_i(M) L^{n-i}$ (this easily follows from the fact that L is a companion matrix). Thus, relation (4.12) reads $L^n = \sum_{i=1}^n g_i L^{n-i}$. Comparing with $L^n = \sum_{i=1}^n f_i L^{n-i}$ (Cayley–Hamilton theorem) and using gl-regularity of L, we come to the system of n algebraic relations

$$f_i = g_i(x, f).$$

To make sure that these relations can be resolved with respect to f and find $f_i = f_i(x)$ as a real analytic function of x (for small x), it is sufficient to check that $\frac{\partial g_i}{\partial f_\alpha}(0, \dots, 0, x^n, f) = 0$, which is obviously true as $g_i(0, \dots, 0, x^n, f)$ coincides with $v_i(x^n)$ and therefore does not depend on f_α. This proves the first statement of Theorem 1.6, and also shows that the initial conditions are indeed fulfilled: if $x^1 = \dots = x^{n-1} = 0$, then $f_i(0, \dots, 0, x^n) = g_i(0, \dots, 0, x^n, f) = v_i(x^n)$, as required.

It remains to show that the coefficients $f_1, \dots, f_n$ of the characteristic polynomial of L satisfying (4.12) solve the PDE system (4.1). This easily follows from Lemmas 4.1 and 4.3. Indeed, these lemmas provide implicit formulas for the eigenvalues of L_{comp1}, and we only need to show that these formulas are equivalent to (4.12).

Lemma 4.4. *An algebraically generic operator L satisfies (4.12) if and only if the eigenvalues $\lambda_1, \dots, \lambda_n$ of L satisfy*

$$\lambda_i = \mu_i(x^1 \lambda_i^{n-1} + x^2 \lambda_i^{n-2} + \dots + x^{n-1} \lambda_i + x^n), \quad i = 1, \dots, n, \tag{4.13}$$

where the functions $\mu_i(t)$ are the roots of the equation

$$\lambda^n - v_1(t) \lambda^{n-1} - v_2(t) \lambda^{n-2} - \dots - v_{n-1}(t) \lambda - v_n(t) = 0. \tag{4.14}$$

Proof. Assume that a gl-regular operator $L(x)$ satisfies relation (4.12). Then on the initial line $x(t) = (0, \dots, 0, t)$ we have

$$L^n - v_1(t) L^{n-1} - \dots - v_{n-1}(t) L - v_n(t) \,\mathrm{Id} = 0.$$

Consider the polynomial equation (4.14) with coefficients depending on $t \in U(0) \subset \mathbb{R}$. The multiplicities of its roots (perhaps complex) depend on t, but since the functions $v_i(t)$ are real analytic, these multiplicities are constant everywhere except for a discrete subset $\mathsf{Sing} \subset U(0)$. If necessary, we can choose a smaller neighbourhood $V(0) \subset U(0)$ such that all points in $V(0)$, except perhaps for 0, are non-singular. This implies that in the punctured neighbourhood $V(0) \setminus \{0\}$, the roots of (4.14) are defined by real analytic functions.

More precisely, on $V(0) \setminus \{0\}$ there exist s analytic functions (perhaps complex-valued) $\mu_1(t), \dots, \mu_s(t)$ such that

$$\lambda^n - v_1(t)\lambda^{n-1} - v_2(t)\lambda^{n-2} - \dots - v_{n-1}(t)\lambda - v_n(t) = \prod_{i=1}^{s}(\lambda - \mu_i(t))^{k_i},$$

where $\mu_i(t) \neq \mu_j(t)$ for $t \in V(0) \setminus \{0\}$. Notice that $\mu_i(t)$ are exactly the eigenvalues of L on the initial line $x(t) = (0, \dots, 0, t)$. (In general, there is no natural relation between the μ_i's for $t > 0$ and $t < 0$, and we treat these two disconnected intervals independently.) The matrix relation (4.12) can therefore be rewritten in the form $\prod_{i=1}^{s}(L - \mu_i(M))^{k_i} = 0$ for any point $(x^1, \dots, x^n)$ sufficiently close to $(0, \dots, 0, t)$. This, in turn, implies that each eigenvalue $\lambda_\alpha = \lambda_\alpha(x)$ of $L(x)$ satisfies the relation

$$\prod_{i=1}^{s}\big(\lambda_\alpha - \mu_i\,(x^1\lambda_\alpha^{n-1} + \dots + x^{n-1}\lambda_\alpha + x^n)\big)^{k_i} = 0.$$

Taking into account the fact that the eigenvalues of $L(x)$ depend on x continuously and for $x = (0, \dots, 0, t)$ they are $\mu_i(t)$ with multiplicities k_i, we conclude that at every point $x = (x^1, \dots, x^n)$ sufficiently close to $(0, \dots, 0, t)$, the operator $L(x)$ has s eigenvalues $\lambda_i(x)$ with multiplicities k_i and, moreover, these eigenvalues satisfy (4.13), as required.

The proof of the converse statement (which is not important for our purposes) is similar. ∎

We are now ready to complete the proof of Theorem 1.6. Let L satisfy (4.12). Then, by Lemma 4.4, its eigenvalues satisfy (4.13) and therefore (by Lemma 4.3) are solutions of the PDE system (4.3). By Lemma 4.1, the coefficients $f_1, \dots, f_n$ of the characteristic polynomial of $L(x)$ satisfy (4.1) as required. Strictly speaking, this proof works in a small neighbourhood of the set $\{(0, \dots, 0, t),\ t \in V(0) \setminus \{0\}\}$. However, the final conclusion still holds in a neighbourhood of the origin $(0, \dots, 0, 0)$ due to the analyticity of f. This completes the proof of Theorem 1.6.

Our next goal is to discuss Nijenhuis perturbations of a Jordan block J_0, that is, Nijenhuis operators of the form $L(x) = J_0 +$ *higher order terms*. Recall that a generic Nijenhuis perturbation of J_0 is described by the following result.

Proposition 4.5 ([8], see also Remark 1.2). *Let L be a Nijenhuis operator such that at a point* p, *the operator $L(\mathsf{p})$ is similar to the (nilpotent) Jordan block J_0. Assume that the differentials of the coefficients of the characteristic polynomial of L are linearly independent at* p. *Then in a neighbourhood of* p, *there exist local coordinates $x^1, \dots, x^n$ with* $\mathsf{p} \simeq (0, \dots, 0)$ *in which $L(x)$ is given by* (1.6).

It is easily seen that, under the assumptions of Proposition 4.5, for a generic point $\mathsf{q} \in U(\mathsf{p})$, the operator $L(\mathsf{q})$ is semisimple with distinct eigenvalues. Moreover, for any collection of real and complex conjugate numbers $S = \{\lambda_1, \dots, \lambda_k, \mu_1, \bar\mu_1, \dots, \mu_s, \bar\mu_s\}$ $(k + 2s = n)$ sufficiently close to zero and not necessarily distinct, there exists a unique point $\mathsf{q} \in U(\mathsf{p})$ such that S is the spectrum of $L(\mathsf{q})$. This follows immediately from the fact that the local coordinates $x^1, \dots, x^n$ in $U(\mathsf{p})$ are the coefficients of the characteristic polynomial of L, which can be reconstructed from S by Vieta's formulas. In particular, we see that in $U(\mathsf{p})$ we can find operators of all possible algebraic types that are potentially allowed for gl-regular operators (this means that for repeated eigenvalues there will be only one Jordan block).

It is natural to ask about other scenarios for Nijenhuis perturbations, for instance, with a prescribed algebraic structure of L at a generic point q. Let us show that all scenarios (in the sense of Theorem 1.10) are possible. Notice that the proof of Lemma 4.4 implies the following.

Corollary 4.6. *Let $L_{\mathsf{comp1}}(x)$ be a Nijenhuis operator written in the first companion form. Let $x = (0, \dots, 0, t_0)$ be a point on the initial line such that the multiplicities of the eigenvalues of $L_{\mathsf{comp1}}(0, \dots, 0, t)$ are constant for t sufficiently close to t_0. Then the multiplicities of the eigenvalues of $L_{\mathsf{comp1}}(x^1, \dots, x^{n-1}, x^n)$ are constant for $(x^1, \dots, x^{n-1}, x^n)$ sufficiently closed to $(0, \dots, 0, t)$. In other words, the point $(0, \dots, 0, t_0)$ is algebraically generic.*

Now to prove Theorem 1.10, it is enough to choose the initial conditions $f_i(0, \dots, 0, t) = v_i(t)$ in such a way that $v_i(0) = 0$ and the polynomial

$$\chi_{L(0,\dots,0,t)}(\lambda) = \lambda^n - v_1(t)\lambda^{n-1} - \dots - v_{n-1}(t)\lambda - v_n(t)$$

for all $t \neq 0$ has s distinct roots with multiplicities $k_1, \dots, k_s$. In terms of Theorem 1.10, this means that $L(0, \dots, 0, t) \in W_{k_1,\dots,k_s}$. Corollary 4.6 implies that the same holds true for all points $x = (x^1, \dots, x^n)$ sufficiently closed to $(0, \dots, 0, t)$, $t \neq 0$, i.e., $L(x) \in W_{k_1,\dots,k_s}$. Since this inclusion takes place on an open non-empty subset, then due to the analyticity of $L(x)$ we conclude that $L(x) \in \overline{W}_{k_1,\dots,k_s}$ for all x, which completes the proof of Theorem 1.10.

Let us finally discuss an example showing how Theorem 1.6 works in practice to construct explicit examples of Nijenhuis operators with non-trivial singularities.

Example 4.7. For $n = 3$, in the settings of Theorem 1.6, define the initial conditions in such a way that on the initial line $x(t) = (0, 0, t)$ the characteristic polynomial of L takes the form

$$\chi_{L(x(t))}(\lambda) = (\lambda - t)^2(\lambda - 2t) = \lambda^3 - 4t\lambda^2 + 5t^2\lambda - 2t^3,$$

or, equivalently,

$$f_1(0, 0, t) = 4t = v_1(t), \quad f_2(0, 0, t) = -5t^2 = v_2(t), \quad f_3(0, 0, t) = 2t^3 = v_3(t).$$

The algorithm described in Theorem 1.6 allows us to reconstruct the functions f_1, f_2 and f_3. To that end, we need to use the matrix relation

$$L^3 - (4M)L^2 + (5M^2)L - 2M^3 = 0, \quad \text{with } M = x^1L^2 + x^2L + x^3\,\mathrm{Id},$$

to express the coefficients of the characteristic polynomial of L in terms of x^1, x^2 and x^3.

Notice that this relation can be rewritten as $(L - M)^2(L - 2M) = 0$. This factorisation immediately allows us to find the eigenvalues of L by taking the roots of the polynomial

$$(\lambda - x^1\lambda^2 - x^2\lambda - x^3)^2(\lambda - 2x^1\lambda^2 - 2x^2\lambda - 2x^3) = 0.$$

Recall that we are interested in those of them which, on the initial line, coincide with the above prescribed roots, that is,

$$\lambda_1(0,0,x^3) = \lambda_2(0,0,x^3) = x^3 \quad \text{and} \quad \lambda_3(0,0,x^3) = 2x^3.$$

Thus, we just need to choose the suitable root (one of the two) of the corresponding quadratic equation. Namely,

$$\lambda - x^1\lambda^2 - x^2\lambda - x^3 = 0 \quad \Rightarrow \quad \lambda = \frac{2x^3}{(1-x^2) + \sqrt{(1-x^2)^2 - 4x^1x^3}}, \tag{4.15}$$

$$\lambda - 2x^1\lambda^2 - 2x^2\lambda - 2x^3 = 0 \quad \Rightarrow \quad \lambda = \frac{4x^3}{(1-2x^2) + \sqrt{(1-2x^2)^2 - 16x^1x^3}}. \tag{4.16}$$

The root of the first equation is an eigenvalue of L of multiplicity 2, whereas the root of the second equation is an eigenvalue of multiplicity one. As a result we have found explicit expressions for the eigenvalues of the Nijenhuis operator L in coordinates x^1, x^2, x^3, and therefore

$$L_{\mathsf{comp1}} = \begin{pmatrix} f_1(x) & 1 & 0 \\ f_2(x) & 0 & 1 \\ f_3(x) & 0 & 0 \end{pmatrix}, \quad \text{with} \quad \begin{cases} f_1 &= \lambda_1 + \lambda_2 + \lambda_3, \\ f_2 &= -\lambda_1\lambda_2 - \lambda_2\lambda_3 - \lambda_3\lambda_1, \\ f_3 &= \lambda_1\lambda_2\lambda_3, \end{cases}$$

where $\lambda_1 = \lambda_2 = \lambda$ from (4.15) and $\lambda_3 = \lambda$ from (4.16). This is an example of a Nijenhuis perturbation of the nilpotent 3×3 Jordan block J_0 under which J_0 splits into two Jordan blocks of size 2 and 1 with non-constant eigenvalues.

5. Local classification of gl-regular Nijenhuis operators in dimension two and global applications

The goal of this section is to describe local normal forms for gl-regular Nijenhuis operators at singular points in dimension 2. However, for the sake of completeness we first recall the list of (algebraically) generic types of such operators along with their *local* canonical forms:

- Two distinct real eigenvalues: $L = \begin{pmatrix} f(x) & 0 \\ 0 & g(y) \end{pmatrix}$, where $f(x)$ and $g(y)$ are smooth functions such that $f(x) \neq g(y)$ for all (x, y). In the real analytic case, $f(x)$ is either constant or can be reduced, by an appropriate local change of coordinates, to $f(x) = f_0 \pm x^{2m}$ or $f(x) = f_0 + x^{2m-1}$, $m \in \mathbb{N}$, and similarly for $g(y)$.
- Two complex conjugate eigenvalues: $L = \begin{pmatrix} f(x,y) & -g(x,y) \\ g(x,y) & f(x,y) \end{pmatrix}$, where $h = f + \mathrm{i}\, g$ is a holomorphic function of the complex variable $z = x + \mathrm{i}\, y$, $g(x, y) \neq 0$ for all (x, y). This function $h(z)$ is either constant or can be reduced, by an appropriate local change of coordinates, to $h(z) = h_0 + z^m$, $m \in \mathbb{N}$.
- Jordan block: $L = \begin{pmatrix} f(y) & 1 \\ 0 & f(y) \end{pmatrix}$, where $f(y)$ is a smooth function. As above, in the real analytic case, $f(y)$ is either constant or can be reduced, by an appropriate local change of coordinates, to $f(y) = f_0 \pm y^{2m}$ or $f(y) = f_0 + y^{2m-1}$, $m \in \mathbb{N}$.

This classification is easy and well known (see e.g. [8]). A non-trivial problem is to describe the local behaviour of L near a singular point p at which the algebraic type of L changes. In dimension 2, under the gl-regularity assumption, there is only one possibility for $L(\mathsf{p})$, namely, this operator (after an appropriate change of coordinates) is a Jordan block:

$$L(\mathsf{p}) = \lambda \,\mathrm{Id} + J_0, \quad \text{where } J_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ \lambda = \mathrm{const} \in \mathbb{R}.$$

Since $L - \lambda \,\mathrm{Id}$ is still a Nijenhuis operator, we will assume without loss of generality that $L(\mathsf{p}) = J_0$, and our problem reduces to the classification of Nijenhuis perturbations of the nilpotent Jordan block J_0. Below we will describe all possible normal forms for such perturbations, i.e., for Nijenhuis operators L such that $L(\mathsf{p}) = J_0$. To our great surprise, they are all polynomial. Before stating our classification result, we notice that there are two essentially different cases depending on the coefficients of the characteristic polynomial

$$\chi_L(\lambda) = \det(\lambda \cdot \mathrm{Id} - L) = \lambda^2 - v\lambda - u, \quad v = \mathrm{tr}\, L,\ u = -\det L.$$

In the real analytic case, there are two possibilities: either $\mathrm{d}v \wedge \mathrm{d}u \equiv 0$, or $\mathrm{d}v \wedge \mathrm{d}u \neq 0$ on an open everywhere dense subset. In the latter case, the operator L can be completely reconstructed from v and u and the following fundamental relation for Nijenhuis operators (see Corollary 2.2 in [8]):

$$L = \begin{pmatrix} v_x & v_y \\ u_x & u_y \end{pmatrix}^{-1} \begin{pmatrix} v & 1 \\ u & 0 \end{pmatrix} \begin{pmatrix} v_x & v_y \\ u_x & u_y \end{pmatrix}, \quad v = \mathrm{tr}\, L,\ u = -\det L. \tag{5.1}$$

At those points where the Jacobi matrix is not invertible, we define L by continuity. In other words, in the above formula we should automatically observe "cancellation of the denominator" $v_x u_y - v_y u_x$ involved in the formula of the inverse matrix. For this reason, in Theorem 5.1 below, when appropriate, instead of the matrix of L we will give formulas for $v(x, y)$ and $u(x, y)$, as they are much simpler and more intuitive. The reader may easily "reconstruct" L from (5.1) and, in particular, see the above mentioned cancellation.

If $\mathrm{d}v \wedge \mathrm{d}u \equiv 0$, then (5.1) makes no sense, but we may still use another, even more general relation (see Proposition 2.2 in [8]):

$$\begin{pmatrix} v_x & v_y \\ u_x & u_y \end{pmatrix} \begin{pmatrix} l^1_1 & l^1_2 \\ l^2_1 & l^2_2 \end{pmatrix} = \begin{pmatrix} v & 1 \\ u & 0 \end{pmatrix} \begin{pmatrix} v_x & v_y \\ u_x & u_y \end{pmatrix}, \quad v = \mathrm{tr}\, L,\ u = -\det L,\ L = \begin{pmatrix} l^1_1 & l^1_2 \\ l^2_1 & l^2_2 \end{pmatrix}. \tag{5.2}$$

We will assume that L is defined in a neighbourhood of the origin $\mathsf{p} = (0, 0) \in \mathbb{R}^2(x, y)$ and coordinate transformations always leave the origin fixed. The theorem below provides the complete list of normal forms for L which are divided into several series.

Theorem 5.1. *Let L be a Nijenhuis operator such that $L(\mathsf{p}) = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. Then in suitable local coordinates (x, y), this operator takes one of the following forms*:

(1) *Series L, M and N (for $k \geq 1, \varepsilon = \pm 1$)*:

$$L_{\mathrm{nil}} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad L_{\mathrm{nd}} = \begin{pmatrix} x & 1 \\ y & 0 \end{pmatrix}, \quad M_{2k-1} = \begin{pmatrix} 0 & 1 \\ 0 & y^{2k-1} \end{pmatrix},$$

$$M^{\varepsilon}_{2k} = \begin{pmatrix} 0 & 1 \\ 0 & \varepsilon y^{2k} \end{pmatrix}, \quad N_{2k-1} = \begin{pmatrix} y^{2k-1} & 1 \\ 0 & y^{2k-1} \end{pmatrix}, \quad N^{\varepsilon}_{2k} = \begin{pmatrix} \varepsilon y^{2k} & 1 \\ 0 & \varepsilon y^{2k} \end{pmatrix}.$$

(2) *Series* $O^{d,\varepsilon}_{k,\mathrm{c}}$, $k \geq 1$, $d \geq 2k+1$, $\varepsilon = \pm 1$, $\mathrm{c} = (c_0, \dots, c_{k-1}) \in \mathbb{R}^k$ *and we set* $\varepsilon = 1$, *if* $d = 2m+1$ *is odd.*
The operator L *is defined by* (5.1) *with* $v = \operatorname{tr} L$ *and* $u = -\det L$ *given by*

$$v = \alpha x y^{2k-1} + y^k (c_{k-1} y^{k-1} + \dots + c_1 y + c_0) \quad \text{and} \quad u = \varepsilon y^d,$$

where $\alpha = k c_0^2 (1 - k/d) \neq 0$.

(3) *Series* $P^{k,\varepsilon}_{s,\mathrm{c}}$, $k \geq 1$, $s \geq 2k$, $\varepsilon = \pm 1$, $\mathrm{c} = (c_0, \dots, c_{k-1}) \in \mathbb{R}^k$.
The operator L *is defined by* (5.1) *with* $v = \operatorname{tr} L$ *and* $u = -\det L$ *given by*

$$v = \alpha x y^s + y^{s-k+1} (c_{k-1} y^{k-1} + \dots + c_1 y + c_0) + 2\varepsilon\, y^k \quad \text{and} \quad u = -y^{2k},$$

where $\alpha = 2\varepsilon k c_0 \neq 0$.

(4) *Series* $S^{2k,\varepsilon}_{\mathrm{c}}$ *and* S^{2k+1}_{c}, $k \geq 1$, $\mathrm{c} = (c_0, \dots, c_{k-1}) \in \mathbb{R}^k$.
The operator L *is defined by* (5.1) *with* $v = \operatorname{tr} L$ *and* $u = -\det L$ *given respectively by*

$$v = \alpha x y^{2k-1} + y^k (c_{k-1} y^{k-1} + \dots + c_1 y + c_0) \quad \text{and} \quad u = \varepsilon y^{2k},$$

where $\alpha = \frac{k}{2}(c_0^2 + 4\varepsilon) \neq 0$, *and*

$$v = \alpha x y^{2k} + y^{k+1} (c_{k-1} y^{k-1} + \dots + c_1 y + c_0) \quad \text{and} \quad u = y^{2k+1},$$

where $\alpha = 2k + 1$.

Proof. The idea of the proof is natural: since L is basically defined by its trace and determinant, we will be looking for local coordinates x, y in which $v = \operatorname{tr} L$ and $u = -\det L$ have their "simplest" possible form. Our proof involves several coordinate transformations $(x, y) \mapsto (x_{\text{new}}, y_{\text{new}})$. In order to avoid complicated notations, each time after we do such a transformation, we return to the initial notation (x, y), the previous coordinate system is then understood as $(x_{\text{old}}, y_{\text{old}})$.

We start with two technical lemmas.

Lemma 5.2. *Under the assumptions of Theorem* 5.1, *there exist local coordinates* (x, y) *such that for* $u = -\det L$ *one of the following holds*:

$$\text{(i) } u \equiv 0, \quad \text{(ii) } u = \pm\, y^{2k}, \quad \text{(iii) } u = y^{2k-1}, \quad k \in \mathbb{N}.$$

Proof. In companion coordinates (see (1.3)), the function $u = -\det L$ satisfies the equation

$$u_x = g(x, y) u, \tag{5.3}$$

where $g(x, y) = \partial_y \operatorname{tr} L$. Hence $u = f(y) \exp(\int_0^x g(t, y)\, dt)$ for some real analytic function $f(y)$. If $f(y) \equiv 0$, we have Case (i). Otherwise, writing f in the form $f(y) = \varepsilon y^m h(y)$

with $\varepsilon = \pm 1$, $h(0) > 0$, $m \in \mathbb{N}$, we get for $m = 2k$ and $m = 2k-1$ respectively,

$$u = \pm\Big(y \sqrt[2k]{h(y)\exp\Big(\int_0^x g(t,y)\,dt\Big)}\Big)^{2k}, \quad \text{or}$$
$$u = \Big(y \sqrt[2k-1]{\pm h(y)\exp\Big(\int_0^x g(t,y)\,dt\Big)}\Big)^{2k-1}.$$

Letting y_{new} be the expression in brackets gives $u = \pm\, y_{\text{new}}^{2k}$ or $u = y_{\text{new}}^{2k-1}$, as required. ∎

This lemma brings $\det L$ to its simplest canonical form. After this we may keep the y-coordinate fixed and simplify $v = \operatorname{tr} L$ by changing the x-coordinate only.

The next statement applies to any gl-regular operator in dimension 2.

Lemma 5.3. *There exists a coordinate change of the form $(x_{\text{old}}, y) \mapsto (x, y)$ such that the l_2^1-component of L in new coordinates equals identically* 1.

Proof. Setting $x_{\text{old}} = b(x,y)$ and applying the standard transformation rule for components of an operator, we observe that the required condition is

$$\frac{l_2^1(b,y) + b_y\, l_1^1(b,y) - b_y\, l_2^2(b,y) - b_y^2\, l_1^2(b,y)}{b_x} = 1,$$

where l_j^i are the components of L in the old coordinate system. Writing this relation in the form $b_x = F(b_y, b, y)$, we can locally solve it by the Cauchy–Kovalevskaya theorem. Since L is gl-regular, we can choose initial conditions $b(0,y) = f(y)$ in such a way that $b_x(0,0) \neq 0$, so that the coordinate transformation is invertible. ∎

Now let us discuss all the cases one by one. First assume that $u \equiv 0$ and $v \equiv 0$. Then L is a nilpotent Jordan block and its companion form coincides with L_{nil}.

Next suppose $u \equiv 0$, while v is not. In companion coordinates (see (1.3)), v satisfies the Hopf equation $v v_y - v_x = 0$. This relation can be rewritten as

$$v_x = g(x,y)\,v, \quad \text{with } g = v_x, \tag{5.4}$$

which is similar to the above equation (5.3) for u. Just in the same way as in Lemma 5.2, we find a coordinate system in which $v = y^{2k-1}$ or $v = \varepsilon y^{2k}$ for $k \geq 1, \varepsilon = \pm 1$. By Lemma 5.3, we may also assume that $l_2^1 = 1$. Now $L = (l_j^i)$ can be reconstructed from relation (5.2). This yields series M_{2k-1} and M_{2k}^{ε} for different v respectively.

Now let $u \not\equiv 0$, but $\mathrm{d}v \wedge \mathrm{d}u \equiv 0$. Combining Lemmas 5.2 and 5.3, we may assume that $u = y^{2m-1}$ or $u = \pm\, y^{2m}$ for $m \geq 1$, and $l_2^1 = 1$. Since $\mathrm{d}v \wedge \mathrm{d}u \equiv 0$, we also know that $v_x \equiv 0$. Relation (5.2) implies that $l_1^2 = 0$ and we come to the operator of the form

$$L = \begin{pmatrix} f(y) & 1 \\ 0 & g(y) \end{pmatrix}, \quad \text{with } v = f + g \text{ and } u = -fg.$$

It is straightforward to check that the Nijenhuis condition in this case reads $f_y'(f-g) = 0$. In our case, f cannot be constant as in this case, since L is nilpotent at the origin, we would necessarily have $f \equiv 0$, which contradicts our assumption that $u = -fg \not\equiv 0$. Therefore, we conclude that $f - g = 0$, meaning that L is a Jordan block at each point. This yields series N_{2k-1} and N^{ε}_{2k}.

If $\mathrm{d}v \wedge \mathrm{d}u \neq 0$ at the point p, then L is differentially non-degenerate and its normal form is L_{nd} (see Theorem 4.4 in [8]). Notice that in terms of Lemma 5.2, the non-degeneracy condition corresponds exactly to the case $u = y$, and below we exclude this case.

Finally, we consider the most interesting case when $\mathrm{d}v \wedge \mathrm{d}u \not\equiv 0$ (but $\mathrm{d}v \wedge \mathrm{d}u = 0$ at p). As previously, we assume that $u = y^{2m+1}$ or $u = \varepsilon y^{2m}$ for $m \geq 1, \varepsilon = \pm 1$ and $l^1_2 = 1$. Computing l^1_2 from the matrix relation (5.1) yields the following equation on v:

$$v_x = v v_y - \frac{1}{d}\, y (v_y)^2 + u_y, \tag{5.5}$$

where $d = 2m + 1$ or $2m$. This equation implies the following.

Lemma 5.4. *The function $v(x, y)$ can be written as $v = v_0(y) + y^s(\alpha x + F)$, where $\alpha \neq 0$, $s \geq 1$ and $F(x, y)$ is a real analytic function with no constant or linear part.*

Proof. Let $v(x, y) = v_0(y) + v_1(y)x + v_2(y)x^2 + \cdots$ be a solution of (5.5). Differentiating (5.5) with respect to x, we get

$$v_{xx} = v_x v_y + v v_{xy} - \frac{2}{d}\, y v_y v_{xy}.$$

Note that $v = v_0(y)$ satisfies this equation for initial conditions $v(0, y) = v_0(y)$ and $v_x(0, y) = v_1(y) \equiv 0$. By the Cauchy–Kovalevskaya theorem, this solution is unique. Hence, if $v_1 \equiv 0$, then $\mathrm{d}v \wedge \mathrm{d}u \equiv 0$, which is wrong. Thus, in our case $v_1(y) = y^s r_1(y)$, where $r_1(0) = \alpha \neq 0$.

Assume that $s = 0$. This means that $\mathrm{d}v$ and $\mathrm{d}y$ are linearly independent. We can introduce $x_{\mathrm{new}} = v = \operatorname{tr} L$, leaving y the same. In these new coordinates, relation (5.1) gives

$$L = \begin{pmatrix} 1 & 0 \\ 0 & u_y^{-1} \end{pmatrix} \begin{pmatrix} x_{\mathrm{new}} & 1 \\ u(y) & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & u_y \end{pmatrix} = \begin{pmatrix} x_{\mathrm{new}} & u_y \\ u u_y^{-1} & 0 \end{pmatrix}.$$

It is easy to see that L at the origin p is similar to the nilpotent Jordan block only for $u = y$. But in this case we get $L = L_{\mathrm{nd}}$, falling into the previous case.

Thus, we have $s \geq 1$. Equating the coefficients of x^i in both sides of (5.5) yields

$$\begin{aligned}(i+1)\, v_{i+1}(y) &= v_0(y)\, v_i'(y) + v_0'(y)\, v_i(y) - \frac{2}{d}\, y\, v_0'(y)\, v_i'(y) \\ &\quad + \sum_{j=1}^{i-1} \Big(v_j(y)\, v_{i-j}'(y) - \frac{1}{d}\, y\, v_j'(y)\, v_{i-j}'(y) \Big).\end{aligned} \tag{5.6}$$

If $v_1, \ldots, v_i$ are divisible by y^s, then $v_1', \ldots, v_i'$ are divisible by y^{s-1}. As $v(0, 0) = 0$, then v_0 is divisible by y. By formula (5.6) the coefficient v_{i+1} is divisible by y^s. Thus, by induction all the coefficients $v_1, v_2, \ldots$ are divisible by y^s, and one writes $v = v_0 + y^s(x r_1(y) + \cdots) = v_0 + y^s(\alpha x + F)$, where F is analytic and has no constant or linear parts. The lemma is proved. ■

Using Lemma 5.4, we introduce new coordinates $x_{\mathrm{new}} = x + \frac{1}{\alpha} F + \tilde{v}_0$, $y_{\mathrm{new}} = y$, where $\tilde{v}_0$ contains all the terms of v_0 of order $\geq s+1$. In this new coordinate system (for which we continue using old notation x and y), we have

$$v = p_s(y) + \alpha x y^s, \quad u = y^{2m+1} \text{ or } u = \varepsilon y^{2m}, \tag{5.7}$$

where $\alpha \neq 0$, $m, s \geq 1$, and p_s is polynomial of degree at most s.

This coordinate system is optimal in the sense that $v = \operatorname{tr} L$ and $u = -\det L$ cannot be simplified further. The last step is to distinguish those pairs of functions $v(x, y)$ and $u(x, y)$ from the family (5.7) that indeed *generate* analytic perturbations of the nilpotent Jordan block J via relation (5.1). The point is that (5.1) will generate a Nijenhuis operator L for any v and u, but we need only those of them for which the entries of L so obtained are smooth and, moreover, $L(\mathsf{p})$ is similar to J_0.

A straightforward reconstruction of L, from (5.1) with v and u given by (5.7), shows that all the components of L are non-singular and vanish at the origin except for l^1_2:

$$L = \begin{pmatrix} v - \frac{y v_y}{d} & \frac{v v_y - \frac{1}{d} y v_y^2 + u'}{v_x} \\ \frac{y v_x}{d} & \frac{y v_y}{d} \end{pmatrix},$$

where $d = 2m+1$ or $d = 2m$ (power of y in the formula for u).

The "troublesome" component, in more detail, reads

$$l^1_2 = s\alpha x^2 y^{s-1} \left(1 - \frac{s}{d}\right) + x \left(\frac{1}{y} p_s + \left(1 - \frac{2}{d}\right) p_s'\right) + \frac{p_s p_s' - \frac{1}{d} y (p_s')^2 + u'}{\alpha y^s}.$$

Notice that $p_s(0) = 0$, and therefore $\frac{1}{y} p_s$ is analytic. Hence, we only need to analyse the fraction

$$\frac{p_s p_s' - \frac{1}{d} y (p_s')^2 + u'}{\alpha y^s}. \tag{5.8}$$

This fraction must define an analytic function having value 1 at the origin (in order for $L(\mathsf{p})$ to be the standard nilpotent Jordan block). Thus, we need to solve a purely algebraic problem: find all polynomials p_s, for which the numerator of (5.8) is divisible by αy^s so that this fraction is, in fact, a polynomial with free term equal to 1. We rewrite (5.8) as

$$p_s p_s' - \frac{1}{d} y (p_s')^2 = -u' + \alpha y^s + \alpha_1 y^{s+1} + \dots + \alpha_{s-1} y^{2s-1}, \tag{5.9}$$

where $\alpha \neq 0$ and α_i are, in general, arbitrary.

Let p_s starts with a term of order $k \geq 1$, that is,

$$p_s = y^k (c_0 + c_1 y + \dots + c_{s-k} y^{s-k}).$$

Then the smallest degree term in the left-hand side of (5.9) is $k c_0^2 (1 - k/d) y^{2k-1}$. On the other hand, the term of the smallest degree in the right-hand side is either $u' = \pm d y^{d-1}$ or αy^s (or both of them).

First, assume $s < d-1$. Then we get $2k-1 = s$ and furthermore $p_{2k-1} = c_0 y^k + \cdots + c_{k-1} y^{2k-1}$, where $c_1, \dots, c_{k-1}$ are arbitrary and $c_0 \neq 0$. We also have $\alpha = k c_0^2 (1 - k/d)$ obtaining, as a result, the series $O^{d,\varepsilon}_{k,\mathrm{c}}$.

Next, assume $d-1 < s$. Then we get $d-1 = 2k-1$ and, thus, $u = \varepsilon y^{2k}$. Equating the coefficients of y^{2k-1} on both sides of (5.9) we get $\frac{k}{2} c_0^2 = -2k\varepsilon$ and, thus, $u = -y^{2k}$ and $c_0 = \pm 2$. We write

$$p_s = \pm 2 y^k + c_1 y^{k+1} + \cdots + c_{s-k} y^s$$

and substitute it into (5.9). Equating the coefficients of $y^{2k}, \dots, y^{s-1}$ in the left-hand side of (5.9) to zero we get, step by step, that $c_1 = c_2 = \cdots = c_{s-2k} = 0$. Hence, re-denoting $c_{s-2k+j} \mapsto c_{j-1}$ for $j = 1, \dots, k$, we have

$$p_s = y^{s-k+1} (c_{k-1} y^{k-1} + \cdots + c_1 y + c_0) \pm 2 y^k,$$

and equating the coefficients of y^s in both sides of (5.9), we obtain $\alpha = \pm 2k c_0 \neq 0$. This yields series $P^{k,\varepsilon}_{s,c}$.

Finally, consider $d-1 = s$. We have two possibilities. First, assume that $d = 2m$, i.e., $u = \varepsilon y^{2m}$. We get that $2k-1 = 2m-1$, $k = m$ and

$$v = \alpha x\, y^{2m-1} + c_0\, y^m + \cdots + c_{m-1}\, y^{2m-1},$$

with $\alpha = \frac{m}{2}(c_0^2 + 4\varepsilon) \neq 0$. Now assume that $d = 2m+1$, i.e., $u = y^{2m+1}$. This yields

$$v = \alpha x y^{2m} + c_0 y^{m+1} + \cdots + c_{m-1} y^{2m} \quad \text{and} \quad \alpha = 2m+1.$$

This yields $S^{2m,\varepsilon}_{\mathrm{c}}$ and S^{2m+1}_{c} respectively (in the statement of the theorem we replace m by k). ■

Remark 5.5. For the series O, P and S, the canonical coordinate system is essentially unique (in some cases one can simultaneously change the sign of x and y). Indeed, these coordinates are those in which $u = -\det L$ and $v = \operatorname{tr} L$ are given by (5.7). The integer parameters m and s involved in (5.7) are uniquely defined for given u and v. Hence, y can be reconstructed from u (sometimes up to sign), and x is determined, up to a constant factor, by the condition that $v(0, y)$ is a polynomial of degree $\leq s$. Finally, the rescaling of x is chosen in such a way that at the origin we have $L(0,0) = J_0$.

This implies that Nijenhuis operators from different series (or from the same series but with different parameters) are not equivalent to each other. The only exception is related to the above mentioned “canonical” transformation $(x, y) \mapsto (-x, -y)$, that changes the parameter $\mathrm{c} \in \mathbb{R}^k$, but this change is easy to control.

We now apply the local classification of gl-regular Nijenhuis operators to study the existence (and examples) of such operators on closed two-dimensional surfaces.

Let (M^2, L) be a gl-regular Nijenhuis manifold of dimension 2 (recall that we always assume them to be real analytic). Consider the set Sing of singular points of L where the algebraic type of L changes. In our case, this means that the eigenvalues of L collide, i.e.,

$$\mathsf{Sing} = \{\mathsf{p} \in \mathsf{M}^2 \mid v^2 + 4u = 0\}, \quad \text{where } v = \operatorname{tr} L,\ u = -\det L,$$

unless $v^2 + 4u \equiv 0$ on M^2 meaning that L is similar to a Jordan block at each point.

From Theorem 5.1 we immediately obtain a *local* description of Sing in canonical coordinates x, y (below $\mathsf{Sing}_{\mathrm{loc}}$ denotes the intersection of Sing with a small neighbourhood of a singular point):

- for L_{nil}, N_{2k-1} and N_{2k}^{ε}, the singular set is empty;
- for L_{nd} the singular set is $\mathsf{Sing}_{\mathrm{loc}} = \{x^2 + 4y = 0\}$;
- for all the other series M, O, P and S, $\mathsf{Sing}_{\mathrm{loc}} = \{y = 0\}$.

Thus, locally Sing is a smooth curve. Since $\mathsf{Sing} \subset \mathsf{M}^2$ is closed, we may think of it as a submanifold consisting, perhaps, of several connected components:

$$\mathsf{Sing} = \cup_i \mathsf{S}_i.$$

If M is compact, then the number of components is finite and each of them is an embedded circle. Next, we can easily observe that all points from S_i relate to the same series (different components may, of course, relate to different series). However, the parameters of the series may change. This happens for the series O, P and S. Indeed, moving along $\mathsf{Sing}_{\mathrm{loc}} = \{y = 0\}$ leads to the shift $x_{\mathrm{new}} = x - x_0$, resulting in the following modification for $v = \operatorname{tr} L$ (whereas $\det L$ remains unchanged):

$$\begin{aligned} v &= \alpha x y^s + c_{k-1} y^s + \cdots = \alpha (x_{\mathrm{new}} + x_0) y^s + c_{k-1} y^s + \cdots \\ &= \alpha x_{\mathrm{new}} y^s + (c_{k-1} + \alpha x_0) y^s + \cdots \end{aligned}$$

In other words, all parameters remain fixed except for c_{k-1}, which undergoes the shift $c_{k-1} \mapsto c_{k-1} + \alpha x_0$. Notice that if we move along S_i in a certain direction, then c_{k-1} is either strictly increasing or strictly decreasing. This leads us to the following conclusion.

Proposition 5.6. *Singular points from the series O, P and S may not occur on closed* gl*-regular Nijenhuis* 2*-manifolds.*

According to Corollary 6.1 in [8], the same conclusion holds for differentially nondegenerate singular points (series L_{nd}) and therefore we obtain the following.

Proposition 5.7. *Let* (M^2, L) *be a closed* gl*-regular Nijenhuis* 2*-manifold. Then*

- *either* Sing *is empty* (*i.e., all points of* M^2 *are of the same algebraic type*),
- *or each* $\mathsf{p} \in \mathsf{Sing}$ *belongs to the series M and then automatically one of the eigenvalues of L is constant on* M^2.

We are now ready to prove our final result.

Proof of Theorem 1.7. Consider the two options from Proposition 5.7. First assume that $\mathsf{Sing} = \emptyset$. Then L belongs to one of three generic types listed in the beginning of this section:

(i) either L has two distinct real eigenvalues at each point of M^2;

(ii) or L has two complex conjugate eigenvalues at each point of M^2;

(iii) or L is similar to a Jordan block at each point of M^2.

In Case (i), at each point $\mathsf{p} \in \mathsf{M}^2$, we have an eigenbasis $e_1, e_2 \in T_{\mathsf{p}}\mathsf{M}^2$, where e_1 corresponds to the maximal eigenvalue at a given point. If we fix some Riemannian metric on M^2, we may assume that the e_i are normalised so that $|e_i| = 1$. Since such e_i are defined up to $\pm$, we have 4 different bases at each point. A priori, it is not clear whether or not we can choose a smooth "moving frame" field on the whole manifold, but this can obviously be done on a finite sheeted covering $\widetilde{\mathsf{M}}^2$ of M (number of sheets is at most four). This implies that $\widetilde{\mathsf{M}}^2$ is parallelisable and hence is a torus. Therefore, M^2 is either a torus or a Klein bottle, and we obtain Case (2) of Theorem 1.7.

In Case (ii), according to Theorem 6.1 in [8], the complex eigenvalues λ and $\bar{\lambda}$ of the Nijenhuis operator L are constant, and we obtain Case (1) from Theorem 1.7.

In Case (iii), at each point $\mathsf{p} \in \mathsf{M}^2$ we have a non-zero eigenvector $e \in T_{\mathsf{p}}\mathsf{M}^2$, and the same argument as above shows that, on M^2 or on its two sheeted covering, one can define a smooth vector field with no singular points. Hence M^2 is either a torus or a Klein bottle. However, in this case we have one additional property that the automorphism group of a Jordan block consists of orientation preserving transformations, which allows us to define orientation on M^2. Hence, the Klein bottle is forbidden, and we are led to Case (3) of Theorem 1.7.

Thus, the condition $\mathsf{Sing} = \emptyset$ necessarily implies one of the first three cases of Theorem 1.7.

Finally, we consider the second option from Proposition 5.7. This option implies that one of the eigenvalues of L is constant, allowing us to consider a non-zero eigenvector related to this eigenvalue at each point and, in the same way as above, to construct a smooth vector field with no zeros either on M^2 or on its two-sheeted covering. This implies that M^2 is either a torus or a Klein bottle, and we obtain Case (4) of Theorem 1.7. Thus, the list of possibilities presented in Theorem 1.7 is complete. ∎

We conclude this section with examples of Nijenhuis operators listed in Theorem 1.7.

Example 5.8. Let T^2 be a torus with standard angle coordinates ϕ_1 and ϕ_2 defined modulo 2π. For an operator L with two distinct eigenvalues at each point (ϕ_1, ϕ_2), we can distinguish three essentially different possibilities.

- Two constant eigenvalues λ_1 and λ_2. Let ξ and η be two vector fields on T^2 that are linearly independent at each point (NB: there are many non-equivalent examples of such vector fields). Then we define L by setting

$$L(\xi) = \lambda_1 \xi \quad \text{and} \quad L(\eta) = \lambda_2 \eta. \tag{5.10}$$

- One constant eigenvalue (without loss of generality, $\lambda_1 = 0$), the other λ_2 is not. In coordinates (ϕ_1, ϕ_2), we define L as

$$L = \begin{pmatrix} 0 & g(\phi_1, \phi_2) \\ 0 & f(\phi_2) \end{pmatrix}, \tag{5.11}$$

  with $f(\phi_2) > 0$ or $f(\phi_2) < 0$. Here $\xi = \left(1, -\frac{g(\phi_1,\phi_2)}{f(\phi_2)}\right)$ is an eigenvector field related to the non-constant eigenvalue $\lambda_2 = f(\phi_2)$.

- Two non-constant eigenvalues λ_1 and λ_2. An obvious example is

$$L = \begin{pmatrix} f(\phi_1) & 0 \\ 0 & g(\phi_2) \end{pmatrix}, \quad f(\phi_1) < c < g(\phi_2). \tag{5.12}$$

This example can be modified by taking a finite-sheeted covering over this "standard" torus. On the covering torus, the above *global diagonalisation* of L is not always possible.

Example 5.9. Each of the above examples (5.10), (5.11) and (5.12) can be naturally "transferred" to the Klein bottle K^2, that can be thought of as the quotient of T^2 with respect to the involution $\sigma\colon \mathsf{T}^2 \to \mathsf{T}^2$ given by $(\phi_1, \phi_2) \overset{\sigma}{\mapsto} (-\phi_1, \phi_2 + \pi)$. We only need to make sure that L is invariant with respect to σ. Namely, in the above three cases from Example 5.8, we assume in addition that

- ξ is σ-invariant, whereas η changes the direction under the action of σ, i.e., $\mathrm{d}\sigma(\xi) = \xi$ and $\mathrm{d}\sigma(\eta) = -\eta$,
- $f(\phi_2)$ is π-periodic, $g(\phi_1, \phi_2)$ is even with respect to ϕ_1,
- $g(\phi_2)$ is π-periodic and $f(\phi_1)$ is even.

If these conditions are fulfilled, then the operators L given by (5.10), (5.11) and (5.12) naturally descend to the quotient $\mathsf{K}^2 = \mathsf{T}^2/\sigma$.

The next is an example of a Nijenhuis operator on T^2 of Jordan block type (see Case (3) in Theorem 1.7).

Example 5.10. Assume that L is a gl-regular operator L on T^2 with a single eigenvalue λ of multiplicity 2. The cases with constant and non-constant λ are essentially different. If λ is constant, then without loss of generality we may assume that $\lambda = 0$, i.e., that L is nilpotent.

- Consider two vector fields ξ and η on T^2 which are linearly independent at each point, and define L as follows:
$$L(\xi) = 0, \quad L(\eta) = \xi.$$
Then L is a gl-regular nilpotent Nijenhuis operator on T^2 (notice that any nilpotent operator in dimension 2 is automatically Nijenhuis).
- The case with a non-constant eigenvalue on T^2 can be modelled as follows:
$$\begin{pmatrix} f(\phi_2) & g(\phi_1, \phi_2) \\ 0 & f(\phi_2) \end{pmatrix}, \quad g(\phi_1, \phi_2) > 0,$$
where ϕ_1, ϕ_2 denote usual angle coordinates on the torus as above.

Finally, we notice that the examples corresponding to Case (4) of Theorem 1.7 on the torus T^2 and the Klein bottle $\mathsf{K}^2 = \mathsf{T}^2/\sigma$ can be defined by the same formula as (5.11). The only difference is that now $f(\phi_2)$ vanish for some ϕ_2 (but then $g(\phi_1, \phi_2)$ does not!). The operator L will become nilpotent at such points, which will be automatically singular from series M. Notice that the topological structure of the eigenvector field ξ related to the eigenvalue $f(\phi_2)$ may now be rather non-trivial in contrast to the case when $f \neq 0$.

We conjecture that the above list of examples essentially exhausts all possible *real-analytic* Nijenhuis operators on closed two-dimensional surfaces. In the smooth case, however, there are essentially different possibilities.

A. Proof of Proposition 3.4

Let $n = \dim \mathsf{M} > 2$ and L be a gl-regular operator. Recall that for two commuting operators A and B (i.e., such that $AB = BA$), we may define a vector-valued quadratic form $\langle A, B\rangle$ by setting

$$\langle A, B\rangle(\xi,\xi) \stackrel{\text{def}}{=} [A\xi, B\xi] - B[A\xi,\xi] - A[\xi, B\xi] = 0.$$

Our goal is to prove the following:

(1) If $\langle L^i, L^j\rangle = 0$ for $1 \le i < j \le n-1$, then L is a Nijenhuis operator.

(2) If $\langle M_i, M_j\rangle = 0$ for $1 \le i < j \le n-1$, where M_i is defined as in (3.6), then L is a Nijenhuis operator.

Proof. (1) It is easily seen that for any three commuting operators L, A and B, the following (algebraic) identity holds:

$$\mathcal{N}_L(A\xi, B\xi) = \big(\langle LA, LB\rangle - L\langle LA, B\rangle - L\langle A, LB\rangle + L^2\langle A, B\rangle\big)(\xi,\xi) \tag{A.1}$$

Hence, for $A = L^i$, $B = L^j$, $0 \le i, j < n-1$, we have

$$\mathcal{N}_L(L^i\xi, L^j\xi) = \big(\langle L^{i+1}, L^{j+1}\rangle - L\langle L^{i+1}, L^j\rangle - L\langle L^i, L^{j+1}\rangle + L^2\langle L^i, L^j\rangle\big)(\xi,\xi) = 0$$

for any ξ. Replacing ξ with $\eta = \xi + L\xi$ and setting $j = n-2$ in this formula, we get

$$\begin{aligned}0 &= \mathcal{N}_L(L^i(\xi + L\xi), L^{n-2}(\xi + L\xi)) = \mathcal{N}_L(L^i\xi, L^{n-2}\xi) + \mathcal{N}_L(L^i(L\xi), L^{n-2}(L\xi))\\ &\quad + \mathcal{N}_L(L^{i+1}\xi, L^{n-2}\xi) + \mathcal{N}_L(L^i\xi, L^{n-1}\xi) = 0 + 0 + 0 + \mathcal{N}_L(L^i\xi, L^{n-1}\xi).\end{aligned}$$

Thus, $\mathcal{N}_L$ vanishes for any pair of vectors from the set $\xi, L\xi, \dots, L^{n-1}\xi$. As L is gl-regular, one can choose ξ in a way that $\xi, L\xi, \dots, L^{n-1}\xi$ form a basis in the tangent space. Hence, $\mathcal{N}_L = 0$, as stated.

(2) In what follows, we assume that $M_0 = \mathrm{Id}$ and $M_n = 0$, which perfectly agrees with the above definition of M_i's (due to the Cayley–Hamilton theorem). We start with:

Lemma A.1. *If $\langle M_i, M_j\rangle = 0$ for $1 \le i < j \le n-1$, then the following identities hold:*

$$\mathrm{d} f_{j+1}(M_i\xi) - \mathrm{d} f_{i+1}(M_j\xi) = 0, \quad i, j = 0, \dots, n-1,$$

where the f_i are coefficients of the characteristic polynomial of L, and ξ is an arbitrary tangent vector.

Proof. In formula (3.9), the expression $\langle A, B\rangle$ is treated as a (vector-valued) quadratic form on the tangent bundle (one assumes that A and B commute). We can also naturally interpret it as a symmetric bilinear form by setting

$$\langle A, B\rangle(\xi,\eta) = \frac{1}{2}\big([A\xi, B\eta] - A[\xi, B\eta] - B[A\xi,\eta] + [A\eta, B\xi] - A[\eta, B\xi] - B[A\eta,\xi]\big).$$

Obviously $\langle A, B\rangle(\xi,\xi) \equiv 0$ implies $\langle A, B\rangle(\xi,\eta) \equiv 0$.

First we observe the following (purely algebraic) identity:

$$\langle M_i L, M_j\rangle(\xi,\xi) + \langle M_i, M_j L\rangle(\xi,\xi) = M_i\langle L, M_j\rangle(\xi,\xi) - M_j\langle L, M_i\rangle(\xi,\xi) - 2\langle M_j, M_i\rangle(L\xi,\xi).$$

In our case we have $L = M_1 + f_1\mathrm{Id}$ and, in addition, $\langle M_i, M_j\rangle \equiv 0$, which gives

$$\begin{aligned}\langle M_i L, M_j\rangle + \langle M_i, M_j L\rangle &= M_i\langle L, M_j\rangle - M_j\langle L, M_i\rangle \\ &= M_i\langle M_1 + f_1\mathrm{Id}, M_j\rangle - M_j\langle M_1 + f_1\mathrm{Id}, M_i\rangle = M_i\langle f_1\mathrm{Id}, M_j\rangle - M_j\langle f_1\mathrm{Id}, M_i\rangle.\end{aligned} \tag{A.2}$$

Using (A.2) and the definition of the M_i's, we now compute the right-hand side of the identity $0 = \langle M_{i+1}, M_j\rangle + \langle M_i, M_{j+1}\rangle$:

$$\begin{aligned}0 &= \langle M_{i+1}, M_j\rangle + \langle M_i, M_{j+1}\rangle = \langle LM_i - f_{i+1}\mathrm{Id}, M_j\rangle + \langle M_i, LM_j - f_{j+1}\mathrm{Id}\rangle \\ &= \langle LM_i, M_j\rangle + \langle M_i, LM_j\rangle - \langle f_{i+1}\,\mathrm{Id}, M_j\rangle - \langle M_i, f_{j+1}\,\mathrm{Id}\rangle \\ &= M_i\langle f_1\,\mathrm{Id}, M_j\rangle - M_j\langle f_1\,\mathrm{Id}, M_i\rangle - \langle f_{i+1}\,\mathrm{Id}, M_j\rangle + \langle f_{j+1}\,\mathrm{Id}, M_i\rangle.\end{aligned} \tag{A.3}$$

Notice that

$$\langle f\,\mathrm{Id}, A\rangle(\xi,\xi) = [f\xi, A\xi] - f[\xi, A\xi] - A[f\xi,\xi] = \mathrm{d}f(A\xi)\xi - \mathrm{d}f(\xi)A\xi \tag{A.4}$$

for an arbitrary function f and operator A. Applying this relation to (A.3) gives

$$\begin{aligned}0 &= M_i\langle f_1\,\mathrm{Id}, M_j\rangle(\xi,\xi) - M_j\langle f_1\,\mathrm{Id}, M_i\rangle(\xi,\xi) \\ &\quad - \langle f_{i+1}\,\mathrm{Id}, M_j\rangle(\xi,\xi) + \langle f_{j+1}\,\mathrm{Id}, M_i\rangle(\xi,\xi) \\ &= (\mathrm{d}f_1(M_j\xi) - \mathrm{d}f_{j+1}(\xi))\,M_i\xi - (\mathrm{d}f_1(M_i\xi) - \mathrm{d}f_{i+1}(\xi))\,M_j\xi \\ &\quad + (\mathrm{d}f_{j+1}(M_i\xi) - \mathrm{d}f_{i+1}(M_j\xi))\,\xi.\end{aligned} \tag{A.5}$$

Recall that L is gl-regular. Hence $\xi, L\xi, \dots, L^{n-1}\xi$ are linearly independent for almost all tangent vectors ξ. By formula (3.6) for M_i, this is still true for $\xi, M_1\xi, \dots, M_{n-1}\xi$. Therefore ξ, $M_i\xi$ and $M_j\xi$ are linearly independent in (A.5), and the coefficients of this linear combination vanish. Thus, $\mathrm{d}f_{j+1}(M_i\xi) - \mathrm{d}f_{i+1}(M_j\xi) = 0$ for *almost all* vectors ξ, and by continuity, for *all* vectors. The lemma is proved. ∎

Similar to the first case, our goal is to show that $\mathcal{N}_L(M_i\xi, M_j\xi) = 0$ for all $i, j = 0, \dots, n-1$. Since $M_0\xi, M_1\xi, \dots, M_{n-1}\xi$ form a basis for a generic vector ξ, this will imply $\mathcal{N}_L = 0$.

As above, we use (A.1) with $A = M_i$ and $B = M_j$:

$$\begin{aligned}&\mathcal{N}_L(M_i\xi, M_j\xi) \\ &\qquad = \big(\langle LM_i, LM_j\rangle - L\langle LM_i, M_j\rangle - L\langle M_i, LM_j\rangle + L^2\langle M_i, M_j\rangle\big)(\xi,\xi).\end{aligned} \tag{A.6}$$

Substituting $LM_i = M_{i+1} + f_{i+1}\mathrm{Id}$ and using the relations $\langle M_i, M_j\rangle = 0$ $(i, j = 0, \dots, n)$ and the identity $\langle f_{i+1}\mathrm{Id}, f_{j+1}\mathrm{Id}\rangle = 0$, we can rewrite the vector-valued quadratic form in the right-hand side of (A.6) as follows:

$$\begin{aligned}
&\langle LM_i, LM_j\rangle - L\langle LM_i, M_j\rangle - L\langle M_i, LM_j\rangle + L^2\langle M_i, M_j\rangle \\
&= \langle M_{i+1}+f_{i+1}\mathrm{Id}, M_{j+1}+f_{j+1}\mathrm{Id}\rangle - L\langle M_{i+1}+f_{i+1}\mathrm{Id}, M_j\rangle - L\langle M_i, M_{j+1}+f_{j+1}\mathrm{Id}\rangle \\
&= \langle f_{i+1}\mathrm{Id}, M_{j+1}\rangle + \langle M_{i+1}, f_{j+1}\mathrm{Id}\rangle - L\langle f_{i+1}\mathrm{Id}, M_j\rangle - L\langle M_i, f_{j+1}\mathrm{Id}\rangle \\
&= \langle f_{i+1}\mathrm{Id}, LM_j - f_{j+1}\mathrm{Id}\rangle + \langle LM_i - f_{i+1}\mathrm{Id}, f_{j+1}\mathrm{Id}\rangle - L\langle f_{i+1}\mathrm{Id}, M_j\rangle - L\langle M_i, f_{j+1}\mathrm{Id}\rangle \\
&= \langle f_{i+1}\mathrm{Id}, LM_j\rangle + \langle LM_i, f_{j+1}\mathrm{Id}\rangle - L\langle f_{i+1}\mathrm{Id}, M_j\rangle - L\langle M_i, f_{j+1}\mathrm{Id}\rangle.
\end{aligned}$$

Hence, using (A.4), we get

$$\begin{aligned}
&\mathcal{N}_L(M_i\xi, M_j\xi) \\
&\quad = \big(\langle f_{i+1}\mathrm{Id}, LM_j\rangle + \langle LM_i, f_{j+1}\mathrm{Id}\rangle - L\langle f_{i+1}\mathrm{Id}, M_j\rangle - L\langle M_i, f_{j+1}\mathrm{Id}\rangle\big)(\xi,\xi) \\
&\quad = \mathrm{d}f_{i+1}(LM_j\xi)\,\xi - \mathrm{d}f_{i+1}(\xi)\,LM_j\xi - \mathrm{d}f_{j+1}(LM_i\xi)\,\xi - \mathrm{d}f_{j+1}(\xi)\,LM_i\xi \\
&\qquad - L(\mathrm{d}\,f_{i+1}(M_j\xi)\,\xi - \mathrm{d}f_{i+1}(\xi)\,M_j\xi) + L(\mathrm{d}\,f_{j+1}(M_i\xi)\,\xi - \mathrm{d}f_{j+1}(\xi)\,M_i\xi) \\
&\quad = (\mathrm{d}f_{i+1}(M_j(L\xi)) - \mathrm{d}f_{j+1}(M_i(L\xi)))\,\xi - (\mathrm{d}f_{i+1}(M_j\xi) - \mathrm{d}f_{j+1}(M_i\xi))\,L\xi.
\end{aligned}$$

It remains to notice that the coefficients in front of ξ and $L\xi$ vanish by Lemma A.1, which completes the proof. ■

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