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On the topology of leaves of singular Riemannian foliations

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Abstract. In this paper, we establish a number of results about the topology of the leaves of a closed singular Riemannian foliation $(M, \mathcal{F})$. If M is simply connected, we prove that the leaves are finitely covered by nilpotent spaces, and characterize the fundamental group of the generic leaves. If M has virtually nilpotent fundamental group, we prove that the leaves have virtually nilpotent fundamental group as well.

1. Introduction

The study of isometric group actions on Riemannian manifolds has seen a number of important applications in Riemannian geometry.

Many of them fall under the umbrella of the so-called *Grove's program*, whose goal is to study the properties of Riemannian manifolds with non-negative (or even almost non-negative) sectional curvature in the presence of symmetry. This program has been extremely fruitful both in producing new examples of manifolds with non-negative sectional curvature, and in proving important conjectures in the area when some symmetry is added (cf., for instance, [4, 6, 9–11, 13, 17]).

The concept of an isometric group action can be generalized by a *singular Riemannian foliation*, which roughly speaking is the partition of a Riemannian manifold into smooth and equidistant submanifolds of possibly varying dimensions, called leaves (and the leaves can be thought as a generalization of the orbits of an isometric group action). It turns out that, while being more flexible than group actions (cf. for example [23]), singular Riemannian foliations still retain a lot of the same structure of isometric group actions (cf., for instance, [2, 7, 8, 20, 22]).

Given the action of a compact Lie group, the orbits are homogeneous spaces and thus have a very restricted topology, which can be employed to extrapolate topological properties of the ambient manifold (e.g., [14] and [12]). In [12], the authors ask to what extent the leaves of a singular Riemannian foliation on a non-negatively curved space are also topologically restricted. In [7], Galaz-Garcia and the first author proved that if $(M, \mathcal{F})$ is a closed singular Riemannian foliation on a compact, simply connected Riemannian

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manifold M, then the fundamental group of a generic leaf is a product $A \times K_2$ of an abelian group A and a 2-step nilpotent 2-group K_2 – in particular, it is nilpotent. In the present paper, we continue exploring the topology of the leaves of singular Riemannian foliations $(M, \mathcal{F})$.

The first result states that if M is simply connected, then a generic leaf L_0 of $\mathcal{F}$ is a *nilpotent space*, i.e., $\pi_1(L_0)$ acts nilpotently on $\pi_n(L_0)$ for all $n > 1$.

Theorem A. *If $(M, \mathcal{F})$ is a closed singular Riemannian foliation on a compact, simply connected Riemannian manifold M, then the principal leaves of $\mathcal{F}$ are nilpotent spaces. Furthermore, all leaves are finitely covered by a nilpotent space.*

This answers the first part of Problem 4.8 in [12]:

Question. *Let $\mathcal{F}$ be a closed singular Riemannian foliation on a closed (simply connected) Riemannian manifold M of almost non-negative curvature. Are the leaves of $\mathcal{F}$ finitely covered by a nilpotent space, which moreover is rationally elliptic?*

Our result does not in fact use the curvature assumption. On the rationally elliptic part of the question, we make the following remarks:

(1) The very question of whether the leaves are rationally elliptic, only makes sense the moment we know that the leaves are (virtually) nilpotent spaces: these are in fact the spaces on which rational homotopy theory applies, and the rational dichotomy of rationally elliptic vs. rationally hyperbolic spaces holds.

(2) Assuming the question above to be true, and applying it to the product foliation $(M \times \mathbb{S}^n, M \times \{\text{pts.}\})$ with M simply connected and almost non-negatively curved, would imply that every simply connected, almost non-negatively curved Riemannian manifold is rationally elliptic, which is the statement of the celebrated (and out of reach) Bott–Halperin–Grove conjecture. In particular, the rationally elliptic part of the question is so far out of reach.

The second result analyzes more in detail the structure of the fundamental group of a generic leaf L_0 of a singular Riemannian foliation $(M, \mathcal{F})$ with M simply connected:

Theorem B. *Let $(M, \mathcal{F})$ be a closed singular Riemannian foliation on a compact, simply connected Riemannian manifold M. If L_0 is a principal leaf of $\mathcal{F}$, then the non-abelian part K_2 of the fundamental group of L_0 is of the form*

$$K_2 \cong \Big(\prod_{j=1}^{s} \mathbb{Z}_{2^{a_j}} \times \mathbb{Z}_2^b \times \prod_{i=1}^{k} G_i \Big) / (\mathbb{Z}_2^c \times \mathbb{Z}_4^d),$$

where each G_i is isomorphic to a central product of copies of Q_8, with possibly one copy of D_8 or $\mathbb{Z}_4$.

The groups G_i in the theorem are called *generalized extraspecial*. These 2-groups already occur as fundamental groups of orbits of orthogonal representations and hence are impossible to avoid (e.g., SO(3) acting on $\mathbb{S}^4$), see also a family of examples from Section 4.2.

Finally, we extend Theorem A from [7] by showing that when M has virtually nilpotent fundamental group, the leaves of any closed singular Riemannian foliation $(M, \mathcal{F})$ have virtually nilpotent fundamental group as well:

Theorem C. *Suppose $(M, \mathcal{F})$ is a closed singular Riemannian foliation on compact Riemannian manifold M with virtually nilpotent fundamental group. Then the leaves of $\mathcal{F}$ have virtually nilpotent fundamental group as well.*

In the fundamental paper [18], the authors show that every Riemannian manifold with almost non-negative sectional curvature is finitely covered by a nilpotent space. With this in mind, Theorem C gives the following straightforward corollary.

Corollary D. *Given a closed singular Riemannian foliation $(M, \mathcal{F})$ on an almost non-negatively curved manifold M, the leaves have virtually nilpotent fundamental group.*

This paper is organized as follows. In Section 2, we collect some preliminaries about topological results for singular Riemannian foliations, and the main notation for bilinear and quadratic forms we need in the proof of Theorem B. In Section 3, we prove Theorem A. In Section 4, we prove Theorem B and provide a family of examples showing that the generalized extraspecial groups can indeed appear in the fundamental group of principal orbits of orthogonal representations. Finally, in Section 5, we prove Theorem C.

2. Preliminaries

2.1. Singular Riemannian foliations

Let M be a Riemannian manifold. A singular Riemannian foliation on M is a partition $\mathcal{F}$ of M into connected, injectively immersed submanifolds called leaves such that every geodesic that starts perpendicular to a leaf remains perpendicular to all the leaves it meets, and moreover, M admits a family of smooth vector fields that spans the leaves at all points.

A singular Riemannian foliation is called closed if all of its leaves are closed in M. Given a singular Riemannian foliation $(M, \mathcal{F})$ on a complete manifold M we define the *dimension of* $\mathcal{F}$, denoted $\dim \mathcal{F}$, as the maximal dimension of its leaves. The codimension of $\mathcal{F}$ is defined by $\dim M - \dim \mathcal{F}$.

A leaf L of the foliation $\mathcal{F}$ is called regular if its dimension is maximal, or equivalently, $\dim L = \dim \mathcal{F}$. The union of all regular leaves is an open, dense and connected submanifold, which is called the principal stratum of M and is denoted by M_0. The union of all other leaves is called the singular stratum of $(M, \mathcal{F})$ and the connected components of the singular stratum are called singular strata.

For a closed singular Riemannian foliation $(M, \mathcal{F})$, the canonical projection $\pi\colon M \to M/\mathcal{F}$ induces a metric space structure on the leaf space $M/\mathcal{F}$, where the metric is given by $d_{M/\mathcal{F}}(\pi(p), \pi(q)) = d_M(L_p, L_q)$. If in addition all the leaves of $\mathcal{F}$ are regular, then the leaf space is a Riemannian orbifold. In particular, given a closed singular Riemannian foliation $(M, \mathcal{F})$, the quotient space $M_0/\mathcal{F}$ is an orbifold.

We then call a leaf $L \subset M_0$ *principal* if it projects to a manifold point of $M_0/\mathcal{F}$. Clearly, the set of principal leaves is open and dense in M_0.

2.2. Slice theorem

In this section we describe the structure of a singular Riemannian foliation around a leaf. For more details, we refer the interested reader to [20].

Let $(M, \mathcal{F})$ be a closed singular Riemannian foliation, let $p \in M$, and let L_p denote the leaf through p. Define the *horizontal space to* $\mathcal{F}$ *at* p, $\nu_p L_p \subseteq T_p M$, as the subspace perpendicular to $T_p L_p$. Then there exists a singular Riemannian foliation $(\nu_p L_p, \mathcal{F}_p)$, called the *infinitesimal foliation of* $\mathcal{F}$ *at* p, with two important properties:

(1) $\mathcal{F}_p$ is invariant under rescalings,

(2) in an ε-neighbourhood $\nu_p^\varepsilon L_p$ of the origin in $\nu_p L_p$, the exponential map $\exp_p \colon \nu_p^\varepsilon L_p \to M$ takes the leaves of $\mathcal{F}_p$ onto the connected components of the intersections $L \cap \exp \nu_p^\varepsilon L_p$, with $L \in \mathcal{F}$.

Furthermore, there is a group of isometries $K \subseteq O(\nu_p L_p)$, sending leaves of L_p to (possibly different) leaves of $\mathcal{F}_p$, with the property that for any $v \in \nu_p^\varepsilon L_p$, the leaf $L_v \in \mathcal{F}_p$ satisfies the following:

$$\exp_p(K \cdot L_v) = L_{\exp_p(v)} \cap \exp_p \nu_p^\varepsilon L_p.$$

In other words, two leaves of $\mathcal{F}_p$ are in the same K-orbit if and only if they exponentiate to different connected components of an intersection $L \cap \exp_p \nu_p^\varepsilon L_p$, for some $L \in \mathcal{F}$.

In [20], the following slice theorem establishes a model for a singular Riemannian foliation around a leaf:

Theorem (Foliated slice theorem). *Given a closed singular Riemannian foliation* $(M, \mathcal{F})$ *and a point* $p \in M$, *let* $(\nu_p L_p, \mathcal{F}_p)$ *be the infinitesimal foliation of* $\mathcal{F}$ *at* p. *Then there exist a compact Lie group* $K \subset O(\nu_p L_p)$ *and a principal* K*-bundle* $P \to L_p$ *such that the foliation* $\mathcal{F}$ *in an* ε*-neighbourhood of* L_p *is foliated diffeomorphic to*

$$(P \times_K \nu_p L, P \times_K \mathcal{F}_p)$$

It follows directly from the slice theorem that all principal leaves are diffeomorphic to each other, and for any leaf L_p, there is a locally trivial fiber bundle $L_0 \to L_p$ from a principal leaf L_0, whose fiber is an orbit $K \cdot L_v$ for some principal point $v \in (\nu_p L_p, \mathcal{F}_p)$, and it consists of a finite disjoint union of principal leaves of $\mathcal{F}_p$.

2.3. The Molino bundle

Let $(M, \mathcal{F})$ be a closed singular Riemannian foliation of codimension q on a compact Riemannian manifold M. The principal $\mathrm{O}(q)$-bundle $\hat{M} \to M_0$, where $\hat{M}$ is the collection of orthonormal frames of $TM_0/T\mathcal{F}$, is called the Molino bundle. The foliation $\mathcal{F}$ lifts to a foliation $\hat{\mathcal{F}}$ on $\hat{M}$ whose leaves are diffeomorphic to the leaves of $\mathcal{F}$ on an open dense set. Moreover, the leaves of $\hat{\mathcal{F}}$ are given by fibers of a submersion $\theta \colon \hat{M} \to W$, where W is the frame bundle of the orbifold $M_0/\mathcal{F}$.

Consider the fibration $\hat{\theta} \colon \hat{M}_{\mathrm{O}(q)} \to W_{\mathrm{O}(q)}$ induced by θ, where $\hat{M}_{\mathrm{O}(q)} = \hat{M} \times_{\mathrm{O}(q)} \mathrm{EO}(q)$ and $W_{\mathrm{O}(q)} = W \times_{\mathrm{O}(q)} \mathrm{EO}(q)$ denote the Borel constructions of $\hat{M}$ and W, respectively. Note that $\hat{\theta} \colon \hat{M}_{\mathrm{O}(q)} \to W_{\mathrm{O}(q)}$ and $\theta \colon \hat{M} \to W$ have the same fibers and hence the fiber of $\hat{\theta}$ is diffeomorphic to L_0, where L_0 is a principal leaf of $\mathcal{F}$. In addition, $\hat{M}_{\mathrm{O}(q)}$ is homotopy equivalent to $\hat{M}/\mathrm{O}(q) = M_0$ and $W_{\mathrm{O}(q)}$ coincides with the Haefliger's classifying space B of $M_0/\mathcal{F}$. Therefore, we get the following fibration (up to homotopy):

$$L_0 \xrightarrow{\iota_0} M_0 \xrightarrow{\hat{\theta}} B.$$

2.4. Bilinear and quadratic forms over $\mathbb{Z}_2$

Let V be a finite dimensional vector space over a field F. A quadratic form on V is a map $Q: V \to F$ such that $Q(\lambda v) = \lambda^2 Q(v)$ for all $\lambda \in F$ and $v \in V$, and moreover, the map $B_Q: V \times V \to F$ defined by $B_Q(u, v) = Q(u+v) - Q(u) - Q(v)$ is a bilinear form. Given a basis $\{v_1, \dots, v_\ell\}$ of V, it follows that

$$Q(x_1 v_1 + \cdots + x_\ell v_\ell) = \sum_{i=1}^{\ell} Q(v_i) x_i^2 + \sum_{1 \le i < j \le \ell} B_Q(v_i, v_j) x_i x_j, \tag{2.1}$$

Two quadratic forms $Q: V \to F$ and $Q': V \to F$ are called *isometric* (or equivalent) if there exists an invertible linear map $f: V \to V$ such that $Q(v) = Q'(f(v))$ for all $v \in V$.

Finally, given quadratic forms $Q: V \to F$ and $Q': V' \to F$, one defines the *orthogonal sum* $Q \oplus Q': V \oplus V' \to F$ by $(Q \oplus Q')(v, v') := Q(v) + Q'(v')$.

3. The topology of leaves

Let $(M, \mathcal{F})$ be a closed singular Riemannian foliation on a compact, simply connected Riemannian manifold M. The goal is to prove Theorem A, that the principal leaves are nilpotent manifolds.

We begin by collecting some of the results proved in [7] about the fundamental group of the principal leaves of $\mathcal{F}$.

3.1. Known results on the topology of leaves

Since the fundamental group of M does not change if we delete the strata of $\mathcal{F}$ with codimension > 2, we can assume that we only have singular strata of codimension ≤ 2. Furthermore, it is known that there are no strata of codimension one, which reduces $\mathcal{F}$ to only having strata of codimension two.

Let $\Sigma_1, \dots, \Sigma_m$ denote the singular strata of $\mathcal{F}$ of codimension two. For $i = 1, \dots, m$, choose a singular leaf L_i' in Σ_i, and let L_i be a principal leaf at some distance ε_i from L_i'. For ε_i small enough, the foot-point projection $\pi_i: L_i \to L_i'$ is a circle bundle. Fix a point $p_i \in L_i$, and let $[c_i] \in \pi_1(L_i, p_i)$ be the element represented by the fiber c_i of π_i through p_i.

Fixing a principal leaf L_0 and $p_0 \in L_0$, we can choose, for each $i = 1, \dots, m$, a diffeomorphism $h_i: L_i \to L_0$, and define $k_i = (h_i)_*([c_i]) \in \pi_1(L_0, p_0)$. The group K generated by the elements k_i is then a normal subgroup of $\pi_1(L_0, p_0)$. Furthermore, there exists a homotopy fibration

$$L_0 \xrightarrow{\iota_0} M_0 \xrightarrow{\hat{\theta}} B,$$

as described in Section 2.3. One has the following (see the proof of Theorem A in [7]):

(1) $\pi_1(L_0, p_0)$ is generated by the subgroup K and the image of the boundary map $\partial\colon \pi_2(B, b_0) \to \pi_1(L_0, p_0)$.

(2) $H := \mathrm{im}(\partial)$ is central in $\pi_1(L_0, p_0)$

(3) Any two non-commuting generators k_i and k_j of K satisfy $k_i k_j = k_j^{-1} k_i$.

(4) Let $N \subseteq K$ be the subgroup generated by the non-central k_i's, and let $Z_{(2)}$ denote the Sylow 2-subgroup of $Z(K)$. Then $\pi_1(L_0, p_0)$ is nilpotent, and equal to $A \times K_2$, where A is abelian and $K_2 = N \cdot Z_{(2)}$.

3.2. Proof of Theorem A

As discussed in Section 3.1, the principal leaves of $\mathcal{F}$ have nilpotent fundamental groups. As a first step towards the proof of Theorem A, we prove that the principal leaves are nilpotent spaces:

Proposition 3.1. *Suppose $(M, \mathcal{F})$ is a closed singular Riemannian foliation on a compact, simply connected Riemannian manifold M. Let L_0 denote a principal leaf of $\mathcal{F}$ and let $p_0 \in L_0$. Then $\pi_1(L_0, p_0)$ acts trivially on $\pi_n(L_0, p_0)$ for $n \geq 2$.*

Proof. Let $[\gamma] \in \pi_1(L_0, p_0)$ and $[\omega] \in \pi_n(L_0, p_0)$. The goal is to prove that $[\gamma]$ acts trivially on $[\omega]$. By the discussion in Section 3.1, we may assume that either $[\gamma] \in H$ or $[\gamma] = k_i$ for some i.

First consider the case in which $[\gamma] = k_i$ for some i. Note that $\mathbf{p}_i := \pi_i \circ h_i^{-1}\colon L_0 \to L_i'$ is a circle bundle whose fiber is represented by k_i. This means that $k_i \in \ker((\mathbf{p}_i)_*)$, where $(\mathbf{p}_i)_*$ is the induced map on π_n. Hence we have

$$(\mathbf{p}_i)_*([\gamma] \cdot [\omega]) = (\mathbf{p}_i)_*(k_i \cdot [\omega]) = ((\mathbf{p}_i)_*(k_i)) \cdot ((\mathbf{p}_i)_*([\omega])) = (\mathbf{p}_i)_*([\omega]).$$

By the long exact sequence of homotopy groups associated to the fibration $\mathbb{S}^1 \to L_0 \xrightarrow{\mathbf{p}_i} L_i'$, it follows that the homomorphism $(\mathbf{p}_i)_*$ is injective in π_n for $n \geq 2$. This, together with $(\mathbf{p}_i)_*([\gamma] \cdot [\omega]) = (\mathbf{p}_i)_*([\omega])$, implies that $[\gamma]$ acts trivially on $[\omega]$.

Suppose now that $[\gamma] \in H = \mathrm{im}(\partial)$ and choose $[\beta] \in \pi_2(B, b_0)$ such that $[\gamma] = \partial([\beta])$. Consider the fibration

$$L_0 \xrightarrow{\iota_0} M_0 \xrightarrow{\hat{\theta}} B.$$

Note that the action of $\pi_1(L_0, p_0)$ on $\pi_n(L_0, p_0)$ satisfies $[\gamma] \cdot [\omega] = (\iota_0)_*([\gamma]) \cdot [\omega]$ (see Exercise 4.3.10 in [16]). Therefore,

$$[\gamma] \cdot [\omega] = (\iota_0)_*([\gamma]) \cdot [\omega] = (\iota_0)_*(\partial([\beta])) \cdot [\omega] = e \cdot [\omega] = [\omega].$$

This completes the proof. ∎

Moving to the non-principal leaves, we first prove that every leaf has a virtually nilpotent fundamental group.

Lemma 3.2. *Suppose $(M, \mathcal{F})$ is a closed singular Riemannian foliation with principal leaf L_0. If $\pi_1(L_0)$ is virtually nilpotent, then so is the fundamental group $\pi_1(L)$ of every leaf L of $\mathcal{F}$.*

Proof. For any leaf L of $\mathcal{F}$, the foliated slice theorem (cf. Section 2.2) implies that there is a fibration $L_0 \to L$ whose fiber F has finitely many connected components. From the long exact sequence in homotopy one then has

$$\pi_1(L_0) \to \pi_1(L) \to \pi_0(F),$$

from which it follows that $\pi_1(L)$ is a finite extension of a quotient of $\pi_1(L_0)$, therefore it is virtually nilpotent as well. ∎

Proof of Theorem A. The statement about principal leaves has been proved in Proposition 3.1, so we now have to only consider non-principal leaves.

Given a leaf L, choose $p \in L$. Recall that, by the foliated slice theorem (cf. Section 2.2), there is a locally trivial fibration $\phi\colon L_0 \to L$ whose fiber F has finitely many connected components, all diffeomorphic to a principal leaf of the infinitesimal foliation $(\nu_p L_p, \mathcal{F}_p)$. Furthermore, the action $\pi_1(L) \to \mathrm{Diff}(F)$ induces an action $\pi_1(L) \to \mathrm{Aut}(\pi_*(F))$, which factors as $\pi_1(L) \xrightarrow{\psi} \pi_0(K) \to \mathrm{Aut}(\pi_*(F))$. In particular,

(1) the subgroup $G_1 := \ker \psi \subseteq \pi_1(L)$ has finite index in $\pi_1(L)$ and it acts trivially on $\pi_*(F)$,

(2) the fibration induces a map $\pi_1(L_0) \xrightarrow{\phi_*} \pi_1(L) \to \pi_0(F)$. Thus $G_2 := \phi_*(\pi_1(L_0))$ is a nilpotent subgroup of $\pi_1(L)$ with finite index.

Consider $G := G_1 \cap G_2 \subseteq \pi_1(L)$, which is by the points above a nilpotent subgroup with finite index. We will now show that G acts nilpotently on each $\pi_n(L)$, i.e., the *lower central series* $\Gamma_G^m(\pi_n(L)) \subseteq \pi_n(L)$ defined iteratively by

$$\Gamma_G^1(\pi_n(L)) = \pi_n(L), \quad \Gamma_G^{m+1}(\pi_n(L)) = \{\gamma \cdot \alpha - \alpha \mid \gamma \in G,\ \alpha \in \Gamma_G^m(\pi_n(L))\}$$

eventually becomes trivial.

Consider the long exact sequence

$$\cdots \to \pi_n(F) \to \pi_n(L_0) \xrightarrow{\phi_*} \pi_n(L) \xrightarrow{\partial} \pi_{n-1}(F) \to \cdots$$

Let $\alpha \in \pi_n(L)$, and $\gamma = \phi_*(\gamma_0) \in G$, where $\gamma_0 \in \pi_1(L_0)$. Recall that $\partial(\gamma \cdot \alpha) = \gamma \cdot \partial(\alpha)$, where the action on the left is $\pi_1(L)$ acting on $\pi_*(L)$, while on the right we have the $\pi_1(L)$-action on $\pi_*(F)$. Since $G \subseteq G_1$, we have

$$\partial(\gamma \cdot \alpha) = \partial(\alpha) \quad \Longrightarrow \quad \partial(\gamma \cdot \alpha - \alpha) = 0,$$

and therefore

$$\Gamma_G^2(\pi_n(L)) \subseteq \ker(\partial) = \phi_*(\pi_n(L_0)) = \phi_*(\Gamma^1_{\pi_1(L_0)}(\pi_n(L_0))).$$

Finally, we notice that if $\alpha = \phi_*(\alpha_0)$ with $\alpha_0 \in \pi_n(L_0)$, then

$$\gamma \cdot \alpha = (\phi_*(\gamma_0)) \cdot (\phi_*(\alpha_0)) = \phi_*(\alpha_0) \quad \Longrightarrow \quad \gamma \cdot \alpha - \alpha = \phi_*(\gamma_0 \cdot \alpha_0 - \alpha_0).$$

By induction on m, one then has

$$\Gamma_G^{m+1}(\pi_n(L)) \subseteq \phi_*\big(\Gamma^m_{\pi_1(L_0)}(\pi_n(L_0))\big).$$

Since by Proposition 3.1, $\Gamma^2_{\pi_1(L_0)}(\pi_n(L_0)) = 0$, we have $\Gamma^3_G(\pi_n(L)) = 0$, which proves that G acts nilpotently on $\pi_n(L)$, hence finishing the proof. ∎

4. Fundamental groups of the principal leaves

This section consists of two parts. The first part is devoted to the proof of Theorem B. In the second part, we provide examples of singular Riemannian foliations whose principal leaves have fundamental groups of the form discussed in Theorem B.

Suppose that $(M, \mathcal{F})$ is a closed singular Riemannian foliation on a compact, simply connected Riemannian manifold M. Fix a principal leaf L_0 of $\mathcal{F}$ and $p_0 \in L_0$. Let N and K_2 be the subgroups of $\pi_1(L_0, p_0)$ discussed in Section 3.1.

Consider the graph Γ with vertices the generators of N and an edge between k_i and k_j if and only if $k_i k_j k_i^{-1} = k_j^{-1}$. Note that for every generator k_i of N, there exists another generator which does not commute with k_i. Therefore, Γ does not contain any isolated vertices. Note moreover that for every connected component Γ_i of Γ, all vertices of Γ_i square to the same element c_i. In addition, by the proof of Theorem A in [7], for any generator k_i of N, we have $k_i^4 = 1$ and k_i^2 is central in K. Therefore, c_i is a central element of N of order two. Altogether, we get that there is a map $C\colon \pi_0(\Gamma) \to Z(N)$ defined by $C(\Gamma_i) = c_i$.

Notation 4.1. From now on, we fix an element c of $Z(N)$ which is of the form k_i^2 for some generator k_i of N. Moreover, N_c denotes the subgroup of N that is generated by all the vertices in $\Gamma_c := C^{-1}(c)$.

Recall that given a group G, the *Frattini subgroup* $\Phi(G)$ is the intersection of all the maximal subgroups of G. Furthermore, we recall the following.

Definition 4.2. A 2-group G is called *generalized extraspecial* if $\Phi(G)$ is central, and $\Phi(G) = [G, G] = \mathbb{Z}_2$.

We prove two important properties of the groups N_c.

Lemma 4.3. *Let $\{N_c\}_{c \in Im(C)}$ be the collection of groups defined above. Then*

(1) *for $c \neq c'$, the groups N_c and $N_{c'}$ commute,*

(2) *each N_c is a generalized extraspecial 2-group.*

Proof. First, we prove (1). Let $k_1, \dots, k_\ell$ be the generators of N_c, and let $k'_1, \dots, k'_r$ be the generators of $N_{c'}$. As vertices of Γ, there is no edge between any k_i and any k'_j, which means that each k_i commutes with any k'_j in K. Hence the result follows.

As for statement (2), if $k_1, \dots, k_\ell$ denote the generators of N_c, then $V = N_c/\langle c\rangle$ is isomorphic to $\mathbb{Z}_2^\ell$ and is generated by $[k_1], \dots, [k_\ell]$. It follows that N_c fits into a short exact sequence

$$1 \to \langle c\rangle \to N_c \to V \to 1$$

and in particular one has that both $N_c^2 := \langle g^2 \mid g \in N_c\rangle$ and the commutator subgroup $[N_c, N_c]$ coincides with $\langle c\rangle \simeq \mathbb{Z}_2$. Therefore, the same is true for the Frattini subgroup $\Phi(N_c)$ since for a 2-group G, one has $\Phi(G) = G^2 \cdot [G, G]$. ∎

Given generalized extraspecial groups G_1 and G_2, with Frattini subgroups generated by c_1 and c_2, respectively, define the *central product* $G_1 * G_2$ by

$$G_1 * G_2 := (G_1 \times G_2)/\langle (c_1, c_2)\rangle.$$

This is again a generalized extraspecial group, since

$$\Phi(G_1 * G_2) = \Phi(G_1) \times_{\mathbb{Z}_2} \Phi(G_2) \cong \mathbb{Z}_2.$$

The $*$ operation is furthermore associative, and thus it makes sense to define, for a generalized extraspecial group G, the central product powers

$$(G)^{*m} := \underbrace{G * G * \ldots * G}_{m \text{ times}}$$

Generalized extraspecial 2-groups are, as the name suggests, a generalization of *extraspecial* 2-*groups*, that is 2-groups such that $\Phi(G) = Z(G) = [G, G] \cong \mathbb{Z}_2$. These groups have been thoroughly studied at least since the 60's, see [15]. They are extremely simple: an extraspecial group has the form $(Q_8)^{*m}$ or $(Q_8)^{*(m-1)} * D_8$ for some $m \geq 1$, where Q_8 is the quaternion group and D_8 is the dihedral group of order 8 (cf. Theorem 2.2.11 of [19]). It then follows from Lemma 3.2 in [27] that:

Theorem 4.4. *A generalized extraspecial* 2*-group is of the form* $G \times \mathbb{Z}_2^n$*, where* G *is one of*

$$Q_8^{*m}, \quad Q_8^{*(m-1)} * D_8 \quad \textit{or} \quad Q_8^{*(m-1)} * \mathbb{Z}_4.$$

4.1. The associated quadratic form

Let G be a generalized extraspecial 2-group with $\Phi(G) = G^2 = \langle c \rangle$, and let $V := G/\langle c \rangle$. It is easy to check that V is a vector space over $\mathbb{Z}_2$.

Define the function $Q_G \colon V \to \mathbb{Z}_2$ by $Q_G([g]) = k$, where $g^2 = c^k$. Since c is central in G and has order two, for any $g \in G$, we have $(cg)^2 = cgcg = c^2g^2 = g^2$ and thus $Q_G([cg]) = Q_G([g])$. Therefore, $Q := Q_G$ is well-defined and in fact a quadratic form as defined in Section 2.4. Furthermore, the bilinear form B_Q associated to Q (cf. Section 2.4) satisfies

$$ghg^{-1}h^{-1} = c^{B_Q([g],[h])}, \quad \text{for } g, h \in G.$$

In order to see this, note that both g^2 and h^2 are central elements of G. Therefore,

$$c^{B_Q([g],[h])} = c^{Q([g]+[h])}\, c^{-Q([g])}\, c^{-Q([h])} = (gh)^2 g^{-2} h^{-2} = ghg^{-1}h^{-1}.$$

The quadratic form of each generalized extraspecial group can be explicitly computed. For this, consider the quadratic forms:

$$\begin{aligned} &H_+ : \mathbb{Z}_2^2 \to \mathbb{Z}_2, && H_- : \mathbb{Z}_2^2 \to \mathbb{Z}_2, && Q_1 : \mathbb{Z}_2 \to \mathbb{Z}_2, \\ &H_+(x, y) = xy, && H_-(x, y) = x^2 + y^2 + xy, && Q_1(x) = x^2. \end{aligned}$$

We have the following.

Proposition 4.5. *Suppose* G *is a generalized extraspecial* 2*-group and let* $V := G/\Phi(G)$*.*

(1) *If* $G = (Q_8)^{*m}$*, then* $V \simeq \mathbb{Z}_2^{2m}$ *and*

$$Q_G = H_-^{\oplus m} = \begin{cases} H_+^{\oplus m} & \text{for } m \text{ even}, \\ H_- \oplus H_+^{\oplus (m-1)} & \text{for } m \text{ odd}. \end{cases}$$

(2) *If* $G = (Q_8)^{*(m-1)} * D_8$, *then* $V \simeq \mathbb{Z}_2^{2m}$ *and*

$$Q_G = H_-^{\oplus(m-1)} \oplus H_+ = \begin{cases} H_+^{\oplus m} & \text{for } m \text{ odd,} \\ H_- \oplus H_+^{\oplus(m-1)} & \text{for } m \text{ even.} \end{cases}$$

(3) *If* $G = (Q_8)^m * \mathbb{Z}_4$, *then* $V \simeq \mathbb{Z}_2^{2m+1}$ *and*

$$Q_G = H_+^{\oplus m} \oplus Q_1 = H_-^{\oplus m} \oplus Q_1.$$

(4) *If* $G = G' \times \mathbb{Z}_2^n$ *with* G' *as in the previous points, then* $V \simeq V' \oplus \mathbb{Z}_2^n$ *and* $Q_G = Q_{G'} \oplus 0^{\oplus n}$.

Proof. This proposition follows easily from the following straightforward facts:

(1) For $G = Q_8$, $G/\Phi(G) \simeq \mathbb{Z}_2^2$ and $Q_G = H_-$.
(2) For $G = D_8$, $G/\Phi(G) \simeq \mathbb{Z}_2^2$ and $Q_G = H_+$.
(3) For $G = \mathbb{Z}_4$, $G/\Phi(G) \simeq \mathbb{Z}_2$ and $Q_G = Q_1$.
(4) Given G_1 and G_2 with quotients $V_i = G_i/\Phi(G_i)$, one has

$$(G_1 * G_2)/\Phi(G_1 * G_2) = V_1 \oplus V_2 \quad \text{and} \quad Q_{G_1 * G_2} = Q_{G_1} \oplus Q_{G_2}.$$

(5) Given G with quotient $V = G/\Phi(G)$, one has

$$(G \times \mathbb{Z}_2^n)/\Phi(G \times \mathbb{Z}_2^n) \simeq V \oplus \mathbb{Z}_2^n \quad \text{and} \quad Q_{G\times\mathbb{Z}_2^n} = Q_G \oplus 0^{\oplus n}.$$ ■

Remark 4.6. The group N_c discussed above is generated by elements of order four, that is, the k_i's. Moreover, for each k_i, there exists k_j such that $k_i k_j k_i^{-1} k_j^{-1} = c$. This is reflected in the corresponding quadratic form $Q: V \to \mathbb{Z}_2 = \{0, 1\}$ as follows. There exists a basis $\{v_1, \dots, v_\ell\}$ of $V \cong \mathbb{Z}_2^\ell$ with the property that $Q(v_i) = 1$ for all i, and for each v_i, there exists v_j such that $B_Q(v_i, v_j) = 1$. We call such quadratic forms *admissible*.

The next step consists of understanding which of the quadratic forms in Proposition 4.5 are admissible. We start by reducing the problem to quadratic forms without trivial summands:

Lemma 4.7. *Let* $Q: V \to \mathbb{Z}_2$ *be a quadratic form. If there exists a splitting* $V = V_1 \oplus V_2$ *such that* Q *splits as* $Q = q \oplus 0^{\oplus n}$, *with* $Q|_{V_1} = q$ *and* $Q|_{V_2} = 0^{\oplus n}$, *then* Q *is admissible if and only if* q *is admissible.*

Proof. Suppose that Q is admissible and choose a basis

$$\{(v_1, w_1), \dots, (v_{m+n}, w_{m+n})\}$$

of $V_1 \oplus V_2$ with the property that $Q(v_i, w_i) = 1$, and for every (v_i, w_i) there exists (v_j, w_j) with $B_Q((v_i, w_i), (v_j, w_j)) = 1$. After possibly rearranging basis elements of $V_1 \oplus V_2$, we may assume that $\{v_1, \dots, v_m\}$ forms a basis for V_1. Since $Q(v_i, w_i) = q(v_i)$ and $B_Q((v_i, w_i), (v_j, w_j)) = B_q(v_i, v_j)$, the basis $\{v_1, \dots, v_m\}$ of V_1 is admissible for q. On the other hand, if $\{v_1, \dots, v_m\}$ is admissible for q and $\{w_1, \dots, w_n\}$ is any basis of V_2, then

$$\{(v_i, \mathbf{0}) \mid i = 1, \dots, m\} \cup \{(v_1, w_j) \mid j = 1, \dots, n\},$$

forms an admissible basis for Q. ■

We now apply Lemma 4.7 to classify the admissible quadratic forms.

Theorem 4.8. *Any admissible quadratic form* $Q\colon \mathbb{Z}_2^\ell \to \mathbb{Z}_2$ *is isometric to one of the following, up to orthogonal sum with* $0^{\oplus n}$ *:*

$$H_+^{\oplus m}\ (m \geq 2), \quad H_- \oplus H_+^{\oplus m-1} \quad \text{or} \quad H_+^{\oplus m} \oplus Q_1\ \ (m \geq 2). \tag{4.1}$$

Proof. Since the quadratic forms over $\mathbb{Z}_2$ are classified (see Proposition A.1), we only need to check the admissibility condition. By Lemma 4.7, we may assume that Q does not split as $q \oplus 0^{\oplus m}$. We break the proof into cases.

Case 1. $Q = H_- \oplus H_+^{\oplus m-1}$, where $2m = \ell$.

The quadratic form Q is given by

$$Q(x, y, z_1, z_2, \ldots, z_{2m-2}) = x^2 + xy + y^2 + z_1 z_2 + \cdots + z_{2m-3}\, z_{2m-2}.$$

Let $e_1, \ldots, e_\ell$ denote the standard basis elements of $\mathbb{Z}_2^\ell$ and consider the following basis:

$$\begin{aligned}
&v_1 = e_1 + e_2, \quad v_2 = e_3 + e_4, \quad \ldots, \quad v_m = e_{2m-1} + e_{2m},\\
&v_{m+1} = e_1,\\
&v_{m+2} = e_1 + e_3, \quad v_{m+3} = e_1 + e_5, \quad \ldots, \quad v_{2m} = e_1 + e_{2m-1}.
\end{aligned}$$

Then $Q(v_i) = 1$ for all i, and for every v_i, there exists v_j such that $B_Q(v_i, v_j) = 1$. Hence Q is admissible.

Case 2. $Q = H_+^{\oplus m}$, where $2m = \ell$.

Note that the only element of $\mathbb{Z}_2^2$ that is mapped to 1 by H_+ is $(1,1)$. Therefore, H_+ is not admissible. However, if $m \geq 2$, then the following basis of $\mathbb{Z}_2^\ell$ is admissible for Q:

$$\begin{aligned}
&v_1 = e_1 + e_2, \quad v_2 = e_3 + e_4, \quad \ldots, \quad v_m = e_{2m-1} + e_{2m},\\
&v_{m+1} = e_1 + e_{2m-1} + e_{2m},\\
&v_{m+2} = e_1 + e_2 + e_4, \quad v_{m+3} = e_3 + e_4 + e_6, \quad \ldots, \quad v_{2m} = e_{2m-3} + e_{2m-2} + e_{2m}.
\end{aligned}$$

Case 3. $Q = H_+^{\oplus m} \oplus Q_1$, where $m \geq 2$ and $2m + 1 = \ell$.

Let $\{v_1, \ldots, v_{2m}\}$ denote the basis constructed for $H_+^{\oplus m}$ in Case 2, and let $v_{2m+1} = e_1 + e_{2m+1}$. Then $\{v_1, \ldots, v_{2m+1}\}$ forms an admissible basis for Q.

For $Q = H_+ \oplus Q_1$, the elements with non-zero quadratic form are $(1,1,0)$, $(1,0,1)$, $(0,1,1)$, $(0,0,1)$. Among these, the only vectors with non-zero bilinear form are the first three, which are linearly dependent and thus do not form a basis. Hence $H_+ \oplus Q_1$ is not admissible. ∎

Recall that the group N_c (cf. Notation 4.1) is a generalized extraspecial group with an admissible basis. From the previous theorem, we then get:

Corollary 4.9. *If* N_c *is a generalized extraspecial group whose corresponding quadratic form is admissible, then, up to a direct product with copies of* $\mathbb{Z}_2$*, the group* N_c *is isomorphic to one of the following*:

$$(Q_8)^{*m_1}, \quad (Q_8)^{*(m_1-1)} * D_8\ \ (m_1 \geq 2) \quad \text{or} \quad (Q_8)^{*m_1} * \mathbb{Z}_4\ \ (m_1 \geq 2). \tag{4.2}$$

Proof. This follows trivially by comparing the quadratic forms in Proposition 4.5 with the classification of admissible quadratic forms in Theorem 4.8. ∎

Finally, we prove Theorem B.

Proof of Theorem B. Fix $p_0 \in L_0$. As discussed in Section 3.1, the non-abelian part K_2 of $\pi_1(L_0, p_0)$ is a 2-group of the form $K_2 = N \cdot Z_{(2)}$, where N is generated by the non-central generators of K and $Z_{(2)}$ denotes the Sylow 2-subgroup of $Z(K)$. Furthermore, by the discussion in Section 4, $N = N_{c_1} \cdots N_{c_k}$, where the elements $c_i \in Z(K)$ have order two. By Corollary 4.9, each N_{c_i} is of the form $G_i \times \mathbb{Z}_2^{a_i}$, where G_i is one of the groups listed in equation (4.2). Let $a = \sum_i a_i$. Finally, since all the groups N_{c_i} commute with one another by Lemma 4.3, one has

$$N_{c_i} \cap N_{c_j} \subseteq Z(N_{c_i}) \cap Z(N_{c_j}) \quad \text{and} \quad Z(N_{c_i}) \subseteq Z(K_2).$$

Therefore

$$K_2 \cong \Big(Z_{(2)} \times \prod_{i=1}^{k} N_{c_i}\Big)/Z' = \Big(Z_{(2)} \times \mathbb{Z}_2^a \times \prod G_i\Big)/Z',$$

where $Z' \subseteq Z_{(2)} \times \prod_i Z(N_{c_i})$ is the subgroup of K_2 generated by the intersections $H_{ij} = N_{c_i} \cap N_{c_j}$ and $H_{0j} = Z_{(2)} \cap N_{c_j}$. Since the groups H_{ij}, H_{0j} are all abelian and central, commute with one another, and have elements of order 2 or 4 (because $Z(N_{c_i}) = \mathbb{Z}_2^{a_i} \times \mathbb{Z}_2$ or $\mathbb{Z}_2^{a_i} \times \mathbb{Z}_4$), it follows that $Z' = \mathbb{Z}_2^{\alpha} \times Z_4^{\beta}$ for some α and β. ∎

4.2. Examples of fundamental groups of principal leaves

The family of examples below shows that the non-abelian groups G_i discussed in Theorem B actually arise as fundamental groups of principal leaves of homogeneous singular Riemannian foliations.

Let $\{e_1, \ldots, e_n\}$ be the standard basis of $\mathbb{R}^n$. The Clifford algebra $\mathrm{Cl}(0, n)$ on $\mathbb{R}^n$ is defined as the associative algebra generated by $e_1, \ldots, e_n$, where multiplication of the elements e_i is given by

$$e_i^2 = -1, \quad e_i e_j = -e_j e_i.$$

Consider the subset $E(n) = \{\pm e_{i_1} \ldots e_{i_{2k}}\} \subseteq \mathrm{Cl}(0, n)$ containing products of even numbers of the e_i's. This is easily seen to be a group under the product of $\mathrm{Cl}(0, n)$. In [3], Czarnecki, Howe, and McTavish prove that for the action of $G = \mathrm{SO}(n) \times \mathrm{SO}(n)$ on $M_{n\times n}(\mathbb{R})$ defined by $(g, h) \cdot A = g^T A h$, the fundamental group of a principal orbit is of the form $E(n) \times \mathbb{Z}_2$. In this section, we investigate the structure of $E(n)$.

Lemma 4.10. *Let $G_{0,n-1}$ be the group defined by the generators $-1, e_1, \ldots, e_{n-1}$ and the relations*

$$(-1)^2 = 1, \quad (e_i)^2 = -1, \quad [e_i, e_j] = -1 \ (i \neq j), \quad [e_i, -1] = 1.$$

Then the groups $E(n)$ and $G_{0,n-1}$ are isomorphic.

Proof. We have

$$G_{0,n-1} = \{\pm e_{i_1} \dots e_{i_\ell} \mid 1 \le i_j \le n-1,\, e_i^2 = -1,\, e_i e_j = -e_j e_i\}.$$

Given an ordered set $I = (i_1, \dots, i_m)$ with indices i_j in $\{1, \dots, n-1\}$, let $e_I = e_{i_1} \dots e_{i_m}$. Notice that if $I = (i_1, \dots, i_m)$ and $J = (j_1, \dots, j_p)$, then $e_I e_J = e_{I \cup J}$, where $I \cup J = (i_1, \dots, i_m, j_1, \dots, j_p)$. Now, define the map $\psi\colon G_{0,n-1} \to E(n)$ by

$$\psi(e_I) = \begin{cases} e_I & \text{for } |I| \text{ even,} \\ e_{I \cup \{n\}} & \text{for } |I| \text{ odd.} \end{cases}$$

First, we claim that $\psi(e_I e_J) = \psi(e_I)\psi(e_J)$ for multi-indices I and J.

Case 1. $|I|$ and $|J|$ are both even. In this case, we have

$$\psi(e_I e_J) = \psi(e_{I \cup J}) = e_{I \cup J} = e_I e_J = \psi(e_I)\psi(e_J).$$

Case 2. $|I|$ and $|J|$ are both odd. In this case, we have

$$\psi(e_I e_J) = \psi(e_{I \cup J}) = e_{I \cup J} = e_I e_J = e_I e_J(-e_n e_n) = e_{I \cup \{n\}} e_{J \cup \{n\}} = \psi(e_I)\psi(e_J).$$

Case 3. If $|I|$ is even and $|J|$ is odd, then

$$\psi(e_I e_J) = \psi(e_{I \cup J}) = e_{I \cup J \cup \{n\}} = e_I e_{J \cup \{n\}} = \psi(e_I)\psi(e_J).$$

Case 4. If $|I|$ is odd and $|J|$ is even, then

$$\psi(e_I e_J) = \psi(e_{I \cup J}) = e_{I \cup J \cup \{n\}} = e_{I \cup \{n\}} e_J = \psi(e_I)\psi(e_J).$$

Therefore, ψ is a homomorphism. It is easy to see that ψ is injective, and hence an isomorphism since the groups $G_{0,n-1}$ and $E(n)$ have the same order. ∎

The groups $G_{0,n-1}$ have been classified by Salingaros [24–26] (cf. [1]). We use this classification to write the group $E(n) \cong G_{0,n-1}$ as a central product. This gives rise to the following list for fundamental groups of the principal orbits of the G-action on $M_{n\times n}(\mathbb{R})$:

$$E(n) \times \mathbb{Z}_2 \cong \begin{cases} ((Q_8)^{*\frac{n-4}{2}} * D_8) \times \mathbb{Z}_2^2 & \text{for } n \equiv 0 \pmod 8, \\ (Q_8)^{*\frac{n-1}{2}} \times \mathbb{Z}_2 & \text{for } n \equiv 1, 3 \pmod 8, \\ ((Q_8)^{*\frac{n-2}{2}} * \mathbb{Z}_4) \times \mathbb{Z}_2 & \text{for } n \equiv 2, 6 \pmod 8, \\ (Q_8)^{*\frac{n-2}{2}} \times \mathbb{Z}_2^2 & \text{for } n \equiv 4 \pmod 8, \\ ((Q_8)^{*\frac{n-3}{2}} * D_8) \times \mathbb{Z}_2 & \text{for } n \equiv 5, 7 \pmod 8. \end{cases}$$

We do not know, however, whether *all* groups in Theorem B do in fact arise as fundamental groups of principal leaves in a simply connected manifold.

5. Virtually nilpotent fundamental group

In this section, we consider singular Riemannian foliations $(M, \mathcal{F})$, where the fundamental group of M is virtually nilpotent. As the following example shows, the fundamental group of a principal leaf is not necessarily nilpotent in this case.

Example 5.1. Let $\hat{M} = \mathbb{C}^2 \times \mathbb{S}^1$ and consider the homogeneous foliation $\hat{\mathcal{F}}$ on $\hat{M}$ induced by the linear action of $T^3 = T^2 \times S^1$. Let $M = \hat{M}/\mathbb{Z}_2$, where the non-trivial element g of $\mathbb{Z}_2$ acts by $g \cdot (z_1, z_2, t) = (\bar{z}_1, \bar{z}_2, t + \frac{1}{2})$. Note that M inherits a singular Riemannian foliation $\mathcal{F} = \hat{\mathcal{F}}/\mathbb{Z}_2$.

The manifold M is orientable, and is homotopy equivalent to $\mathbb{S}^1$. In particular, M is nilpotent. However, the principal leaf of $\mathcal{F}$ is $T^3/\mathbb{Z}_2$, which has fundamental group

$$G = \mathbb{Z}^2 \rtimes \mathbb{Z} = \langle a, b, c : cac^{-1} = a^{-1}, cbc^{-1} = b^{-1}, ab = ba \rangle.$$

Since $G_\ell = \langle a^{2^\ell}, b^{2^\ell} \rangle$ for any ℓ, G is not nilpotent.

Nevertheless, in what follows, we prove that the fundamental groups of the leaves contain a nilpotent subgroup of finite index.

Notation 5.2. Throughout the rest of this section, L_0 denotes a principal leaf of $\mathcal{F}$. Furthermore, we fix $p_0 \in L_0$, and $K = \langle k_1, \dots, k_m \rangle$ denotes the normal subgroup of $\pi_1(L_0, p_0)$ discussed at the beginning of Section 4. Recall that there is a homotopy fibration

$$L_0 \stackrel{\iota_0}{\to} M_0 \stackrel{\hat{\theta}}{\to} B,$$

which induces a long exact sequence

$$0 \to H \to \pi_1(L_0, p_0) \stackrel{(\iota_0)_*}{\to} \pi_1(M_0, p_0) \stackrel{\hat{\theta}_*}{\to} \pi_1(B, b) \to 1,$$

where $H = \partial(\pi_2(B))$, as well as an action of $\pi_1(B, b)$ on L_0. Denote by $\hat{K}$ the group generated by H and $c \cdot K$, for $c \in \pi_1(B, b)$. Notice that for every $\gamma \in \pi_1(M_0, p_0)$ with $c = \hat{\theta}_*(\gamma)$, and every $g \in \pi_1(L_0, p_0)$,

$$(\iota_0)_*(c \cdot g) = \gamma (\iota_0)_*(g) \gamma^{-1}.$$

Lemma 5.3. *Let $(M, \mathcal{F})$ be a closed singular Riemannian foliation on a compact Riemannian manifold M. If $\pi_1(M)$ is n-step nilpotent, then $(\pi_1(L_0, p_0))_{n+1} \subseteq \hat{K}$, where $(\pi_1(L_0, p_0))_{n+1}$ denotes the $(n+1)$-th group in the lower central series of $\pi_1(L_0, p_0)$.*

Proof. Since removing strata of codimension > 2 does not change the fundamental group of M, we can assume that M only contains singular strata of codimension ≤ 2. In particular, we use the notation and results in Section 3.1.

Letting $\iota\colon L_0 \to M$ denote the inclusion, one then has

$$\iota_*((\pi_1(L_0, p_0))_{n+1}) \subseteq (\pi_1(M, p_0))_{n+1} = 1.$$

Therefore, given any curve α representing an element of $(\pi_1(L_0, p_0))_{n+1}$, there exists a disk $\bar{\iota}\colon \mathbb{D}^2 \to M$ extending $\iota(\alpha)$. By transversality, this can be deformed to only intersect, transversely, the singular strata $\Sigma_1, \dots, \Sigma_m$ of codimension 2, and the intersection consists of finitely many points $\{q_1, \dots, q_r\}$ with $q_j \in \Sigma_{i_j}$. For each $j = 1, \dots, r$, let q_j' be a point in $\bar{\iota}(\mathbb{D}^2)$ close to q_j, let u_j be a curve in $\bar{\iota}(\mathbb{D}^2)$ connecting p_0 to q_j', and let ψ_j be a small loop in $\bar{\iota}(\mathbb{D}^2)$ based at q_j', turning once around q_j. Finally, let $\gamma_j = u_j \star \psi_j \star u_j^{-1}$. Then:

(1) For each $i = 1, \dots, r$, $[\gamma_j] \in \pi_1(M_0, p_0)$ is conjugate to $(\iota_0)_*(k_{i_j})$ with $k_{i_j} \in K \subseteq \pi_1(L_0, p_0)$. By the discussion in Notation 5.2, it follows that $[\gamma_j] = (\iota_0)_*(c_j \cdot k_{i_j})$ for some $c_j \in \pi_1(B, b)$.

(2) $(\iota_0)_*[\alpha] = [\gamma_1] \star \cdots \star [\gamma_r] = (\iota_0)_*((c_1 \cdot k_{i_1}) \star \cdots \star (c_r \cdot k_{i_r}))$ in $\pi_1(M_0, p_0)$.

Since $H = \ker((\iota_0)_*)$, it follows that

$$[\alpha] = h((c_1 \cdot k_{i_1}) \star \cdots \star (c_r \cdot k_{i_r}))$$

for some $h \in H$. In particular, $[\alpha] \in \hat{K}$, and therefore $(\pi_1(L_0, p_0))_{n+1} \subseteq \hat{K}$. ■

We are finally ready to prove that if $(M, \mathcal{F})$ is a closed singular Riemannian foliation with $\pi_1(M)$ virtually nilpotent, then the fundamental group of every leaf is virtually nilpotent as well.

Proof of Theorem C. Notice that if $\pi\colon \hat{M} \to M$ is a finite cover, and $(\hat{M}, \hat{\mathcal{F}})$ is the lifted singular Riemannian foliation, one has that a leaf $\hat{L} \in \hat{\mathcal{F}}$ has virtually nilpotent fundamental group if and only if the corresponding leaf $\pi(\hat{L}) \in \mathcal{F}$ does. Therefore, up to replacing M with a finite cover $\hat{M}$, we can assume that $\pi_1(M)$ is nilpotent.

Let L_0 be a principal leaf, and consider the Hurewicz homomorphism $h\colon \pi_1(L_0, p_0) \to H_1(L_0; \mathbb{Z})$ and let $G = h^{-1}(2 \cdot H_1(L_0; \mathbb{Z}))$. Clearly, G has finite index in $\pi_1(L_0, p_0)$, Since $\pi_1(L_0, p_0)/G \cong H_1(L_0; \mathbb{Z})/2 \cdot H_1(L_0; \mathbb{Z})$ is finite. We claim that if $\pi_1(M)$ is n-step nilpotent, then G is $(n+1)$-step nilpotent.

By Lemma 5.3, $G_{n+1} \subseteq G \cap \hat{K}$. The proof is complete once we prove that G commutes with $\hat{K}$. Notice that $\hat{K}$ is generated by H, and by elements of the form $c \cdot k_i$ for $c \in \pi_1(B, b)$ and k_i one of the generators of K. Recall that H is central in $\pi_1(L_0, p_0)$ (in particular, G commutes with H), and for each $g \in \pi_1(L_0, p_0)$, $g k_i g^{-1} = k_i^{\pm 1}$. Since $\pi_1(B, b)$ acts on $\pi_1(L_0, p_0)$ by group automorphisms, it also follows that for every $g \in \pi_1(L_0, p_0)$, $g(c \cdot k_i)g^{-1} = (c \cdot k_i)^{\pm 1}$.

Notice that if $g(c \cdot k_i)g^{-1} = (c \cdot k_i)^{\varepsilon}$ (for $\varepsilon = \pm 1$), then $g^{-1}(c \cdot k_i)g = (c \cdot k_i)^{\varepsilon}$ as well. In particular, for every $g_1, g_2 \in \pi_1(L_0, p_0)$, and every $(c \cdot k_i) \in \hat{K}$, one has

$$[g_1, g_2] \cdot (c \cdot k_i)[g_1, g_2]^{-1} = (c \cdot k_i).$$

The main observation is that, by definition, any element $g \in G$ can be written as

$$g = g_3^2 [g_1, g_2] \cdots [g_{2k-1}, g_{2k}]$$

for some $g_1, \dots, g_{2k} \in \pi_1(L_0, p_0)$ and therefore, for any generator $(c \cdot k_i)$ of $\hat{K}$, one has

$$\begin{aligned} g(c \cdot k_i)g^{-1} &= g_3^2 [g_1, g_2] \cdots [g_{2k-1}, g_{2k}](c \cdot k_i)[g_{2k-1}, g_{2k}]^{-1} \cdots [g_1, g_2]^{-1} g_3^{-2} \\ &= g_3^2 (c \cdot k_i) g_3^{-2} = g_3 (c \cdot k_i)^{\varepsilon} g_3^{-1} \\ &= (c \cdot k_i)^{\varepsilon^2} = (c \cdot k_i). \end{aligned}$$

Therefore, G commutes with $\hat{K}$ and hence $G_{n+2} = [G, G_{n+1}] \subseteq [G, \hat{K}] = \{1\}$.

This proves that the principal leaves of $\mathcal{F}$ have virtually nilpotent fundamental group. The corresponding statement for the non-principal leaves then follows from Lemma 3.2. ∎

A. Classification of quadratic forms over $\mathbb{Z}_2$

The classification of quadratic forms over $\mathbb{Z}_2$ is well known. However, what appears usually in the literature is the classification of *nondegenerate* quadratic forms, which is not what interests us here. Therefore, we provide the details of the classification.

Proposition A.1. *Every non-trivial quadratic form on $\mathbb{Z}_2^\ell$ is isometric to one of the following*:

$$H_+^{\oplus m_1} \oplus 0^{m_2}, \quad H_- \oplus H_+^{\oplus m_1 - 1} \oplus 0^{m_2} \quad \textit{or} \quad H_+^{\oplus m_1} \oplus Q_1 \oplus 0^{m_2 - 1},$$

where $2m_1 + m_2 = \ell$.

Proof. Let $H\colon \mathbb{Z}_2^2 \times \mathbb{Z}_2^2 \to \mathbb{Z}_2$ be the bilinear form given by

$$H((x, y), (z, w)) = xw + yz.$$

By the classification of bilinear forms over $\mathbb{Z}_2$ (cf. for example Proposition 1.8, Corollary 1.9 and the discussion below in [5]), every symmetric bilinear form on a vector space V over $\mathbb{Z}_2$ is isometric to $H^{m_1} \oplus 0^{m_2}$, where $2m_1 + m_2 = \ell$. By equation (2.1) in Section 2.4, it is easy to see that there are two equivalence classes of quadratic forms associated to H^{m_1}, that is, the quadratic forms $Q = H_+^{m_1}$ and $Q = H_- \oplus H_+^{m_1 - 1}$, where $H_\pm\colon \mathbb{Z}_2^2 \to \mathbb{Z}_2$ are given by

$$H_+(x, y) = xy \quad \text{and} \quad H_-(x, y) = x^2 + y^2 + xy.$$

Similarly, to 0^{m_2} correspond the quadratic forms $Q_0 = 0$ and $Q_\alpha(x_1, \dots, x_{m_2}) = \sum_{i=1}^{\alpha} x_i^2$ for any $1 \le \alpha \le m_2$. However, Q_α is isometric to $Q_1 \oplus 0^{m_2 - 1}$. Moreover, one has well known isometries

$$H_+^{\oplus m_1} \oplus Q_1 \simeq H_- \oplus H_+^{\oplus m_1 - 1} \oplus Q_1 \quad \text{and} \quad H_+^{\oplus 2} \simeq H_-^{\oplus 2},$$

which conclude the proof. ∎

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