Horizontally quasiconvex envelope in the Heisenberg group

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Abstract. This paper is concerned with a PDE-based approach to the horizontally quasiconvex (h-quasiconvex, for short) envelope of a given continuous function in the Heisenberg group. We provide a characterization for upper semicontinuous, h-quasiconvex functions in terms of the viscosity subsolution to a first-order nonlocal Hamilton–Jacobi equation. We also construct the corresponding envelope of a continuous function by iterating the nonlocal operator. One important step in our arguments is to prove the uniqueness and existence of viscosity solutions to the Dirichlet boundary problem for the nonlocal Hamilton–Jacobi equation. Applications of our approach to the h-convex hull of a given set in the Heisenberg group are discussed as well.

1. Introduction

1.1. Background and motivation

Convex analysis is a classical and fundamental topic with numerous applications in various fields of mathematics and beyond. In contrast to the extensive literature on convex analysis in the Euclidean space, less is known about the case of a general geometric setting such as sub-Riemannian manifolds. This paper is mainly concerned with a PDE method to deal with a certain weak type of convexity for both sets and functions in the first Heisenberg group $\mathbb{H}$.

The Heisenberg group $\mathbb{H}$ is $\mathbb{R}^3$ endowed with the non-commutative group multiplication

$$(x_p, y_p, z_p) \cdot (x_q, y_q, z_q) = \left(x_p + x_q, y_p + y_q, z_p + z_q + \frac{1}{2}(x_p y_q - x_q y_p) \right),$$

for all $p = (x_p, y_p, z_p)$ and $q = (x_q, y_q, z_q)$ in $\mathbb{H}$. The differential structure of $\mathbb{H}$ is determined by the left-invariant vector fields

$$X_1 = \frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial z}, \quad X_2 = \frac{\partial}{\partial y} + \frac{x}{2}\frac{\partial}{\partial z} \quad \text{and} \quad X_3 = \frac{\partial}{\partial z}.$$

One may easily verify the commutation relation $X_3 = [X_1, X_2] = X_1 X_2 - X_2 X_1$. Let

$$\mathbb{H_0} = \{h \in \mathbb{H} : h = (x, y, 0) \text{ for } x, y \in \mathbb{R}\}.$$

For any $p \in \mathbb{H}$, the set

$$\mathbb{H}_p = \{p \cdot h : h \in \mathbb{H_0}\}$$

is called the horizontal plane through p. It is clear that $\mathbb{H}_p = \text{span}\{X_1(p), X_2(p)\}$ for every $p \in \mathbb{H}$. See [9] for a detailed introduction of the Heisenberg group.

Our primary interest is to understand how to convexify a given bounded set in the Heisenberg group $\mathbb{H}$, that is, we aim to find the smallest convex set that contains the given set. Let us first clarify the meaning of convex sets we consider in this work. We shall actually focus on the notion of weakly h-convex sets, which was first introduced in [14], and later studied in [1, 6, 28] etc. A set $E \subset \mathbb{H}$ is said to be weakly h-convex if the horizontal segment connecting any two points in E lies in E; see also Definition 2.1. Hereafter we call such a set an h-convex set for simplicity of terminology.

There are several other types of set convexity in $\mathbb{H}$ defined with different kinds of convex combination of two points. One seemingly natural notion is based on geodesics in $\mathbb{H}$. A set $E \subset \mathbb{H}$ is said to be geodetically convex if, for every pair of points $p, q \in E$, the set E contains all geodesics joining p and q. This notion is known to be a very strong one; the geodetically convex hull of any three points that are not on the same geodesic in $\mathbb{H}$ has to be the whole group [27]. A different notion, called strong h-convexity [14] or twisted convexity [6], uses the dilation of group quotient to combine two points. It is still a quite strong notion, much stronger than the Euclidean convexity. In general, a strongly h-convex hull of a bounded set consisting of more than two points could be unbounded [6]. We refer the reader to [6, 28] for related discussions on these convexity notions.

The notion of (weak) h-convexity is obviously weaker than the Euclidean convexity as well as the other aforementioned notions because of the restriction on horizontal segments. Such relaxation gives rise to unexpected properties. An h-convex set can even be disconnected, as pointed out in [7]. One simple example of h-convex sets is the union of two points $(0, 0, 1)$ and $(0, 0, -1)$ in $\mathbb{H}$. It is h-convex, because no horizontal segments exist to connect the points. A less trivial example of disconnected h-convex sets is presented in Example 2.2. Such an unusual character makes it challenging to find the h-convex hull of a given set in $\mathbb{H}$. In contrast to the Euclidean situation, where one can generate the convex hull of a set by simply connecting every pair of points in the set with a segment, building the h-convex hull of a given set in the Heisenberg group requires possibly infinite iterations to include all necessary horizontal segments; see the proof of Lemma 4.1 in [29] by Rickly about this method. In general, it is not straightforward to describe and construct the h-convex hull of a set. This motivates us to search possible analytic methods to solve this problem.

A closely related problem we also intend to discuss is constructing the horizontally quasiconvex (or simply h-quasiconvex) envelope of a given function in an h-convex domain $\Omega \subset \mathbb{H}$. An h-quasiconvex function in Ω is defined to be a function whose sublevel sets are all h-convex. It is equivalent to saying that

(1.1) $$u(w) \leq \max\{u(p), u(q)\}$$

holds for any $p \in \Omega$, $q \in \mathbb{H}_p \cap \Omega$ and $w \in [p, q]$. This notion is introduced in [32]. It is also studied in [7], where such functions are called weakly h-quasiconvex functions. We again suppress the term "weakly" in this paper to avoid redundancy. Similar to the case of sets, for any function in Ω, it is not trivial in general how one can find its h-quasiconvex envelope in a constructive way.

We remark that a stronger notion of function convexity in $\mathbb{H}$, called horizontal convexity, is introduced by [14] and [23]. Various properties and generalizations of such convex functions are discussed in [2, 3, 8, 19, 24, 25, 33] etc. The corresponding convex envelope and its applications to convexity properties of sub-elliptic equations are studied in [22].

1.2. Main results

Inspired by the Euclidean results in [5], in this work we provide a PDE-based approach to investigate h-convex hulls and h-quasiconvex envelopes. Our study starts from an improved characterization of h-quasiconvex functions. It is known (see Theorem 4.5 in [7]) that any function $u \in C^1(\Omega)$ is h-quasiconvex if and only if

$$(1.2) \qquad u(\xi) < u(p) \Rightarrow \langle \nabla_H u(p), (p^{-1} \cdot \xi)_h \rangle \leq 0$$

for any $\xi \in \mathbb{H}_p \cap \Omega$. Here, $\nabla_H u$ denotes the horizontal gradient of u, given by $\nabla_H u = (X_1 u, X_2 u)$. Also, for each $\zeta = (x_\zeta, y_\zeta, z_\zeta) \in \mathbb{H}$, ζ_h represents its horizontal coordinates, that is, $\zeta_h = (x_\zeta, y_\zeta)$. For the sake of our applications to construct h-convex hulls, we generalize this characterization to functions which are not necessarily of C^1 class. Extending the Euclidean arguments in [5] to $\mathbb{H}$, we show in Theorem 2.6 that an upper semicontinuous (USC) function u is h-quasiconvex if and only if (1.2) holds in the viscosity sense. It is equivalent to saying that

$$(1.3) \qquad \sup \left\{ \langle \nabla_H \varphi(p), (p^{-1} \cdot \xi)_h \rangle : \xi \in \mathbb{H}_p \cap \Omega, \ u(\xi) < u(p) \right\} \leq 0$$

whenever there exist $p \in \Omega$ and $\varphi \in C^1(\Omega)$ such that $u - \varphi$ attains a maximum at p.

Further developing the generalized characterization, we adopt an iterative scheme to find the h-quasiconvex envelope of a continuous function f, denoted by $Q(f)$ (see Definition 2.9), in a bounded h-convex domain Ω under a particular Dirichlet boundary condition. The iteration is implemented by solving a sequence of nonlocal Hamilton–Jacobi equations, where the Hamiltonian is given by the left-hand side of (1.3), that is,

$$H(p, u(p), \nabla u(p)) = \sup \left\{ \langle \nabla_H u(p), (p^{-1} \cdot \xi)_h \rangle : \xi \in \mathbb{H}_p \cap \Omega, \ u(\xi) < u(p) \right\}.$$

We briefly describe our scheme in what follows.

Let $f \in C(\overline{\Omega})$ be a given function satisfying

$$(1.4) \qquad\qquad\qquad \begin{cases} f = K & \text{on } \partial\Omega, \\ (1.5) \qquad\qquad\qquad\quad\ f \leq K & \text{in } \overline{\Omega}, \end{cases}$$

with $K \in \mathbb{R}$. This set of conditions resembles the coercivity assumption on f when $K > 0$ is taken large. It guarantees the existence of an h-quasiconvex function $\underline{f} \in C(\overline{\Omega})$ such that $\underline{f} \leq f$ in $\overline{\Omega}$ and $\underline{f} = K$ on $\partial\Omega$; see Proposition 4.1. This in turn implies the existence of $Q(f)$ taking the same boundary value.

We set $u_0 = f$ in Ω and take u_n ($n = 1, 2, \ldots$) to be the unique viscosity solution of

$$(1.6) \qquad\qquad u_n + H(p, u_n, \nabla_H u_n) = u_{n-1} \quad \text{in } \Omega$$

satisfying the same set of conditions as in (1.4)–(1.5), that is,

$$(1.7) \qquad\qquad\qquad\qquad \begin{cases} u_n = K & \text{on } \partial\Omega, \\ u_n \leq K & \text{in } \overline{\Omega}. \end{cases}$$
$$(1.8)$$

It turns out that u_n is a nonincreasing sequence and that it converges uniformly to $Q(f)$ as $n \to \infty$. This is in fact our main theorem.

Theorem 1.1 (Iterative scheme for envelope). *Suppose that Ω is a bounded h-convex domain in $\mathbb{H}$ and that $f \in C(\overline{\Omega})$ satisfies (1.4)–(1.5) for some $K \in \mathbb{R}$. Let $Q(f)$ be the h-quasiconvex envelope of f in Ω. Let $u_0 = f$ in Ω and let u_n be the unique solution of (1.6) satisfying (1.7) and (1.8) for $n \geq 1$. Then $u_n \to Q(f)$ uniformly in $\overline{\Omega}$ as $n \to \infty$.*

Such type of nonlocal schemes are proposed in [5] in the Euclidean case for general Dirichlet data. We remark that in the Euclidean space a similar class of nonlocal equations depending on the level sets of the unknown is also studied for applications in geometric evolutions and front propagation [10, 11, 20, 30]. Although our PDE looks analogous to theirs, the well-posedness in the sub-Riemannian case is not straightforward at all. The main difference from the Euclidean case lies in an additional constraint that requires $\xi \in \mathbb{H}_p \cap \Omega$, which depends on the space variable p. This extra constraint brings us much difficulty in proving the comparison principle for (1.6). It is the coercivity-like setting (1.7)–(1.8) that enables us to overcome the difficulty and obtain the uniqueness of solutions. More details can be found in Section 3.2.

The existence of viscosity solutions, on the other hand, can be handled in a standard way by adapting Perron's method [13]. Since the existence in the Euclidean case is not explicitly discussed in [5], we give full details of the arguments for our sub-Riemannian version in Section 3.3. Once the sequence u_n is determined iteratively for all $n = 1, 2, \ldots$, Theorem 1.1 can be proved by applying a stability argument for viscosity solutions.

It is worth mentioning that, instead of adopting the restrictive setting (1.7)–(1.8), one can solve (1.6) with general boundary data and obtain the scheme convergence as in Theorem 1.1 if there exists an appropriate h-quasiconvex barrier $\underline{f}$ from below compatible with the boundary value; see Theorem 3.15 and Remark 4.4.

Another possible modification of the scheme is to consider the maximal subsolution of (1.6) rather than its solutions at each step. Although we only get pointwise convergence of the scheme in this case, it allows us to avoid the uniqueness issue and to construct the h-quasiconvex envelopes for a general class of upper semicontinuous functions, even in an unbounded domain Ω. See Theorem 4.5 for results in this relaxed setting.

We would like to point out that, besides the PDE-based approach described above, there is a more direct constructive method to build $Q(f)$, which employs the following convexification operator:

$$(1.9) \quad T[f](w) = \inf\big\{\max\{f(p), f(q)\} : w \in [p, q],\ p \in \Omega,\ q \in \Omega \cap \mathbb{H}_p\big\}, \quad \text{for } w \in \Omega.$$

It turns out that $T[f]$ itself may not be h-quasiconvex in Ω, but iterated application of T yields a pointwise approximation of the h-quasiconvex envelope $Q(f)$; see Theorem 2.12

for details. A similar idea is used in the proof of Lemma 4.1 in [29] for h-convex functions. It is also adopted in [22] to construct the h-convex envelope of a given function f.

As an application of our constructive methods for h-quasiconvex envelopes, we study the h-convex hull of a given bounded set in $\mathbb{H}$. A key ingredient is the so-called level set formulation, which plays an important role in the study of geometric evolutions, see [12, 17, 18]. We can apply the same idea to our problem, since the nonlocal Hamiltonian is actually a geometric operator, homogeneous in u. Suppose that E is a bounded open set in $\mathbb{H}$. Take a bounded h-convex domain Ω such that $E \subset \Omega$. We next choose a defining function $f \in C(\overline{\Omega})$ such that

$$(1.10) \qquad E = \{p \in \Omega : f(p) < 0\}$$

and (1.4)–(1.5) hold for some $K > 0$. It turns out that the h-convex hull of E coincides with the zero sublevel set of $Q(f)$, i.e.,

$$(1.11) \qquad \mathrm{co}(E) = \{p \in \Omega : Q(f) < 0\}.$$

We remark that $\mathrm{co}(E)$ is independent of the choices of f and Ω. As long as (1.10) together with (1.4)–(1.5) holds, the $\mathrm{co}(E)$ obtained in (1.11) will not change. See Theorem 5.2 for more precise statements.

This PDE approach leads us to a better understanding about h-convex hulls. One application is about the inclusion principle. By definition, it is easily seen that $\mathrm{co}(D) \subset \mathrm{co}(E)$ holds for any sets $D, E \subset \mathbb{H}$ satisfying $D \subset E$. In Theorem 5.4, we establish a quantitative version of the inclusion principle in $\mathbb{H}$. For any bounded open (or closed) sets $D, E \subset \mathbb{H}$, we obtain

$$(1.12) \quad \inf\left\{\tilde{d}_H(p,q) : p \in \mathrm{co}(D), q \in \mathbb{H} \setminus \mathrm{co}(E)\right\} \geq \inf\left\{\tilde{d}_H(p,q) : p \in D, q \in \mathbb{H} \setminus E\right\},$$

where $\tilde{d}_H$ denotes the right invariant gauge metric in $\mathbb{H}$; see (1.13) below. This property amounts to saying that taking h-convex hulls of two sets in $\mathbb{H}$, one contained in the other, does not reduce the shortest $\tilde{d}_H$ distance between their boundaries. If E contains the right invariant δ-neighborhood of D for some $\delta > 0$, then $\mathrm{co}(E)$ also contains the right invariant δ-neighborhood of $\mathrm{co}(D)$.

While such a result can be obtained comparatively easily in the Euclidean case, the proof is more involved in the Heisenberg group. Our proof is based on comparing the h-quasiconvex envelopes of defining functions of both sets combined with arguments involving sup-convolutions. It is not clear to us whether one can replace $\tilde{d}_H$ by the left invariant gauge metric d_H. This problem is related to the h-convexity preserving property for solutions of evolution equations in the Heisenberg group; see some partial results in [21, 22].

Another natural question is on the continuity (or stability) of $\mathrm{co}(E)$ with respect to the set E, which we discuss in the last part of this paper. In contrast to the Euclidean case, in general the Hausdorff distance $d_H(\mathrm{co}(E_j), \mathrm{co}(E))$ between $\mathrm{co}(E_j)$ and $\mathrm{co}(E)$ does not necessarily converge to zero when $d_H(E_j, E) \to 0$ in the Heisenberg group, see Example 5.6. One can show rather easily that such stability result holds under a strict star-shapedness assumption on the set E; see Proposition 5.9.

1.3. Notations

We conclude the introduction by listing several notations that are often used in the work. Throughout this paper, $|\cdot|_G$ stands for the Korányi gauge, i.e., for $p = (x, y, z) \in \mathbb{H}$,

$$|p|_G = ((x^2 + y^2)^2 + 16z^2)^{1/4}.$$

The Korányi gauge induces a left invariant metric d_H on $\mathbb{H}$, with

$$d_H(p, q) = |p^{-1} \cdot q|_G, \quad \text{for } p, q \in \mathbb{H}.$$

We also use the right invariant metric $\tilde{d}_H$, defined by

$$(1.13) \qquad\qquad \tilde{d}_H(p, q) = |p \cdot q^{-1}|_G, \quad \text{for } p, q \in \mathbb{H}.$$

The associated distances between a point $p \in \mathbb{H}$ and a set $E \subset \mathbb{H}$ are respectively denoted by $d_H(p, E)$ and $\tilde{d}_H(p, E)$. For two sets $D, E \subset \mathbb{H}$, we write $d_H(D, E)$ and $\tilde{d}_H(D, E)$ to denote, respectively, the Hausdorff distances between D and E with respect to the metrics d_H and $\tilde{d}_H$, i.e., for $d = d_H$ or $d = \tilde{d}_H$,

$$d(D, E) = \max \left\{ \sup_{p \in D} d(p, E), \ \sup_{p \in E} d(p, D) \right\}.$$

We denote by $B_r(p)$ the open gauge ball in $\mathbb{H}$ centered at $p \in \mathbb{H}$ with radius $r > 0$, that is,

$$B_r(p) = \{q \in \mathbb{H} : |p^{-1} \cdot q|_G < r\},$$

while $\tilde{B}_r(p)$ represents the corresponding right-invariant metric ball.

Let δ_λ denote the non-isotropic dilation in $\mathbb{H}$ with $\lambda \geq 0$, that is, $\delta_\lambda(p) = (\lambda x, \lambda y, \lambda^2 z)$ for $p = (x, y, z) \in \mathbb{H}$. We write $\delta_\lambda(E)$ to denote the dilation of a given set $E \subset \mathbb{H}$, that is, $\delta_\lambda(E) = \{\delta_\lambda(p) : p \in E\}$.

The rest of the paper is organized in the following way. In Section 2, we first give a review on the definitions and basic properties of h-convex sets and h-quasiconvex functions, and then present the viscosity characterization of upper semicontinuous h-quasiconvex functions. We also show how to construct the h-quasiconvex envelope by iterated application of the operator in (1.9). Section 3 is devoted to the well-posedness of the nonlocal Hamilton–Jacobi equation, including the uniqueness and existence of viscosity solutions. Our PDE-based iterative scheme is introduced in Section 4. We finally discuss applications of our results to the h-convex hull of a given open or closed set in Section 5.

2. H-quasiconvex functions

2.1. Definition and basic properties

Let us first go over the definition of h-convex sets. We restrict the original definition proposed in [14] for general Carnot groups to the case of $\mathbb{H}$.

Definition 2.1 (Definition 7.1 in [14]). We say that a set $E \subset \mathbb{H}$ is h-convex if for every $p \in E$ and $q \in \mathbb{H}_p \cap E$, the horizontal segment $[p, q]$ joining p and q stays in E.

As pointed out in Proposition 7.4 of [14], any gauge ball $B_R(p)$ with $p \in \mathbb{H}$ and $R > 0$ is h-convex. The notion of h-convex sets is in fact very weak. There are numerous h-convex sets in $\mathbb{H}$ that are obviously not convex in the Euclidean sense.

Example 2.2 (Disconnected h-convex sets). Denote by $\pi(0, \rho)$ the planar open disk centered at the origin with radius $\rho > 0$, i.e.,

$$(2.1) \qquad \pi(0, \rho) := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < \rho^2\}.$$

Let us consider a disconnected set $E = (\pi(0, r) \times \{0\}) \cup (\pi(0, R) \times \{t\})$, where $r, R, t > 0$ are given. Such a set E is h-convex under appropriate conditions on r and R. To see this, we take the horizontal plane

$$Z_t = \{(x, y, z) \in \mathbb{H} : z = t\}$$

and compute the distance between $q_t = (0, 0, t)$ and $\mathbb{H}_p \cap Z_t$ for each point $p = (x_p, y_p, 0)$ $\in \pi(0, r) \times \{0\}$. It turns out that

$$d_H(q_t, \mathbb{H}_p \cap Z_t) = \frac{2t}{\sqrt{x_p^2 + y_p^2}} \geq \frac{2t}{r}.$$

If $d_H(q_t, \mathbb{H}_p \cap Z_t) \geq R$, then none of the horizontal planes of points $p \in \pi(0, r) \times \{0\}$ in the lower disk will intersect the upper disk $\pi(0, R) \times \{t\}$. This means that E is h-convex if $2t \geq rR$. It is obvious that in general E is not connected and thus cannot be convex as a subset of $\mathbb{R}^3$. It is also clear that E is no longer h-convex if $2t < rR$.

Let us also recall from [7] the definition of h-quasiconvex functions in $\mathbb{H}$.

Definition 2.3 (Definition 4.3 in [7]). Suppose that $\Omega \subset \mathbb{H}$ is h-convex. We say that a function $u : \Omega \to \mathbb{R}$ is h-quasiconvex if (1.1) holds for every $p \in \Omega$, $q \in \mathbb{H}_p \cap \Omega$ and $w \in [p, q]$. In other words, u is h-quasiconvex if for every $\lambda \in \mathbb{R}$ the sublevel set $\{w \in \Omega : u(w) \leq \lambda\}$ is h-convex.

Remark 2.4. We remark that it is equivalent to define h-convex functions with strict sublevel set $\{w \in \Omega : u(w) < \lambda\}$. In fact, first note that

$$\{w \in \Omega : u(w) \leq \lambda\} = \bigcap_{\varepsilon > 0} \{w \in \Omega : u(w) < \lambda + \varepsilon\},$$

and that the intersection of h-convex sets is still h-convex. On the other hand, if $\{w \in \Omega : u(w) \leq \lambda\}$ is h-convex, then (1.1) holds for every $p \in \Omega$, $q \in \mathbb{H}_p \cap \Omega$ and $w \in [p, q]$. If $p, q \in \{w \in \Omega : u(w) < \lambda\}$ and $q \in \mathbb{H}_p \cap \Omega$, it follows that $u(w) < \lambda$ when $w \in [p, q]$ and thus $\{x \in \Omega : u(w) < \lambda\}$ is h-convex.

For our later applications, below we provide a typical h-quasiconvex function associated to a given h-convex set. The construction is based on the right invariant metric $\tilde{d}_H$, as given in (1.13).

Proposition 2.5 (A metric-based h-quasiconvex function). *Suppose that $\Omega \subset \mathbb{H}$ is an h-convex domain in $\mathbb{H}$ and E is an h-convex open subset of Ω. Then $\psi_E \in C(\Omega)$ given by*

$$\psi_E(p) = -\tilde{d}_H(p, \mathbb{H} \setminus E), \quad \text{for } p \in \Omega,$$

is an h-quasiconvex function in Ω.

Proof. It suffices to show that

$$E_\lambda := \{p \in \Omega : \psi_E(p) < \lambda\}$$

is h-convex for every $\lambda \in \mathbb{R}$. We only need to consider the case for $\lambda < 0$, since $E_\lambda = E$ if $\lambda = 0$ and $E_\lambda = \Omega$ if $\lambda > 0$. Assume by contradiction that there exist $p, q \in E_\lambda$ with $q \in \mathbb{H}_p$ as well as a point $w \in [p, q] \cap (\Omega \setminus E_\lambda)$. This means that

$$\tilde{d}_H(w, \mathbb{H} \setminus E) \leq -\lambda$$

and that there exists $\zeta \in \mathbb{H} \setminus E$ such that $|\zeta \cdot w^{-1}|_G \leq -\lambda$. Taking $\xi = \zeta \cdot w^{-1} \cdot p$ and $\eta = \zeta \cdot w^{-1} \cdot q$, we can easily verify that $\eta \in \mathbb{H}_\xi$ and $\zeta \in [\xi, \eta]$.

Suppose that $\xi \in \mathbb{H} \setminus E$ holds. Then, since

$$\tilde{d}_H(\xi, p) = |\xi \cdot p^{-1}|_G = |\zeta \cdot w^{-1}|_G \leq -\lambda,$$

we have $\tilde{d}_H(p, \mathbb{H} \setminus E) \leq -\lambda$ or equivalently $\psi_E(p) \geq \lambda$, which is a contradiction to the condition $p \in E_\lambda$. We can similarly derive a contradiction if $\eta \in \mathbb{H} \setminus E$ holds.

The remaining case when $\xi, \eta \in E$ is an obvious contradiction to the assumption that E is h-convex, since $\zeta \in [\xi, \eta]$ and $\zeta \in \mathbb{H} \setminus E$. ∎

2.2. Viscosity characterization of h-quasiconvexity

The following characterization of h-quasiconvexity is known for smooth functions, see Theorem 4.5 in [7]. We provide a generalized result in the nonsmooth case by extending Proposition 2.2 in [5] to the Heisenberg group.

Theorem 2.6 (Viscosity characterization of h-quasiconvexity). *Let $\Omega \subset \mathbb{H}$ be open and h-convex, and let $u \colon \Omega \to [-\infty, \infty)$ be upper semicontinuous. Then, u is h-quasiconvex if and only if whenever there exist $p \in \Omega$ and $\varphi \in C^1(\Omega)$ such that $u - \varphi$ attains a maximum at p,*

$$(2.2) \quad \langle \nabla_H \varphi(p), (p^{-1} \cdot \xi)_h \rangle \leq 0 \quad \text{holds for any } \xi \in \mathbb{H}_p \cap \Omega \text{ satisfying } u(\xi) < u(p).$$

Remark 2.7. It is worth mentioning that, as in [7], we can also express the inner product term $\langle \nabla_H \varphi(p), (p^{-1} \cdot \xi)_h \rangle$ in (2.2) by $\langle \nabla \varphi(p), \xi - p \rangle$. This is possible because of the condition that $\xi \in \mathbb{H}_p$. We shall maintain the expression on the left-hand side to suggest possible generalization of our results in general Carnot groups, which is not elaborated in this paper.

Proof of Theorem 2.6. Let us prove the necessity of (2.2) by contradiction. Suppose, that u is upper semicontinuous h-quasiconvex function, $\varphi \in C^1(\Omega)$ is such that $u - \varphi$ attains a maximum at $p \in \Omega$, and there exists $\xi \in \mathbb{H}_p \cap \Omega$ with $u(\xi) < u(p)$ such that

$$\langle \nabla_H \varphi(p), (p^{-1} \cdot \xi)_h \rangle > 0.$$

Then, for $\lambda > 0$ small enough and $w = p \cdot \delta_\lambda(\xi^{-1} \cdot p)$ there holds $u(w) < u(p)$. Indeed, the directional derivative of φ at p in the direction $p^{-1} \cdot \xi$ is positive, and hence $\varphi(w) < \varphi(p)$ for $\lambda > 0$ small enough. Since $u - \varphi$ attains a maximum at p we obtain $u(p) > u(w)$. We conclude with $\xi \in \mathbb{H}_w$, $p \in [w, \xi]$ and $u(p) > \max\{u(w), u(\xi)\}$, which contradicts the h-quasiconvexity of u.

Now we are left with proving sufficiency of (2.2). Suppose that u is not h-quasiconvex. Then, without loss of generality there exists a point $\xi = (x_\xi, y_\xi, 0) \in \mathbb{H}_0$ such that

$$\max_{[0,\xi]} u > \max\{u(0), u(\xi)\}.$$

Denote by Z the set of maximizers of u on the segment $[0, \xi]$. By the upper semicontinuity of u, there exists $R \in (0, |\xi|/4)$ small enough such that

$$\min\{d_H(0, Z), d_H(\xi, Z)\} > R$$

(i.e., Z is in the relative interior of $[0, \xi]$), and

$$u(q) < \max_{[0,\xi]} u \quad \text{for any } q \in B_R(0) \cup B_R(\xi).$$

Let

$$\mathcal{C} := \{q \in \Omega : d_H(q, [0, \xi]) < R, 0 < \langle q, \xi \rangle < |\xi|^2\}$$

be a cylindrical neighborhood of the segment $[0, \xi]$. We define φ_n by

$$\varphi_n(p) = \frac{1}{n} \langle p, \xi \rangle + n \left((x_p\, y_\xi - y_p\, x_\xi)^2 + z_p^2\right), \quad p \in \Omega.$$

As $n \to \infty$, $u - \varphi_n \to u$ pointwise in $[0, \xi]$ and $u - \varphi_n \to -\infty$ elsewhere in $\overline{\mathcal{C}}$. Then there exists a sequence $p_n = (x_{p_n}, y_{p_n}, z_{p_n}) \in \mathcal{C}$ such that

$$\max_{\overline{\mathcal{C}}}(u - \varphi_n) = u(p_n) - \varphi_n(p_n),$$

which converges to a point in Z via a subsequence. Let us index the subsequence still by n for notational simplicity. Suppose that the subsequence $p_n \to (t x_\xi, t y_\xi, 0)$ as $n \to \infty$ for some $t \in (R/|\xi|, 1 - R/|\xi|)$. Let us consider $w_n = (x_{p_n}/t, y_{p_n}/t, z_{p_n}) \in \mathbb{H}_{p_n}$. Observing that

$$d_H(\xi, w_n) = |\xi^{-1} \cdot w_n|_G \to 0,$$

we get $u(w_n) < u(p_n)$ for n large enough.

Let us compute $\nabla_H \varphi_n(p_n)$ as follows:

$$
\begin{aligned}
&\nabla_H \varphi_n(p_n) \\
&= \left(\frac{x_\xi}{n} + 2n y_\xi(x_{p_n} y_\xi - y_{p_n} x_\xi) - n y_{p_n} z_{p_n}, \frac{y_\xi}{n} - 2n x_\xi(x_{p_n} y_\xi - y_{p_n} x_\xi) + n x_{p_n} z_{p_n} \right).
\end{aligned}
$$

Since

$$(p_n^{-1} \cdot w_n)_h = \left(\frac{1}{t} - 1\right)(x_{p_n}, y_{p_n}),$$

we have

$$\langle \nabla_H \varphi_n(p_n), (p_n^{-1} \cdot w_n)_h \rangle = \left(\frac{1}{t} - 1\right) \left(\frac{\langle p_n, \xi \rangle}{n} + 2n(x_{p_n} y_\xi - y_{p_n} x_\xi)^2 \right) > 0,$$

which contradicts (2.2). ∎

Theorem 2.6 amounts to saying that $u \in \mathrm{USC}(\Omega)$ is h-quasiconvex if (1.3) holds in the viscosity sense, that is,

$$\sup \left\{ \langle \nabla_H \varphi(p), (p^{-1} \cdot \xi)_h \rangle : \xi \in \mathbb{H}_p \cap \Omega, \, u(\xi) < u(p) \right\} \leq 0$$

whenever there exist $p \in \Omega$ and $\varphi \in C^1(\Omega)$ such that $u - \varphi$ attains a maximum at p. As a standard remark in the viscosity solution theory, the maximum here can be replaced by a local maximum or a strict maximum.

In spite of the nonlocal nature, we can obtain the following property by using the geometricity of the operator.

Lemma 2.8 (Invariance with respect to composition). *Let $\Omega \subset \mathbb{H}$ be h-convex and let $u \colon \Omega \to \mathbb{R}$ be bounded and upper semicontinuous. Assume that $g \colon \mathbb{R} \to \mathbb{R}$ is a nondecreasing continuous function. If u is h-quasiconvex in Ω, then so is $g \circ u$.*

Proof. It is clear that $g \circ u$ is still upper semicontinuous. Assume first that $g \in C^1(\mathbb{R})$ is strictly increasing. It is clear that $g \circ u$ is still upper semicontinuous. Suppose that there exists $p \in \Omega$ and $\varphi \in C^1(\Omega)$ such that $g \circ u - \varphi$ attains a maximum at p_0. Then using the inverse function g^{-1} of g, we see that $u - g^{-1} \circ \varphi$ also attains a maximum at p_0. Applying the characterization of h-quasiconvexity in Theorem 2.6, we get

$$\sup \left\{ \langle \nabla_H (g^{-1} \circ \varphi)(p_0), (p_0^{-1} \cdot \xi)_h \rangle : \xi \in \mathbb{H}_{p_0} \cap \Omega, \, u(\xi) < u(p_0) \right\} \leq 0,$$

which implies

$$(2.3) \quad \sup \left\{ \langle \nabla_H \varphi(p_0), (p_0^{-1} \cdot \xi)_h \rangle : \xi \in \mathbb{H}_{p_0} \cap \Omega, \, (g \circ u)(\xi) < (g \circ u)(p_0) \right\} \leq 0.$$

This shows that $g \circ u$ is h-quasiconvex.

For the general case when $g \in C(\mathbb{R})$ is nondecreasing, we can take a sequence $g_k \in C^1(\mathbb{R})$ increasing such that $g_k \to g$ locally uniformly in $\mathbb{R}$ as $k \to \infty$. In this case, suppose that there exists $p \in \Omega$ and $\varphi \in C^1(\Omega)$ such that $g \circ u - \varphi$ attains a strict maximum at p_0. Then there exist $p_k \in \Omega$ such that $p_k \to p_0$ as $k \to \infty$ and $g_k \circ u - \varphi$ attains a local maximum at p_k.

Adopting the argument above, we obtain the h-quasiconvexity of $g_k \circ u$, that is,

$$\sup \left\{ \langle \nabla_H \varphi(p_k), (p_k^{-1} \cdot \xi)_h \rangle : \xi \in \mathbb{H}_{p_k} \cap \Omega, \, (g_k \circ u)(\xi) < (g_k \circ u)(p_k) \right\} \leq 0.$$

By the uniform convergence of $g_k \circ u$ to $g \circ u$ in Ω and the continuity of $\mathbb{H}_p$ with respect to p, we can pass to the limit as $k \to \infty$ and obtain the relation (2.3) again. ∎

2.3. H-quasiconvex envelope

In what follows, we introduce the h-quasiconvex envelope of a given function.

Definition 2.9 (Definition of h-quasiconvex envelope). Let Ω be an h-convex domain in $\mathbb{H}$ and let $f \colon \Omega \to \mathbb{R}$ be a given function. We say that $Q(f)$ is the h-quasiconvex envelope of f if it is the greatest h-quasiconvex function majorized by f, that is,

$$Q(f)(p) := \sup \{ g(p) : g \leq f \text{ and } g \text{ is h-quasiconvex in } \Omega \}, \quad \text{for } p \in \Omega,$$

where we adopt the convention that $\sup \emptyset = -\infty$.

By definition, $Q(f)$ is monotone in f; namely, $Q(f) \leq Q(g)$ in Ω holds provided that $f \leq g$ in Ω. Moreover, $Q(f)$ is stable with respect to f in the following sense.

Proposition 2.10 (Stability of h-quasiconvex envelope). *Suppose that Ω is an h-convex domain in $\mathbb{H}$, and let $f, g \colon \Omega \to \mathbb{R}$ be given functions. Assume that both $Q(f)$ and $Q(g)$ exist in Ω. Then there holds*

$$(2.4) \qquad \sup_{\Omega} |Q(f) - Q(g)| \leq \sup_{\Omega} |f - g|.$$

Proof. Let $M := \sup_{\Omega} |f - g|$. Since $f - M \leq g$ in Ω, by the monotonicity of Q, we get

$$(2.5) \qquad Q(f - M) \leq Q(g) \quad \text{in } \Omega.$$

Noticing that $Q(f) - M$ is h-quasiconvex and $Q(f) \leq f$ in Ω, by Definition (2.9), we deduce that

$$Q(f) - M \leq Q(f - M) \quad \text{in } \Omega,$$

which, by (2.5), yields

$$Q(f) - M \leq Q(g) \quad \text{in } \Omega.$$

Exchanging the roles of f and g, we conclude the proof of (2.4). ∎

Let us now discuss how to find the h-convex envelope of a given function. A straightforward method is to employ a convexification operator. For an h-convex domain $\Omega \subset \mathbb{H}$ and $f \colon \Omega \to \mathbb{R}$, let $T[f]$ be given by (1.9). It is clear that $\inf_{\Omega} f \leq T[f] \leq f$ in Ω. Also, it is easily seen that $T[f] = f$ in Ω if and only if f is h-quasiconvex.

This operator is inspired by its Euclidean analogue, which is given by

$$T_{\mathrm{eucl}}(w) = \inf \{\max\{f(p), f(q)\} : w \in [p, q], p \in \Omega, q \in \Omega\}, \quad w \in \mathbb{R}^3.$$

In the Euclidean case, where the quasiconvex envelope, written as $Q_{\mathrm{eucl}}(f)$, satisfies

$$Q_{\mathrm{eucl}}(f) = T_{\mathrm{eucl}}[f]$$

in a bounded convex domain Ω. In contrast, the following example shows that in the Heisenberg group, in general $T[f]$ is not necessarily an h-quasiconvex function.

Example 2.11. Let $f \colon \mathbb{H} \to \mathbb{R}$ be defined as $f(p) = |1 - z^2|$ for $p = (x, y, z) \in \mathbb{H}$. One can compute directly to get, for $p = (x, y, z)$,

$$T[f](p) = \begin{cases} z^2 - 1, & |z| \geq 1, \\ 0, & |z| < 1 \text{ and } (x, y) \neq (0, 0), \\ 1 - z^2, & |z| < 1 \text{ and } (x, y) = (0, 0), \end{cases}$$

which fails to be quasiconvex. In fact, letting $p = (x, y, t)$, $q = (-x, -y, t)$ with $|t| < 1$, we see that $q \in \mathbb{H}_p$ and at $w = (0, 0, t) \in [p, q]$, we have

$$T[f](w) = 1 - t^2 > 0 = \max\{T[f](p), T[f](q)\}.$$

However, if we apply the operator one more time, then we have, for $p = (x, y, z) \in \mathbb{H}$,

$$T^2[f](p) = \begin{cases} z^2 - 1, & |z| \geq 1, \\ 0, & |z| < 1. \end{cases}$$

It is not difficult to see that $Q(f) = T^2[f]$ in $\mathbb{H}$. Indeed, Noticing that $T^2[f]$ is h-quasiconvex, by definition we have $T^2[f] \leq Q(f)$ in $\mathbb{H}$. On the other hand, the reverse inequality $Q(f) \leq T^2[f]$ can be obtained by applying the operator T twice to the inequality $Q(f) \leq f$.

It turns out that in general one can obtain the quasiconvex envelope by iterating the operator T. Such type of iteration is also used in [22] to construct the h-convex envelope of a given continuous function in the Heisenberg group.

Theorem 2.12 (Iterative scheme with direct convexification). *Let Ω be an h-convex domain in $\mathbb{H}$. Suppose that f is bounded from below. Let T be the operator given by* (1.9). *Then $T^n[f] \to Q(f)$ pointwise in Ω as $n \to \infty$.*

Proof. Notice that by the monotonicity of $T^n[f]$ in n and the boundedness of f from below, the pointwise limit of $T^n[f]$ exists. Let us denote it by F, i.e.,

$$F := \lim_{n \to \infty} T^n[f].$$

Let us fix $\varepsilon > 0$, $p \in \Omega$, $q \in \Omega \cap \mathbb{H}_p$ and $w \in [p, q]$. For n sufficiently large, there holds

$$F(p) \geq T^n[f](p) - \varepsilon \quad \text{and} \quad F(q) \geq T^n[f](q) - \varepsilon.$$

Moreover, we have

$$\max\{T^n[f](p), T^n[f](q)\} \geq T^{n+1}[f](w) \geq F(w),$$

and therefore

$$\max\{F(p), F(q)\} \geq F(w) - \varepsilon.$$

Letting $\varepsilon \to 0$, we deduce that F is h-quasiconvex and thus $F \leq Q(f)$ in Ω.

On the other hand, for any $w \in \Omega$, and $p \in \Omega$ and $q \in \Omega \cap \mathbb{H}_p$ such that $w \in [p, q]$, there holds

$$\max\{f(p), f(q)\} \geq \max\{Q(f)(p), Q(f)(q)\} \geq Q(f)(w).$$

It follows that $T[f] \geq Q(f)$ in Ω. We can iterate this argument to obtain $T^n[f] \geq Q(f)$ in Ω for every n. Hence, sending $n \to \infty$, we conclude that $F \geq Q(f)$ holds in Ω, which completes the proof. ∎

As shown in Example 2.11, $T^n[f] = Q(f)$ may hold for a finite n. We do not know in general how many iterations one needs to run to obtain $Q(f)$. It would be interesting to find a condition to guarantee the finiteness of n.

3. Nonlocal Hamilton–Jacobi equation

Inspired by [5], we would like to focus our attention on a PDE-based approach to build the h-quasiconvex envelope of a given function f in a bounded domain $\Omega \subset \mathbb{H}$. We develop the idea in Theorem 2.6, which provides a characterization of h-quasiconvexity in terms of viscosity subsolutions of a nonlocal PDE. For our convenience of notations below, for any function $u: \Omega \to \mathbb{R}$ and any $p \in \Omega$, we denote

$$S_p(u) = \{\xi \in \mathbb{H}_p \cap \Omega : u(\xi) < u(p)\}.$$

We study the following nonlocal Hamilton–Jacobi equation:

$$(3.1) \qquad u(p) + H(p, u(p), \nabla_H u(p)) = f(p) \quad \text{in } \Omega,$$

to which the subsolutions are defined with

$$H(p, u(p), \nabla_H u(p)) = \sup \left\{ \langle \nabla_H u(p), (p^{-1} \cdot \xi)_h \rangle : \xi \in S_p(u) \right\},$$

while the supersolutions are defined with

$$H(p, u(p), \nabla_H u(p)) = \sup \left\{ \langle \nabla_H u(p), (p^{-1} \cdot \xi)_h \rangle : \xi \in \hat{S}_p(u) \right\}.$$

Here we set

$$\hat{S}_p(u) := \{\xi \in \mathbb{H}_p \cap \Omega : u(\xi) \le u(p)\}.$$

The major difficulty lies at the degeneracy and nonlocal nature of the first order operator. We mainly study uniqueness and existence of solutions to this equation in a slightly restrictive setting, assuming that the solutions u satisfy

$$(3.2) \qquad \qquad \left\{ \begin{array}{ll} u = K & \text{on } \partial\Omega, \\ (3.3) \qquad \qquad u \le K & \text{in } \overline{\Omega} \end{array} \right.$$

for some $K \in \mathbb{R}$. It turns out that we can obtain a unique solution if f satisfies the same conditions.

These conditions can be viewed as a bounded-domain variant of coercivity assumption on u. If $u \in C(\mathbb{H})$ is coercive in $\mathbb{H}$, that is, if

$$\lim_{R \to \infty} \inf_{|p|_G \ge R} u(p) \to \infty,$$

then we can take $K \in \mathbb{R}$ large and $\Omega = \{p : u(p) < K\}$ such that both (3.2) and (3.3) hold.

3.1. Definition and basic properties of solutions

Let us first present the definition of subsolutions of (3.1).

Definition 3.1 (Subsolutions). Let $f \in \mathrm{USC}(\Omega)$ be locally bounded in Ω. A locally bounded upper semicontinuous function $u: \Omega \to \mathbb{R}$ is called a subsolution of (3.1) if whenever there exist $p \in \Omega$ and $\varphi \in C^1(\Omega)$ such that $u - \varphi$ attains a maximum at p,

$$(3.4) \qquad u(p) + \sup \left\{ \langle \nabla_H \varphi(p), (p^{-1} \cdot \xi)_h \rangle : \xi \in S_p(u) \right\} \le f(p).$$

Note that the supremum in (3.4) does make sense. We consider naturally

$$\sup \left\{ \left\langle \nabla_H \varphi(p), (p^{-1} \cdot \xi)_h \right\rangle : \xi \in S_p(u) \right\} = 0$$

if $\nabla_H \varphi(p) = 0$. It is also easily seen that $S_p(u) \neq \emptyset$ provided that $\nabla_H \varphi(p) \neq 0$ and $u - \varphi$ attains a local maximum at p.

We next give a definition of supersolutions.

Definition 3.2 (Supersolutions). Let $f \in \mathrm{LSC}(\Omega)$ be locally bounded in Ω. A locally bounded lower semicontinuous function $u: \Omega \to \mathbb{R}$ is called a weak supersolution of (3.1) if whenever there exist $p \in \Omega$ and $\varphi \in C^1(\Omega)$ such that $u - \varphi$ attains a minimum at p,

$$(3.5) \qquad u(p) + \sup \left\{ \left\langle \nabla_H \varphi(p), (p^{-1} \cdot \xi)_h \right\rangle : \xi \in \hat{S}_p(u) \right\} \geq f(p).$$

For any point $p \in \Omega$, we also say that $u \in \mathrm{USC}(\Omega)$ (respectively, $\mathrm{LSC}(\Omega)$) satisfies the subsolution (respectively, supersolution) property at p if (3.4) (respectively, (3.5)) holds for any $\varphi \in C^1(\Omega)$ such that $u - \varphi$ attains a maximum (respectively, minimum) at p.

As a standard remark in the theory of viscosity solutions, we may replace the maximum in the definitions above by local maximum or strict maximum.

Lemma 3.3 (Upper bound). *Suppose that Ω is a domain in $\mathbb{H}$. Let $f \in \mathrm{USC}(\Omega)$ be locally bounded in Ω. If $u \in \mathrm{USC}(\Omega)$ is a subsolution of (3.1), then $u \leq f$ in Ω.*

Proof. Let us first show $u \leq f$ at all points where u can be tested. Assume that there exists $\varphi \in C^1(\Omega)$ such that $u - \varphi$ attains a maximum at some $p_0 \in \Omega$. If $\nabla_H \varphi(p_0) = 0$, then we immediately obtain the desired inequality $u(p_0) \leq f(p_0)$ by Definition 3.1.

If $\nabla_H \varphi(p_0) \neq 0$ and thus $S_{p_0}(u) \neq \emptyset$, we can take a sequence $\xi_j \in S_{p_0}(u)$ such that $\xi_j \to p_0$ as $j \to \infty$. This yields

$$\sup \left\{ \left\langle \nabla_H \varphi(p_0), (p_0^{-1} \cdot \xi)_h \right\rangle : \xi \in S_{p_0}(u) \right\} \geq \limsup_{j \to \infty} \left\langle \nabla_H \varphi(p_0), (p_0^{-1} \cdot \xi_j)_h \right\rangle = 0,$$

As a result, we get $u(p_0) \leq f(p_0)$ by Definition 3.1 again.

It remains to show that $u \leq f$ holds also at those points where u cannot be tested. Fix $p_0 \in \Omega$ arbitrarily. Since u is locally bounded, for $\varepsilon > 0$ small, we can find $p_\varepsilon \in \Omega$ in a neighborhood of p_0 such that

$$p \mapsto u(p) - \frac{1}{\varepsilon} |p_0^{-1} \cdot p|_G^4$$

attains a local maximum in Ω at p_ε. In particular, we have

$$(3.6) \qquad u(p_0) \leq u(p_\varepsilon) - \frac{1}{\varepsilon} |p_0^{-1} \cdot p_\varepsilon|_G^4.$$

By the local boundedness of u, we have $p_\varepsilon \to p_0$ as $\varepsilon \to 0$. Noticing that u is tested from above by a smooth function at p_ε, we may apply our result shown in the first part to deduce $u(p_\varepsilon) \leq f(p_\varepsilon)$. It follows from (3.6) that $u(p_0) \leq f(p_\varepsilon)$. Sending $\varepsilon \to 0$ and applying the upper semicontinuity of f, we end up with $u(p_0) \leq f(p_0)$. ∎

Let us present more properties for (3.1).

Lemma 3.4 (Basic properties). *Suppose that Ω is a domain in $\mathbb{H}$. For each locally bounded $f \in \mathrm{USC}(\Omega)$, let $\mathcal{A}[f]$ denote the set of all subsolutions of (3.1). Then the following properties hold.*

(i) *(Monotonicity) For any locally bounded $f_1, f_2 \in \mathrm{USC}(\Omega)$ satisfying $f_1 \leq f_2$ in Ω, $\mathcal{A}[f_1] \subset \mathcal{A}[f_2]$ holds.*

(ii) *(Constant addition invariance) For any $c \in \mathbb{R}$ and $u \in \mathcal{A}[f]$, $u + c \in \mathcal{A}[f + c]$ holds.*

(iii) *(Left translation invariance) For any $\eta \in \mathbb{H}$, $u_\eta \in \mathcal{A}[f_\eta]$ holds, where u_η and f_η are given by*

$$u_\eta(p) = u(\eta \cdot p) \quad and \quad f_\eta(p) = f(\eta \cdot p), \quad for\ p \in \Omega_\eta.$$

Analogous properties to the above also hold for supersolutions of (3.1).

We omit the details of the proof, since it is quite straightforward from the structure of the Hamiltonian.

3.2. Comparison principle

We next establish a comparison principle for (3.1) under the conditions (3.2) and (3.3). General Dirichlet boundary problems will also be briefly discussed later.

Theorem 3.5 (Comparison principle with constant boundary data). *Let Ω be a bounded domain in $\mathbb{H}$ and let $f \in C(\Omega)$. Let $u \in \mathrm{USC}(\overline{\Omega})$ and $v \in \mathrm{LSC}(\overline{\Omega})$ be respectively a subsolution and a supersolution of (3.1). Assume in addition that*

$$(3.7) \qquad\qquad u \leq v = K \quad on\ \partial\Omega$$

and $u \leq K$ in $\overline{\Omega}$ for some $K \in \mathbb{R}$. Then $u \leq v$ in $\overline{\Omega}$.

Proof. Assume by contradiction that $\max_{\overline{\Omega}}(u - v) = \sigma$ for some $\sigma > 0$. For $\varepsilon > 0$ small, we consider

$$\Phi_\varepsilon(p, q) = u(p) - v(q) - \frac{|p \cdot q^{-1}|_G^4}{\varepsilon}$$

for $p, q \in \overline{\Omega}$. It is clear that Φ_ε attains a positive maximum in $\overline{\Omega} \times \overline{\Omega}$. Let $(p_\varepsilon, q_\varepsilon) \in \overline{\Omega} \times \overline{\Omega}$ be a maximizer. We thus obtain

$$(3.8) \qquad\qquad \Phi_\varepsilon(p_\varepsilon, q_\varepsilon) \geq \sigma,$$

which implies that

$$(3.9) \qquad\qquad u(p_\varepsilon) - v(q_\varepsilon) \geq \sigma$$

and

$$(3.10) \qquad\qquad \frac{|p_\varepsilon \cdot q_\varepsilon^{-1}|_G^4}{\varepsilon} \leq u(p_\varepsilon) - v(q_\varepsilon) - \sigma$$

for all $\varepsilon > 0$. Due to the boundedness of u and v, it follows from the inequality (3.10) that $|p_\varepsilon \cdot q_\varepsilon^{-1}|_G \to 0$, which in turn implies that $d_H(p_\varepsilon, q_\varepsilon) = |p_\varepsilon^{-1} \cdot q_\varepsilon|_G \to 0$ as $\varepsilon \to 0$.

By taking a subsequence, still indexed by ε, we have $p_\varepsilon, q_\varepsilon \to p_0$ as $\varepsilon \to 0$ for some point $p_0 \in \overline{\Omega}$. Thanks to (3.8) and the assumption that $u \leq v$ on $\partial\Omega$, we deduce that $p_0 \in \Omega$ and $p_\varepsilon, q_\varepsilon \in \Omega$ for all $\varepsilon > 0$ small.

We now apply the definition of subsolutions and supersolutions. Note that $u - \varphi_1$ attains a maximum at p_ε and $v - \varphi_2$ attains a minimum at q_ε, where we take

$$\varphi_1(p) = v(q_\varepsilon) + \frac{|p \cdot q_\varepsilon^{-1}|_G^4}{\varepsilon} \quad \text{and} \quad \varphi_2(q) = u(p_\varepsilon) - \frac{|p_\varepsilon \cdot q^{-1}|_G^4}{\varepsilon}$$

for $p, q \in \mathbb{H}$. Writing

$$p_\varepsilon = (x_{p_\varepsilon}, y_{p_\varepsilon}, z_{p_\varepsilon}) \quad \text{and} \quad q_\varepsilon = (x_{q_\varepsilon}, y_{q_\varepsilon}, z_{q_\varepsilon}),$$

we see, by direct calculations, that

$$\nabla_H \varphi_1(p_\varepsilon) = \nabla_H \varphi_2(q_\varepsilon)$$

$$(3.11) \quad = \frac{1}{\varepsilon}\Big(4A_\varepsilon(x_{p_\varepsilon} - x_{q_\varepsilon}) - 16B_\varepsilon(y_{p_\varepsilon} + y_{q_\varepsilon}), \, 4A_\varepsilon(y_{p_\varepsilon} - y_{q_\varepsilon}) + 16B_\varepsilon(x_{p_\varepsilon} + x_{q_\varepsilon})\Big),$$

where

$$A_\varepsilon = (x_{p_\varepsilon} - x_{q_\varepsilon})^2 + (y_{p_\varepsilon} - y_{q_\varepsilon})^2 \quad \text{and} \quad B_\varepsilon = z_{p_\varepsilon} - z_{q_\varepsilon} - \frac{1}{2}x_{p_\varepsilon}y_{q_\varepsilon} + \frac{1}{2}y_{p_\varepsilon}x_{q_\varepsilon}.$$

We next discuss two different cases.

Case 1. Suppose that there exists a subsequence, again indexed by ε for simplicity of notation, such that

$$\sup\big\{\langle \nabla_H \varphi_2(q_\varepsilon), (q_\varepsilon^{-1} \cdot \eta)_h \rangle : \eta \in \hat{S}_{q_\varepsilon}(v)\big\} = 0.$$

Then applying the definition of supersolutions to v, we get

$$(3.12) \qquad\qquad\qquad\qquad v(q_\varepsilon) \geq f(q_\varepsilon).$$

On the other hand, by Lemma 3.3, we obtain

$$(3.13) \qquad\qquad\qquad\qquad u(p_\varepsilon) \leq f(p_\varepsilon).$$

Combining (3.12) and (3.13), we have

$$u(p_\varepsilon) - v(q_\varepsilon) \leq f(p_\varepsilon) - f(q_\varepsilon)$$

for $\varepsilon > 0$ small. Letting $\varepsilon \to 0$, we use the continuity of f to get

$$\limsup_{\varepsilon \to 0} (u(p_\varepsilon) - v(q_\varepsilon)) \leq 0,$$

which is a contradiction to (3.9).

Case 1. Suppose that

$$\sup\big\{\langle \nabla_H \varphi_2(q_\varepsilon), (q_\varepsilon^{-1} \cdot \eta)_h \rangle : \eta \in \hat{S}_{q_\varepsilon}(v)\big\} > 0$$

for all $\varepsilon > 0$ small. We then can find a sequence $\eta_{\varepsilon,n} \in \hat{S}_{q_\varepsilon}(v)$ such that

$$(3.14) \qquad \langle \nabla_H \varphi_2(q_\varepsilon), (w_n)_h \rangle \geq \sup\big\{\langle \nabla_H \varphi_2(q_\varepsilon), (q_\varepsilon^{-1} \cdot \eta)_h \rangle : \eta \in \hat{S}_{q_\varepsilon}(v)\big\} - \frac{1}{n}$$

as $n \to \infty$, where $w_n = q_\varepsilon^{-1} \cdot \eta_{\varepsilon,n} \in \mathbb{H}_0$. In particular, we have $v(\eta_{\varepsilon,n}) \leq v(q_\varepsilon)$.

In view of (3.3) and (3.9), we get $v(q_\varepsilon) \leq K - \sigma$. On the other hand, since $v = K$ due to (3.7), we see that as $n \to \infty$ and $\varepsilon \to 0$, $\eta_{\varepsilon,n}$ cannot converge to a boundary point. In other words, there exists $r > 0$ such that $B_r(\eta_{\varepsilon,n}) \subset \Omega$. We may assume that ε is small enough so that $r > |p_\varepsilon \cdot q_\varepsilon^{-1}|_G$.

Besides, it also follows from (3.14) that

$$(3.15) \qquad \langle \nabla_H \varphi_2(q_\varepsilon), (w_n)_h \rangle > 0$$

for any $n \geq 1$ large. Taking

$$\mu(s) = |p_\varepsilon \cdot (sw_n) \cdot \eta_{\varepsilon,n}^{-1}|_G^4 = |p_\varepsilon \cdot (s-1)w_n \cdot q_\varepsilon^{-1}|_G^4, \quad \text{for } s \in \mathbb{R},$$

with sw_n denoting the usual constant multiple of w_n by the factor s, i.e.,

$$sw_n = (sx_{w_n}, sy_{w_n}, 0) \quad \text{for } w_n = (x_{w_n}, y_{w_n}, 0),$$

we get, by direct calculations,

$$\mu'(s) = 4A_{\varepsilon,s}(x_{p_\varepsilon} - x_{q_\varepsilon} + (s-1)x_{w_n})x_{w_n} + 4A_{\varepsilon,s}(y_{p_\varepsilon} - y_{q_\varepsilon} + (s-1)y_{w_n})y_{w_n}$$
$$- 16 B_{\varepsilon,s}(y_{p_\varepsilon} + y_{q_\varepsilon})x_{w_n} + 16 B_{\varepsilon,s}(x_{p_\varepsilon} + x_{q_\varepsilon})y_{w_n},$$

where we let

$$A_{\varepsilon,s} = (x_{p_\varepsilon} - x_{q_\varepsilon} + (s-1)x_{w_n})^2 + (y_{p_\varepsilon} - y_{q_\varepsilon} + (s-1)y_{w_n})^2,$$
$$B_{\varepsilon,s} = z_{p_\varepsilon} - z_{q_\varepsilon} - \frac{1}{2}x_{p_\varepsilon}y_{q_\varepsilon} + \frac{1}{2}y_{p_\varepsilon}x_{q_\varepsilon} - \frac{1}{2}(s-1)x_{w_n}y_{q_\varepsilon} + \frac{1}{2}(s-1)y_{w_n}x_{q_\varepsilon}.$$

It is then clear that

$$\mu'(1) = (4A_\varepsilon(x_{p_\varepsilon} - y_{q_\varepsilon}) - 16B_\varepsilon(y_{p_\varepsilon} + y_{q_\varepsilon}))x_{w_n} + (4A_\varepsilon(y_{p_\varepsilon} - y_{q_\varepsilon}) + 16B_\varepsilon(x_{p_\varepsilon} + x_{q_\varepsilon}))y_{w_n}$$
$$= \varepsilon \langle \nabla_H \varphi_2(q_\varepsilon), w_n \rangle.$$

Owing to (3.15), we are led to $\mu'(1) > 0$, which implies

$$\mu(s) = |p_\varepsilon \cdot (sw_n) \cdot \eta_{\varepsilon,n}^{-1}|_G^4 < |p_\varepsilon \cdot q_\varepsilon^{-1}|_G^4$$

for any $s < 1$ sufficiently close to 1. This amounts to saying that we can take $s_n \in (0, 1)$ such that $s_n \to 1$ as $n \to \infty$ and

$$(3.16) \qquad |\xi_{\varepsilon,n} \cdot \eta_{\varepsilon,n}^{-1}|_G^4 < |p_\varepsilon \cdot q_\varepsilon^{-1}|_G^4$$

for $\xi_{\varepsilon,n} = p_\varepsilon \cdot (s_n w_n)$. This yields $\xi_{\varepsilon,n} \in B_r(\eta_{\varepsilon,n})$ and thus $\xi_{\varepsilon,n} \in \Omega$. It is also clear that $\xi_{\varepsilon,n} \in \mathbb{H}_{p_\varepsilon}$ and

$$(3.17) \qquad \left| (p_\varepsilon^{-1} \cdot \xi_{\varepsilon,n})_h - (q_\varepsilon^{-1} \cdot \eta_{\varepsilon,n})_h \right| \to 0 \quad \text{as } n \to \infty.$$

In view of the maximality of Φ_ε at $(p_\varepsilon, q_\varepsilon)$, we have

$$u(\xi_{\varepsilon,n}) - v(\eta_{\varepsilon,n}) - \frac{1}{\varepsilon}|\xi_{\varepsilon,n} \cdot \eta_{\varepsilon,n}^{-1}|_G^4 \leq u(p_\varepsilon) - v(q_\varepsilon) - \frac{1}{\varepsilon}|p_\varepsilon \cdot q_\varepsilon^{-1}|_G^4$$

which by (3.16) yields

$$u(\xi_{\varepsilon,n}) - u(p_{\varepsilon}) < v(\eta_{\varepsilon,n}) - v(q_{\varepsilon}) \leq 0.$$

We have shown that $\xi_{\varepsilon,n} \in S_{p_{\varepsilon}}(u)$. Recalling the definition of subsolutions, we deduce that

$$u(p_{\varepsilon}) + \langle \nabla_H \varphi_1(p_{\varepsilon}), (p_{\varepsilon}^{-1} \cdot \xi_{\varepsilon,n})_h \rangle \leq f(p_{\varepsilon}).$$

On the other hand, we can also apply the definition of supersolutions, together with (3.14), to get

$$(3.18) \qquad v(q_{\varepsilon}) + \langle \nabla_H \varphi_2(q_{\varepsilon}), (q_{\varepsilon}^{-1} \cdot \eta_{\varepsilon,n})_h \rangle \geq f(q_{\varepsilon}) - \frac{1}{n}$$

for $n \geq 1$ large. Combining these two inequalities, we use (3.11) to obtain

$$u(p_{\varepsilon}) - v(q_{\varepsilon}) \leq f(p_{\varepsilon}) - f(q_{\varepsilon}) - \langle \nabla_H \varphi_1(p_{\varepsilon}), (p_{\varepsilon}^{-1} \cdot \xi_{\varepsilon,n})_h - (q_{\varepsilon}^{-1} \cdot \eta_{\varepsilon,n})_h \rangle + \frac{1}{n}.$$

Sending $n \to \infty$ and applying (3.17), we get

$$u(p_{\varepsilon}) - v(q_{\varepsilon}) \leq f(p_{\varepsilon}) - f(q_{\varepsilon}),$$

which is a contradiction to (3.9) and the continuity of f. ∎

It is possible to give a slightly different comparison theorem for more general Dirichlet boundary problems under the h-convexity assumption on Ω. To this end, we prove the following lemma using the intrinsic cone property of h-convexity given in Theorem 1.4 of [1]; see also [26] for regularity results related to this property.

Lemma 3.6. *Let $\Omega \subset \mathbb{H}$ be an open set. If Ω is h-convex, then for every $p \in \partial\Omega$ and $q \in \mathbb{H}_p \cap \Omega$, the horizontal segment $(p, q]$ joining p and q stays in Ω.*

Proof. Assume by contradiction that there exist $p \in \partial\Omega$, $q \in \mathbb{H}_p \cap \Omega$ such that the half-open horizontal segment $(p, q]$ does not stay in Ω. Let w be the closest point to q on $[q, p)$ that is not in Ω, i.e., $w = p \cdot \lambda_0(p^{-1} \cdot q)$, where

$$\lambda_0 = \sup\{\lambda \geq 0 : p \cdot \lambda(p^{-1} \cdot q) \notin \Omega\}.$$

Then $w \in \partial\Omega$, $q \in \mathbb{H}_w$ and $w \cdot \lambda(w^{-1} \cdot q) \in \Omega$ for all $\lambda \in (\lambda_0, 1)$. Such w is a so-called non-characteristic point defined in [1]. In view of Theorem 1.4 in [1], the h-convexity of Ω implies the existence of an intrinsic cone in the exterior of Ω with vertex w and axis along the horizontal segment $(p, w]$, which further enables us to find a point $z \in (p, w)$ and $r > 0$ such that $B_r(z) \subset \mathbb{H} \setminus \Omega$. Noticing that there exists a sequence $p_j \in \Omega$ such that $p_j \to p$ as $j \to \infty$, we can take $q_j = p_j \cdot p^{-1} \cdot q$ and $z_j = p_j \cdot p^{-1} \cdot z$ so that $q_j, z_j \in \mathbb{H}_{p_j}$ and $q_j \to q$, $z_j \to z$ as $j \to \infty$. We choose $j > 0$ large such that $q_j \in \Omega$ and $z_j \in B_r(z)$. This contradicts the h-convexity of Ω, since $p_j, q_j \in \Omega$, but the point z_j on the horizontal segment $[p_j, q_j]$ is not in Ω. ∎

Theorem 3.7 (Comparison principle under domain convexity). *Let Ω be a bounded h-convex domain in $\mathbb{H}$ and let $f \in C(\overline{\Omega})$. Let $u \in \mathrm{USC}(\overline{\Omega})$ and $v \in \mathrm{LSC}(\overline{\Omega})$ be respectively a subsolution and a supersolution of (3.1). If $u \leq v$ on $\partial\Omega$, then $u \leq v$ in $\overline{\Omega}$.*

Proof. The proof is almost the same as that of Theorem 3.5 except for some necessary modifications to handle this general boundary condition. The only difference lies at the argument by contradiction in Case 2 on the occasion when $\eta_{\varepsilon,n} \in \hat{S}_{q_\varepsilon}$ converges to a boundary point $\eta_0 \in \partial\Omega$ as $n \to \infty$ and $\varepsilon \to 0$. (This gives a contradiction to the conditions (3.3) and (3.7) in the setting of Theorem 3.5.)

Let us derive a contradiction only in this case. Thanks to the condition $u \le v$ on $\partial\Omega$, we have $u(\eta_0) \le v(\eta_0)$, which by (3.9) and the upper semicontinuity of u yields the existence of $r > 0$ such that

$$(3.19) \qquad B_r(\eta_0) \cap \Omega \subset \{z \in \Omega : u(z) < u(p_\varepsilon)\}$$

for any $\varepsilon > 0$ small.

Since $p_\varepsilon, q_\varepsilon \to p_0 \in \Omega$, by Lemma 3.6 we can utilize the h-convexity of Ω to deduce that

$$\eta^\lambda = (1-\lambda)p_0 + \lambda\eta_0 \in \Omega$$

for all $0 \le \lambda < 1$. We may also choose λ close to 1, depending only on r, such that $\eta^\lambda \in B_r(\eta_0) \cap \Omega$. Letting

$$\eta_{\varepsilon,n}^\lambda = (1-\lambda)q_\varepsilon + \lambda\eta_{\varepsilon,n} = q_\varepsilon \cdot (\lambda w_n),$$

we have $\eta_{\varepsilon,n}^\lambda \in B_r(\eta_0) \cap \Omega$ when n is sufficiently large and ε is sufficiently small. Let us now take $\xi_{\varepsilon,n}^\lambda = p_\varepsilon \cdot (\lambda w_n)$. It is easily seen that

$$|\xi_{\varepsilon,n}^\lambda \cdot (\eta_{\varepsilon,n}^\lambda)^{-1}|_G \to 0 \quad \text{as } n \to \infty \text{ and } \varepsilon \to 0,$$

and thus $\xi_{\varepsilon,n}^\lambda \in B_r(\eta_0)$ for n large and ε small. By (3.19), we have $\xi_{\varepsilon,n}^\lambda \in S_{p_\varepsilon}(u)$. It follows from the definition of subsolutions that

$$(3.20) \qquad u(p_\varepsilon) + \langle \nabla_H \varphi_1(p_\varepsilon), \lambda(w_n)_h \rangle \le f(p_\varepsilon).$$

Combining this with (3.18) and using (3.9) and (3.11), we obtain

$$\sigma \le u(p_\varepsilon) - v(q_\varepsilon) \le \frac{1}{n} + (1-\lambda)\langle \nabla_H \varphi_1(p_\varepsilon), (w_n)_h \rangle.$$

Applying (3.20) and (3.9) again, we are led to

$$\sigma \le \frac{1}{n} + \frac{1-\lambda}{\lambda}(f(p_\varepsilon) - u(p_\varepsilon)) \le \frac{1}{n} + \frac{1-\lambda}{\lambda}(f(p_\varepsilon) - v(p_\varepsilon)).$$

Sending $n \to \infty$ and $\varepsilon \to 0$, we get

$$\sigma \le \frac{1-\lambda}{\lambda}(f(p_0) - v(p_0)).$$

Passing to the limit as $\lambda \to 1$ immediately yields a contradiction. ∎

3.3. Existence of solutions

We also adapt Perron's method to show the existence of solutions of (3.1) with f satisfying (1.4) and (1.5). In the sequel, for a locally bounded function $u : \Omega \to \mathbb{R}$, we denote by u^* and u_* its upper and lower semicontinuous envelopes, respectively.

Theorem 3.8 (Existence of solutions). *Let $\Omega \subset \mathbb{H}$ be a bounded domain. Assume that $f \in C(\overline{\Omega})$ satisfies* (1.4) *and* (1.5) *for some $K \in \mathbb{R}$. Assume that there exists a subsolution $\underline{u} \in C(\overline{\Omega})$ of* (3.1) *satisfying $\underline{u} = K$ on $\partial\Omega$. For any $p \in \overline{\Omega}$, let*

(3.21) $$U(p) = \sup\{u(p) : u \in \mathrm{USC}(\overline{\Omega}) \text{ is a subsolution of } (3.1)\}.$$

Then U^* is continuous in $\overline{\Omega}$ and is the unique solution of (3.1) *satisfying $U^* \leq f \leq K$ in $\overline{\Omega}$ and $U^* = f = K$ on $\partial\Omega$.*

Remark 3.9. In the theorem above, if the domain Ω is further assumed to be h-convex, then the existence of $\underline{u}$ is guaranteed; see Proposition 4.1 below for an explicit construction of $\underline{u}$ satisfying the required conditions.

Remark 3.10. In view of Lemma 3.3, we see that any subsolution u satisfies $u \leq f$ in Ω. Therefore $\underline{u} \leq u \leq f$ in $\overline{\Omega}$ holds and U in (3.21) is well-defined.

Let us first show that U is a subsolution. To this purpose, we prove the following result, where we do not need the coercivity-like conditions (3.2) and (3.3).

Proposition 3.11 (Maximum subsolution). *Suppose that Ω is a domain in $\mathbb{H}$ and let $f \in \mathrm{USC}(\Omega)$. Let $\mathcal{A}$ be a family of subsolutions of* (3.1). *Let v be given by*

$$v(p) = \sup\{u(p) : u \in \mathcal{A}\}, \quad \text{for } p \in \Omega.$$

Then v^* is a subsolution of (3.1).

Proof. Suppose that there exist $\varphi \in C^1(\Omega)$ and $p_0 \in \Omega$ such that $v^* - \varphi$ attains a strict maximum at p_0. Then we can find a sequence $u_j \in \mathcal{A}$ and $p_j \in \Omega$ such that $u_j - \varphi$ attains a maximum at p_j, and, as $j \to \infty$,

(3.22) $$p_j \to p_0 \quad \text{and} \quad u_j(p_j) \to v^*(p_0).$$

Let us first consider the case when $\nabla_H \varphi(p_0) \neq 0$. It follows immediately that $S_{p_0}(v^*) \neq \emptyset$. For any $\delta > 0$ small, there exists $q_\delta \in S_{p_0}(v^*)$ such that

(3.23) $$\sup\left\{\langle \nabla_H \varphi(p_0), (p_0^{-1} \cdot \xi)_h \rangle : \xi \in S_{p_0}(v_*)\right\} \leq \langle \nabla_H \varphi(p_0), (p_0^{-1} \cdot q_\delta)_h \rangle + \delta.$$

Let us take $\xi_j \in \mathbb{H}_{p_j} \cap \Omega$ such that $\xi_j \to q_\delta$ as $j \to \infty$. Since $v^*(q_\delta) < v^*(p_0)$ and

$$\limsup_{j \to \infty} u_j(\xi_j) \leq \limsup_{j \to \infty} v^*(\xi_j) \leq v^*(q_\delta),$$

we get $u_j(\xi_j) < v^*(p_0)$ and thus $\xi_j \in S_{p_j}(u_j)$ for $j \geq 1$ sufficiently large.

Applying Definition 3.1, we have

$$u_j(p_j) + \sup\left\{\langle \nabla_H \varphi(p_j), (p_j^{-1} \cdot \xi)_h \rangle : \xi \in S_{p_j}(u_j)\right\} \leq f(p_j),$$

which implies

$$u_j(p_j) + \langle \nabla_H \varphi(p_j), (p_j^{-1} \cdot \xi_j)_h \rangle \leq f(p_j).$$

Letting $j \to \infty$, by (3.22) and the upper semicontinuity of f, we are led to

$$v^*(p_0) + \langle \nabla_H \varphi(p_0), (p_0^{-1} \cdot q_\delta)_h \rangle \leq f(p_0).$$

It follows from (3.23) that

$$v^*(p_0) + \sup \left\{ \langle \nabla_H \varphi(p_0), (p_0^{-1} \cdot \xi)_h \rangle : \xi \in S_{p_0}(v^*) \right\} \leq f(p_0) + \delta.$$

Due to the arbitrariness of $\delta > 0$, we obtain

$$v^*(p_0) + \sup \left\{ \langle \nabla_H \varphi(p_0), (p_0^{-1} \cdot \xi)_h \rangle : \xi \in S_{p_0}(v^*) \right\} \leq f(p_0).$$

Let us discuss the case when $\nabla_H \varphi(x_0) = 0$. Since $u_j - \varphi$ attains a maximum at p_j, by Lemma 3.3, we get $u_j(p_j) \leq f(p_j)$. Sending $j \to \infty$, we have $v^*(p_0) \leq f(p_0)$, as desired. Our proof is now complete. ∎

Remark 3.12. Proposition 3.11 can be used in more general circumstances than the setting in Theorem 3.8. Note that f is only assumed to be upper semicontinuous in Ω in contrast to the continuity assumption in Theorem 3.8.

Remark 3.13. The proof of Proposition 3.11, which is just an adaptation of Perron's argument, is certainly not restricted to (3.1). It can be extended with ease to a more general class of nonlocal Hamilton–Jacobi equation including

$$\sup \left\{ \langle \nabla_H u(p), (p^{-1} \cdot \xi)_h \rangle : \xi \in \hat{S}_p(u) \right\} = 0.$$

In this case, in view of Theorem 2.6, our result simply means that the upper semicontinuous envelope of the pointwise supremum of a class of upper semicontinuous h-quasiconvex functions is also h-quasiconvex.

The following result shows that the maximal subsolution is a solution.

Proposition 3.14 (Locally greater subsolutions). *Let $\Omega \subset \mathbb{H}$ be a bounded domain and let $f \in \mathrm{LSC}(\Omega)$. Let u be a subsolution of (3.1). Assume that $u_* \geq f$ on $\partial\Omega$. If u_* fails to satisfy the supersolution property at $p_0 \in \Omega$, then there exist $r > 0$ and a subsolution U_r of (3.1) such that*

$$\tag{3.24} \sup_{B_r(p_0)} (U_r - u) > 0$$

and

$$\tag{3.25} U_r = u \quad in \ \Omega \setminus B_r(p_0).$$

Proof. Since u_* fails to satisfy the supersolution property at p_0, there exists $\varphi \in C^1(\Omega)$ such that $u_* - \varphi$ attains a minimum at p_0 but

$$\tag{3.26} u_*(p_0) + \sup \left\{ \langle \nabla_H \varphi(p_0), (p_0^{-1} \cdot \xi)_h \rangle : \xi \in \hat{S}_{p_0}(u_*) \right\} < f(p_0).$$

Since the supremum is nonnegative, it is then clear that

$$\tag{3.27} u_*(p_0) < f(p_0).$$

By adding $a|p \cdot p_0^{-1}|_G^4 + b$ to φ with $a < 0$ and $b \in \mathbb{R}$, we may additionally assume that there exists $r > 0$ small satisfying

$$(u_* - \varphi)(p) > (u_* - \varphi)(p_0) = 0$$

for all $p \in B_r(p_0) \setminus \{p_0\}$. We also take $\delta > 0$ accordingly small such that

$$\varphi(p) + \delta \leq u(p) \quad for \ p \in B_r(p_0) \setminus B_{r/2}(p_0).$$

Letting

$$U_r(p) = \begin{cases} \max\{\varphi(p) + \delta, u(p)\}, & \text{if } p \in B_r(p_0), \\ u(p), & \text{if } p \notin B_r(p_0), \end{cases}$$

we see that (3.24) and (3.25) hold. In what follows we show that U_r is a subsolution of (3.1) for such $r > 0$ and $\delta > 0$ small.

Suppose that $\psi \in C^1(\Omega)$ is a test function such that $U_r - \psi$ attains a maximum at some $q_0 \in \Omega$. We divide our argument into two cases.

Case 1. Suppose that $U_r(q_0) = u(q_0)$. It follows that $u - \psi$ attains a maximum at q_0. Since u is a subsolution, we have

$$(3.28) \qquad u(q_0) + \sup\left\{\langle \nabla_H \psi(q_0), (q_0^{-1} \cdot \eta)_h \rangle : \eta \in S_{q_0}(u)\right\} \leq f(q_0).$$

Noticing that $U_r \geq u$ in Ω, we have $S_{q_0}(U_r) \subset S_{q_0}(u)$, which by (3.28) yields

$$U_r(q_0) + \sup\left\{\langle \nabla_H \psi(q_0), (q_0^{-1} \cdot \eta)_h \rangle : \eta \in S_{q_0}(U_r)\right\} \leq f(q_0),$$

as desired.

Case 2. Suppose that $U_r(q_0) = \varphi(q_0) + \delta$. In this case, we have $q_0 \in B_r(p_0)$. Also, it is easily seen that $\varphi + \delta - \psi$ attains a maximum at q_0, which yields

$$\nabla_H \psi(q_0) = \nabla_H \varphi(q_0).$$

It thus suffices to show that there exists $r, \delta > 0$ small such that

$$U_r(q) + \sup\left\{\langle \nabla_H \varphi(q), (q^{-1} \cdot \eta)_h \rangle : \eta \in S_q(U_r)\right\} \leq f(q)$$

for all $q \in B_r(p_0)$ satisfying $U_r(q) = \varphi(q) + \delta$.

Arguing by contradiction, we take $r_n, \delta_n > 0$ and $q_n \in B_{r_n}(p_0)$ such that $r_n, \delta_n \to 0$ as $n \to \infty$,

$$U_{r_n}(q_n) = \varphi(q_n) + \delta_n,$$

and

$$(3.29) \qquad \varphi(q_n) + 2\delta_n + \sup\left\{\langle \nabla_H \varphi(q_n), (q_n^{-1} \cdot \eta)_h \rangle : \eta \in S_{q_n}(U_r)\right\} > f(q_n).$$

Note that if there exists a subsequence of q_n at which $\nabla_H \varphi(q_n) = 0$, then sending $n \to \infty$ in (3.29) yields $\varphi(p_0) \geq f(p_0)$, which is clearly a contradiction to (3.27).

We thus only need to consider the case when $\nabla_H \varphi(q_n) \neq 0$ for all $n \geq 1$. By (3.29), this implies the existence of $\eta_n \in S_{q_n}(U_{r_n})$ such that

$$(3.30) \qquad \varphi(q_n) + 2\delta_n + \langle \nabla_H \varphi(q_n), (q_n^{-1} \cdot \eta_n)_h \rangle > f(q_n).$$

In particular, we have $\eta_n \in \mathbb{H}_{q_n} \cap \Omega$ and

$$(3.31) \qquad u(\eta_n) \leq U_{r_n}(\eta_n) < \varphi(q_n) + \delta_n.$$

Due to the boundedness of Ω, we may find a subsequence, still indexed by n, such that $\eta_n \to \eta_0$ for some $\eta_0 \in \overline{\Omega}$. Since the horizontal plane $\mathbb{H}_p$ is continuous in p, we have $\eta_0 \in \mathbb{H}_{p_0}$. Moreover, passing to the limit in (3.31), we obtain

$$u_*(\eta_0) \leq \varphi(p_0) = u_*(p_0),$$

which by (3.27) implies $u_*(\eta_0) < f(p_0)$. In view of the assumption that $u_* \geq f$ on $\partial\Omega$, we immediately have $\eta_0 \in \Omega$. In other words, we have shown that

$$(3.32) \qquad \eta_0 \in \hat{S}_{p_0}(u_*).$$

Finally, taking $\liminf_{n\to\infty}$ in (3.30), we are led to

$$u_*(p_0) + \left\langle \nabla_H \varphi(p_0), (p_0^{-1} \cdot \eta_0)_h \right\rangle \geq f(p_0),$$

which, together with (3.32), yields a contradiction to (3.26). ∎

Let us complete the proof of Theorem 3.8.

Proof of Theorem 3.8. Let U be given by (3.21). By Proposition 3.11, U^* is a subsolution of (3.1). Also, by Lemma 3.3 and the continuity of f in $\overline{\Omega}$, we obtain

$$(3.33) \qquad U^* \leq f \leq K \quad \text{in } \overline{\Omega}.$$

By the assumptions on $\underline{u}$, we also have $U \geq \underline{u}$ in $\overline{\Omega}$ and

$$(3.34) \qquad (U^*)_* \geq U_* \geq \underline{u} = K \quad \text{on } \partial\Omega.$$

Applying Proposition 3.14, we see that $(U^*)_*$ is a supersolution of (3.1), for otherwise we can construct a subsolution locally larger than U^*, which contradicts the definition of U. Noticing that (3.33) and (3.34) are combined to imply that

$$U^* = (U^*)_* = K \quad \text{on } \partial\Omega,$$

by the comparison principle, Theorem 3.5, we get $U^* \leq (U^*)_*$ in $\overline{\Omega}$. It follows that U^* is continuous in $\overline{\Omega}$ and is the unique solution of (3.1) satisfying $U^* = f = K$ on $\partial\Omega$. ∎

Following the proof above, one can obtain an existence result for general Dirichlet boundary problems under the h-convexity of Ω.

Theorem 3.15 (Existence for general Dirichlet problems). *Let $\Omega \subset \mathbb{H}$ be a bounded h-convex domain and let $f \in C(\overline{\Omega})$. Assume that there exist a subsolution $\underline{u} \in C(\overline{\Omega})$ of (3.1) such that $\underline{u} = f$ on $\partial\Omega$. Let U be given by (3.21). Then U^* is continuous in $\overline{\Omega}$ and is the unique solution of (3.1) satisfying $U^* = f$ on $\partial\Omega$.*

We omit the detailed proof but remark that Theorem 3.7 enables us to handle general boundary data following similar arguments above.

4. H-quasiconvex envelope via PDE-based iteration

In Section 1, an iteration was introduced to find the h-quasiconvex envelope $Q(f)$ of a given function f in a bounded h-convex domain $\Omega \subset \mathbb{H}$ with f satisfying (1.4) and (1.5). In this section, we first prove our main result, Theorem 1.1. We shall later study a more general case when Ω is possibly unbounded without boundary data prescribed on $\partial\Omega$.

We begin with an easy construction of an h-quasiconvex $\underline{f}$ as a lower bound of the whole scheme.

Proposition 4.1 (Existence of h-quasiconvex barriers). *Let $\Omega \subset \mathbb{H}$ be a bounded h-convex domain. Assume that $f \in C(\overline{\Omega})$ satisfies (1.4) and (1.5) for some $K \in \mathbb{R}$. Then there exists $\underline{f} \in C(\overline{\Omega})$ h-quasiconvex in Ω such that $\underline{f} \leq f$ in $\overline{\Omega}$ and $\underline{f} = K$ on $\partial\Omega$.*

Proof. By Proposition 2.5, we see that

$$\psi := -\tilde{d}_H(\cdot, \mathbb{H} \setminus \Omega)$$

is an h-quasiconvex function in $\mathbb{H}$. Since f is continuous and Ω is bounded, there exists a modulus of continuity ω_f, strictly increasing, such that

$$f(p) \geq -\omega_f(\tilde{d}_H(p, \mathbb{H} \setminus \Omega)) + K \quad \text{for all } p \in \overline{\Omega}.$$

Taking

(4.1) $$g(s) = -\omega_f(-s) + K, \quad \text{for } s \leq 0,$$

we immediately get

$$f(p) \geq g(-\tilde{d}_H(p, \mathbb{H} \setminus E)) \geq g(\psi(p)) \quad \text{for all } p \in \Omega.$$

Let $\underline{f} = g \circ \psi$. We easily see that $\underline{f} \leq f$ in $\overline{\Omega}$ and $\underline{f} = K$ on $\partial\Omega$. Noticing that g is continuous and nondecreasing, we deduce by Lemma 2.8 that $\underline{f}$ is h-quasiconvex in Ω. ∎

Note that the existence of the h-quasiconvex envelope $Q(f)$ is guaranteed by the existence of $\underline{f}$. Moreover, due to the conditions above, we have $Q(f) = f = K$ on $\partial\Omega$.

Let $u_0 = \overline{f}$. By Theorem 3.8, for $n = 1, 2, \ldots$ we can find a unique solution u_n of (1.6) satisfying (1.7) and (1.8). By Lemma 3.3, one can also see that

(4.2) $$\underline{f} \leq \cdots \leq u_n \leq u_{n-1} \leq \cdots \leq u_0 = f \quad \text{in } \Omega \text{ for } n = 1, 2, \ldots$$

We proceed to prove Theorem 1.1. To this end, we mention a fundamental fact on monotone sequences of functions.

Remark 4.2. Since u_n is non-increasing in n, the pointwise limit $\lim_{n\to\infty} u_n$ is equal to $\limsup^*_{n\to\infty} u_n$, defined by

$$\limsup_{n\to\infty}{}^* u_n(p) = \lim_{k\to\infty} \sup\{u_n(q) : q \in B_{1/k}(p), n \geq k\}, \quad p \in \Omega.$$

In fact, for any fixed $p \in \Omega$, $\varepsilon > 0$ and any $n \geq 1$, by the upper semicontinuity of u_n, we can take $k \geq 1$ sufficiently large to get

$$u_n(p) \geq \sup\{u_n(q) : q \in B_{1/k}(p)\} - \varepsilon.$$

By the monotonicity of u_n in n, we have

$$u_n(p) \geq \sup\{u_n(q) : q \in B_{1/k}(p), n \geq k\} - \varepsilon,$$

which yields

$$u_n(p) \geq \limsup_{n\to\infty}{}^* u_n(p) - \varepsilon.$$

Letting $n \to \infty$ and $\varepsilon \to 0$, we are led to

$$\lim_{n \to \infty} u_n(p) \geq \limsup_{n \to \infty}{}^* u_n(p).$$

Since the reverse inequality clearly holds, we thus obtain the equality.

Proposition 4.3 (Stability of h-quasiconvexity). *Suppose that Ω is a bounded h-convex domain in $\mathbb{H}$. Let $u_0 = f \in \mathrm{USC}(\Omega)$ and let $u_n \in \mathrm{USC}(\Omega)$ be a subsolution of* (1.6). *Then*

$$u = \limsup_{n \to \infty}{}^* u_n$$

is h-quasiconvex in Ω.

Proof. Our argument below in based on the characterization given in Theorem 2.6. Suppose that there exist $p_0 \in \Omega$ and $\varphi \in C^1(\Omega)$ such that $u - \varphi$ attains a strict maximum in Ω at p_0. Then by Remark 4.2, there exists $p_n \in \Omega$ such that $u_n - \varphi$ attains a local maximum at p_n and $p_n \to p_0$, $u_n(p_n) \to u(p_0)$ as $n \to \infty$.

We may assume that $\nabla_H \varphi(p_0) \neq 0$, for otherwise it is clear that

$$\langle \nabla_H \varphi(p_0), (p_0^{-1} \cdot \xi)_h \rangle = 0$$

holds for all $\xi \in S_{p_0}(u)$. We thus have $S_{p_0}(u) \neq \emptyset$. For any $\varepsilon > 0$, let $\xi_0 \in S_{p_0}(u)$ satisfy

(4.3) $\quad \langle \nabla_H \varphi(p_0), (p_0^{-1} \cdot \xi_0)_h \rangle \geq \sup \left\{ \langle \nabla_H \varphi(p_0), (p_0^{-1} \cdot \xi)_h \rangle : \xi \in S_{p_0}(u) \right\} - \varepsilon.$

Since $u(\xi_0) < u(p_0)$, we have $u_n(\xi_0) < u_n(p_n)$ when n is sufficiently large. Then there is a point $\xi_n \in \mathbb{H}_{p_n}$ satisfying $u_n(\xi_n) < u_n(p_n)$ for n large and $p_n^{-1} \cdot \xi_n = p_0^{-1} \cdot \xi_0$. Applying the definition of subsolutions of (1.6), we get

$$u_n(p_n) + \langle \nabla_H \varphi(p_n), (p_n^{-1} \cdot \xi_n)_h \rangle \leq u_{n-1}(p_n),$$

which is equivalent to

$$u_n(p_n) + \langle \nabla_H \varphi(p_n), (p_0^{-1} \cdot \xi_0)_h \rangle \leq u_{n-1}(p_n).$$

Passing to the limit as $n \to \infty$, we have

$$u(p_0) + \langle \nabla_H \varphi(p_0), (p_0^{-1} \cdot \xi_0)_h \rangle \leq u(p_0).$$

It follows from (4.3) that

$$\sup \left\{ \langle \nabla_H \varphi(p_0), (p_0^{-1} \cdot \xi)_h \rangle : \xi \in S_{p_0}(u) \right\} \leq \varepsilon.$$

Letting $\varepsilon \to 0$, we get

$$\sup \left\{ \langle \nabla_H \varphi(p_0), (p_0^{-1} \cdot \xi)_h \rangle : \xi \in S_{p_0}(u) \right\} \leq 0.$$

The proof for h-quasiconvexity of u is now complete. ∎

We are now in a position to prove Theorem 1.1.

Proof of Theorem 1.1. In view of Proposition 4.1, there exists $\underline{f} \in C(\overline{\Omega})$ h-quasiconvex such that $\underline{f} \leq f$ in $\overline{\Omega}$ and $\underline{f} = f = K$ on $\partial\Omega$. Since $u_n \in C(\overline{\Omega})$ is a monotone sequence and $u_n \geq \underline{f}$ in $\overline{\Omega}$, it converges to $u \in \mathrm{USC}(\overline{\Omega})$ pointwise as $n \to \infty$. We thus get $u = Q(f) = f = K$ on $\partial\Omega$. By Proposition 4.3, we see that u is h-quasiconvex in Ω. It follows immediately that $u \leq Q(f)$ in Ω.

Note that $Q(f)$ is a subsolution of (1.6) for every n satisfying $Q(f) = f$ on $\partial\Omega$. Then by the comparison principle, we have $Q(f) \leq u_n$ in $\overline{\Omega}$. We thus have $Q(f) \leq u$ in $\overline{\Omega}$ and therefore $u_n \to Q(f)$ pointwise in $\overline{\Omega}$ as $n \to \infty$.

The uniform convergence of u_n requires extra work. Let us extend the definitions of u_n ($n = 0, 1, 2, \ldots$) by a constant K to the whole space $\mathbb{H}$, that is,

$$u_n^K(p) = \begin{cases} u_n(p) & \text{if } p \in \Omega, \\ K & \text{if } p \in \mathbb{H} \setminus \Omega. \end{cases}$$

Let us verify that u_n^K is a solution of

(4.4) $$u_n^K(p) + H(p, u_n^K(p), \nabla_H u_n^K(p)) = u_{n-1}^K(p) \quad \text{in } \mathbb{H}.$$

In fact, it is obvious that u_n^K is a supersolution. Below we show that it is a subsolution. Suppose that there is a test function $\varphi \in C^1(\mathbb{H})$ such that $u_n^K - \varphi$ attains a maximum at $p_0 \in \mathbb{H}$. We easily see that

(4.5) $$u_n^K(p_0) + \sup\left\{\left\langle \nabla_H \varphi(p_0), (p_0^{-1} \cdot \xi)_h \right\rangle : \xi \in S_{p_0}(u_n^K)\right\} \leq u_{n-1}^K(p_0)$$

holds if $p_0 \in \mathbb{H} \setminus \overline{\Omega}$ or $p_0 \in \Omega$. In the case that $p_0 \in \partial\Omega$, extending $\underline{f}$ to get

$$\underline{f}^K(p) = \begin{cases} \underline{f}(p) & \text{if } p \in \Omega, \\ K & \text{if } p \in \mathbb{H} \setminus \Omega. \end{cases}$$

we observe that $\underline{f}^K$ is h-quasiconvex in $\mathbb{H}$ and $\underline{f}^K - \varphi$ also attains a maximum at p_0; the latter is due to the facts that $\underline{f}^K \leq u_n^K$ in $\mathbb{H}$ and $\underline{f}^K = u_n^K = K$ on $\partial\Omega$. It follows from Theorem 2.6 that

$$\sup\left\{\left\langle \nabla_H \varphi(p_0), (p_0^{-1} \cdot \xi)_h \right\rangle : \xi \in S_{p_0}(\underline{f}^K)\right\} \leq 0,$$

which implies that

$$\sup\left\{\left\langle \nabla_H \varphi(p_0), (p_0^{-1} \cdot \xi)_h \right\rangle : \xi \in S_{p_0}(u_n^K)\right\} \leq 0.$$

It is then clear that (4.5) holds again, since $p_0 \in \partial\Omega$ and thus $u_n^K(p_0) = u_{n-1}^K(p_0) = K$. Hence, u_n^K is a subsolution of (4.4).

We next translate the extended solutions. For any fixed $h \in \mathbb{H}$, we have that

(4.6) $$u_{n-1,h}^K(p) - \omega_f(|h|_G) \leq u_{n-1}^K(p) \quad \text{in } \mathbb{H}$$

holds for $n = 1$, where ω_f is the modulus of continuity of f in $\overline{\Omega}$. Using Lemma 3.4, we can also deduce that $u_{n,h}^K(p) = u_n^K(h \cdot p) - \omega_f(|h|_G)$ is a subsolution of

$$u_{n,h}^K(p) + H(p, u_{n,h}^K(p), \nabla_H u_{n,h}^K(p)) = u_{n-1,h}^K(p) - \omega_f(|h|_G) \quad \text{in } \mathbb{H},$$

and therefore by (4.6) is a subsolution of

$$u_{n,h}^K(p) + H(p, u_{n,h}^K(p), \nabla_H u_{n,h}^K(p)) = u_{n-1}^K(p) \quad \text{in } \mathbb{H}$$

in the case that $n = 1$. Take a bounded h-convex domain Ω' such that

$$\{p \in \mathbb{H} : |p^{-1} \cdot q|_G \leq |h|, \; q \in \overline{\Omega}\} \subset \Omega'.$$

Applying the comparison principle, Theorem 3.5, in Ω', we are led to

$$u_{1,h}^K(p) - \omega_f(|h|_G) \leq u_1^K(p) \quad \text{in } \Omega',$$

which implies (4.6) with $n = 2$.

We can repeat the argument to show that (4.6) holds for all $n \geq 1$. Exchanging the roles of u_n^K and $u_{n,h}^K$, we deduce that

$$|u_{n,h}^K - u_n^K| \leq \omega_f(|h|_G) \quad \text{in } \mathbb{H},$$

which yields

$$|u_n(h \cdot p) - u_n(p)| \leq \omega_f(|h|_G)$$

for all for all $p \in (h^{-1} \cdot \overline{\Omega}) \cap \overline{\Omega}$ and $n \geq 1$. Due to the arbitrariness of h, we get the equi-continuity of u_n in $\overline{\Omega}$ with modulus ω_f. It follows that $Q(f) = \inf_{n \geq 1} u_n$ is also continuous in $\overline{\Omega}$ with the same modulus. The uniform convergence of u_n to $Q(f)$ in $\overline{\Omega}$ as $n \to \infty$ is an immediate consequence of Dini's theorem. ∎

Remark 4.4. It is possible to use the same scheme to approximate the h-quasiconvex envelope $Q(f)$ of a function $f \in C(\overline{\Omega})$ that takes general boundary values. In this case, the uniqueness and existence of u_n are guaranteed by Theorem 3.7 and Theorem 3.15. The pointwise convergence of u_n to $Q(f)$ can be shown in the same way as in the proof of Theorem 1.1. It is also possible to obtain uniform convergence if $\underline{f}$ can be extended to an h-quasiconvex function in a neighborhood of $\overline{\Omega}$.

We conclude this section by providing a relaxed version of Theorem 1.1 with no assumptions on boundedness or convexity of Ω. Based on Proposition 3.11, for any $f \in \text{USC}(\Omega)$ that is bounded below, one can still obtain a sequence u_n that converges pointwise to $Q(f)$ in Ω without even using the boundary value of f. In this general case, we can take u_n to be the maximum subsolution of (1.6).

More precisely, let $u_0 = f$ and, for $n = 1, 2, \ldots$, let u_n be the maximum subsolution of (1.6), that is,

$$(4.7) \qquad\qquad\qquad u_n = v_n^* \quad \text{in } \Omega,$$

where, for any $p \in \Omega$,

$$(4.8) \qquad v_n(p) = \sup\{u(p) : u \in \text{USC}(\Omega) \text{ is a subsolution of } (3.1) \text{ with } f = u_{n-1}\}.$$

By Proposition 3.11, u_n is indeed a subsolution of (3.1) with $f = u_{n-1}$.

Theorem 4.5 (Iterative scheme for envelope without boundary data). *Let Ω be an h-convex domain in $\mathbb{H}$ and let $f \in \text{USC}(\Omega)$. Assume that there exists an h-quasiconvex function $\underline{f} \in \text{USC}(\Omega)$ such that $f \geq \underline{f}$ in Ω. Let $u_n \in \text{USC}(\Omega)$ be iteratively defined by (4.7)–(4.8). Then $u_n \to Q(f)$ pointwise in Ω as $n \to \infty$, where $Q(f) \in \text{USC}(\Omega)$ is the h-quasiconvex envelope of f.*

Proof. By Lemma 3.3, we still have (4.2). It then follows from Remark 4.2 and Proposition 4.3 that $\lim_{n\to\infty} u_n \in \mathrm{USC}(\Omega)$ and

$$\lim_{n\to\infty} u_n \leq Q(f) \quad \text{in } \Omega.$$

We can also get the reversed inequality by showing that $Q(f) \leq u_n$ in Ω for all $n \geq 1$. Note that here we cannot prove it by the comparison principle, Theorem 3.5, due to the possible unboundedness of Ω and loss of the boundary data. Instead, we use the maximality of u_n among all subsolutions of (1.6). ∎

5. H-convex hull

In this section, we study the h-convex hull of a given set in $\mathbb{H}$. Using the definition of h-convex sets introduced in Definition 2.1, we can define the h-convex hull of a set in the following natural way.

Definition 5.1 (H-convex hull). For a set $E \subset \mathbb{H}$, we denote by $\mathrm{co}(E)$ the h-convex hull of E defined to be the smallest h-convex set in $\mathbb{H}$ containing E, i.e.,

$$\mathrm{co}(E) = \bigcap \{D \subset \mathbb{H} : D \text{ is h-convex and satisfies } E \subset D\}.$$

Below we attempt to understand several basic properties of h-convex hulls.

5.1. Level set formulation

We first establish connection between h-quasiconvex envelopes and h-convex hulls. Our general process of convexifying an open or closed set is an adaptation of the so-called level set method, which can be summarized as follows.

(1) For a given open (respectively, closed) set $E \subset \mathbb{H}$, we take a function $f \in C(\mathbb{H})$ such that

$$(5.1) \qquad E = \{p \in \mathbb{H} : f(p) < 0\} \quad (\text{respectively, } E = \{p \in \mathbb{H} : f(p) \leq 0\}).$$

(2) We construct the h-quasiconvex envelope $Q(f)$.

(3) The h-convex hull turns out to be the 0-sublevel set of $Q(f)$, that is,

$$\mathrm{co}(E) = \{p \in \mathbb{H} : Q(f)(p) < 0\} \quad (\text{resp., } \mathrm{co}(E) = \{p \in \mathbb{H} : Q(f)(p) \leq 0\}).$$

Let us prove the result stated in the step (3) above under more precise assumptions. We first examine the case when E is a bounded open or closed set in $\mathbb{H}$. Then we can take $\Omega = B_R(0) \supset \overline{\mathrm{co}(E)}$ with $R > 0$ large and use Theorem 1.1 to construct the h-quasiconvex envelope $Q(f) \in C(\Omega)$ for a defining function $f \in C(\Omega)$ of the set E.

Theorem 5.2 (Level set method for h-convex hull). *Let $E \subset \mathbb{H}$ be a bounded open (respectively, closed) set. Let $R > 0$ large such that $\overline{\mathrm{co}(E)} \subset B_R(0)$. Assume that $f \in C(\mathbb{H})$ satisfies (5.1). Assume also that there exists $K > 0$ such that*

$$(5.2) \qquad \begin{cases} f \leq K & \text{in } B_R(0), \\ f \equiv K & \text{in } \mathbb{H} \setminus B_R(0). \end{cases}$$

Let $Q(f)$ be the h-quasiconvex envelope of f. Then, $Q(f) \in C(\mathbb{H})$ and $Q(f)$ satisfies

(5.3) $$Q(f) \equiv K \quad in \ \mathbb{H} \setminus B_R(0)$$

and

(5.4) $\mathrm{co}(E) = \{p \in \Omega : Q(f)(p) < 0\}$ *(respectively,* $\mathrm{co}(E) = \{p \in \Omega : Q(f)(p) \leq 0\}$).

Proof. Let us only give a proof in the case when E is a bounded open set. One can use the same argument to prove the result for a bounded closed set E. In our current setting, we take $\Omega = B_R(0)$ and choose

$$\underline{f}(p) = \min\{L(|p|_G - R), 0\} + K, \quad \text{for } p \in \mathbb{H},$$

with $L > 0$ large so that all of the assumptions in Theorem 1.1 are satisfied for any $n > 0$. We thus have $Q(f) \in C(\overline{\Omega})$ and (5.3).

Note that $\mathrm{co}(E) \subset \{p \in \mathbb{H} : Q(f)(p) < 0\}$ holds, since the right-hand side is h-quasiconvex and contains E. In order to show (5.4), it thus suffices to prove the reverse implication. To this end, we recall from Proposition 2.5 that

$$\psi_{\mathrm{co}(E)} := -\tilde{d}_H(\cdot, \mathbb{H} \setminus \mathrm{co}(E))$$

is an h-quasiconvex function in $\mathbb{H}$. We adopt the same proof of Proposition 4.1 to construct an h-quasiconvex function $\underline{f} := g \circ \psi_{\mathrm{co}(E)}$ in Ω satisfying $\underline{f}(p) \geq \underline{f}(p)$ for all $p \in \Omega$, where g is determined by the modulus of continuity ω_f of f, given by (4.1). It follows that

$$Q(f) \geq Q(g \circ \psi_{\mathrm{co}(E)}) = g \circ \psi_{\mathrm{co}(E)}.$$

Since g is actually strictly increasing, we are led to

$$\{p \in \mathbb{H} : Q(f)(p) < 0\} \subset \{p \in \mathbb{H} : g(\psi_{\mathrm{co}(E)}(p)) < 0\}$$
$$= \{p \in \mathbb{H} : \psi_{\mathrm{co}(E)}(p) < 0\} = \mathrm{co}(E).$$

The proof is now complete. ∎

As an immediate consequence of the result above, if $E \subset \mathbb{H}$ is a bounded open set, its h-convex hull $\mathrm{co}(E)$ is also bounded and open as well. Likewise, $\mathrm{co}(E)$ is bounded and closed provided that $E \subset \mathbb{H}$ is bounded and closed.

The h-convex hull $\mathrm{co}(E)$ essentially does not depend on the choices of Ω and $f \in C(\mathbb{H})$ as long as Ω is large enough to contain $\mathrm{co}(E)$ and the 0-sublevel set of f agrees with E.

Following a similar argument as in the proof of Theorem 5.2, one can construct the h-convex hull of a general, possibly unbounded open set $E \subset \mathbb{H}$. In this case, we study $\mathrm{co}(E)$ in $\Omega = \mathbb{H}$ without imposing technical assumptions like (5.2). We can use Theorem 4.5 to get $Q(f) \in \mathrm{USC}(\mathbb{H})$.

Theorem 5.3. *Let $E \subset \mathbb{H}$ be an open set. Assume that $f \in C(\mathbb{H})$ satisfies*

$$E = \{p \in \mathbb{H} : f(p) < 0\}.$$

Let $Q(f)$ be the h-quasiconvex envelope of f. Then $Q(f) \in \mathrm{USC}(\mathbb{H})$ and

$$\mathrm{co}(E) = \{p \in \mathbb{H} : Q(f)(p) < 0\}.$$

The proof is omitted, since it resembles that of Theorem 5.2.

5.2. Quantitative inclusion principle

The definition of h-convex hulls, Definition 5.1, immediately yields the inclusion principle: $\mathrm{co}(D) \subset \mathrm{co}(E)$ if $D \subset E$. Using the formulation via h-quasiconvex envelopes, we can quantify this result for bounded open or closed sets.

Theorem 5.4 (Quantitative inclusion principle). *Let D and E be two bounded open (or closed) sets in $\mathbb{H}$. If $D \subset E$, then $\mathrm{co}(D) \subset \mathrm{co}(E)$ and (1.12) holds.*

In order to prove this result, let us introduce the following sup-convolution for a bounded continuous function u with $\delta > 0$:

$$(5.5) \qquad u^\delta(p) = \sup_{\tilde{d}_H(p,q) < \delta} u(q), \quad \text{for } p \in \mathbb{H}.$$

Due to the continuity of u, we also have

$$u^\delta(p) = \max_{\tilde{d}_H(p,q) \le \delta} u(q), \quad \text{for } p \in \mathbb{H}.$$

It turns out that this sup-convolution also preserves h-quasiconvexity.

Lemma 5.5 (H-quasiconvexity preserving by sup-convolution). *Assume that u is a bounded continuous function in $\mathbb{H}$. Let $u^\delta \in C(\mathbb{H})$ be given by (5.5). Then u^δ is h-quasiconvex in $\mathbb{H}$ if u is h-quasiconvex in $\mathbb{H}$.*

Proof. Note that u^δ is bounded and continuous in $\mathbb{H}$, since u is bounded and continuous. We approximate u^δ by

$$(5.6) \qquad u^\delta_\beta(p) = \sup_{q \in \mathbb{H}} \left\{ u(q) - \frac{\tilde{d}_H(p,q)^\beta}{\delta^\beta} \right\}$$

with $\beta > 0$ large. It is easily seen that u^δ_β is also bounded and continuous in $\mathbb{H}$. For any $p_0 \in \mathbb{H}$, we can find $q_{\delta,\beta} \in \mathbb{H}$ such that

$$(5.7) \qquad u^\delta_\beta(p_0) = u(q_{\delta,\beta}) - \frac{\tilde{d}_H(p_0, q_{\delta,\beta})^\beta}{\delta^\beta},$$

which yields

$$\frac{\tilde{d}_H(p_0, q_{\delta,\beta})^\beta}{\delta^\beta} \le u(q_{\delta,\beta}) - u^\delta_\beta(p_0).$$

Due to the boundedness of u, sending $\beta \to \infty$, we obtain

$$\limsup_{\beta \to \infty} \tilde{d}_H(p_0, q_{\delta,\beta}) \le \delta.$$

We thus can take a subsequence of $q_{\delta,\beta}$ converging to a point $q_\delta \in \mathbb{H}$, which satisfies $\tilde{d}_H(p_0, q_\delta) \le \delta$. It follows that

$$(5.8) \qquad \limsup_{\beta \to \infty} u^\delta_\beta(p_0) \le u(q_\delta) \le \max_{\tilde{d}_H(p_0,q) \le \delta} u(q) = u^\delta(p_0).$$

On the other hand, noticing that

$$u_\beta^\delta(p_0) \geq \sup\left\{u(q) - \frac{\tilde{d}_H(p_0,q)^\beta}{\delta^\beta} : \tilde{d}_H(p_0,q) < \delta\right\},$$

we deduce that

$$(5.9) \qquad \sup_{\beta \geq 1} u_\beta^\delta(p_0) \geq \sup\{u(q) : \tilde{d}_H(p,q) < \delta\} = u^\delta(p_0).$$

Combining (5.8) and (5.9) as well as the arbitrariness of p_0, we are led to

$$(5.10) \qquad u^\delta(p) = \sup_{\beta \geq 1} u_\beta^\delta(p) \quad \text{for all } p \in \mathbb{H}.$$

Suppose that there exist $\varphi \in C^1(\mathbb{H})$ and $p_0 \in \mathbb{H}$ such that $u_\beta^\delta - \varphi$ attains a local maximum at p_0. Then

$$(5.11) \qquad (p,q) \mapsto u(q) - \varphi(p) - \frac{\tilde{d}_H(p,q)^\beta}{\delta^\beta}$$

attains a maximum at $(p_0, q_{\delta,\beta})$, where $q_{\delta,\beta} \in \mathbb{H}$ is the point satisfying (5.7). The maximality of (5.11) implies that $u - \psi$ attains a maximum at $q_{\delta,\beta}$, where $\psi \in C^1(\mathbb{H})$ is given by

$$\psi(q) = \varphi(p_0) + \frac{\tilde{d}_H(p_0,q)^\beta}{\delta^\beta}.$$

Since u is h-quasiconvex, we have

$$(5.12) \qquad \sup\left\{\langle \nabla_H \psi(q_{\delta,\beta}), (q_{\delta,\beta}^{-1} \cdot \eta)_h \rangle : \eta \in S_{q_{\delta,\beta}}(u)\right\} \leq 0.$$

Note that for any $\xi \in S_{p_0}(u_\beta^\delta)$, we can choose $\eta = q_{\delta,\beta} \cdot p_0^{-1} \cdot \xi$ so that $\eta \in S_{q_{\delta,\beta}}(u)$. Indeed, we can easily verify that $\eta \in \mathbb{H}_{q_{\delta,\beta}}$ and by (5.6) and (5.7) we obtain

$$u(\eta) - \frac{\tilde{d}_H(\xi,\eta)^\beta}{\delta^\beta} \leq u_\beta^\delta(\xi) < u_\beta^\delta(p_0) = u(q_{\delta,\beta}) - \frac{\tilde{d}_H(p_0,q_{\delta,\beta})^\beta}{\delta^\beta},$$

which reduces to $u(\eta) < u_\beta^\delta(p_0)$ due to the fact that

$$\tilde{d}_H(\xi,\eta) = \tilde{d}_H(p_0, q_{\delta,\beta}).$$

Hence, we have $\eta \in S_{q_{\delta,\beta}}(u)$.

Since $p_0^{-1} \cdot \xi = q_{\delta,\beta}^{-1} \cdot \eta$ and $\nabla_H \varphi(p_0) = \nabla_H \psi(q_{\delta,\beta})$ (similar to the calculations for (3.11)), (5.12) can be rewritten as

$$\sup\left\{\langle \nabla_H \varphi(p_0), (p_0^{-1} \cdot \xi)_h \rangle : \xi \in S_{p_0}(u_\beta^\delta)\right\} \leq 0.$$

By Theorem 2.6, we see that u_β^δ is h-quasiconvex in $\mathbb{H}$. Thanks to (5.10), we conclude the proof of the h-quasiconvexity of u^δ by applying the result in Remark 3.13. ∎

Proof of Theorem 5.4. Since $E \subset \mathrm{co}(E)$, we get $D \subset \mathrm{co}(E)$. Due to the h-convexity of $\mathrm{co}(E)$, by Definition 5.1 we obtain $\mathrm{co}(D) \subset \mathrm{co}(E)$. We next prove (1.12), assuming that E and D are open. The case for closed sets can be similarly treated. Take $\Omega = B_R(0)$ and $f \in C(\mathbb{H})$ as in Theorem 5.2. In this case, $Q(f) \in C(\mathbb{H})$ and (5.4) hold. Since $Q(f)$ is constant outside $B_R(0)$, we can also obtain the boundedness of $Q(f)$.

Adopting Lemma 5.5, we see that the sup-convolution $Q(f)^\delta$ of $Q(f)$ is h-quasi-convex in $\mathbb{H}$ for any $\delta > 0$. It follows that

$$\mathrm{co}(E)_\delta := \{p \in \mathbb{H} : Q(f)^\delta(p) < 0\} = \{p \in \mathbb{H} : \tilde{B}_p(\delta) \subset \mathrm{co}(E)\}$$

is h-convex, where we recall that $\tilde{B}_p(\delta)$ denotes the right-invariant metric ball centered at p with radius δ. By taking $\delta = \inf_{p \in \mathbb{H} \setminus E} \tilde{d}_H(p, D)$, we have

$$D \subset \{p \in \mathbb{H} : \tilde{B}_p(\delta) \subset E\} \subset \mathrm{co}(E)_\delta,$$

which yields $\mathrm{co}(D) \subset \mathrm{co}(E)_\delta$ by the h-convexity of $\mathrm{co}(E)_\delta$. This gives (1.12) immediately. ∎

5.3. Continuity under star-shapedness

To conclude this work, we briefly mention the continuity of $\mathrm{co}(E)$ with respect to E. Let us recall that for any two sets $D, E \subset \mathbb{H}$, the distance $d_H(D, E)$ is defined as

$$d_H(D, E) = \max \left\{ \sup_{p \in D} d_H(p, E), \sup_{p \in E} d_H(p, D) \right\}.$$

We are interested in the following question: for a bounded open or closed set $E \subset \mathbb{H}$ and a sequence of sets $E_j \subset \mathbb{H}$, is it true that

(5.13) $$d_H(\mathrm{co}(E), \mathrm{co}(E_j)) \to 0 \quad \text{if} \quad d_H(E, E_j) \to 0$$

holds?

In contrast to the Euclidean case, where the answer is affirmative, the situation in the Heisenberg group is less straightforward. In general, we cannot expect (5.13) to hold, as indicated by the following example.

Example 5.6. We slightly change the set E in Example 2.2 by taking

$$E = (\pi(0, r) \times (-\delta, 0)) \cup (\pi(0, r) \times (t, t + \delta))$$

with $r, t, \delta > 0$. In other words, E is set to be the union of two circular cylinders of height $\delta > 0$. We may use the same argument as in Example 2.2 to show that E is h-convex if $r^2 \le 2t$ but not h-convex if $r^2 > 2t$.

Consider the critical case when $r^2 = 2t$. Since E is h-convex in this case, we have $\mathrm{co}(E) = E$. On the other hand, it is not difficult to see that any ε-neighborhood of E (with respect to d_H or other equivalent metrics), denoted by $\mathcal{N}_\varepsilon(E)$, is not h-convex. Its h-convex hull $\mathrm{co}(\mathcal{N}_\varepsilon(E))$ at least contains horizontal segments joining the upper and lower cylinders. Hence, we can get $c > 0$ such that

$$d_H(\mathrm{co}(E), \mathrm{co}(\mathcal{N}_\varepsilon(E))) > c$$

for all $\varepsilon > 0$. This shows that (5.13) fails to hold in general.

We have seen that the h-convex hull $\mathrm{co}(E)$ is not stable with respect to perturbations of the set E. It is however worth emphasizing that the h-quasiconvex envelope is stable, as shown in Proposition 2.10. The main reason for such a discrepancy is that the quasiconvexification process may fatten the level sets of functions, that is, a level set of $Q(f)$ may contain interior points while the level set of f at the same level initially does not. See for example [4, 18, 31] on the fattening phenomenon for the level-set formulation of geometric evolution equations in different contexts.

We need additional assumptions to guarantee (5.13). One sufficient condition is the following type of star-shapedness of the set E.

Definition 5.7 (Strict star-shapedness). A connected set $E \subset \mathbb{H}$ is strictly star-shaped with respect to the group identity $0 \in H$ if for any $0 \leq \lambda < 1$ we have $\delta_\lambda(E) \subset E$ and

$$(5.14) \qquad \inf_{x \in \delta_\lambda(E)} d_H(x, \mathbb{H} \setminus E) > 0.$$

We say E is strictly star-shaped if there exists a point $p_0 \in \mathbb{H}$ such that the left translation $p_0^{-1} \cdot E$ of E is strictly star-shaped with respect to 0.

This is a stronger version than the star-shapedness property studied in [15, 16], where all conditions except (5.14) are required. Note that h-convex set can be disconnected and does not imply strict or weak star-shapedness, as observed in Example 2.2. Below, we present three examples of connected sets to compare these notions. We refer the interested reader to [15] for more examples of star-shaped sets.

Example 5.8. Let us take

$$E_1 = \bigcup_{\lambda \in [0,1]} \delta_\lambda((1,1,1)) \cup \delta_\lambda((-1,-1,1)),$$
$$E_2 = (\pi(0,1) \times \{0\}) \cup (\pi(0,1) \times \{t\}) \cup (\{(0,0)\} \times (0,1)),$$
$$E_3 = \{(x,0,0) : x \in (-1,1)\},$$

where $\pi(0,1)$ denotes the unit disk centered at the origin as defined in (2.1).

All of these sets are bounded and connected. The set E_1 is strictly star-shaped with respect to the group identity but not h-convex; it is easily seen that for any $\delta_s(E_1) \subset E_1$ for all $0 \leq s < 1$ but the horizontal segment between $(1,1,1)$ and $(-1,-1,1)$ does not stay in E_1. In view of Example 2.2, we see that E_2 is h-convex. But it is not star-shaped, since for any $p_0 \in E_2$, there exists $p \in E_2$ such that the curve $\{p_0 \cdot \delta_\lambda(p_0^{-1} \cdot p) : \lambda \in [0,1)\}$ contains points outside E_2. The horizontal segment E_3 is obviously h-convex and star-shaped with respect to every point in E_3. It is however not star-shaped with respect to points in $\mathbb{H} \setminus E_3$. It is not strictly star-shaped in $\mathbb{H}$ either. Indeed, for any $p_0 \in E_3$, we can find $\lambda < 1$ close to 1 such that $0 \in \delta_\lambda(p_0^{-1} \cdot E_3)$ and $d_H(0, \mathbb{H} \setminus (p_0^{-1} \cdot E_3)) = 0$; in other words, $p_0^{-1} \cdot E_3$ is not strictly star-shaped with respect to 0.

Assuming strict star-shapedness, we prove stability of h-convex hull.

Proposition 5.9 (Stability of h-convex hulls of star-shaped sets). *Let $E \subset \mathbb{H}$ be a bounded open (respectively, closed) set that is strictly star-shaped. Let E_j be a sequence of bounded open (respectively, closed) sets in $\mathbb{H}$ such that $d_H(E, E_j) \to 0$ as $j \to \infty$. Then there holds*

$$(5.15) \qquad d_H(\mathrm{co}(E), \mathrm{co}(E_j)) \to 0 \quad \text{as } j \to \infty.$$

Proof. By definition of h-convex hull and $\delta_\lambda(p) \in \mathbb{H}_{\delta_\lambda(q)}$ provided $p \in \mathbb{H}_q$, it is not difficult to verify that

$$\text{(5.16)} \qquad \qquad \text{co}(\delta_\lambda E) = \delta_\lambda(\text{co}(E))$$

for any $\lambda > 0$.

Owing to the strict star-shapedness condition on E, for each fixed $0 < \lambda < 1$ we have $\delta_\lambda(E) \subset E_j$ when $j \geq 1$ is sufficiently large. It then follows that $\text{co}(\delta_\lambda(E)) \subset \text{co}(E_j)$. By (5.16), we get $\delta_\lambda(\text{co}(E)) \subset \text{co}(E_j)$ for all $j \geq 1$ large.

Since the strict star-shapedness also implies $E \subset \delta_{1/\lambda}(E)$ and

$$d_H(E, \delta_{1/\lambda}(E)) > 0$$

for any $0 < \lambda < 1$, we can use a symmetric argument to show that $\text{co}(E_j) \subset \delta_{1/\lambda}(\text{co}(E))$ for $j \geq 1$ large. Hence, $\delta_\lambda(\text{co}(E)) \subset \text{co}(E_j) \subset \delta_{1/\lambda}(\text{co}(E))$ and $\delta_\lambda(\text{co}(E)) \subset \text{co}(E) \subset \delta_{1/\lambda}(\text{co}(E))$ imply that

$$d_H(\text{co}(E), \text{co}(E_j)) \leq d_H(\delta_\lambda(\text{co}(E)), \delta_{1/\lambda}(\text{co}(E)))$$

for $j \geq 1$ large. Letting $j \to \infty$ and then $\lambda \to 1$ yields (5.15). ∎

The strict star-shapedness is only a sufficient condition to guarantee the stability of h-convex hull. It would be interesting to find a sufficient and necessary condition for this property.

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