Non-isotopic transverse tori in Engel manifolds

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Abstract. In every Engel manifold, we construct an infinite family of pairwise non-isotopic transverse tori that are all smoothly isotopic. To distinguish the transverse tori in the family, we introduce a homological invariant of transverse tori that is similar to the self-linking number for transverse knots in contact 3-manifolds. Analogous results are presented for Legendrian tori in even contact 4-manifolds.

1. Introduction

In recent years, some spectacular breakthroughs in the theory of Engel structures were obtained that have renewed interest in the field. After the solution of the existence question for Engel structures on parallelizable 4-manifolds by Vogel [25], a number of flexibility results, formulated in the language of h-principles, were proven, see [5, 7, 22, 23]. (For more details, consult Section 2 and the references therein.)

In [23], del Pino and Vogel have introduced the notion of an overtwisted Engel structure, fulfilling an h-principle. The 2-tori transverse to an Engel structure play a crucial role in this construction, similar as transverse knots played a key role in the construction of overtwisted contact structures by Martinet [12, 18]. This suggests that transverse tori in Engel manifolds will play a similar prominent role as transverse knots in contact 3-manifolds as started with Bennequin's work [2]. This would fit with the observation that in many cases a lot of the underlying topology and geometry of manifolds is encoded in its codimension-2 knot theory. The aim of this article is to show that the knot theory of transverse tori is rich in examples that are smoothly isotopic but not isotopic as transverse tori.

Theorem 1.1. *In any Engel manifold* $(M, \mathcal{D})$, *there exist infinitely many pairwise non-isotopic transverse* 2-*tori that are all smoothly isotopic.*

This result was independently obtained by Gompf in [16].

To prove Theorem 1.1, we will define in Section 3 a stabilization operation for transverse tori coming from the del Pino–Vogel construction [23], that does not change the smooth isotopy class. In order to distinguish a transverse torus from its stabilization, we construct in Sections 4 and 5 a homological invariant, depending only on the formal data

of a transverse torus, that we will use to distinguish stabilized transverse tori. This invariant can be thought of as an analogue of the self-linking number for transverse knots in contact 3-manifolds.

We have a full h-principle for transverse immersions of surfaces [22]. On the other hand, it remains open if transverse tori fulfill a full h-principle. Although, in recent work of Gompf [16] it is shown that transverse tori fulfill an existence h-principle: every 2-torus with trivial normal bundle in an Engel manifold is isotopic to a transverse torus. For 1-dimensional submanifolds of Engel manifolds, the situation is solved. The subspace of tangent knots (also called horizontal or Engel knot) that are not everywhere tangent to $\mathcal{W}$ fulfill a full h-principle [6], cf. [1, 11, 13]. Similarly, transverse curves (i.e., knots always transverse to $\mathcal{D}$) fullfill a full h-principle, see Theorem 4.6.2 in [10].

Any Engel structure $\mathcal{D}$ induces naturally an even contact structure by $\mathcal{E} = [\mathcal{D}, \mathcal{D}]$ (see Section 2 for details). Motivated by Theorem 1.1, we want to study codimension-2 embeddings in even contact manifolds that fit to the even contact structure. Here the natural objects are *Legendrian tori*, embedded 2-tori that are everywhere tangent to the even contact structure $\mathcal{E}$. With similar ideas as in the proof of Theorem 1.1, we will prove an analogous statement for Legendrian tori in even contact structures in Section 7.

Theorem 1.2. *There exist even contact* 4-*manifolds containing infinite families of pairwise non-isotopic Legendrian* 2-*tori that are all smoothly isotopic.*

First, we will give some relevant background on Engel and even contact structures in Section 2. In Section 3, we will recall some facts about transverse tori from [23]. In particular, we will review a method of constructing transverse tori in Engel manifolds from transverse knots in contact 3-manifolds and define a stabilization operation that will not change the smooth isotopy class of the transverse torus. To show that this stabilization operation changes the isotopy class as a transverse torus, we devote Section 4 to *the linking class*, a homology class, which can be seen as a generalization of the linking number, defined for general subsets in topological spaces. From this we can define the self-linking class in Section 5, and deduce a way of computing it for transverse tori coming from transverse knots via the del Pino–Vogel construction of transverse tori. Using this method, we will distinguish transvere tori that are all smoothly isotopic. The remaining section is devoted to Legendrian tori in even contact structures and to prove Theorem 1.2.

Conventions. We work in the smooth and oriented category, i.e., all manifolds, maps, and ancillary objects are assumed to be smooth. In addition, all manifolds and geometric structures on them like Engel and contact structures are assumed to be oriented and cooriented. Homology groups are always understood over the integers.

2. Background

2.1. Topologically stable distributions

An *Engel structure* is a maximally non-integrable tangential 2-plane field $\mathcal{D}$ on a 4-manifold M, i.e., its iterated Lie-brackets $[\mathcal{D}, [\mathcal{D}, \mathcal{D}]]$ give the full tangent bundle TM. It is well known that Engel structures admit a local normal form, which means that Engel structures can only be studied globally. Moreover, the set of Engel structures on a 4-mani-

fold M is open in the set of 2-dimensional distributions on M. In general, a class of k-dimensional distributions on an n-manifold M is called *topologically stable* if it is open and admits a local normal form. The classification of such distributions in terms of geometric structures on manifolds can be traced back to Cartan [20, 24]: every topologically stable distribution is

- a line field ($k = 1$ and n arbitrary),
- a *contact structure* ($k = n - 1$, $n = 2m + 1$ odd; locally written as the kernel of a 1-form α with $\alpha \wedge (d\alpha)^m$ non-vanishing),
- an *even contact structure* ($k = n - 1$, $n = 2m + 2$ even; locally written as the kernel of a 1-form α with $\alpha \wedge (d\alpha)^m$ non-vanishing),
- or an Engel structure ($k = 2$ and $n = 4$).

Although Engel structures play a distinguished role in the above classification, since they are the only class that exists only in the special dimension 4, they are the least understood class of topologically stable distributions. In dimension 4, all these geometries are closely related. An Engel structure $\mathcal{D}$ on a 4-manifold M induces an even contact structure $\mathcal{E}$ by $\mathcal{E} = [\mathcal{D}, \mathcal{D}]$ and the so-called *characteristic line field* $\mathcal{W} \subset \mathcal{D}$ defined by $[\mathcal{W}, \mathcal{E}] \subset \mathcal{E}$. Finally, every 3-dimensional hypersurface N in M transverse to $\mathcal{W}$ carries a canonical contact structure defined by $\xi = \mathcal{E} \cap TN$.

It is therefore not surprising that Engel structures, contact structures and even contact structures often behave in the same way structurally. In the following short summary, we will outline a few points. First of all, by definition all these structures look locally like a standard model

$$\left(\mathbb{R}^3, \xi_{\mathrm{st}} = \ker(dz - y\,dx) = \langle \partial_y, \partial_x + y\partial_z \rangle\right)$$

in the contact case,

$$\left(\mathbb{R}^4, \mathcal{D}_{\mathrm{st}} = \ker(dz - y\,dx) \cap \ker(dy - w\,dx) = \langle \partial_w, \partial_x + y\partial_z + w\partial_y \rangle\right)$$

in the Engel case, and

$$\left(\mathbb{R}^4, \mathcal{E}_{\mathrm{st}} = [\mathcal{D}_{\mathrm{st}}, \mathcal{D}_{\mathrm{st}}] = \ker(dz - y\,dx) = \langle \partial_w, \partial_y, \partial_x + y\partial_z \rangle\right)$$

in the case of an even contact structure.

The existence question for these structures has been solved, and can be simplified by the statement that such a structure exists whenever there is no obvious obstruction coming from the underlying algebraic topology. Some relevant references are [3, 5, 8, 10, 12, 17–19, 23, 25]. On the other hand, the uniqueness question up to homotopy through such structures is in general open.

2.2. Contact structures

We first discuss the 3-dimensional contact case. We have to distinguish the cases where the underlying 3-manifold is open or closed. From Gromov's work [17], cf. [10], it follows that there is an *h*-principle for contact structures on open manifolds. Roughly speaking, an *h-principle* holds if the inclusion of a genuine structure into the formal data of the structure is a weak homotopy equivalence. In this situation, it implies that two positive

contact structures on an open manifold are homotopic through positive contact structures if and only if they are homotopic through tangential 2-plane fields. Thus, the problem of classifying contact structures on open manifolds up to homotopy reduces completely to algebraic topology methods.

However, for open manifolds, there is in general a difference between homotopy through contact structures and isotopy. The first such example was found by Bennequin, see [2]. He showed that there exists a so-called overtwisted contact structure ξ_{ot} on $\mathbb{R}^3$ that is homotopic through contact structures but non-isotopic to the standard structure ξ_{st}. This can be proven by studying *Legendrian knots*, always tangent to the contact structures, or *transverse knots*, always transverse to the contact structures, and their *classical invariants*: the *self-linking number* sl for transverse knots, and the *Thurston–Bennequin invariant* tb and the *rotation number* rot for Legendrian knots.

For closed manifolds, the situation is different. Eliashberg [8] distinguishes two classes of contact structures: the *overtwisted* contact structures (containing a Legendrian unknot with vanishing Thurston–Bennequin invariant), and the *tight* contact structures (containing no such unknot). Then he shows that the overtwisted contact structures fulfill an *h*-principle, implying that two positive overtwisted contact structures are homotopic through positive contact structures if and only if they are homotopic as tangential 2-plane fields. On closed manifolds we have Gray's theorem, implying that two contact structures that are homotopic through contact structures are also isotopic [12].

On the other hand, the tight contact structures are closely linked to the underlying topology of the manifold, and do not only depend on the algebraic topology of the underlying 2-plane field. As in the case of $\mathbb{R}^3$, tight contact structures are closely related to their Legendrian and transverse knot theory. In fact, a contact manifold is tight if and only if every nullhomologous transverse knot T fulfills the Bennequin bound $sl(T) \leq 2g(T) - 1$, or equivalently, if every nullhomologous Legendrian knot L fulfills $tb(L) + rot(L) \leq 2g(L) - 1$, where g denotes the genus of the underlying smooth knot type [9].

Recent developments show that the situation in higher dimensions is formally the same, see [3].

2.3. Even contact structures

For even contact structures and Engel structures, we can ask how much of the above discussion carries over. First, we discuss the case of even contact structures on 4-manifolds. For open manifolds, there is again an *h*-principle by the work of Gromov [17]. Surprisingly, we also have a full *h*-principle for even contact structures on closed manifolds by work of McDuff [19], so we do not need to restrict to a subclass. However, Gray stability does not hold in general, not even for closed manifolds [21]. The reason is that every even contact structure $\mathcal{E}$ comes together with a so-called *characteristic line field* $\mathcal{W} \subset \mathcal{E}$ uniquely defined by $[\mathcal{W}, \mathcal{E}] \subset \mathcal{E}$ and a small perturbation of $\mathcal{E}$ can change the dynamics of $\mathcal{W}$ completely. (A version of Gray's theorem holds if we can assume the characteristic line field to stay fixed during the homotopy between the even contact structures [15].)

In conclusion, we can say that even contact structures on open or closed manifolds will behave structurally similar to contact structures on open manifolds, i.e., the study of even contact structures up to homotopy reduces completely to algebraic topology, but even contact structures up to isotopy are more interesting to study.

2.4. Engel structures

The situation for Engel structures is less well understood. First, we observe that any Engel structure D on a 4-manifold M induces a flag of its tangent bundle

$$\mathcal{W} \subset \mathcal{D} \subset \mathcal{E} \subset TM,$$

where $\mathcal{E}$ is the associated even contact structure and $\mathcal{W}$ its characteristic line field. Thus, in the oriented case, a 4-manifold admitting an Engel structure has to be parallelizable. On the other hand, this is the only obstruction to the existence of an Engel structure: every parallelizable 4-manifold admits an Engel structure. This follows for open manifolds again by Gromov's h-principle, and was proven for closed manifolds by Vogel [25], cf. [5].

If a full h-principle holds is unknown, but recently ([23]) the subclass of *overtwisted* Engel structures fulfilling a full h-principle was introduced. One of the main questions in the field of Engel structures asks if there are Engel structures that are not overtwisted. Moreover, it is not clear if overtwistedness is a property that is preserved under homotopy of Engel structures. And thus we can more generally ask if there exist Engel structures that are not homotopic to overtwisted Engel structures. For Engel structures, there also exists the class of *loose* Engel structures that satisfy an h-principle as well [7]. Unlike the overtwisted Engel structures, that are characterized by a local property, these loose Engel structures are given by a global property. It is not clear how loose Engel structures are related to overtwisted ones.

Since Gray's theorem for Engel structures does not hold [21], there can be also Engel structures that are homotopic but not isotopic. The first such examples were found on $\mathbb{R}^4$ by Gershkovich [14]. One possibility to construct such an example is by using an overtwisted contact structure ξ_{ot} on $\mathbb{R}^3$ as follows. Let X_1 and X_2 be two linearly independent vector fields in ξ_{ot}. Then it is easy to see that $Y_1 := \partial_t$ and $Y_2 := \cos(t)X_1 + \sin(t)X_2$ generate an Engel structure on $\mathbb{R}^4$, where t is a coordinate on the extra $\mathbb{R}$ factor. In [14], Gershkovich used contact geometry to distinguish this Engel structure from the standard Engel structure on $\mathbb{R}^4$. Such Engel structures could potentially also be distinguished by the classical invariants of transverse tori.

3. Transverse tori

In [23], del Pino and Vogel used embedded transverse tori in Engel manifolds as generalizations of transverse knots in contact manifolds. In this section, we recall the relevant definitions and constructions from [23].

Definition 3.1. An embedded 2-torus T in an Engel manifold $(M, \mathcal{D})$ is called a *transverse torus* if it is everywhere transverse to $\mathcal{D}$.

In principle, this definition would make sense for any embedded surface, but in the oriented case any such surface would inherit a parallelization and a trivialization of its normal bundle (both coming from the parallelization of the Engel structure). Thus, if we restrict to oriented closed surfaces, every transverse surface is a torus. We remark that the same reasoning applies for non-orientable closed surfaces, and we observe that any non-orientable closed transverse surface is a Klein bottle.

Example 3.2. We consider the 4-torus T^4 with angular coordinates $(\theta_1, \theta_2, \theta_3, \theta_4)$ and Engel structure given by the span of ∂_{θ_1} and $\cos \theta_1 \partial_{\theta_2} + \sin \theta_1 \partial_{\theta_3} + \partial_{\theta_4}$. Then the 2-torus given by $\{\theta_1 = \theta_2 = 0\}$ is a transverse torus.

In [23], the following construction of transverse tori in standard neighborhoods of transverse curves is described. A *transverse curve C* is an embedded S^1 in an Engel manifold $(M, \mathcal{D})$ that is everywhere transverse to the induced even contact structure $\mathcal{E} = [\mathcal{D}, \mathcal{D}]$. By standard methods, one can prove the following two lemmas, see Proposition 2.7 and Lemma 2.8 in [23] for details.

Lemma 3.3. *Let $C: S^1 \to (M, \mathcal{D})$ be an embedding. Then C is isotopic to a transverse curve. Moreover, if C_1 and C_2 are two transverse curves in $(M, \mathcal{D})$ that are smoothly isotopic, then they are isotopic as transverse curves.*

Lemma 3.4. *Any transverse curve C in $(M, \mathcal{D})$ admits a neighborhood $(\nu C, \mathcal{D})$ (where here $\mathcal{D}$ denotes the restriction of the Engel structure of M to νC) which is, for some $R > 0$, Engel diffeomorphic to the following standard neighborhood:*

$$\left(S^1 \times D_R^3, \mathcal{D}_{st} = \ker(d\theta - y\,dx) \cap \ker(dy - w\,dx)\right),$$

where θ is an angular coordinate of S^1, (x, y, w) are Cartesian coordinates on the disk D_R^3 with radius R in $\mathbb{R}^3$, and the Engel-diffeomorphism sends C to $S^1 \times \{0\}$.

In particular, there exist many transverse curves, and in any standard neighborhood we can construct transverse tori with the following lemma, see Lemma 3.3 in [23] for details.

Lemma 3.5. *Let K be a transverse knot in $(D_R^3, \xi = \ker dy - w\,dx)$. Then*

$$S^1 \times K \subset (S^1 \times D_R^3, \mathcal{D}_{st})$$

is a transverse torus.

With Lemmas 3.3, 3.4 and 3.5, we get transverse tori in any Engel manifold. We call a transverse torus arising like this a *del Pino–Vogel torus*. On the other hand, not every transverse torus is a del Pino–Vogel torus. Indeed, if a transverse torus arises as above, it is nullhomologous, since it bounds the embedded 3-manifold $S^1 \times F$, where F is a Seifert surface of K in D_R^3. In particular, we see that the transverse torus from Example 3.2 is not a del Pino–Vogel torus. Moreover, it is not clear if the *core C* and the *profile K* in the above construction are unique.

There are different types of natural equivalence relations on transverse tori. We briefly discuss them to fix our notation. Two transverse tori T_1 and T_2 in an Engel manifold are called *(ambient) isotopic* if there exists an Engel diffeomorphism $f: (M, \mathcal{D}) \to (M, \mathcal{D})$ that maps T_1 to T_2 and is isotopic (through Engel diffeomorphisms) to the identity. Two transverse tori T_1 and T_2 in an Engel manifold are called *equivalent* if there exists an Engel diffeomorphism $f: (M, \mathcal{D}) \to (M, \mathcal{D})$ that maps T_1 to T_2. Obviously, ambient isotopy implies equivalence, the reverse implication is in general false (since not every Engel diffeomorphism will be isotopic to the identity). Two transverse tori $T_i: T^2 \to (M, \mathcal{D})$ are called *isotopic* if there exists a homotopy between them through transverse tori. Ambient isotopy implies isotopy, but other than in the smooth category, the isotopy extension theorem does not hold, i.e., there exist transverse tori that are isotopic but not ambient isotopic.

Note that $\mathcal{E}$ intersects a transverse torus in a line field, the so called *characteristic foliation*, which will be preserved under ambient isotopy but not under isotopy. Two transverse tori $T_i \colon T^2 \to (M, \mathcal{D})$ are called *formal isotopic* if there exists a homotopy between them through formal transverse tori. Every isotopy can be also seen as a formal isotopy. In Section 6, we will properly define formal transverse tori and demonstrate that our invariants will distinguish transverse tori only up to formal isotopy. It remains open if transverse tori fulfill a full *h*-principle, i.e., if any two formal isotopic transverse tori are isotopic.

In Theorem 1.1, we show that in any Engel manifold there are infinitely many non-formally isotopic transverse tori that are all smoothly isotopic. The construction of these examples is via the obvious stabilization operation. Let T be a del Pino–Vogel transverse torus with core C and profile K. We define the *stabilization* T_{stab} of T to be $C \times K_{\text{stab}}$, where K_{stab} is the stabilization of K. Since it is not clear if the profile is unique, the stabilization could depend on K and C, and it is not clear if the transverse knot K is an invariant of T. However, we would expect that T_{stab} is not isotopic to T. In the next sections we will develop an invariant, similar to the self-linking number of transverse knots, that will enable us to distinguish stabilizations of transverse tori.

4. The linking class for embedded tori

In this section, we will introduce a generalization of the linking number to higher dimensional embedded submanifolds. We believe that this invariant, which we call linking class, is known to the experts, but we could not find a discussion in the literature. Other definitions of linking numbers in higher dimensions are mostly given for embeddings of spheres and differ from our discussion here, see for example [4].

Let L and T be submanifolds of dimension l and t of a topological m-manifold M. Then we define the l-th *linking class* $\mathrm{lk}(L, T)$ to be the homology class

$$\mathrm{lk}(L, T) := [L] \in H_l(M \setminus T).$$

We remark that for nullhomologous knots L and T in a 3-manifold M, we can identify $H_1(M \setminus T)$ canonically with $H_1(M) \oplus \mathbb{Z}$, where the first linking class $\mathrm{lk}(L, T)$ lies in the $\mathbb{Z}$-summand and can thus be identified with an integer, the linking number.

To make a useful invariant out of the linking class, we first need to compute the homology of $M \setminus T$. With a view to our application and for concreteness, we consider two special cases in the following lemmas.

Lemma 4.1. *Let T and T' be embedded 2-tori with trivial normal bundle in a 4-manifold M. If T and T' are nullhomologous in $H_2(M)$ and if $H_3(M) = 0$, then the linking class $\mathrm{lk}(T', T)$ takes values in a free abelian group of rank 2, generated by the meridians α and β of T.*

Proof. We choose an identification of νT with $S^1 \times S^1 \times D^2$ such that $S^1 \times S^1 \times \text{pt}$ is nullhomologous in $H_2(M \setminus \overset{\circ}{\nu} T)$. Then we consider the following part of the Mayer–

Vietoris sequence and the images of the generators under the homomorphisms:

$$
\begin{array}{ccccccc}
0 & \longrightarrow & \overbrace{H_2(\partial \nu T)}^{\cong \mathbb{Z}^3} & \xrightarrow{\ f\ } & \overbrace{H_2(\nu T)}^{\cong \mathbb{Z}} \oplus H_2(M \setminus \overset{\circ}{\nu}T) & \longrightarrow & H_2(M). \\
& & S^1 \times S^1 \times \mathrm{pt} & \longmapsto & (S^1 \times S^1 \times 0, 0) & \longmapsto & 0 \\
& & \alpha := S^1 \times \mathrm{pt} \times \partial D^2 & \longmapsto & (0, \alpha) & \longmapsto & 0 \\
& & \beta := \mathrm{pt} \times S^1 \times \partial D^2 & \longmapsto & (0, \beta) & \longmapsto & 0
\end{array}
$$

The map f is injective, and we write for simplicity again α and β for their images in $H_2(M \setminus \overset{\circ}{\nu}T)$ (see Figure 1). Since $T = S^1 \times S^1 \times 0$ is nullhomologous, it maps to 0 in $H_2(M)$. The generators α and β are boundaries in νT and thus also vanish in $H_2(M)$. Therefore, f acts on the generators, as stated. Since f is injective, it follows that α and β are primitive elements in $H_2(M \setminus \overset{\circ}{\nu}T)$ and thus generate a free abelian group of rank 2. Since T' is nullhomologous in $H_2(M)$, it follows that T' seen as a homology class in $H_2(M \setminus \overset{\circ}{\nu}T)$ is a linear combination of α and β, which implies the statement. ∎
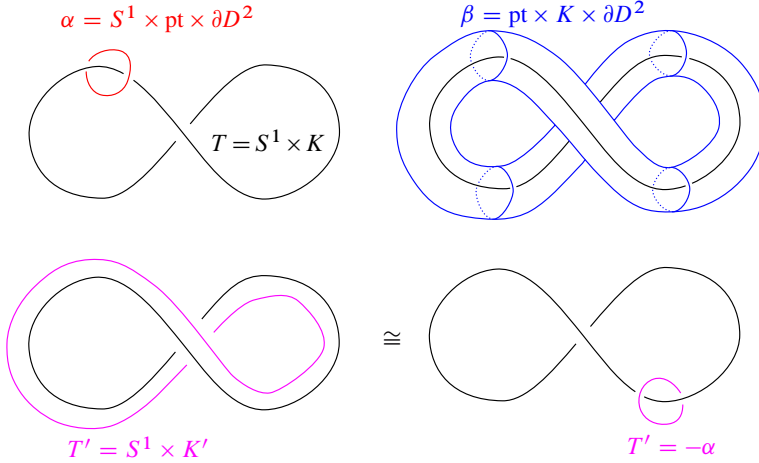

Figure 1. Top row: the generators α and β are presented in the front projection of an $\mathbb{R}^3$-slice. Bottom row: the push-off T' is isotopic to $\mathrm{sl}(K)\alpha$.

Lemma 4.2. *Let K and K' be nullhomologous knots in a closed 3-manifold N. We consider $L = S^1 \times K$ and $L' = S^1 \times K'$ in $M = S^1 \times N$. Then $H_2(M \setminus L)$ is isomorphic to $H_1(N) \oplus H_2(N) \oplus \mathbb{Z}$ and the linking class $\mathrm{lk}(L', L)$ takes values in the last free abelian group generated by $S^1 \times \mu_K$, where μ_K is the meridian of K.*

Proof. First we combine Poincaré duality, excision and the universal coefficient theorem, to deduce that

$$H_2(M \setminus \overset{\circ}{\nu}L) \cong F_2(M, L) \oplus T_1(M, L),$$

where $F_2(M, L)$ denotes the free part of $H_2(M, L)$ and $T_1(M, L)$ the torsion of $H_1(M, L)$. By applying the relative Künneth formula to $S^1 \times (N, K)$, we compute

$$H_k(M, L) \cong H_{k-1}(N, K) \oplus H_k(N, K).$$

Next, we consider the long exact sequence of the pair (N, K),

$$0 \longrightarrow H_2(N) \longrightarrow H_2(N, K) \longrightarrow \mathbb{Z} \overset{0}{\longrightarrow} H_1(N) \longrightarrow H_1(N, K) \longrightarrow 0,$$

where the map $\mathbb{Z} \to H_1(N)$ is the 0-map since K is nullhomologous in N. Thus

$$H_1(N, K) = H_1(N) \quad \text{and} \quad H_2(N, K) \cong H_2(N) \oplus \mathbb{Z}.$$

Putting everything together and using that $H_2(N)$ is a free group, we get that $H_2(M \setminus L)$ is isomorphic to $H_1(N) \oplus H_2(N) \oplus \mathbb{Z}$, where we can follow the above isomorphisms to see that the $\mathbb{Z}$-summand is generated by $S^1 \times \mu_K$. Since L' is nullhomologous in $H_2(M)$ and K' is nullhomologous in $H_1(N)$, it follows that L' seen as a homology class in $H_2(M \setminus L)$ is a multiple of $S^1 \times \mu_K$. ∎

5. The self-linking class for transverse tori

Let T be a transverse 2-torus embedded in an Engel 4-manifold $(M, \mathcal{D})$. We denote by T' the embedded 2-torus that is obtained by pushing T into the $\mathcal{W}$-direction. The smoothly embedded link of 2-tori $T \cup T'$ in M is an invariant of the transverse torus T. Similar as for the definition of the classical invariants for Legendrian or transverse knots in contact manifolds, we define the following invariant for transverse tori.

Definition 5.1. Let T be a transverse 2-torus in an Engel 4-manifold $(M, \mathcal{D})$. Then we define the self-linking class $\mathrm{sl}(T)$ as the homology class

$$\mathrm{sl}(T) := \mathrm{lk}(T', T) \in H_2(M \setminus T).$$

Next, we will see that for a del Pino–Vogel torus, the self-linking class is related to the self-linking number of its profile.

Lemma 5.2. *Let T be a del Pino–Vogel transverse torus in $(M, \mathcal{D})$ with a fixed choice of a core C and a profile K. In the notation introduced in the proof of Lemma 4.1, we can express the self-linking class of T as*

$$\mathrm{sl}(T) = \mathrm{sl}(K) \cdot \alpha \in H_2(M \setminus T).$$

Proof. We get an identification of T with

$$S^1 \times K \subset (S^1 \times D_R^3, \mathcal{D}_{\mathrm{st}}) \subset (M, \mathcal{D}).$$

In this local model, T' is defined to be the push-off of T in the w-direction. In this situation, T' is $S^1 \times K'$, where K' is the push-off of K in w-direction. If we draw the transverse knot $K \subset (D_R^3, \ker dy - w dx)$ in its front projection to the wx-plane, we see that K' is isotopic to the blackboard framing of this projection (see Figure 1).

By an easy homology computation in $H_2(M \setminus \overset{\circ}{\nu} T)$ and the definition of the self-linking number for transverse knots, it follows that

$$\mathrm{sl}(T) = \mathrm{lk}(T, T') = [T'] = [S^1 \times K'] = \mathrm{sl}(K) \cdot [S^1 \times \mathrm{pt} \times \partial D^2] = \mathrm{sl}(K) \cdot \alpha. \quad ∎$$

With this preparation, we are ready to give the proof of Theorem 1.1.

Proof of Theorem 1.1. By the local normal form for Engel structures, it is enough to work in $(\mathbb{R}^4, \mathcal{D}_{st})$. Let T_0 be a del Pino–Vogel transverse torus of the form $C \times K_0$, where C is a transverse curve in $(\mathbb{R}^4, \mathcal{D}_{st})$ and K_0 a transverse knot with self-linking number $sl(K_0)$ in the $(D_R^3, \ker dy - w dx)$-slice of a standard neighborhood $(\nu C, \mathcal{D}_{st})$ of C. We denote by T_n the n-fold stabilization of T_0, i.e., the transverse torus given by $C \times K_n$, where K_n denotes the n-fold stabilization of K_0 with self-linking number $sl(K_n) = sl(K_0) - n$.

Note that since K_n is smoothly isotopic to K_0, it follows that T_n is smoothly isotopic to T_0, for all $n \in \mathbb{N}_0$. Next, we want to show that the T_n are pairwise non-isotopic as transverse tori by comparing their self-linking classes. For that we use Lemma 5.2 to compute the self-linking classes for some choice of α and β as

$$sl(T_n) = sl(K_n) \cdot \alpha = (sl(K_0) - n) \cdot \alpha.$$

But since α is a primitive element in a free abelian group, its divisibility $sl(K_n)$ is independent of the choice of α and β, and thus distinguishes the transverse tori T_n pairwise. ∎

6. Formal transverse tori and their self-linking classes

We recall that a formal Engel structure on M (in the oriented setting) consists of three oriented and cooriented subbundles $(\mathcal{W}^1, \mathcal{D}^2, \mathcal{E}^3)$ of TM such that $\mathcal{W} \subset \mathcal{D} \subset \mathcal{E} \subset TM$. A *formal transverse torus* in a formal Engel manifold $(M; \mathcal{W}, \mathcal{D}, \mathcal{E})$ is an embedding $f: T^2 \to M$ covered by a family of injective bundle maps $(F_s)_{s \in [0,1]}: TT^2 \to TM$ such that $F_0 = Tf$ and $F_1(TT^2)$ is transverse to $\mathcal{D}$. We have obvious inclusions of the space of Engel structures and the space of transverse tori into their formal counterparts. Next, we generalize the definition of the self-linking class to formal transverse tori.

Let $T = (f, (F_s)_{s \in [0,1]})$ be a formal transverse torus in a formal Engel manifold $(M; \mathcal{W}, \mathcal{D}, \mathcal{E})$, and set $\mathcal{W}_1 := \mathcal{W}$. Since $\mathcal{W}_1$ is transverse to $F_1(TT^2)$, we can apply the homotopy lifting property to obtain unique (up to homotopy) line fields $\mathcal{W}_s$ that are transverse to $F_s(TT^2)$, for all $s \in [0,1]$. Then $\mathcal{W}_0$ is transverse to $T = f(T^2)$, and we denote by T' the 2-torus which is obtained by pushing T into $\mathcal{W}_0$-direction. We define the *self-linking class* $sl(T)$ of T as the homology class

$$sl(T) := lk(T', T) \in H_2(M \setminus T).$$

Since $\mathcal{W}_0$ is unique up to homotopy, this is well-defined and generalizes the previous definition.

We will now demonstrate that the self-linking class does not change under homotopy through formal transverse tori, and also does not change through formal homotopy of the ambient Engel manifold.

Lemma 6.1. *Let $(\mathcal{W}_t, \mathcal{D}_t, \mathcal{E}_t)_{t \in [0,1]}$ be a family of formal Engel structures on a closed 4-manifold M, and let $(T_t)_{t \in [0,1]}$ be a family of formal transverse 2-tori such that T_t is formally transverse to $\mathcal{D}_t$. Then there is a smooth ambient isotopy that identifies $H_2(M \setminus T_0)$ and $H_2(M \setminus T_1)$. Under this identification, the self-linking class $sl(T_0)$ of T_0 equals the self-linking class $sl(T_1)$ of T_1.*

Proof. Since T_t is a smooth family of embedded 2-tori in M, the isotopy extension theorem guarantees the existence of a smooth ambient isotopy of the T_t, i.e., a family of diffeomorphisms $(G_t)_{t \in [0,1]} \colon M \to M$ such that $G_0 = \mathrm{Id}_M$ and $G_t(T_0) = T_t$. By restricting this ambient isotopy to the complements, we obtain a family of diffeomorphisms, for simplicity again denoted by $(G_t)_{t \in [0,1]} \colon M \setminus T_0 \to M \setminus T_t$, such that $G_0 = \mathrm{Id}_{M \setminus T_0}$. We use the induced isomorphism on homology to identify $H_2(M \setminus T_t)$ with $H_2(M \setminus T_0)$, for any $t \in [0,1]$.

Now let $T_t = (f_t, (F_s^t)_{s \in [0,1]})$, where $f_t \colon T^2 \to M$ is an embedding, and where $(F_s^t)_{s \in [0,1]} \colon TT^2 \to TM$ denotes a family of injective bundle maps with $F_0^t = Tf_t$ and $F_1^t(TT^2)$ transverse to $\mathcal{D}_t$. As in the definition of the self-linking number for formal transverse tori, we obtain a family of line fields $\mathcal{W}_t^s$ such that $\mathcal{W}_t^1 = \mathcal{W}_t$ and $\mathcal{W}_t^s$ is transverse to $F_s^t(TT^2)$. Applying again the homotopy lifting property, we see that $\mathcal{W}_t^0$ is homotopic to $TG_t(\mathcal{W}_0^0)$. But this implies that the homology class $[T_0']$ gets mapped under $(G_t)_*$ to $[T_t']$, and thus the self-linking classes of T_0 and T_1 get identified. ∎

7. Legendrian tori in even contact manifolds

In this section, we will briefly initiate the study of tangential embeddings of maximal dimension in even contact structures.

Definition 7.1. An embedded 2-torus L in an even contact 4-manifold $(M, \mathcal{E})$ is called *Legendrian* if it is everywhere tangent to $\mathcal{E}$, i.e., $TL \subset \mathcal{E}$.

It is possible to construct examples coming from Legendrian knots in contact 3-manifolds as follows.

Example 7.2. Let K be a Legendrian knot in a contact 3-manifold (N, ξ). Then

$$L = S^1 \times K \subset (S^1 \times N, \mathcal{E} := \partial_\theta \oplus \xi)$$

is a Legendrian torus, where θ denotes an angular coordinate on S^1.

Observe that in these examples the characteristic line field $\mathcal{W}$ is tangent to L. This is always the case: if p is a point in L, where L is transverse to $\mathcal{W}$, then L is transverse in a whole open neighborhood of p. But this neighborhood projects along $\mathcal{W}$ to a surface F in a contact 3-manifold (N, ξ) such that F is tangent to ξ, implying that such a point p cannot exist. In particular, we do not have any Legendrian tori in the local normal form $(\mathbb{R}^4, \mathcal{E}_{\mathrm{st}})$ of an even contact structure. More generally, it is known that the $\mathcal{W}$-orbits of a generic Engel manifolds are isolated (Theorem 27 in [22]), cf. [21], and thus a generic Engel manifold contains no Legendrian 2-torus.

By a similar argument as for transverse surfaces, we argue that a general closed Legendrian surface has to be a 2-torus if it is orientable, and a Klein bottle if it is non-orientable. We now define a homological invariant for Legendrian tori from the linking class as follows.

Definition 7.3. Let L be a Legendrian torus with trivial normal bundle in $(M, \mathcal{E})$. We denote by L' the embedded 2-torus obtained by pushing L into a direction transverse

to $\mathcal{E}$, i.e., in the $TM/\mathcal{E}$-direction. The *Thurston–Bennequin class* tb(L) is defined as

$$\text{tb}(L) = \text{lk}(L', L) \in H_2(M \setminus L).$$

Proof of Theorem 1.2. Let K_0 be a nullhomologous Legendrian knot with Thurston–Bennequin invariant tb(K_0) in a contact 3-manifold (N, ξ), and let L_0 be the Legendrian torus $S^1 \times K_0$ in $(M = S^1 \times N, \mathcal{E} = \partial_\theta \oplus \xi)$. We write L_n for the Legendrian torus $S^1 \times K_n$, where K_n is some n-fold stabilization of K_0 with Thurston–Bennequin invariant tb(K_n) = tb(K_0) − n. We note that the L_n are all smoothly isotopic, since the K_n are smoothly isotopic.

By Lemma 4.2, we know that tb(L_n) is a multiple of μ_{K_0}. Next we compute this multiple. The push-off L_n' is given by $S^1 \times K_n'$, where K_n' is the push-off of K_n in the Reeb-direction of ξ. A computation in homology yields (see Figure 2)

$$\text{tb}(L_n) = [L_n'] = [S^1 \times K_n'] = \text{tb}(K_n) \cdot \mu_{K_0} = (\text{tb}(K_0) - n) \cdot \mu_{K_0} \in H_2(M \setminus L_0),$$

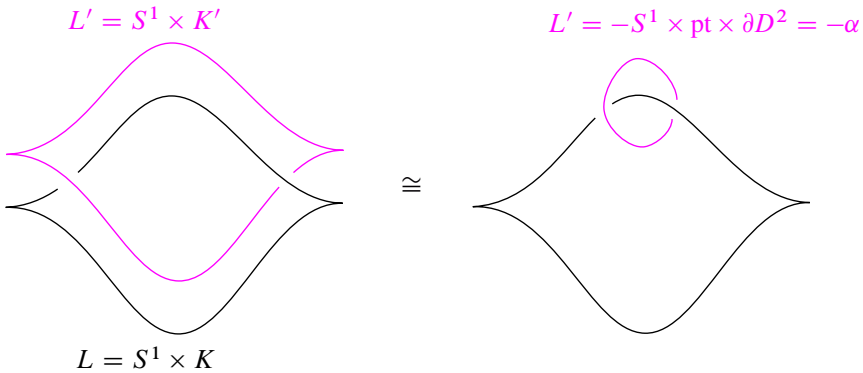and thus the Legendrian tori L_n are all pairwise non-isotopic. ∎

Figure 2. The push-off L' is isotopic to tb(K)α.

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