On the Uniqueness of Solution to the Cauchy Problem for Elliptic Equations in Two Variables

Dedicated to Professor Shigetake Matsuura on the sixtieth anniversary of his birthday

By

Shin-ichi Fujii*

Contents

§1.   Introduction

The famous example of Pliš ([11] Theorem 4) has shown us that the uniqueness in the Cauchy problem for four-fold characteristics does not hold in general even in smooth coefficients and two independent variables. Similar but more general examples of two independent variables have been constructed by Nakane [7]. But when we consider triple characteristics case the results are not ap-

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plicable and we have few of information. There exists only a counter ex-
ample with non-smooth coefficients. (See [3]).

In this paper we shall give a positive result for the uniqueness in case of
equations with smooth coefficients of two independent variables and with triple
characteristics, in spite of the existence of such a counterexample. This result
will show us the uniqueness almost always holds in such a case.

In two dimensional case, there is an effective reduction technique to a
simple case that the set, on which the multiplicity of characteristic roots vary
coincides with the initial surface. This will work well in our proof. (See [13],
[14], [15] for other related results.) Our main result is the following.

Theorem. *Let P be a linear elliptic operator of order m ($m \geq 1$) with C^{∞}
coefficients in an open neighbourhood ω of the origin in R^2. Under the follow-
ing assumptions* (a), (b) *one can find an open neighbourhood $\omega' \subset \omega$ of the origin
such that every $u \in C^{\infty}(\omega)$ satisfying $Pu=0$ and $u|_{t \leq 0}=0$ vanishes in ω'.*

Assumptions. *We denote by $P_m(t, x, \tau, \xi)$ the principal symbol.*
(a) $p_m(t, x, \tau, \xi) = \prod_{i=1}^{3} \prod_{j=0}^{k_i} p_{ij}(t, x, \tau, \xi)$, *where* $k_i \geq 0$, $p_{ij}(t, x, \tau, \xi) = (\tau - a_{ij}\xi)^i + \sum_{l=0}^{i-1} a_{ijl}(t, x) \xi^{i-l}(\tau - a_{ij}\xi)^l$ *for $j \neq 0$, and $p_{i0} = 1$,*
with distinct non real constants a_{ij} and $a_{ijl} \in C^{\infty}(\omega)$ satisfying $a_{ijl}(0, 0) = 0$.

(b) *Let $\Delta_{ij}(t, x)$, $i \neq 1, j \neq 0$ (resp. $\Delta_j'(t, x), j \neq 0$) is the discriminant of
$p_{ij}(t, x, \tau, 1)$ (resp. $\partial_{\tau} p_{3j}(t, x, \tau, 1)$) as a polynomial in τ. We assume that
$\Delta_{ij}(t, x)$ (resp. $\Delta_j'(t, x)$) has a finite order zero at $(0, 0)$ unless $\Delta_{ij} \equiv 0$ (resp. $\Delta_j' \equiv 0$)
in ω, where we say that $g \in C^{\infty}(\omega)$ has a finite order zero at $(0, 0)$ when $g(0, 0) = 0$
and $\partial_{t,x}^{\alpha} g(0, 0) \neq 0$ for some α.*

We shall mention the organization of this paper. In section two we carry
out a reduction as above and the proof of the main theorem is reduced to that of
Proposition 2.3. We devote ourselves to prove this from section three to section
ten. The essential part of proof is section three and section eight. The idea of
proof is a factorization of the symbol of the operator modulo terms of order
less than its order minus one, which is the same as in papers Matsumoto [6],
Ōkaji [9]. Another sections except for section nine is those for tedious tech-
nicalities. The argument in section nine is routine one.

Remark on notations. The set of all positive integers and all non-negative
integers are denoted by N and Z_+ respectively.

Acknowledgement

§2. Reduction of the Proof of Theorem

Reduction mentioned above works well also in our case by virtue of Watanabe's theorem in [12].

First of all, it is easy to see that by making a linear change of coordinates which preserves x-axis one can reduce (b) to a special case that $\Delta_{ij}(t, 0)$ (resp. $\Delta'_j(t, 0)$) has a finite order zero at $t=0$ (namely, $\Delta_{ij}(0, 0)=0$ and $(\partial_t^{k_{ij}} \Delta_{ij})(0, 0)$ $\neq 0$ for some k_{ij} for example) unless $\Delta_{ij} \equiv 0$ (resp. $\Delta'_j \equiv 0$) in ω, since (b) is invariant under such change of coordinates. Indeed, we note that for $a(t, x) \in C^\infty$ near the origin in $\mathbf{R}^2$ and $c \in \mathbf{R}$,

$$\partial_t^j(a(t, x+ct))|_{(0,0)} = \sum_{k=0}^j \binom{j}{k} (\partial_t^{j-k} \partial_x^k a)(0, 0)\, c^k .$$

By the change of coordinates $(t, x) \to (t', x')=(t, x-ct)$, the principal symbol of P is transformed to $\prod_{i=1}^3 \prod_{j=0}^{k_i} \tilde{p}_{ij}$ where $\tilde{p}_{ij}=p_{ij}(t', x'+ct', \tau'+c\xi', \xi')$ so that (a) clearly holds for $\tilde{p}_{ij}$ and the discriminant of $\tilde{p}_{ij}(t, x, \tau', 1)$ and $\partial_\tau \tilde{p}_{3j}(t, x, \tau', 1)$ are $\Delta_{ij}(t', x'+ct')$ and $\Delta'_j(t', x'+ct')$ respectively. Therefore the above special case of (b) holds for a suitable choise of c by virtue of (b). So we may assume this instead of (b). Set $\Lambda=\{(i, j); \Delta_{ij} \not\equiv 0\} \cup \{j; \Delta'_j \not\equiv 0\}$. $\Lambda=\phi$ means that the number of distinct characteristic roots is constant in ω and at most three. So our theorem immediately follows from Watanabe's theorem in [9] when $\Lambda=\phi$. Thus we may assume $\Lambda \neq \phi$. We will prove Theorem by contradiction when $\Lambda \neq \phi$. Suppose that the assertion of Theorem is not true. By Malgrange's preperation theorem one can find $T>0$, $X>0$ such that $\omega_1 \equiv (-T, T) \times (-X, X)$ $\subset \omega$, and that for $(t, x) \in \omega_1$,

$$\Delta_{ij}(t, x) = b_{ij}(t, x)\, r_{ij}(t, x), \quad (i, j) \in \Lambda,$$
$$\Delta'_j(t, x) = b'_j(t, x)\, r'_j(t, x), \quad j \in \Lambda,$$
$$\Delta(t, x) \equiv \prod_{(i, j) \in \Lambda} \Delta_{ij}(t, x) \prod_{j \in \Lambda} \Delta'_j(t, x) = b(t, x)\, r(t, x),$$
$$r_{ij}(t, x) = t^{k_{ij}} + \sum_{l=0}^{k_{ij}-1} d_{ijl}(x)\, t^l,$$
$$r'_j(t, x) = t^{k'_j} + \sum_{l=0}^{k'_j-1} \tilde{d}_{jl}(x)\, t^l,$$
$$r(t, x) = t^k + \sum_{l=0}^{k-1} d_l(x)\, t^l,$$

where $b_{ij}, b'_j, b \in C^\infty(\omega_1)$, $d_{ijl}, \tilde{d}_{jl}, d_l \in C^\infty((-X, X))$ satisfying that $b_{ij}(t, x) \neq 0$,

$b'_j(t, x) \neq 0$, $b(t, x) \neq 0$ for any $(t, x) \in \omega_1$, and that $d_{jji}(0) = 0$, $\tilde{d}_{ji}(0) = 0$, $d_i(0) = 0$, and k_{ij}, k'_j, k are positive integers. We may assume that $r(z, x) \neq 0$ when $z \in C$, $|z| \geq T/2$, since $d_i(0) = 0$. We set

$$J_i = \{x \in (-X, X); \#(\{z \in C; r(z, x) = 0\}) \geq i\}, i = 1, \cdots, k;$$
$$J'_k = J_k; J'_i = J_i \backslash \mathrm{cl}(J_{i+1}), i = 1, \cdots, k-1,$$

where $\mathrm{cl}(J_i)$ denotes the closure of J_i in $(-X, X)$. Then J'_i are open in $(-X, X)$, $\cup_i J'_i$ is dense in $(-X, X)$, and $\#(\{z \in C; r(t, x) = 0\}) = i$ for $x \in J'_i$. Next we set

$$J_{i,j} = \{x \in J'_i; \#(\{z \in C \backslash R; r(t, x) = 0\}) \geq j\}, j = 1, \cdots, i; J'_{i,i} = J_{i,i},$$

$J'_{i,j} = J_{i,j} \backslash \mathrm{cl}(J_{i,j+1}), j = 0, \cdots, i-1$, where $\mathrm{cl}(J_{i,j+1})$ is the closure of $J_{i,j+1}$ in $(-X, X)$. Then $J'_{i,j}$ are open in $(-X, X)$, $\cup_i \cup_j J'_{i,j}$ is dense in $-(X, X)$, and $\#(\{z \in C \backslash R; r(z, x) = 0\}) = j$ for $x \in J'_{i,j}$. By the assumption one can find $u \in C^\infty(\omega)$ satisfying $Pu = 0$ in ω, $u|_{t \leq 0} = 0$, and that supp $u \cap \omega_1 \neq \phi$. So there exists $J'_{i,j}$ and an open interval I such that $I \subset J'_{i,j}$ and $((-T, T) \times I) \cap$ supp $u \neq \phi$. Since both of the numbers of distinct roots and distinct non real roots of the equation in σ, $r(t, x) = 0$, are constant when $x \in I$, it follows that for $i > j$, there exists $\theta_l \in C^\infty(I)$, $l = 1, \cdots, i-j$ with $-T/2 < \theta_1(x) < \cdots < \theta_{i-j}(x) < T/2$, positive integers r_l, $l = 1, \cdots, i$, and $\tilde{b} \in C^\infty((-T, T) \times I)$ with $b(t, x) \neq 0$ for all (t, x) such that $\Delta(t, x) = \tilde{b}(t, x) \prod_{l=1}^{i_1} (t - \theta_l(x))^{r_l}$, and that if $i = j$, $\Delta(t, x) \neq 0$ for all $(t, x) \in (-T, T) \times I$. We set $i_1 = i-j$, $\theta_0 = -T$, $\theta_{i_1+1} = T$, and $A_l = \{(t, x) \in [-T, T] \times I; \theta_l(x) \leq t < \theta_{l+1}(x)\}$, $l = 0, \cdots, i_1$. We can take $i_0 = \inf \{i; A_i \cap$ supp $u \neq \phi\}$ and two cases arise, in one supp $u \cap \{t = \theta_{i_0}(x)\} = \phi$, in the other supp $u \cap \{t = \theta_{i_0}(x)\} \neq \phi$.

In the former case we take $(t_0, x_0) \in A_{i_0} \cap$ supp u and $\varepsilon > 0$ so that $[x_0 - \varepsilon, x_0 + \varepsilon] \subset I$. Consider paraboles $\mathscr{P}_c: \psi_c = t + \varepsilon^{-2}(x - x_0)^2 (c + T) - c = 0$. We can take ε so small that

$$\{(t, x) \in [-T, T] \times I; \psi_{t_0} = 0, t = b_{i_0+1}(x)\} = \phi.$$

We can take

$$c_0 = \inf \{c; \text{supp } u \cap \mathscr{P}_c \neq \phi\} \ (t_0 \geq c_0 \geq 0) \quad \text{and}$$
$$(t_1, x_1) \in \text{supp } u \cap \mathscr{P}_c \cap A_{i_0}.$$

Then $\Delta(t_1, x_1) \neq 0$ implies that the multiplicities of roots of the equation $p_m(t, x, \tau, 1) = 0$ are constant near the point (t_1, x_1). Since in this case by virtue of the fact that the dimension is two, the multiplicities of roots of the charac-

teristic equation of P in σ defined for η, $N \in \boldsymbol{R}^2 \backslash (0)$ with $\langle N, \eta \rangle = 0$, where $\langle \ , \ \rangle$ is the standard inner product in $\boldsymbol{R}^2$, $p_m(t, x, \sigma N + \eta) = 0$, are constant when N, η are fixed and (t, x) is close to (t_1, x_1) and since $u(t, x) = 0$ when $\psi_{c_0}(t, x) \le 0$, after the suitable linear change of coordinates at (t_1, x_1) by an orthogonal matrix Watanabe's theorem can be applied to conclude that $(t_1, x_1) \notin$ supp u which is contradiction.

In the latter case there exists $(t_1, x_1) \in$ supp $u \cap \{t = \theta_{i_0}(x)\}$. Since the number of distinct roots of the equation $r(t, x) = 0$ in t is constant in $x \in I$, any of $\varDelta_{ij}$ and $\varDelta_j'$ is of the form $(t - \theta_{i_0}(x))^r c(t, x)$ with a positive integer r and a smooth c satisfying $c(t_1, x_1) \ne 0$, unless it is identically zero or does not vanish at (t_1, x_1). We need a lemma.

Lemma 2.1. *Suppose that the discriminant of a polynomial in σ,*

$$q(\sigma, s, y) = \sigma^l + a_1(s, y) \sigma^{l-1} + \cdots + a_l(s, y),$$
$$a_i \in C^\infty(U), \ U = [-S, S] \times [-Y, Y]$$

is equal to $s^n D(s, y)$ with a positive integer n and $D \in C^\infty(U)$ satisfying $\inf_U |D(s, y)| > 0$. Then the roots of the equation in σ, $q(\sigma, s, y) = 0$, is expressed by l continuous functions $\mu_j(s, y), j = 1, \cdots, l$ for $(s, y) \in [0, S_1) \times (-Y, Y)$ with a $S_1 \in (0, \min(S, 1))$, satisfying there exists a positive integer h such that $\mu_j(s^h, y)$ belongs to $C^\infty([0, S_1) \times (-Y, Y))$ for all j.

Proof. Set

$$a_i'(z, y, w) = \sum_{j<N}(j!)^{-1} z^j (\partial_s^j a_i)(0, y) + ((N-1)!)^{-1} z^N w_i, \ i = 1, \cdots, l \ \text{for}$$

$z \in \boldsymbol{C}, w = (w_1, \cdots, w_l) \in \boldsymbol{C}^l$, and

$$\tilde{a}_i(s, y) = \int_0^1 (\partial_s^N a_i)(\theta s, y)(1-\theta)^{N-1} d\theta,$$
$$q'(\sigma, z, y, w) = \sigma^l + \sum_{i<l} a_i'(z, y, w) \sigma^i.$$

Since $a_i(s, y) - a_i'(s, y, w) = s^N(\tilde{a}_i(s, y) - w_i)/(N-1)!$, for any $N > n$ there exists $S' \in (0, \min(1, S))$ such that when $|z| \le S'$, $|w| \le M \equiv \max_i \sup_U(|\tilde{a}_i| + 1)$, the discriminant of polynomial in σ is equal to $z^n D'(z, y, w)$ where D' is a polynomial in (z, w) with coefficients in $C^\infty([-Y, Y])$ satisfying $\inf \{|D'(z, y, w)| ; |z| \le S', |y| \le Y, |w| \le M\} > 0$. We set

$$\varGamma = \varGamma_{a,b} = \{(z, y, w); |z| < S', |y| < Y, |w| < M, z \in \{re^{\sqrt{-1}\theta}; \\ 0 < r < S', a < \theta < b\}\} \quad \text{for} \quad a, b \in \boldsymbol{R} \text{ such that } b - a < \pi.$$

Since $\varGamma$ is simply connected, the implicit function theorem and the monodromy

theorem give $\mu_{j,\Gamma}(z, y, w) \in C^\infty(\Gamma), j = 1, \cdots, l$ which are holomorphic in (z, w) and satisfy $q'(\mu_{j,\Gamma}(z, y, w), z, y, w) = 0$ and $\mu_{i,\Gamma}(z, y, w) \neq \mu_{j,\Gamma}(z, y, w)$ if $i \neq j$, for all $(z, y, w) \in \Gamma$. Set $\Phi_\Gamma = \{\mu_{j,\Gamma}; j = 1, \cdots, l\}$, $R = \{(z, y, w);$ $0 < |z| < S', |y| < Y, |w| < M\}$, $\mathcal{U} = \{\Gamma_{a,b}; a, b\}$. Let R^* be the set whose points are determined by a pair $(p, \varphi), p \in R, \varphi \in \Phi_\Gamma$ such that $p \in \Gamma, \Gamma \in \mathcal{U}$ and two pairs $(p_1, \varphi_1), (p_2, \varphi_2)$ are identified if and only if $p_1 = p_2$ and $\varphi_1 = \varphi_2$ in a neighbourhood of p_1. We define the topology of R^* so that the fundamental neighbourhood system of $(p_0, \varphi_0) \in R^*$ with $p_0 \in \Gamma$ and $\varphi_0 \in \Phi_\Gamma$ is a family of sets $\{(p, \varphi); p \in V\}$ such that there exists $\Gamma_1 \in \mathcal{U}$ such that $V \subset \Gamma_1, \varphi \in \Phi_{\Gamma_1}$, and such that $\varphi = \varphi_0$ in a neighbourhood of p_0 and V is a open set with $p_0 \in V$. This definition is well defined. We define a mapping $\pi: R^* \to R$ by $\pi(p^*) = p$ if $p^* = (p, \varphi)$. Then a pair (R^*, π) is a covering space of R. (For the definition of the covering space we refer L. S. Pontryagin: Topological Groups. See also Ahlfors Sario Riemann Surfaces.) Since R is arcwise connected, each component of R^* with the restriction of π to it is a arcwise connected covering space of R. It is clear that the set $V^*_{\varphi,\Gamma} = \{(p, \varphi); p \in \Gamma\}$ is connected for any $\Gamma \in \mathcal{U}$ and $\varphi \in \Phi_\Gamma$. Let $h_i \in \mathbb{Z}$ be the number of sheets of a covering space $(R^*_i, \pi|_{R^*_i})$ of R where R^*_i is the component of R^* containing $V^*_{\varphi,\Gamma}$ with $\Gamma = \Gamma_{-\pi/4,\pi/4}, \varphi = \mu_{i,\Gamma}$. We can construct in the same way as above a covering space (R^*_{i1}, π_i) of R for any $i = 1, \cdots, l$ by the family of functions $\mu_{i,j,\Gamma}(z, y, w)$ $= \mu_{j,\Gamma_i}(z^{h_i}, y, w), j = 1, \cdots, l$ where $\Gamma = \Gamma_{a,b} \in \mathcal{U}$ with $b - a < \pi/h_i$ and $\Gamma_i = \{(z^{h_i}, y, w); (z, y, w) \in \Gamma\}$. Then since any closed curve in R is homotopic to one of curves $C_m: t \to (S'/2\, e^{2\pi\sqrt{-1}\,mt}, 0, 0), t \in [0, 1]; m \in \mathbb{Z}$, for any open interval is simply connected, the number of sheets of a covering space $(R^*_{i2}, \pi_i|_{R^*_{i2}})$ of R where R^*_{i2} is the component of R^*_{i1} containing a connected subset of $\{(p, \varphi);$ $\varphi \in \Gamma_0\}$ with $\Gamma_0 = \Gamma_{-\pi/(4h_i),\pi/(4h_i)}$ and $\varphi = \mu_{i,i,\Gamma}$, is one. So, one can find $F_i \in C^\infty(R)$ which is holomorphic in (z, w) satisfying $F_i = \mu_{i,i,\Gamma}$ on Γ_0 and can be extended to a function $\tilde{F}$ in $C^\infty(\{(z, y, w); |z| < S', |y| < Y, |w| < M\})$ by Cauchy's integral formula. Therefore, $\mu_{i,\Gamma}(s^{h_i}, y, w)$ is C^∞ in $[0, S') \times (-Y, Y)$ $\times \{|w| < M\}$ for $\Gamma = \Gamma_{-\pi/4,\pi/4}$. We will take $h = \prod_{i=1}^l h_i, S_1 = S'$, and with $\Gamma = \Gamma_{-\pi/4,\pi/4}, \mu_i(s, y) = \mu_{i,\Gamma}(s, y, (\tilde{a}_i(s, y)))$ when $(s, y) \in (0, S_1) \times (-Y, Y)$, $= \tilde{F}(0, y, (\tilde{a}_i(0, y)))$ when $s = 0$ and $y \in (-Y, Y)$. Then, these have required properties. The proof is complete.

We need a definition.

Definition 2.2. Let $U_{a,b} = [0, a]_t \times [-b, b]_x$ for $a > 0, b > 0$, and h be a positive integer. We set

$$C_h^\infty(U_{a,b}) = \{g; g \in C(U_{a,b}), g(t^h, x) \in C^\infty(U_{a^{1/h},b})\} \ .$$

Let $\varphi \in C^\infty([-b, b])$ be a non negative function with $\varphi^{1/h} \in C^\infty([-b, b])$. We set

$$F_{h,\varphi}(U_{a,b}) \text{ (or simply } F_{h,\varphi} \text{ if not confused)}$$
$$= \{g; g \in C_h^\infty(U_{a,b}) \text{ with } g \equiv 0, \text{ or } g = (t\varphi)^e \tilde{g}, eh \in \mathbf{Z}, e > 0,$$
$$\tilde{g} \in C^\infty(U_{a,b}) \text{ with } \inf_{U_{a,b}} |\tilde{g}| > 0\} \ ,$$
$$F_{h,\varphi}^*(U_{a,b}) = \{g \in F_{h,\varphi}, g \not\equiv 0\} \ .$$

We say for a function g defined in a neighbourhood of $(0, 0)$ in $[0, \infty)_t \times \mathbf{R}_x$ that g is C_h^∞ (resp. $F_{h,\varphi}, F_{h,\varphi}^*$) near $(0, 0)$ if g is defined in $U_{a,b}$ for some a, b and $g \in C_h^\infty(U_{a,b})$ (resp. $F_{h,\varphi}(U_{a,b}), F_{h,\varphi}^*(U_{a,b})$).

We set $q_{ij}(t, x, \tau) = p_{ij}(t + \theta_{i_0}(x + x_1), x + x_1, \tau, 1)$, $\tilde{\Delta}_{ij}(t, x) = \Delta_{ij}(t + \theta_{i_0}(x + x_1), x + x_1)$, and $\tilde{\Delta}_j'(t, x) = \Delta_j'(t + \theta_{i_0}(x + x_1), x + x_1)$, for $x \in (-\varepsilon_0, \varepsilon_0)$ with a small ε_0. We will factorize q_{ij} when (t, x) close to $(0, 0)$ so that for some $h = h_{ij}$,

$$q_{ij}(t, x, \tau) = \prod_{k=1}^{i} (\tau - \mu_{ijk}(t, x)), \ \mu_{ijk} \in C_h^\infty \quad \text{near the origin, and}$$
$$\mu_{ijk} - \mu_{ijk'} \in F_{h,1} \quad \text{if} \quad \mu_{ijk}(0, 0) - \mu_{ijk'}(0, 0) = 0 \ .$$

It is clear that q_{2j} will cause no problem because of the remark above Lemma 1. We want to consider q_{3j}. When $\tilde{\Delta}_{3j}(0, 0) \neq 0$, there is also no problem. When $\tilde{\Delta}_j'(0, 0) \neq 0$ and $\tilde{\Delta}_{3j}(0, 0) = 0$,

$$q_{3j}(t, x, \tau) = \prod_{k=1}^{2} q_j^{(k)}(t, x, \tau) \quad \text{when } (t, x) \text{ is close to } (0, 0)$$

so that $q_j^{(k)}$ is a polynomial in τ of degree k with C^∞ coefficients and the resultant of $q_j^{(1)}(0, 0, \tau)$ and $q_j^{(2)}(0, 0, \tau)$ is not zero. So if we let $\Delta_j''(t, x)$ be the discriminant of $q_j^{(2)}(t, x, \tau)$ as a polynomial in τ, $\Delta_j'' \in F_{1,1}$ near the origin which means an above factorization of q_{3j} is possible. Let us consider the case that $\tilde{\Delta}_{3j}(0, 0) = 0 = \tilde{\Delta}_j'(0, 0)$. We first note that it is always true in this case that when (t, x) is close to $(0, 0)$,

$$\partial_\tau q_{3j}(t, x, \tau) = 3 \prod_{k=1}^{2} (\tau - \mu_{jk}'(t, x)), \ \mu_{jk}' \in C_2^\infty \quad \text{and} \quad \mu_{j1}' - \mu_{j2}' \in F_{2,1}$$

near $(0, 0)$,

$$q_{3j}(t, x, \tau) = (\tau - \mu_{jk}'(t, x))^2(\tau - \mu_{jk}'(t, x) + (-1)^k c_j(t, x)) + d_{jk}(t, x)$$

where

$$c_j(t, x) = 3/2(\mu'_{j2}(t, x) - \mu'_{j1}(t, x)), \quad d_{jk}(t, x) = q_{3j}(t, x, \mu'_{jk}(t, x)).$$

We also note that the well known relation between the discriminant of q_{3j} and $\partial_\tau q_{3j}$ that

$$(2.1) \qquad 1/27\, \tilde{\Delta}_{3j} = \prod_{k=1}^{2} \prod_{l=1}^{3} (\mu'_{jk} - \mu_{3jl}) = \prod_{k=1}^{2} q_{3j}(t, x, \mu'_{jk}) = \prod_{k=1}^{2} d_{jk}.$$

For this relation we refer B.L. van der Waerden: Algebra I. Clearly $\mu'_{jk} \in C^\infty$ near $(0, 0)$ when $\tilde{\Delta}_{3j} \equiv 0$ and $\tilde{\Delta}'_j \equiv 0$, and then $q_{3j}(t, x, \tau) = (\tau - \mu'_{j1}(t, x))^3$. When $\tilde{\Delta}_{3j} \equiv 0$ and $\tilde{\Delta}'_j \not\equiv 0$, $\prod_{k=1}^{2} d_{jk}(t, x) = 0$ for all $(t, x) \in (0, \varepsilon_1) \times (-\varepsilon_1, \varepsilon_1)$ with a small ε_1. Since both of two sets $\{(t, x) \in (0, \varepsilon_1) \times (-\varepsilon_1, \varepsilon_1); d_{jk}(t, x) = 0\}$, $k = 1, 2$ are open in $(0, \varepsilon_1) \times (-\varepsilon_1, \varepsilon_1)$ if ε_1 is sufficiently small, for $\tilde{\Delta}'_j \in F^*_{1,1}$, one of them is empty. Thus by the above remark a desired factorization of q_{3j} is possible and by a change of numbering if neccessary, we have $d_{j1} = 0$ and $d_{j2}(t, x) = -4/27\, c_j(t, x)^3$ which means $d_{j2} \in F^*_{2,1}$. When $\tilde{\Delta}_{3j} \not\equiv 0$, by Lemma 1 one can find a positive integer h and $\mu_{3jk} \in C^\infty_h$ near $(0, 0)$ such that $q_{3j}(t, x, \tau) = \prod_{k=1}^{3} (\tau - \mu_{3jk}(t, x))$. Note that $\tilde{\Delta}_{3j}(t, x) = \prod_{k > k'} (\mu_{3jk}(t, x) - \mu_{3jk'}(t, x))^2 = 27 \prod_{k=1}^{2} d_{jk}(t, x)$, and that $\mu_{3jk}(t^{2h}, x)$ is C^∞ in a neighbourhood of the origin in $\mathbf{R}^2$. Since $\tilde{\Delta}_{3j} \in F^*_{1,1}$ near $(0, 0)$, $\tilde{\Delta}_{3j}(t, x) = t^s \tilde{\tilde{\Delta}}_{3j}(t, x)$ in a neighbourhood of $(0, 0)$ in $\mathbf{R}^2$ for some positive integer s and C^∞ function $\tilde{\tilde{\Delta}}_{3j}$ with $\tilde{\tilde{\Delta}}_{3j}(0, 0) \neq 0$. So for all $k \neq k'$, $\mu_{3jk}(t^{2h}, x) - \mu_{3jk'}(t^{2h}, x)$ has a finite order zero at $(0, 0)$, and by Margrange's preperation theorem, in a neighbourhood of $(0, 0)$ in $\mathbf{R}^2$

$$\mu_{3jk}(t^{2h}, x) - \mu_{3jk'}(t^{2h}, x) = \tilde{\mu}_{jkk'}(t, x) \left(t^{s_{jkk'}} + \sum_{l < s_{jkk'}} c_{jkk'}(x)\, t^l \right)$$

for some positive integer $s_{jkk'}$ and C^∞ functions $\tilde{\mu}_{jkk'}$, $c_{jkk'}$ with $\tilde{\mu}_{jkk'}(0, 0) \neq 0$. Compairing orders of zeros at $(0, 0)$ in the above equality, we see $hs = \sum_{k > k'} s_{jkk'}$, so we must have $c_{jkk'} \equiv 0$ in a neighbourhood of the origin, since $\tilde{\Delta}_{3j}(t, x) = O(t^s)$ for all x close to 0. This means $\mu_{3jk} - \mu_{3jk'} \in F^*_{2h,1}$ near $(0, 0)$ for all $k \neq k'$. Similar argument shows that $d_{jk} \in F^*_{2h,1}$ near $(0, 0)$ for all k. Therefore a desired factorization of q_{3j} is possible. When the multiplicity of $\mu_{ijk}(t, x)$ is constant in (t, x) near $(0, 0)$ for some (i, j, k), $\mu_{ijk}(t, x)$ is C^∞ in (t, x) from the above argument.

In the latter case we perform following changes of coordinates with some $\delta > 0$ succesively,

$$t = t' + \theta_{i_0}(x' + x_1), \quad x = x' + x_1, \quad |t'| \leq 2\delta, \quad |x'| \leq 2\delta \,;$$
$$t' = \tilde{t} f(\tilde{x}), \quad x' = \tilde{x}, \quad |\tilde{t}| \leq 2\delta, \quad |\tilde{x}| \leq 2\delta,$$

where $f \in C_0^\infty(\mathbf{R})$ defined by

$$(2.2) \qquad f(s) = \exp(-(\delta^2 - s^2)^{-1}) \text{ when } |s| < \delta, = 0 \text{ when } |s| \geq \delta.$$

From now on, δ denotes a fixed positive real number and $f(s)$ always denotes this function in this paper. We shall introduce some notation.

$$(2.3) \quad B_T = \{a(t, x) \in C^\infty([0, T] \times \mathbf{R}); \text{ Derivatives of } a \text{ of all orders are bounded}\},$$

$$(2.4) \quad B_{h,T} = \{a(t, x) \in C([0, T] \times \mathbf{R}); a(t^h, x) \in B_T^{1/h}\},$$

$$(2.5) \quad E_{h,T} = \{a(t, x) \in B_{h,T}; a \equiv 0, \text{ or } a(t, x) = (tf(x))^e \, \tilde{a}(t, x), eh \in N, \\ \tilde{a} \in B_{h,T} \text{ with inf } |\tilde{a}| > 0\}.$$

Then by denoting new variables $(\tilde{t}, \tilde{x})$ by (t, x) anew and by a simple extension argument by the substitution into the variable x of real valued $\varphi \in C^\infty(\mathbf{R})$ such that $|\varphi(s)| = 3\delta/2$ when $|s| \geq 7\delta/4$, $3\delta/2 \geq |\varphi(s)| \geq 5\delta/4$ when $7\delta/4 \geq |s| \geq 5\delta/4$, $\varphi(s) = s$ when $|s| < 5\delta/4$, in the latter case our proof of Theorem is reduced to the following Proposition.

Proposition 2.3. *Let $\zeta \in B_{T_0}$ be real valued, h an integer > 1, and $p_m(t, x, \tau, \xi)$ a homogeneous polynomial in (τ, ξ) of degree $m \geq 1$ with B_{T_0} coefficients such that*

$$p_m(t, x, \tau, \xi) = \prod_{i=1}^{3} \prod_{j=0}^{k_i} p_{ij}(t, x, \tau, \xi), (t, x, \tau, \xi) \in [0, T_0] \times \mathbf{R} \times \mathbf{R}^2, k_i \geq 0$$

(2.6) *where $p_{i0} = 1$, and for $j \neq 1$, p_{ij} is a homogeneous polynomial in (τ, ξ) of degree i with B_{T_0} coefficients,*

for wihch we suppose that

$$p_{ij}(t, x, \tau, \xi) = \prod_{k=1}^{i} (\tau - \mu_{ijk}(t, x) \, \xi), (t, x, \tau, \xi) \in [0, T_0] \times \mathbf{R} \times \mathbf{R}^2 \quad with$$

$\mu_{1j1} \in B_{h,T_0}$, $\mu_{ijk} \in B_{h,T_0}$ *for* $i \neq 1$, *satisfying that*

$(2.7) \quad \inf_{t,x} |\text{Im } \mu_{ijk}(t, x)| > 0 \text{ for all } (i, j, k),$

$\inf_{t,x} |\mu_{ijk}(t, x) - \mu_{i'j'k'}(t, x)| > 0 \text{ if } i \neq i',$

and that

$$(2.8) \qquad \mu_{ijk} - \mu_{ijk'} \in E_{h,T_0} \text{ for all } (i, j, k), (i, j, k').$$

We also suppose that for $j \neq 1$,

$$(2.9) \quad \partial_\tau p_{3j}(t, x, \tau, \xi) = 3 \prod_{k=1}^{2} (\tau - \mu'_{jk}(t, x) \, \xi), (t, x, \tau, \xi) \in [0, T_0] \times \mathbf{R} \times \mathbf{R}^2$$

with

$\mu'_{jk} \in B_{h,T_0}$, $k = 1, 2$ *satisfying that*

$\inf_\omega |\operatorname{Im} \mu'_{jk}(t, x)| > 0$ *for all* k, $\mu'_{j1} - \mu'_{j2} \in E_{h,T_0}$, *and that*

$p_{3j}(t, x, \mu'_{jk}(t, x), 1) \in E_{h,T_0}$ *for all* k.

Then if Q is a differential operator with the symbol $Q(t, x, \tau, \xi) = \tilde{Q}(t, x, \tau, f(x) \xi)$,

$$\tilde{Q}(t, x, \tau, \xi) = p_m(t, x, \tau, \zeta(t, x) \tau + \xi) + f(x)^{1/2} p_{m-1}(t, x, \tau, \xi) + r(t, x, \tau, \xi),$$

where p_{m-1} and r are respectively a homogeneous polynomial in (τ, ξ) of degree $m-1$ with B_{T_0} coefficients and a polynomial in (τ, ξ) of degree $< m-1$ with B_{T_0} coefficients, where we suppose $r=0$ if $m=1$, there exists $T_1 \in (0, T_0)$ such that every $u \in C^\infty([0, T_0] \times R)$ which is flat on $t=0, =0$ when $|x| \geq \delta$, and satisfies $Qu=0$ vanishes when $t \leq T_1$.

Proposition 2.3 follows from a Carleman estimate: there exists $C > 0$ such that for large $n > 0$ and small $T > 0$,

(2.10) $\qquad\qquad C \|Qu\|^{(n)} \geq \|u\|^{(n)}$, $u \in \mathcal{S}_T$,   where

(2.11) $\quad \mathcal{S}_T = \{u \in C^\infty((0, T) \times R); \sup(1 + |x|)^k t^{-k}(T-t)^{-k}|\partial_t^i \partial_x^j u| < \infty$

$\qquad$ for all non negative integers $i, j, k\}$,

(2.12) $\qquad\qquad \|u\|^{(n)} = \|t^{-n} u\|_{L^2((0,T) \times R)}$, $n \geq 1$, $u \in \mathcal{S}_T$.

To deduce (2.10) we shall prove that when under the assumption and notation of Proposition 2.3 with $m \leq 3$, $k_m = 1$, $r = 0$, there exists $C > 0$ such that for large n and small T,

(2.13.m) $\qquad C \|Qu\|^{(n)} \geq \|u\|_m^{(n)}$ for $u \in \mathcal{S}_T$,

(2.14.m) $\quad C (\|Qu\|^{(n-1/(20h))} + n^{-1/2} \|Qu\|^{(n)}) \geq \sum_{|\nu| + |\mu| = 1} \|Q_{\nu\mu} u\|^{(n)}$, $u \in \mathcal{S}_T$

where

(2.15) $\qquad \|u\|_m^{(n)} = \sum_{j + 2k \leq 2m} n^{m/2 - j/2 - k} \|A_{j/2} D_t^k u\|^{(n + m - j/2 - k)}$,

(2.16) $\qquad\qquad A_s = (\langle f\xi \rangle^s)(x, D)$,

that is,

$$A_s u(t, x) = 1/2\pi \iint e^{ix\xi - iy\xi} \langle f(x)\xi \rangle^s u(t, y) \, dy d\xi,$$

(2.17) $\quad Q_{\mu\nu} = [(t \langle t^{1 + 1/(10h)} f\xi \rangle^{-1/2})^{-\mu_1 + \nu_1} \langle f\xi \rangle^{(\mu_2 - \nu_2)/2} f^{\nu_2} \partial_{t,x}^\nu$

$\qquad \{p_m(t, x, 1, \zeta(t, x))^{-1}(\partial_{\tau,\xi}^\mu \tilde{Q})(t, x, \tau, f(x)\xi)\}](t, x, D_t, D_x)$.

Our main task is to prove (2.13.3) which we will devote, and when $m \neq 3$, (2.13.m) seems already known. Once (2.13) and (2.14) is established, one can easily pass to (2.10) using standard partial fraction decomposition.

§3. Factorization of Some Third Degree Polynomial

Definition 3.0. Let $n \in N$, $T > 0$. We define

$$R_T^n := \{x = (x_1, \cdots, x_n) \in R^n; x_1 \in (0, T)\} .$$

Let $p_3(t, x, \tau, \xi)$ and $p_2(t, x, \tau, \xi)$, $(t, x, \tau, \xi) \in R_T^4$, be elements in $C_T^\infty(R^4)$ being homogeneous polynomials in (τ, ξ) respectively of degrees 3 and 2 with coefficients in B_T in (t, x). Let $h \in N$. We suppose that

(3.1) $\partial_\tau p_3(t, x, \tau, \xi) = 3 \prod_{l=1}^{2} (\tau - \mu_l(t, x) \xi)$, $\mu_l \in B_{h,T}$ with inf $|\operatorname{Im} \mu_l| > 0$,

$\qquad c := 3/2(\mu_2 - \mu_1) \in E_{h,T}$, $d_l := p_3(t, x, \mu_l(t, x), 1) \in E_{h,T}$.

Then we have

(3.2) $p_3(t, x, \tau, \xi) = (\tau - \mu_l(t, x) \xi)^2 (\tau - \mu_l(t, x) \xi + (-1)^l c(t, x) \xi) + d_l(t, x) \xi^3$.

Let $\zeta(t, x) \in B_T$ be real valued and put

(3.3) $\qquad L_l = \tau - \mu_l(t, x) (\zeta(t, x) \tau + \xi)$, $l = 1, 2$.

Then we can express uniquely

(3.4) $\qquad p_2 = \sum_{i=0}^{2} a_{il} L_l^i \xi^{2-i}$, $a_{il} \in B_{h,T}$,

(3.5) $\qquad \zeta\tau + \xi = a_l L_l + b_l \xi$, $a_l, b_l \in B_{h,T}$ with $\inf_{t,x} |b_l| > 0$,

since ζ is real and $\inf_{t,x} |\operatorname{Im} \mu_l| > 0$. We set

(3.6) $\qquad q = p_3(t, x, \tau, \zeta(t, x) \tau + \xi) + f(x)^{1/2} p_2(t, x, \tau, \xi)$,

(3.7) $\qquad g_l = d_l b_l^3 \xi^3 + a_{0l} f^{1/2} \xi^2$,

(3.8) $\qquad G_l = \langle g_l \rangle$, $H_l = \langle (cb_l \xi)^3 \rangle$,

where f is the function defined by (2.2) and $\langle z \rangle = (1 + |z|^2)^{1/2}$. Then a direct computation shows

(3.9) $\qquad q = L_l^2(L_l + (-1)^l cb_l \xi) + g_l + A_{0l} L_l^3 + d_l \sum_{i=1}^{3} A_{il} L_l^i \xi^{3-i}$

$\qquad\qquad + f^{1/2} \sum_{i=1}^{2} a_{il} L_l^i \xi^{2-i}$,

$\qquad A_{il} = 3a_l^i b_l^{3-i}$ for $i \neq 0$, $A_{0l} = (-1)^l ca_l + d_l a_l^3$.

Definition 3.1. Let $T \in (0, 1)$ and $h \in N$. We say $a(t, x, \xi) \in C^\infty(\boldsymbol{R}_T^3)$ belongs to $\Sigma_\rho^d = \Sigma_\rho^d(T, h)$ for $\rho \in [0, 1/2]$ and $d = (d_1, d_2, d_3)$ with $d_1 \leq 0$, if a satisfies

$$|\partial^\alpha a(t, x, \xi)| \leq C_\alpha \, \psi_1^{d_1 - \alpha_1} \cdot \psi_2^{d_2 - \alpha_2} \cdot \psi_{3, \rho}^{d_3 - \alpha_3} \quad \text{for all } \alpha$$

where $\partial^\alpha = \partial_t^{\alpha_1} \partial_x^{\alpha_2} \partial_\xi^{\alpha_3}$,

(3.10)
$$\psi_1 = t \langle t^{e_h} f^{1/(10h)} \xi \rangle^{-1/2}, \psi_2 = \langle t^{e_h} \xi \rangle^{-1/2},$$
$$\psi_{3, \rho} = \langle t^{e_h} \xi \rangle^{-\rho} \langle \xi \rangle, e_h = 1 + 1/(10h).$$

Note that

(3.11)
$$ab \in \Sigma_\rho^{d+e} \quad \text{if} \quad a \in \Sigma_\rho^d, b \in \Sigma_\rho^e.$$

Let $\Delta(t, x)$ and $\Delta_1(t, x)$ be respectively the discriminants of $p_3(t, x, \tau, 1)$ and $\partial_\tau p_3(t, x, \tau, 1)$ as polynomials in τ. Then similarly as (2.1),

(3.12)
$$\Delta = 27 \prod_{l=1}^{2} d_l, \quad \Delta_1 = 4/9 c^2.$$

Since $c, d_l \in E_{h, T}$, the possible cases are the followings.

(3.13)   (1) $\Delta \not\equiv 0$.   (2) $\Delta \equiv 0, \Delta_1 \not\equiv 0$.   (3) $\Delta \equiv 0, \Delta_1 \equiv 0$.

So from the definition of $E_{h, T}$,

(3.14)   the only one of d_1 and d_2 is identically zero in case (2), and we have

(3.15) $d_l \equiv 0$ for any l and $\mu_1 \equiv \mu_2$ in case (3)

so that

(3.16)   we delete sub-1 from all above notations in case (3), since in fact they do not depend on l in this case, namely we denote μ_l by μ and L_l by L for example.

Since $c, d_l \in E_{h, T}$ it follows from (3.9)

(3.17) $2 \geq |1 + A_{0l}| \geq 1/2$ when $t \leq T'$ for some $T' \in (0, T)$.

The aim of this section is to prove the following lemma.

Lemma 3.2. *Let T' be as in (3.17). For each cases in (3.13) the following facts holds for some $T_0 \in (0, T')$.*
Case (1). There exists a finite partitions of unity of $\boldsymbol{R}_{T_0}^3$, $\sum_{i \in I} \varphi_i + \sum_{l=1}^{2} \varphi_{0l} = 1$ such that

(3.18)
$$\varphi_i, \varphi_{0l} \in \Sigma_{1/2}^0(T_0, h);$$

(3.19) $$|g_l| \leq Ct^{-\kappa}\langle\xi\rangle^{3/2} \quad \text{on supp } \varphi_{0l}$$

where

$$\kappa = 3/2 - 1/(3h) ;$$

(3.20) $C\langle\xi\rangle \geq t^{-2\kappa/3}$ *on supp φ_i and on supp $d\varphi_{0l}$; for every $i \in I$ there exists $l = l(i) \in \{1, 2\}$ such that*

(3.21) $$q = (1+A_{0l}) \prod_{j=1}^{3} (L_l - \sigma_{ij}) \quad \text{if} \quad (t, x, \xi) \in \text{supp } \varphi_i ,$$

$$C|g_l| \geq t^{-\kappa}\langle\xi\rangle^{3/2} \quad \text{on supp } \varphi_i \text{ with}$$

(3.22) $\sigma_{ij} \in t^{1/(6h)} \Sigma_{1/2}^{(0,-1,1)}$,

$\partial_t \sigma_{ij} \in t^{1/(6h)-1} \Sigma_{1/2}^{(0,-1,1)}, \partial_x \sigma_{ij} \in t^{1/(6h)} \Sigma_{1/2}^{(0,-1,1)}, \partial_\xi \sigma_{ij} \in t^{1/(6h)} \Sigma_{1/2}^{0}$

where by definition for a topological space X

$$aE = \{ae; e \in E\} \quad \text{for} \quad a \in C(X) \quad \text{and} \quad E \subset C(X) .$$

Case (2). *There exists a finite partitions of unity of $\mathbf{R}_{T_0}^3$, $\sum_{i \in I} \varphi_i + \sum_{l=1}^{2} \varphi_{0l} = 1$ such that (3.18)~(3.20) hold, and such that $I = I_1 \cup I_2$ with $I_1 \cap I_2 = \phi$ so that for every $i \in I_1$ (3.21) holds for some $l = l(i)$ and*

(3.23) $$\sigma_{ij} \in \Sigma_{1/2}^{2/3(0,-1,1)} \quad \text{satisfying (3.22)}$$

and for every $i \in I_2$ there exists $l = l(i) \in \{1, 2\}$ and σ_{ij} satisfying (3.23) such that

(3.24) $$q = (1+A_{0l})(L_l - c_l \xi - \sigma_{i1}) \prod_{j=2}^{3} (L_l - \sigma_{ij}) \quad \text{if} \quad (t, x, \xi) \in \text{supp } \varphi_i$$

where

(3.25) $$c_l = (-1)^{l+1}(1+A_{0l})^{-1} b_l c .$$

Moreover the following facts hold.

(3.26) $$|g_{l(i)}| \leq C\langle\xi\rangle^2 \quad \text{on supp } \varphi_i, i \in I ,$$

$$|c| \leq C\langle\xi\rangle^{-1/3} \quad \text{on supp } \varphi_i, i \in I_1 ,$$

$$c\xi \cdot \sigma_{ij} \in \Sigma_{1/2}^{3/2(0,-1,1)} \quad \text{for} \quad j = 1, 2, 3, i \in I_2 .$$

Case (3). *There exists a finite partitions of unity of $\mathbf{R}_{T_0}^3$, $\sum_{i \in I} \varphi_i + \varphi_0 = 1$ such that (3.18)~(3.19) replaced φ_{0l} by φ_0 hold and (3.21) hold for some*

(3.27) $$\sigma_{ij} \in \Sigma_0^{2/3(0,0,1)} \quad \text{satifying}$$

(3.22)′ $\sigma_{ij} \in t^{1/(6h)} \Sigma_0^{(0,0,1)}$,

$$\partial_t \, \sigma_{ij} \in t^{1/(6h)-1} \, \Sigma_0^{(0,0,1)}, \ \partial_x \, \sigma_{ij} \in t^{1/(6h)} \, \Sigma_0^{(0,0,1)}, \ \partial_\xi \, \sigma_{ij} \in t^{1/(6h)} \, \Sigma_0^0 \, .$$

Furthermore there exists $\varphi_i^* \in \Sigma_0^0$ such that

(3.28) $\varphi_i^* = 1$ *on* supp φ_i ;

$C \, |\sigma_{ij} - \sigma_{ik}| \geq \max_s |\sigma_{is}| + t^{-\kappa/3} \langle \xi \rangle^{1/2}$ *on* supp φ_i^* *if* $j \neq k$.

    Proof.   We may assume $T' < 1$. For $S \in (0, T')$ and $\varepsilon \in (0, 1)$ we set

(3.29) $$\Gamma_l(S) = \{(t, x, \xi) \in R_S^3 ; \ t^\kappa \, G_l > \langle \xi \rangle^{3/2}\} \, ,$$
$$\Gamma^{l,i}(S, \varepsilon) = \Gamma_l(S) \cap \mathscr{D}_l^{(i)}(\varepsilon) \cap \mathscr{D}^l \, ,$$

for $l = 1, 2$, $i = 1, 2, 3$ where

(3.30) $\mathscr{D}_l^{(1)} = \{G_l < 10\varepsilon H_l\}$, $\mathscr{D}_l^{(2)} = \{H_l < 10\varepsilon G_l\}$,

$\mathscr{D}_l^{(3)} = \{\dfrac{\varepsilon}{10} \, H_l < G_l < \dfrac{10}{\varepsilon} \, H_l\}$, $\mathscr{D}^l = \{G_{l'} > 10^{-1} \, G_l\}$ with $l \neq l'$.

Then,

(3.31) $2^{-1} \leq |cb_l \, \xi| \, H_l^{-1/2} \leq 1$ on $\Gamma^{l,1}(S, \varepsilon) \cup \Gamma^{l,3}(S, \varepsilon)$ if $\varepsilon S^{-\kappa} > 20$,

$2^{-1} \leq |g_l| \, G_l^{-1} \leq 1$ on $\Gamma_l(S)$ if $S^{-\kappa} > 2$,

$\Gamma^{l,1}(S, \varepsilon) = \Gamma^{l,3}(S, \varepsilon) = \emptyset$ if $\varepsilon S^{-\kappa} > 10$ and $c \equiv 0$;

and there exist constant $C > 0$ and $S_0 \in (0, T')$ such that

(3.32) $\max(2, C^{-1} \, t^{-2\kappa/3}) < \langle \xi \rangle$ on $\Gamma_l(S)$ for $S \in (0, S_0)$,

$S^{\max(1/h, 2\kappa/3)} \, CG_l^{2/3} > |d_l| \langle \xi \rangle^2$ on $\Gamma_l(S)$ for $S \in (0, S_0)$.

Indeed, $H_l > \varepsilon S^{-\kappa}/10$ on $\Gamma^{l,i}(S, \varepsilon)$ for $i = 1, 3$ and $G_l > S^{-\kappa}$ on $\Gamma_l(S)$ which means (3.31), since $\langle z \rangle \geq 2$, $z \in C$ implies $|z| \geq 1$. To see (3.32) we set for $S \in (0, T]$,

$$|a|_{0,S} = \max_{[0,S] \times R} |a(t, x)| \, , \ a \in B_{h,T} \, .$$

It suffices to prove separately for each of (3.32) the existence of C and S_0. Since $d_l \in E_{h,T}$, $d_l = t^{1/h} \, \tilde{d}_l$ for some $\tilde{d}_l \in B_{h,T}$ so that $|g_l| \leq S^{1/h} |\tilde{d}_l|_{0,S} |b|_{0,S}^3 \langle \xi \rangle^3 + |a_{0l}|_{0,S} \langle \xi \rangle^2$ by using $0 \leq f \leq 1$. Thus there exists $C_1 > 0$ such that $C_1 \, t^\kappa \langle \xi \rangle^3 > \langle \xi \rangle^{3/2}$ on $\Gamma_l(S)$ for any S which shows the first one of (3.32) and there exists C_2 such that $|g_l| \leq C_2 \, S^{\min(1/h, 2\kappa/3)} \langle \xi \rangle^3$ on $\Gamma_l(S)$ for any S. On the other hand $|g_l| \geq |b_l|^3 |d_l| |\xi|^3 - |a_{0l}| |\xi|^2$ so that

$$(\inf_{[0,T] \times R} |b_l|)^3 \, |d_l| \, |\xi|^3 - |a_{0l}|_{0,S} \langle \xi \rangle \, t^{2\kappa/3} \, G_l^{2/3}$$
$$\leq C_2^{1/3} \, S^{\min(1/h, 2\kappa/3)} \, G_l^{2/3} \langle \xi \rangle \quad \text{on } \Gamma_l(S)$$

in view of the definition of $\Gamma_l(S)$. This shows the second one of (3.32) in view of $\inf_{[0,T]\times R} |b_l| > 0$ and the first one of that.

Putting $L_l = H_l^{1/3}\sigma$ or $L_l = G_l^{1/3}\sigma$ in (3.9) we get

$$(3.33) \qquad \tilde{q}(H_l^{1/3}\,\sigma,\,\cdot) = (1+A_{0l})\,H_l\,r_1(\sigma,\,z^{l1},\,w^{l1})\,,$$
$$\tilde{q}(G_l^{1/3}\,\sigma,\,\cdot) = (1+A_{0l})\,G_l\,r_2(\sigma,\,z^{l2},\,w^{l2})\,,$$
$$\tilde{q}(H_l^{1/3}\,\sigma,\,\cdot) = H_l\,r_3(\sigma,\,z^{l3},\,w^{l3})\,,$$

where $\tilde{q}(\sigma, t, x, \xi)$ is the polynomial in defined by putting $L_1 = \sigma$ in (3.9),

$$(3.34) \quad r_1(\sigma, z, w) = \sigma^3 + (w_1 + w_2\,z)\,\sigma^2 + w_3\,z^2\,\sigma + w_4\,z^3,\quad (z, w) \in \boldsymbol{C} \times \boldsymbol{C}^4\,,$$
$$r_2(\sigma, z, w) = \sigma^3 + w_1\,\sigma^2 + w_2\,\sigma + z,\quad (z, w) \in \boldsymbol{C} \times \boldsymbol{C}^2\,,$$
$$r_3(\sigma, z, w) = (1+w_1)\,\sigma^3 + (w_2 - z_1)\,\sigma^2 + w_3\,\sigma + z_2,\quad (z, w) \in \boldsymbol{C}^2 \times \boldsymbol{C}^3\;;$$

$$(3.35) \quad z^{l1} = G_l^{1/3}\,H_l^{1/3},\; w^{l1} = (1+A_{0l})^{-1}(-\tilde{c}_l\,\xi H_l^{-1/3},\,G_l^{-1/3}\,v_2,\,G_l^{-2/3}\,v_l,\,g_l\,G_l^{-1})\,,$$
$$z^{l2} = (1+A_{0l})^{-1}\,g_l\,G_l^{-1},\; w^{l2} = (1+A_{0l})^{-1}(-\tilde{c}_l\,\xi G_l^{-1/3} + G_l^{-1/3}\,v_2,\,G_l^{-2/3}\,v_1)\,,$$
$$z^{l3} = (\tilde{c}_l\,\xi H_l^{-1/3},\,g_l\,H_l^{-1}),\; w^{l3} = (A_{0l},\,H_l^{-1/3}\,v_2,\,H_l^{-2/3}\,v_1)\,,$$
$$v_i = d_l\,A_{il}\,\xi^{3-i} + f^{1/2}\,a_{il}\,\xi^{2-i},\;\tilde{c}_l = (-1)^{l+1}\,cb_l\,.$$

Lemma 3.3. *Set for $\eta \in (0, 1)$,*

$$(3.36) \quad X_{1,\eta} = [0, \eta] \times \{w \in \boldsymbol{C}^4;\,\frac{1}{10} \leq |w_j| \leq 10 \quad \text{for} \quad j = 1, 4,\,\max_{j=2,3}|w_j| \leq 1\}\,,$$

$$X_{2,\eta} = \{z \in \boldsymbol{C};\,\frac{1}{10} \leq |z| \leq 10\} \times \{w \in \boldsymbol{C}^2;\,|w_1| \leq \eta,\,|w_2| \leq \eta\}\,,$$

$$X_{3,\eta} = \{(z, w) \in \boldsymbol{C}^2 \times \boldsymbol{C}^3;\,\frac{1}{10} \leq |z| \leq 10,\,\frac{\eta}{100} \leq |z_2| \leq \frac{100}{\eta},\,\max_j|w_j| \leq \eta,$$

$$|D_3(z, w)| \geq \frac{\eta^2}{500}\}\,,$$

where $D_3(z, w)$ denotes the discriminant of r_3 as a polynomial in σ. Then there exists $\eta_0 \in (0, 1)$ such that one can find for any $\eta \leq \eta_0$ finite open coverings U_{ij} of $X_{i,\eta}$ as a subset of $\boldsymbol{C} \times \boldsymbol{C}^4,\,\boldsymbol{C} \times \boldsymbol{C}^2,\,\boldsymbol{C}^2 \times \boldsymbol{C}^3$ respectively for i, holomorphic functions λ_{ijk} on $U_{ij},\,k = 1, 2, 3$ for $i \neq 1$, C^∞ functions $\lambda_{1jk}(t, w)$ on $U_{1j} \cap (0, +\infty) \times \boldsymbol{C}^4$ satisfying

$$(3.37) \quad U_{1j} = B((0, w^{(j)}), \nu),\, w_1^{(j)} \neq 0,\, \nu < 1 \text{ where } B(*, \nu) \text{ denotes the Euclidean}$$
open ball with the center $*$ and the radius ν,

$$r_1(\sigma, t, w) = \prod_{k=1}^3 (\sigma - \lambda_{1jk}(t, w)),\,(t, w) \in U_{1j} \cap (0, +\infty) \times \boldsymbol{C}^4\,,$$
$$|\partial_t^s\,\partial_w^\alpha(\lambda_{1j1}(t, w) + w_1)| \leq C_{s\alpha}\,t^{1-s}\,,$$
$$|\partial_t^s\,\partial_w^\alpha\,\lambda_{1jk}(t, w)| \leq C_{s\alpha}\,t^{3/2-s} \quad \text{for} \quad k \neq 1\,,$$

(3.38) $r_2(\sigma, \cdot) = \prod\limits_{k=1}^{3} (\sigma - \lambda_{2jk})$ on U_{2j} ,

$r_3(\sigma, \cdot) = (1 + w_1) \prod\limits_{k=1}^{3} (\sigma - \lambda_{3jk})$ on U_{3j} ,

$\lambda_{ijk} \neq \lambda_{ijk'}$ everywhere on U_{ij} if $i \neq 1$ and $k \neq k'$,

$|\partial_{z,w}^\alpha \lambda_{ijk}| \leq C_\alpha$ on U_{ij} if $i \neq 1$,

(3.39) $C|\lambda_{2jk} - \lambda_{2jk'}| \geq \max\limits_{s}|\lambda_{2js}| + 1$ on U_{2j} if $k \neq k'$,

(3.40) $\sum_j \varphi_{ij} = 1$ on a neighbourhood of $X_{i,\eta}$ and $\varphi_{ij}^* = 1$ on supp φ_{ij} ,

where the constants $C_{s\alpha}$, C can be chosen independently of η, but it is not important in our argument.

Proof of Lemma 3.3. We first consider the case $i=1$. From the implicit function theorem, for any $w^0 \in C^4$ with $w_j^0 \neq 0$ for $j=1, 4$ one can write near $(0, w^0)$ $r_1 = (\sigma + \lambda)(\sigma^2 + \lambda_1 \sigma + \lambda_2)$ where λ, λ_i are holomorphic and $\lambda(0, w) = -w_1$, because $r_1(\sigma, 0, w) = \sigma^3 + w_1 \sigma^2$. Since $w_4 z^3 = r_1(0, z, w) = \lambda \lambda_2$ and $w_3 z^2 = \partial_\sigma r_1(0, z, w) = \lambda_2$ $+ \lambda \lambda_1$, and since $w_4^0 \neq 0$ and $\lambda(0, w^0) \neq 0$, near $(0, w^0)$, $\lambda_1 = O(z^2)$ as $z \to 0$ and one can write $\lambda_1^2 - 4\lambda_2 = z^3 \lambda_3$ where λ_3 is holomorphic with $\lambda_3(0, w^0) \neq 0$. So the factorization as in (3.37) is possible near $(0, w^0)$, and for $i=1$ Lemma 3.3 follows now by a standard argument by using the compactness of $\{w \in C^4; \frac{1}{10} \leq |w_j|$ ≤ 10 for $j=1, 4, \max\limits_{j=2,3} |w_j| \leq 1\}$. When $i=2$, for any $z^0 \in C \backslash \{0\}$ one can factorize $r_2(\sigma, \cdot)$ near $(z^0, 0)$ so that (3.38), (3.39) hold, since $r_2(\sigma, z, 0) = \sigma^3 + z$, and Lemma 3.3 for $i=2$ follows from the compactness of $\{z \in C; \frac{1}{10} \leq |z| \leq 10\}$. The case that $i=3$ also follows similarly, since $D_3 \neq 0$ on $X_{3,\eta}$ and $X_{3,\eta}$ is compact. The proof is complete.

Lemma 3.4. *There exists ε_0, $\varepsilon_1 \in (0, 1)$ with $\varepsilon_1 < \varepsilon_0$ and $N_0 \in [1, \infty)$ such that ε_0 satisfies the condition for the η_0 in the statement of Lemma 3.3, that $\varepsilon_0^{1-N_0^\kappa} > 20$, and that $\varepsilon_0^{N_0} < S_0$ where S_0 is the constant in (3.32), and such that for the sets*

(3.41) $\tilde{\Gamma}^{l,i}(\varepsilon) = \{(z^{li}(t, x, \xi), w^{li}(t, x, \xi)); (t, x, \xi) \in \Gamma^{l,i}(\varepsilon_0^{N_0}, \varepsilon)\}$, $\varepsilon \in (0, \varepsilon_0)$,

we have for any $\varepsilon \in (0, \varepsilon_1)$ that

(3.42) $\tilde{\Gamma}^{l,i}(\varepsilon) \subset X_{i,\varepsilon_0}$ for $i = 1, 2$;

$\tilde{\Gamma}^{l,3}(\varepsilon) \subset X_{3,\varepsilon}$.

Proof of Lemma 3.4. From (3.32) and the definition of $\Gamma_l(S)$ there exists C_1^- such that

(3.43) $|v_i| G_l^{-(3-i)/3} \leq C_1 S^{(3-i)/6 \min(1/h, 2\kappa/3)}$ on $\Gamma_l(S)$ for $S \in (0, S_0)$, $i = 1, 2$

where S_0 is the constant in (3.32). Using the fact that c, $d_l \in E_{h,T}$ we see as in the proof of (3.32) that there exists C_2 such that

(3.44) $|A_{0l}|_{0,S} \leq C_2 S^{1/h}$, $S \in (0, T]$.

Using (3.17), (3.31), (3.42), (3.44) one can find $S_1 \in (0, S_0)$ and C_3 such that for any ε with $\varepsilon S^{-\kappa} > 20$,

(3.45) $0 < z^{l1} < C_3 \varepsilon^{1/3}$, $1/4 \leq |w_i^{l1}| \leq 2$ for $i = 1, 4$, $\max\limits_{i=2,3} |w_i^{l1}|$

$\leq C_3 S^{1/6 \min(1/h, 2\kappa/3)}$, on $\Gamma^{l,1}(S, \varepsilon)$;

$1/4 \leq |z^{l2}| \leq 2$, $|w_1^{l2}| \leq C_3(\varepsilon^{1/3} + S^{1/6 \min(1/h, 3\kappa/3)})$,

$|w_2^{l2}| \leq C_3 S^{1/3 \min(1/h, 2\kappa/3)}$, on $\Gamma^{l,2}(S, \varepsilon)$;

$1/2 \leq |z_1^{l3}| \leq 1$, $\varepsilon/20 < |z_2^{l3}| < 10/\varepsilon$, $|w_1^{l3}| \leq C_3 S^{1/h}$,

$|w_i^{l3}| \leq C_3(\varepsilon^{-1} S^{1/2 \min(1/h, 2\kappa/3)})^{(i-1)/3}$ for $i = 2, 3$, on $\Gamma^{l,3}(S, \varepsilon)$.

We take $N_0 = (6\deg D_3 + 20)/\min(1/h, 2\kappa/3)$ so that $\varepsilon^{-1} S^{\min(1/h, 2\kappa/3)/2} = \varepsilon^{3(\deg D_3 + 3)}$ if $S = \varepsilon^{N_0}$, where D_3 is as in Lemma 3.3 and $\deg D_3$ is the degree of a polynomial D_3. One can choose $\varepsilon_0 \in (0, 1)$ such that $\varepsilon_0^{N_0} < S_1 \cdot \varepsilon_0^{1-N_0 \kappa} > 20$, and satisfies the condition for η_0 in the statement of Lemma 3.3, since $N_0 > 1$ and $\kappa > 1$. Then it follows from (3.45) that for any $\varepsilon \in (0, \varepsilon_0)$,

(3.46) $0 < z^{l1} \leq C_3 \varepsilon^{1/3}$, $1/4 \leq |w_i^{l1}| \leq 2$

for $i = 1, 4$, $\max\limits_{i=2,3} |w_i^{l1}| \leq C_3 \varepsilon$, on $\Gamma^{l,1}(\varepsilon^{N_0}, \varepsilon)$;

$1/4 \leq |Z^{l2}| \leq 2$, $|w^{l2}| \leq 2C_3 \varepsilon^{1/3}$, on $\Gamma^{l,2}(\varepsilon^{N_0}, \varepsilon)$;

$1/2 \leq |z_1^{l3}| \leq 1$, $\varepsilon/20 \leq |z_2^{l3}| \leq 10/\varepsilon$, $|w^{l3}| \leq C_3 \varepsilon^{3(\deg D_3 + 3)}$, on $\Gamma^{l,3}(\varepsilon^{N_0}, \varepsilon)$;

where $|w| = \max\limits_{1 \leq j \leq n} |w_j|$ if $w = (w_j) \in \mathbf{C}^n$.

So the first one of (3.42) is now clear, and we can choose $\varepsilon^{(1)} \in (0, \varepsilon_0)$ such that this holds for any $\varepsilon \in (0, \varepsilon^{(1)})$. We have to estimate $|D_3(z, w)|$ from below on $\Gamma^{l,3}(\varepsilon^{N_0}, \varepsilon)$ to see the second one of (3.42).

By Taylor expansion at $w = 0$ of D_3 and (3.46) one can find C_4 such that for any $\varepsilon \in (0, \varepsilon_0)$

(3.47) $|D_3(z^{l3}, w^{l3})| \geq |D_3(z^{l3}, 0)| - C_4(1 + \varepsilon^{-1})^{\deg D_3 - 1} \varepsilon^{\deg D_3 + 3}$ on $\Gamma^{l,3}(\varepsilon^{N_0}, \varepsilon)$.

We see as in (2 1),

$$D_3(z, 0) = 27 r_3(0, z, 0) r_3(2z_1/3, z, 0) = (27z_2 - 4z_1^3) z_2$$

so that

$$(3.48) \qquad |D_3(z^{l3}, 0)| = 27|g_l| H_l^{-1} |g_l - \frac{4}{27} (\tilde{c}_l \xi)^3| H_l^{-1} .$$

To compute $g_l - \frac{4}{27} (\tilde{c}_l \xi)^3$ we first note that from the uniquness of the expression in (3.4) and (3.5) and $L_1 - L_2 = \frac{2c}{3} (\zeta \tau + \xi)$,

$$(3.49) \qquad a_{01} - a_{02}, \; b_1 - b_2 \in cB_{h,T} ,$$

for indeed, from (3.4) for $l=1$, (3.5) for $l=2$, and the above equality we have $p_2 = \sum_{j=0}^2 (a_{j1} + c\tilde{a}_j) L_2^j \xi^{2-j}$ for some $\tilde{a}_j \in B_{h,T}$ and the uniqueness implies $a_{j1} + c\tilde{a}_j = a_{j2}$, and the latter follows similarly. Next we verify that

$$d_2 = p_3(t, x, \mu_2, 1) = p_3(t, x, \frac{2c}{3} + \mu_1, 1) = d_1 - \frac{4}{27} c^3 .$$

From this equality and (3.49) we obtain by noting $\inf_{t,x} |b_l| > 0$

$$g_l - \frac{4}{27} (\tilde{c}_l \xi)^3 = (d_l - \frac{4}{27} (-1)^{3(l+1)} c^3) b_1^3 \xi^3 + a_{01} f^{1/2} \xi^2$$

$$= g_{l'} + b_{ll'}(d_l \xi^2)(\tilde{c}_l \xi) + \breve{b}_{ll'}(\tilde{c}_l \xi) \xi$$

$$+ \tilde{\breve{b}}_{ll'} c(\tilde{c}_l \xi)^3$$

for some $b_{ll'}, \breve{b}_{ll'} \in B_{h,T}$ if $l \neq l'$. Therefore by using (3.31) and noting $|g_l| \geq 1/2G_{l'}$ on $\Gamma^{l,3}(\varepsilon^{N_0}, \varepsilon)$ for $\varepsilon \in (0, \varepsilon_0)$ which follows from $G_{l'} > 1/10G_l > \varepsilon^{-N\kappa_0}/10 > 2$ on $\Gamma^{l,3}(\varepsilon^{N_0}, \varepsilon)$ for $\varepsilon \in (0, \varepsilon_0)$,

$$|g_l - \frac{4}{27} (\tilde{c}_l \xi)^3| H_l^{-1} \geq \varepsilon/200 - |b_{ll'}|_{0,T} |d_l \xi^2| H_l^{-2/3}$$

$$- |\breve{b}_{ll'}|_{0,T} |\xi| H_l^{-2/3} \quad \text{on} \quad \Gamma^{l,3}(\varepsilon^{N_0}, \varepsilon), \; \varepsilon \in (0, \varepsilon_0) .$$

By (3.32) and the definition of N_0 there exists C_4 such that for any $\varepsilon \in (0, \varepsilon_0)$ the right hand side is estimated from below by $\varepsilon/200 - C_4 \varepsilon^{2(\deg D_3 + 3)}$ on $\Gamma^{l,3}(\varepsilon^{N_0}, \varepsilon)$ so that one can find $\varepsilon^{(2)} \in (0, \varepsilon_0)$ such that $|g_l - \frac{4}{27} (\tilde{c}_l \xi)^3| H_l^{-1} \geq \varepsilon/210$ on $\Gamma^{l,3}(\varepsilon^{N_0}, \varepsilon)$ for $\varepsilon \in (0, \varepsilon^{(2)})$, and consequently by this estimate, (3.48), (3.30), (3.31), and (3.47) we can conclude that

$$(3.50) \qquad |D_3(z^{l3}, w^{l3})| \geq \varepsilon^2/200 - C_4(1 + \varepsilon^{-1})^{\deg D_3 - 1} \varepsilon^{\deg D_3 + 3}$$
$$\text{on} \; \Gamma^{l,3}(\varepsilon^{N_0}, \varepsilon) \quad \text{for} \quad \varepsilon \in (0, \varepsilon^{(2)}) .$$

Hence one can find $\varepsilon^{(3)} \in (0, \varepsilon^{(2)})$ such that $\tilde{\Gamma}^{l,3}(\varepsilon) \subset X_{3,\varepsilon}$ for $\varepsilon \in (0, \varepsilon^{(3)})$ in view

of (3.46) and the definition of $X_{3,\varepsilon}$. Finally we take $\varepsilon_1 = \min(\varepsilon^{(1)}, \varepsilon^{(3)})$, and N_0, ε_0, and ε_1 in the proof have the desired properties for N_0, ε_0, and ε_1 in the statement. This completes the proof of Lemma 3.4.

Set with the notations in Lemma 3.4,

$$T_0 = (\varepsilon_1/2)^{N_0}, \ \Gamma_l = \Gamma_l(T_0), \ \Gamma^{l,i} = \Gamma^{l,i}(T_0, \varepsilon_1/2) .$$

Let $\chi(s) \in C^\infty(\boldsymbol{R})$ with $\chi = 1$ when $s \geq 1$, $\chi = 0$ when $s \leq 1/2$, $0 < \chi < 1$ when $1/2 < s < 1$, and define the functions on $\boldsymbol{R}_{T_0}^3$ by

(3.51) $\qquad \varphi^{li} = \varphi_l \cdot \chi_{li,\varepsilon_1/2}, \ \varphi_*^{li} = \varphi_l^* \cdot \chi_{li,\varepsilon_1} \quad \text{for} \quad i = 1, 2 ,$

$\qquad\qquad \varphi^{l3} = \varphi_l \cdot \chi_{l3,\varepsilon_1/2}, \ \varphi_*^{l3} = \varphi_l^* \cdot \chi_{l3,\varepsilon_1/4} ,$

$\qquad\qquad \tilde{\varphi}_l = (1-\chi)(160^{-1} t^\kappa G_l \langle\xi\rangle^{-3/2}), \ l = 1, 2$

where $\varphi_l = \Phi_{l,l,80}$, $\varphi_l^* = \Phi_{l,2l,40}$ with

(3.52) $\qquad \Phi_{l,e,R} = \chi(eG_l, G_{l'}^{-1}) \cdot \chi(R^{-1} t^\kappa G_l \langle\xi\rangle^{-3/2}), \ l \neq l', \ e \in \boldsymbol{R}, \quad \text{and}$

$\qquad\qquad \chi_{l1,e} = \chi(eH_l G_{l'}^{-1}), \ \chi_{l2,e} = \chi(eG_l H_l^{-1}) ,$

$\qquad\qquad \chi_{l3,e} = \chi(e^{-1} H_l^{-1} G_l) \cdot \chi(e^{-1} H_l G_l^{-1}), \ e \in \boldsymbol{R} .$

Then we have

(3.53) $\qquad \text{supp } \varphi_*^{li} \subset \Gamma^{l,i}, \ \varphi_*^{li} = 1 \text{ on supp } \varphi^{li}, \ \sum_l (\sum_i \varphi^{li} + \tilde\varphi) \geq 1 .$

We shall explain (3.53). The second one would be easily observed. φ_l, φ_l^*, and $\tilde\varphi_l$ are as in the following:

(3.54) $\quad \tilde\varphi_l = 1$ when $t^\kappa G_l \leq 80 \langle\xi\rangle^{3/2}$, and supp $\tilde\varphi_l \subset \{t^\kappa G_l \leq 160 \langle\xi\rangle^{3/2}\}$;

$\qquad \varphi_1 = 1$ when $t^\kappa G_1 \geq 80 \langle\xi\rangle^{3/2}$ and $G_2 \geq G_1$, and

$\qquad \text{supp } \varphi_1^* \subset \{t^\kappa G_1 \geq 20 \langle\xi\rangle^{3/2}\} \cap \mathscr{D}^1$;

$\qquad \varphi_2 = 1$ when $t^\kappa G_2 \geq 80 \langle\xi\rangle^{3/2}$ and $G_2 \leq 2G$, and supp φ_2^*

$\qquad \subset \{t^\kappa G_2 \geq 20 \langle\xi\rangle^{3/2}\} \cap \mathscr{D}^2$.

So, supp $\varphi_l^* \subset \Gamma_l \cap \mathscr{D}^l$ and $\sum_l (\varphi_l + \tilde\varphi_l) \geq 1$ from which the rest of (3.53) follows, because supp $\chi_{li,\varepsilon_1} \subset \mathscr{D}_l^{(i)}(\varepsilon_1/2)$ for $i = 1, 2$ and supp $\chi_{l3,\varepsilon_1/4} \subset \mathscr{D}_l^{(3)}(\varepsilon_1/2)$, and because $\sum_i \chi_{li,\varepsilon} \geq 1$. From the property of ε_0 in Lemma 3.4 we can choose U_{ij}, λ_{ijk}, φ_{ij}, φ_{ij}^* as in Lemma 3.3 for $\eta = \varepsilon_0$ when $i = 1, 2$ and for $\eta = \varepsilon_1/2$ when $i = 3$. We define

$$\Gamma^{l,i,j} = \{(t, x, \xi) \in \Gamma^{l,i}; (z^{li}, w^{li})(t, x, \xi) \in U_{ij}\} ,$$

$$\tilde\Gamma^{l,i,j} = \{(t, x, \xi) \in \Gamma^{l,i}; (z^{li}, w^{li})(t, x, \xi) \in (\text{supp } \varphi_{ij}^*)^c\} .$$

We regard $\Gamma^{l,i,j}$ and $\tilde{\Gamma}^{l,i,j}$ as subsets of $\boldsymbol{R}^3_{T_0}$.   Then,

(3.55) $\Gamma^{l,i} = \Gamma^{l,i,j} \cup \tilde{\Gamma}^{l,i,j}, \boldsymbol{R}^3_{T_0} = \Gamma^{l,i} \cup (\mathrm{supp}\ \varphi^{li}_*)^c$,

$\Gamma^{l,i,j}$ and $\tilde{\Gamma}^{l,i,j}$ are open in $\boldsymbol{R}^3_{T_0}$.

Now we define the functions on $\boldsymbol{R}^3_{T_0}$, $\varphi_{lij}, \varphi^*_{lij}, \sigma_{lijk}, k = 1, 2, 3$ by the following:

(3.56) $\varphi_{lij}, \varphi^*_{lij}, \sigma_{lijk} = 0$ on $(\Gamma^{l,i,j})^c$,

and on $\Gamma^{l,i,j}$,

(3.57) $\varphi_{lij} = \varphi^{li} \cdot \varphi_{ij}(z^{li}, w^{li}), \varphi^*_{lij} = \varphi^{lj}_* \cdot \varphi^*_{ij}(z^{li}, w^{li})$,

$\sigma_{1lj1} = H^{1/3}_l \cdot \varphi^{l1}_* \cdot (\lambda_{lj1}(z^{l1}, w^{l1}) + w^{l1}_1) \cdot \varphi^*_{1j}(z^{l1}, w^{l1})$,

$\sigma_{l1jk} = H^{1/3}_l \cdot \varphi^{l1}_* \cdot \lambda_{1jk}(z^{l1}, w^{l1}) \cdot \varphi^*_{1j}(z^{l1}, w^{l1})$ for $k \neq 1$,

$\sigma_{l2jk} = G^{1/3}_l \cdot \varphi^{l2}_* \cdot \lambda_{2jk}(z^{l2}, w^{l2}) \cdot \varphi^*_{2j}(z^{l2}, w^{l2})$,

$\sigma_{l3jk} = H^{1/3}_l \cdot \varphi^{l3}_* \cdot \lambda_{3jk}(z^{l3}, w^{l3}) \cdot \varphi^*_{3j}(z^{l3}, w^{l3})$.

Then $\varphi_{lij}, \varphi^*_{lij}, \sigma_{lijk} \in C^\infty(\boldsymbol{R}^3_{T_0})$ by (3.55), because they are C^∞ in $\Gamma^{l,i,j}, \tilde{\Gamma}^{l,i,j}$, and $(\mathrm{supp}\ \varphi^{li}_*)^c$, and we see from (3.53) and (3.40),

(3.58) $\varphi^*_{lij} = 1$ on supp φ_{lij} .

From (3.37), (3.38), (3.58), and (3.33) we obtain

(3.59) $\tilde{q}(\sigma, \cdot) = (1 + A_{0l})\, (\sigma - c_l\, \xi - \sigma_{l1j1}) \prod_{k=2}^{3} (\sigma - \sigma_{l1jk})$

if $(t, x, \xi) \in$ supp φ_{lij}

where $c_l = (1 + A_{0l})^{-1}(-1)^{l+1}\, cb_l$ as in (3.25), and

(3.60) $\tilde{q}(\sigma, \cdot) = (1 + A_{0l}) \prod_{k=1}^{3} (\sigma - \sigma_{lijk})$ if $(t, x, \xi) \in$ supp φ_{lij} for $i = 2, 3$.

From (3.40) and (3.42) $\sum_j \varphi_{ij}(z^{li}, w^{li}) = 1$ on $\Gamma^{l,i}$ so that $\sum_j \varphi_{lij} = \varphi^{li}$ by (3.57), since supp $(\sum_j \varphi_{lij}) \subset \cup_j$ supp $\varphi_{lij} \subset \Gamma^{l,i}$, and since supp $\varphi^{li} \subset \Gamma^{l,i}$ by (3.53). Thus in view of the last of (3.53),

(3.61) $\sum_{l=1}^{2} (\sum_{i=1}^{3} \sum_j \varphi_{lij} + \tilde{\varphi}_l) \geq 1$.

We shall consider the estimates of derivatives of the functions constructed above.   To do so we prove the basic estimates first.   We remark that since $\varepsilon^{1-N_0\kappa}_0 > 20$, $\varepsilon^{N_0}_0 < S_0$, $N_0 > 1$, $\kappa > 1$ with the notation in the statement of Lemma 3.4, $\varepsilon^{1-N\kappa} > 20$ and $\varepsilon^{N_0} < S_0$ for any $\varepsilon \in (0, \varepsilon_0)$ so that from (3.31) and (3.32),

(3.62) $2^{-1} \leq |cb_l\, \xi|\, H^{-1/3}_l \leq 1$ on $\Gamma^{l,1} \cup \Gamma^{l,3}$,

$$2^{-1} \leq |g_l| G_l^{-1} \leq 1 \quad \text{on} \quad \Gamma_l ,$$
$$\Gamma^{l,1} = \Gamma^{l,3} = \emptyset \quad \text{if} \quad c \equiv 0 ;$$

there exists a constant $C > 0$ such that

(3.63) $$\max(2, C^{-1} t^{-2\kappa/3}) < \langle \xi \rangle \quad \text{on} \quad \Gamma_l ,$$
$$CG_l^{2/3} > |d_l| \langle \xi \rangle^2 \quad \text{on} \quad \Gamma_l .$$

We also note that

(3.64) $$\frac{d^k}{dx^k} f(x) = (\delta^2 - x^2)^{-2k} \beta_k(x) f(x), \ |x| < \delta \quad \text{where } \beta_k \text{ is a}$$

polynomial,

and that

(3.65) if $a \in E_{h,T}$,
$$\partial^\alpha a(t, x) = t^{-\alpha_1} \cdot (\delta^2 - x^2)^{-2\alpha_2} \cdot (\textstyle\sum_j \beta_{\alpha j}(x) a_{\alpha j}(t, x)) a(t, x), \ |x| < \delta$$

with a finite number of $a_{\alpha j} \in B_{h,T}$ and polynomials $\beta_{\alpha j}$.

Lemma 3.5. *We denote* $\partial^\alpha = \partial_t^{\alpha_1} \partial_{x^2}^{\alpha_2} \partial_\xi^{\alpha_3}$ *for* $\alpha \in \mathbf{Z}_+^3$. *Then we have the followings.*

(3.66) $$|\partial^\alpha a| \leq C_\alpha t^{1/h \cdot \min(1, \alpha_1) - \alpha_1}, \ t \neq 0 \quad \text{for} \quad a \in B_{h,T} .$$

(3.67) $$|\partial^\alpha g_l| \leq C_\alpha f^{1/(10h)} t^{-\alpha_1} \langle \xi \rangle^{-\alpha_3} \cdot \Psi_l |g_l| \quad \text{on} \quad \Gamma_l, \text{ where}$$

(3.68) $$\Psi_l = t^{1/(6h)} \langle \xi \rangle G_l^{-1/3} .$$

(3.69) $$|\partial_\xi^\alpha g_l| \leq C_k \langle \xi \rangle^{-k} |g_l| \quad \text{on} \quad \Gamma_l \text{ when } d_l \equiv 0 .$$

(3.70) $$|\partial^\alpha(d_l \xi^{3-i})| \leq C_\alpha (t^{1/(2h)} f^{1/(4h)} \langle \xi \rangle G_l^{-1/3})^{\min(3-i, \alpha_2)}$$
$$\times t^{-\alpha_1} \langle \xi \rangle^{-\alpha_3} \cdot G_l^{(3-i)/3} \quad \text{on} \quad \Gamma_l \text{ for} \quad i = 1, 2 .$$

(3.71) $$|\partial^\alpha(c\xi)| \leq C_\alpha (t^{1/(2h)} f^{1/(4h)} \langle \xi \rangle^{1/2} H_l^{-1/6})^{\min(1, \alpha_2)}$$
$$\times t^{-\alpha_1} \langle \xi \rangle^{-\alpha_3} \cdot H_l^{1/3} \quad \text{on} \quad \Gamma_l .$$

We remark that from (3.62), $|x| < \delta$ on Γ_l, since $g_l = 0$ when $|x| \geq \delta$.

Proof of Lemma 3.5. (3.65) is easy. We show (3.67). By (3.64), (3.65), and that $d_l \in E_{h,T}$,

(3.72) $$\partial^\alpha g_l = t^{-\alpha_1} \cdot (\delta^2 - x^2)^{-2\alpha_2} \cdot \{(\textstyle\sum_j \beta_{\alpha j} a_{\alpha j}) g_l + (\textstyle\sum_j \gamma_{\alpha j} b_{\alpha j}) f^{1/2} \xi^2\}$$
$$\times \xi^{-\alpha_3} \text{ if } |x| < \delta \text{ and } \xi \neq 0$$

with finite number of $a_{\alpha j}, b_{\alpha j} \in B_{h,T}$ and polynomials of $x, \beta_{\alpha j}, \gamma_{\alpha j}$.

From (3.72),

(3.73) $|\partial^{\alpha} g_l| \le C_{\alpha} t^{-\alpha_1} \cdot (\delta^2 - x^2)^{-2\alpha_2} |\xi|^{-\alpha_3} \cdot (|g_l| + f^{1/2}|\xi|^2)$ if $|x| < \delta$
and $\xi \neq 0$.

On the other hand,

(3.74) $|\delta^2 - x^2|^{-k} |g_l|^{1/3} \le C_k f^{1/(10h)} (t^{1/(3h)}|\xi| + |\xi|^{2/3})$ if $|x| < \delta$;

(3.75) $C^{-1} \Psi_l \le (t^{1/(6h)}|\xi| + |\xi|^{2/3}) G_l^{-1/3} + |\xi|^2 G_l^{-1} + 1 \le C\Psi_l$ on Γ_l .

In fact, (3.74) follows easily from that $d_l \in E_{h,T}$, and the second inequality of
(3.75) follows from the inequalities $\langle\xi\rangle^{-1/3} < Ct^{2\kappa/9} < Ct^{1/(6h)}$ on Γ_l and $\langle\xi\rangle G_l^{-2/3}$
$< Ct^{2\kappa/3} < Ct^{1/(6h)}$ on Γ_l, both of which are consequences of (3.63) and that
$2\kappa/9 > 1/(6h)$, and from which $G_l^{1/3} \le (1 + |g_l|)^{1/3} \le Ct^{1/(6h)}\langle\xi\rangle$.   The first one of
(3.75) is trivial.   (3.67) is proved by combining (3.73)~(3.75) and using the
second fact of (3.62).   (3.69) is trivially from that $|\xi| > 1$ on Γ_l by (3.63).   We
pass to (3.70).   By (3.65) and the remark before proof,

$$|\partial^{\alpha}(d_l \xi^{3-i})| \le C_{\alpha} t^{-\alpha_1} \cdot |\delta^2 - x^2|^{-\alpha_2} \cdot |d_l| |\xi|^{3-i-\alpha_3} \text{ on } \Gamma_l$$

from which (3.70) for $\alpha_2 > 3-i$ immediately follows, and when $\alpha_2 \le 3-i$,

$$|\delta^2 - x^2|^{-2\alpha_2} \cdot |d_l| \cdot |\xi|^{3-i-\alpha_3}$$
$$\le C(|\delta^2 - x^2|^{-2\alpha_2} \cdot |d_l|^{\alpha_2/2})|d_l\langle\xi\rangle^2|^{(3-i)/2}\langle\xi\rangle^{-\alpha_3} \text{ on } \Gamma_l$$

from which (3.70) for $\alpha_2 \le 3-i$ follows immediately by using (3.63).   (3.71) can
be proved similarly as (3.70) by using that $c \in E_{h,T}$.   The proof of Lemma 3.5
is complete.

Lemma 3.6.   *Let U and V be open sets of $\mathbf{R}_y^m$ and $\mathbf{R}_x^n$ respectively, let
$F \in C^{\infty}(U)$, and let $\tilde{F} = (\tilde{F}_1, \cdots, \tilde{F}_n)$ be C^{∞} mapping from V into U.   We assume
that there exist positive functions $Z(y)$, $N_j(y)$ on U, $j = 1, \cdots, m$, and $M_{\alpha}(x)$ on
V, $\alpha \in \mathbf{Z}_+^n$ satisfying $M_{\alpha+\beta} = M_{\alpha} \cdot M_{\beta}$ such that*

$$\sup_{y \in U} |\partial^{\alpha} F(y)| Z(y)^{-1} N(y)^{\alpha} < \infty \quad \text{for all} \quad \alpha \in \mathbf{Z}_+^n$$

where $N = (N_1, \cdots, N_m)$ and $N^{\alpha} = \prod_{j=1}^{m} N_j^{\alpha_j}$, and such that

$$\sup_{x \in V} |\partial^{\alpha} \tilde{F}_j(x)| M_{\alpha}(x)^{-1} N_j(\tilde{F}(x))^{-1} < \infty \quad \text{for all} \quad \alpha \in \mathbf{Z}_+^n .$$

Then, $\sup_{x \in V} |\partial^{\alpha}(F \circ \tilde{F})(x)| Z(\tilde{F}(x))^{-1} M_{\alpha}(x)^{-1} < \infty$ for all $\alpha \in \mathbf{Z}_+^n$.

This lemma is a simple consequence of the chain rule.   We omit the details.

Definition 3.7. Let U be an open set of $\mathbf{R}^n$, and $Z(x)$; $M_\alpha(x)$, $\alpha \in \mathbf{Z}_+^n$ be positive functions on U satisfying $M_{\alpha+\beta} = M_\alpha \cdot M_\beta$. Then we say that $a \in C^\infty(U)$ belongs to the set $\Xi(U, Z, M_\alpha)$ if

$$\sup_{x \in U} |\partial^\alpha a(x)| M_\alpha(x)^{-1} Z(x)^{-1} < \infty \quad \text{for all} \quad \alpha \in \mathbf{Z}_+^n .$$

Lemma 3.8. *Let U be an open set of $\mathbf{R}^n$. Then we have the followings.*

1. *If $a_i \in \Xi(U, Z_i, M_\alpha)$, $i = 1, 2$, then $a_1 a_2 \in \Xi(U, Z_1 \cdot Z_2, M_\alpha)$.*

2. $\Xi(U, Z, M_\alpha) \subset \Xi(U, \tilde{Z}, \tilde{M}_\alpha)$ *if $Z \leq \tilde{Z}$ and $M_\alpha \leq \tilde{M}_\alpha$ for all α.*

3. *Let $a = (a_1, \cdots, a_m)$ be C^∞ mapping from U into $\mathbf{R}^m$ (resp. $(0, +\infty)^m$) with $a_i \in \Xi(U, 1, M_\alpha)$ (resp. $a_i \in \Xi(U, a_i, M_\alpha)$), and let F be a C^∞ function on an open set of $\mathbf{R}^m$ (resp. $(0, +\infty)^m$) containing the image of a of U such that $F \in \Xi(V, Z, 1)$ (resp. $\Xi(V, Z, (y^\alpha)^{-1})$). Then, $F \circ a \in \Xi(U, Z \circ a, M_\alpha)$.*

4. *Let $a = (a_1, \cdots, a_{m_1}, a_{m_1+1}, \cdots, a_m)$, $m = m_1 + m_2$, $m_i \geq 1$ be C^∞ mapping from U into $(0, +\infty)^{m_1} \times \mathbf{R}^{m_2}$ with $a_i \in \Xi(U, a_i, M_\alpha)$ for $i = 1, \cdots, m_1$ and $a_i \in \Xi(U, 1, M_\alpha)$ for $i = m_1+1, \cdots, m$, and F be a C^∞ function on an open set V of $(0, +\infty)^{m_1} \times \mathbf{R}^{m_2}$ containing the image of a of U such that $F \in \Xi(V, Z, (y'^{\alpha'})^{-1})$ where $y' = (y_1, \cdots, y_{m_1})$ for $y \in \mathbf{R}^m$. Then, $F \circ a \in \Xi(U, Z \circ a, M_\alpha)$.*

5. *Let $a \in C^\infty(U)$, be positive, and $\in \Xi(U, a, M_\alpha)$. Then, $a^s \Xi(U, Z, M_\alpha) = \Xi(U, a^s Z, M_\alpha)$ for all $s \in \mathbf{R}$.*

Proof of Lemma 3.8. 1 follows from Leibniz rule, and 2 is trivial. The former (resp. latter) of 3 of the case that in Lemma 3.6, $V = U$, $U = V$, $Z = Z$, $\tilde{F} = a$, $N_j = 1$ (resp. $N_j = y_j$), $F = F$ where the right hand side denotes the notations in 3. 4 is similar to 3. To see 5 we take $F = t^s$, $t \in (0, +\infty)$, $s \in \mathbf{R}$ in the latter of 3. Then, $a^s \in \Xi(U, a^s, M_\alpha)$ which implies from 1 that $a^s \Xi(U, Z, M_\alpha) \subset \Xi(U, a^s Z, M_\alpha)$ and $a^{-s} \Xi(U, a^s Z, M_\alpha) \subset \Xi(U, Z, M_\alpha)$ as desired. The proof is complete.

Remark on notations. We denote a statement $*$ in a statement $**$ by $** - *$. For example, 1 in Lemma 3.8 is denoted by Lemma 3.8−1.

We set for $l = 1, 2$, $\rho \in [0, 1]$, an open set U of $\mathbf{R}_{T_0}^3$, and a positive function Z on U,

(3.77) $\quad \Xi_\rho^l(U, Z)$ (or simply, $\Xi_\rho^l(Z)$ when $U = \mathbf{R}_{T_0}^3$) $= \Xi(U, Z, M_\alpha^{l,\rho})$ where
$$M_\alpha^{l,\rho} = t^{-\alpha_1} < f^{1/(10h)} \, \Psi_l >^{\alpha_1} \cdot \Psi_l^{\alpha_2 + \rho \alpha_3} \cdot \langle \xi \rangle^{-\alpha_3} .$$

We also define

(3.78) $\quad \rho_l = 1$ when $d_l \not\equiv 0$, $\rho_l = 0$ when $d_l \equiv 0$, for $l = 1, 2$, and

$$\rho_0 = \max(\rho_1, \rho_2).$$

Fact 3.9. 1. $|\partial^{\alpha}_{x,y}\langle z\rangle^s| \leq C_{\alpha}\langle z\rangle^{s-|\alpha|}$, $z = x+\sqrt{-1}\,y$.

2. $|(d/d\xi)^k \varphi(\xi)| \leq C_k\langle\xi\rangle^{-k}$ if $\varphi\in C^{\infty}(\boldsymbol{R})$ and being constant if $|\xi|\geq R$ for some $R>0$.

3. $|(d/dt)^k t^s| \leq C_{sk} t^{s-k}$, $t>0$, $s\in\boldsymbol{R}$.

4. $|\partial^s_t \partial^{\alpha}_w (\lambda_{1j1}+w_1)(t,w)| \leq C_{s\alpha} t^{1-s}$ on $U_{1j}\cap(0,\infty)\times\boldsymbol{C}^4$.

5. $|\partial^s_t \partial^{\alpha}_w \lambda_{1jk}(t,w)| \leq C_{s\alpha} t^{3/2-s}$ on $U_{1j}\cap(0,\infty)\times\boldsymbol{C}^4$ for $k\neq 1$.

6. $|\partial^{\alpha}_{z,w}\lambda_{ijk}(z,w)| \leq C_{\alpha}$ on U_{ij} for $i\neq 1$.

Fact 3.10. 1. $a\in\mathcal{E}^l_0(1)$ if $a\in B_{h,T}$, where we regard a as a constant function in ξ. With similar convention,

2. $a\in\mathcal{E}^l_0(\langle\xi\rangle^m)$ if $a(x,\xi)\in S^m_{1,0}(\boldsymbol{R})$.

In view of Lemma 3.5 and that $\Psi_l\geq C_1>0$ on Γ_l ((3.75)),

(3.79) $g_l\in\mathcal{E}^l_{\rho_l}(\Gamma_l, G_l)$, $c\xi\in\mathcal{E}^l_0(\Gamma^{l,1}\cup\Gamma^{l,3}, H_l^{1/3})$, $d_l\,\xi^{3-i}\in\mathcal{E}^l_0(\Gamma_l, G_l^{(3-i)/3})$

    for $i=1,2$.

By noting Facts 3.11$-$1 and 3.9$-$1 an application of Lemma 3.8$-$1 to $cb_l\xi$ and that of Lemma 3.6 with $F=\langle z\rangle$ and (3.79) give

(3.80) $G^s_l\in\mathcal{E}^l_{\rho_l}(\Gamma_l, G^s_l)$, $H^s_l\in\mathcal{E}^l_0(\Gamma^{l,1}\cup\Gamma^{l,3}, H^s_l)$, $s\in\boldsymbol{R}$.

In the similar way, applications of Lemma 3.9$-$1 to the last of (3.79), the former of (3.80), Fact 3.10$-$1, and those to Fact 3.10$-$1, 3.10$-$2, the former of (3.80) show that

(3.81) $G_l^{-(3-i)/3}\cdot v_i\in\mathcal{E}^l_{\rho_l}(\Gamma_l, 1)$,

and by (3.80), (3.81), Lemma 3.8$-$2,

(3.82) $H_l^{-(3-i)/3}\cdot v_i\in\mathcal{E}^l_{\rho_l}(\Gamma^{l,1}\cup\Gamma^{l,3}, (G_l\,H_l^{-1})^{(3-i)/3})$.

(3.71) shows that $|\partial^{\alpha}(c\xi)| \leq C_{\alpha} t^{1/(2h)-\alpha_1}\cdot f^{1/(4h)}\langle\xi\rangle^{1-\alpha_3}$ on Γ_l when $\alpha_2>0$, and that $|\partial^{\alpha}(c\xi)| \leq C_{\alpha} M_{\alpha}^{l,0}\cdot H_l^{1/3}$ on Γ_l when $\alpha_2=0$. So, from that $M_{\alpha}^{l,\rho}$ is increasing respectively on α_2 and ρ, from (3.80), and from the definition of Ψ_l, Leibniz rule gives

(3.83) $c\xi\cdot G_l^{-1/3}\in\mathcal{E}^l_{\rho_l}(\Gamma^{l,2}, 1)$.

Using (3.84), (3.79)$\sim$(3.80), Fact 3.10$-$1, and Lemma 3.8$-$1,2 it is easy to see that

(3.84) $z^{li}, w^{li}\in\mathcal{E}^l_{\rho_l}(\Gamma^{l,i}, 1)$, $z^{l1}\in\mathcal{E}^l_{\rho_l}(\Gamma^{l,1}, z^{l1})$.

Applications of Lemma 3.8–4 with (3.84) and Fact 3.9–4,5, that of the former of Lemma 3.8–3 with (3.84) and Fact 3.9–6, and those of the former of Lemma 3.8–3 with (3.84) and that φ_{ij}, $\varphi^*_{ij} \in C_0^\infty(U_{ij})$ give

$$(3.85) \qquad (\lambda_{lj1}+w)(z^{l1}, w^{l1}) \in \mathcal{E}^l_{\rho_l}(\Gamma^{l,1}, z^{l1}),$$

$$\lambda_{1jk} \in \mathcal{E}^l_{\rho_l}(\Gamma^{l,1}, (z^{l1})^{3/2}) \quad \text{for} \quad k \neq 1,$$

$$\lambda_{ijk}(z^{li}, w^{li}) \in \mathcal{E}^l_{\rho_l}(\Gamma^{l,i}, 1) \quad \text{for} \quad i \neq 1,$$

$$\varphi_{ij}(z^{li}, w^{li}), \varphi^*_{ij}(z^{li}, w^{li}) \in \mathcal{E}^l_{\rho_l}(\Gamma^{l,1}, 1).$$

Lemma 3.11. *Let U and V be open sets of $\mathbf{R}^n$ with $U \subset V$, and let Z, $\tilde{Z}$, M_α with $\alpha \in \mathbf{Z}^n_+$ be positive functions on V satisfying $M_{\alpha+\beta}=M_\alpha \cdot M_\beta$. Then we have the followings.*

1. *$\partial^\beta a \in \mathcal{E}(V, M_\beta Z, M_\alpha)$ if $a \in \mathcal{E}(V, Z, M_\alpha)$.*
2. *Let $a \in C^\infty(V)$ and assume that $\mathrm{supp}\, da \subset U$, that $a|_U \in \mathcal{E}(U, Z|_U, M_\alpha|_U)$, and that $|a| \leq CZ$ for some constant C. Then $a \in \mathcal{E}(V, Z, M_\alpha)$.*
3. *Let $a, b \in C^\infty(V)$ and assume that $a \in \mathcal{E}(V, Z, M_\alpha)$, that $\mathrm{supp}\, a \cap \mathrm{supp}\, db \subset U$, that $b|_U \in \mathcal{E}(U, \tilde{z}|_U, M_\alpha|_U)$, and that $|b| \leq C\tilde{Z}$ for some constant C. Then $a \cdot b \in \mathcal{E}(V, Z \cdot \tilde{Z}, M_\alpha)$.*

1 and 2 are obvious. 3 is a simple consequence of Leibniz rule. We omit the details.

Lemma 3.12. *Let $l \neq l'$, and define functions on $\mathbf{R}^3_{T_0}$ by*

$$a_1 = G_{l'} \cdot G_l^{-1}, \quad a_2 = t^\kappa G_l \langle \xi \rangle^{-3/2}, \quad a_3 = G_l H_l^{-1}.$$

Let $\chi_1(s) \in C^\infty(\mathbf{R})$ supported in $[1/2, \infty)$ and $=1$ on $[1, \infty)$, and set

$$b_0 = \chi_1(ea_1) \cdot \chi_1(R^{-1}a_2), \quad b_1 = \chi_1(\varepsilon a_3^{-1}), \quad b_2 = \chi_1(\varepsilon a_3),$$

$$b_3 = \chi_1(\varepsilon^{-1}a_3) \cdot \chi_1(\varepsilon^{-1}a_3) \quad \text{for} \quad e \in (0, 5), R \in (20, \infty), \varepsilon \in [\varepsilon_1/4, \varepsilon_1]$$

where ε_1 is the constant in (3.51).
Then we have that

$$(3.86) \quad \mathrm{supp}\, b_0 \subset \Gamma_1 \cap \Gamma_2 \cap \mathcal{D}^l, \ b_0 \in \mathcal{E}^1_{\rho_0}(1),$$

$$b \cdot b_i \in \mathcal{E}^l_{\rho_0}(1) \quad \text{if} \quad b \in \mathcal{E}^l_{\rho_0}(1) \quad \text{with} \ \mathrm{supp} \quad b \subset \Gamma_i \cap \mathcal{D}^l, i = 1, 2, 3.$$

Proof of Lemma 3.12. We note that

$$(3.87) \quad a_1 \in \mathcal{E}^l_{\rho_0}(\Gamma_1 \cap \Gamma_2 \cap \mathcal{D}^l, a_1), a_2 \in \mathcal{E}^l_{\rho_l}(\Gamma_l, a_2), a_3 \in \mathcal{E}^l_{\rho_l}(\Gamma^{l,1} \cap \Gamma^{l,3}, a_3).$$

Indeed, the latter two are clear from (3.80) and the first one follows from (3.80), Lemma 3.8–1, the fact that

$$G_{l'} \in \varXi^l_{p_{l'}}(\Gamma_{l'} \cap \mathscr{D}^l, G_{l'})$$

which follows from Lemma 3.8–1, the fact that $\varPsi_{l'} \leq 10^{1/3} \varPsi_l$ on $\mathscr{D}^l$ which implies $M_\alpha^{l',p_{l'}} \leq C_\alpha M_\alpha^{l,p_0}$. $t^\kappa G_l \langle \xi \rangle^{-3/2} > 10$ and $G_{l'} > G_l/10$ by the definition of χ_1 and the range of e, R, which implies the first one of (3.86). So, the former two of (3.87) and an application of the latter of Lemma 3.8–3 with $F(y) = \chi_1(ey_1) \cdot \chi_1(R^{-1} y_2)$ yields $b_0 \in \varXi^l_{p_0}(\Gamma_1 \cap \Gamma_2 \cap \mathscr{D}^l, 1)$ so that consequently the first one of (3.86) and Lemma 3.11–2 shows $b_0 \in \varXi^l_{p_0}(1)$. To see the last one, we observe from (3.30), the range of ε, and the definition of χ_1 that

(3.88) $\mathscr{D}^l \cap \Gamma_l \cap \text{supp } db_i \subset \Gamma^{l,3}$, $i = 1, 2, 3$.

In fact,

$$\text{supp } db_1 \subset \{(t, x, \xi) \in \boldsymbol{R}^3_{T_0}; a_3(t, x, \xi) \in [\varepsilon, 2\varepsilon]\} ,$$
$$\text{supp } db_2 \subset \{a_3 \in [1/(2\varepsilon), 1/\varepsilon]\}, \text{ supp } b_3 \subset \{a_3 \in [\varepsilon/2, 2/\varepsilon]\} ,$$
$$\mathscr{D}^{(3)}_1 \cap \boldsymbol{R}^3_{T_0} = \{a_3 \in (\varepsilon/10, 10/\varepsilon)\} ,$$

so that $\cup_i \text{supp } db_i \subset \mathscr{D}^{(3)}_1(\varepsilon_1/2)$ which implies (3.88). Since the latter of Lemma 3.8–3 yields $b_i \in \varXi^l_{p_l}(\Gamma^{l,3}, 1)$ from the last one of (3.86) and the definition of b_i, an application of Lemma 3.11–3 with $U = \Gamma^{l,3}$, $V = \boldsymbol{R}^3_{T_0}$, $b = b_i$, $a = b$, where the notations in the proof are on the right, gives the last one of (3.86) in view of $\text{supp } b \subset \Gamma_l \cap \mathscr{D}^l$. The proof is complete.

Lemma 3.12 yields from the definitions (3.51), (3.52),

(3.89) $\varphi^{li}, \varphi^{li}_* \in \varXi^l_{p_0}(1)$,

and similarly an application of the latter of Lemma 3.8–3 with $F = \chi$ in (3.51), $U = \Gamma_l$, $a = a_2$ in Lemma 3.12 gives from (3.86) and Fact 3.9–2 that $\tilde{\varphi}_l \in \varXi^l_{p_l}(\Gamma_l, 1)$ so that from the fact that $d\tilde{\varphi}_l \subset \Gamma_l$ which follows from the definition of $\tilde{\varphi}_l$ and that $d\tilde{\chi} = 0$ on $(-\infty, 1/2]$, Lemma 3.11–2 and the boundedness of $\tilde{\varphi}_l$ yield that

(3.90) $\tilde{\varphi}_l \in \varXi^l_{p_l}(1)$ and supp $d\tilde{\varphi}_l \subset \Gamma_l$.

From the definition (3.57), and (3.85), (3.80), (3.89), Lemma 3.11–2 yield by taking $U = \Gamma^{l,i,j}$,

(3.91) $\varphi_{lij}, \varphi^*_{lij} \in \varXi^l_{p_l}(1)$,

(3.92) $\sigma_{l1jl} \in \varXi^l_{p_0}(G_l^{1/3})$, $\sigma_{l1jk} \in \varXi^l_{p_0}(G_l^{1/2} H_l^{-1/6})$ for $k \neq 1$,
            $\sigma_{l2jk} \in \varXi^l_{p_0}(G_l^{1/3})$, $\sigma_{l3jk} \in \varXi^l_{p_0}(H_l^{1/3})$.

To define φ^*_l as in (3.28) later, in the similar way as (3.51), (3.52), (3.57) we

set with the notations in Lemma 3.12,

$$(3.93) \qquad \varphi_{lj}^* = \left[\begin{array}{ll} \varphi_l^* \cdot \varphi_j^*(z^{l2}, w^{l2}) & \text{on } \Gamma^{l,2,j} \\ 0 & \text{on } (\Gamma^{l,2,j})^c \end{array}\right.$$

where $\varphi_l^* = \tilde{\chi}(2la_1) \cdot \tilde{\chi}(40^{-1}a_2) \cdot \tilde{\chi}(\varepsilon_1 a_3)$ with $\tilde{\chi} \in C^\infty(\mathbf{R})$ supported in $[2/3, \infty)$ and $=1$ on $[3/4, \infty)$, and $\varphi_j^* \in C_0^\infty(U_{2j})$ such that $\varphi_j^* = 1$ on supp φ_{2j} and $\inf\limits_{\text{supp } \varphi_j^*} \varphi_{2j}^* > 0$.
The existence of such φ_j^* is assured by the fact that $\varphi_{2j}^* = 1$ on supp φ_{2j}.

Since for the χ in (3.51) we have that $\inf\limits_{\text{supp } \tilde{\chi}} \chi > 0$, and since the function $\tilde{\chi}(2s)$ equals to 1 on supp χ, we see from (3.93), (3.51), (3.52), (3.57) that

$$(3.94) \qquad \varphi_{lj}^* = 1 \text{ on supp } \varphi_{l2j}, \text{ and there exists } C > 0 \text{ such that}$$
$$\varphi_{l2j}^* \geq C \text{ on supp } \varphi_{lj}^* .$$

As the derivation of (3.89) Lemma 3.8–3, (3.84), Lemma 3.12, and Lemma 3.11–2 show that

$$(3.95) \qquad \varphi_{lj}^* \in \mathcal{Z}_{\rho_0}^l(1) .$$

Now the defintion of Γ_l gives with the notations in Lemma 3.1

$$\Psi_l = t^{1/(6h)}\langle\xi\rangle G_l^{-1/3} \leq t^{1/(6h)+\kappa/3}\langle\xi\rangle^{1/2} \leq t^{1/(18h)-1/(20h)}\psi_2^{-1} \text{ on } \Gamma_l .$$

This easily implies that

$$t^{-1}\langle f^{1/(10h)}\Psi_l\rangle \leq 2\psi_1^{-1} \text{ on } \Gamma_l$$

so that

$$M_\alpha^{l,\rho} \leq 2^{\alpha_1}\psi_{\rho/2,\alpha} \text{ on } \Gamma_l, \text{ where } \psi_{\rho/2,\alpha} = \psi_1^{-1} \cdot \psi_2^{-\alpha_2} \cdot \psi_{3,\rho/2}^{-\alpha_3} .$$

Thus, since $\Psi_l \geq C > 0$ on Γ_l means $G_l^{1/3} \leq C^{-1}t^{1/(6h)}\langle\xi\rangle$ on Γ_l, since the definition of $\Gamma^{l,i}$ and (3.92) imply $\sigma_{lijk} \in \mathcal{Z}_{\rho_0}^l(G_l^{1/3})$, and since supp σ_{lijk}, supp φ_{lij}, supp φ_{lij}^*, supp φ_{lj}^*, supp $d\tilde{\varphi}_l$ are all contained in Γ_l, we have from (3.90) $\sim$ (3.92), (3.95), Lemma 3.8–2, Lemma 3.11–2 that φ_{lij}, φ_{lij}^*, φ_{lj}^*, $\tilde{\varphi}_l \in \mathcal{Z}(\mathbf{R}_{T_0}^3, 1, \psi_{\rho_0/2,\alpha})$ and $\sigma_{lijk} \in \mathcal{Z}(\mathbf{R}_{T_0}^3, t^{1/(6h)}\langle\xi\rangle, \psi_{\rho_0/2,\alpha})$. By the latter of this, Lemma 3.8–5, Fact 3.9–3, we obtain $\sigma_{lijk} \in t^{1/(6h)}\mathcal{Z}(\mathbf{R}_{T_0}^3, \langle\xi\rangle, \psi_{\rho_0/2,\alpha})$. Therefore, noting that

$$(3.96) \qquad (\psi_2)^{-\rho}\psi_{3,\rho/2} = \langle\xi\rangle ,$$

we obtain with $T = T_0$ and the given h

$$(3.97) \qquad \sigma_{lijk} \in t^{1/(6h)}\Sigma_{\rho_0/2}^{(0,-\rho_0,1)}, \ \varphi_{lij}, \varphi_{lij}^*, \varphi_{lj}^* \in \Sigma_{\rho_0/2}^0 .$$

Similarly from Lemmas 3.11–1, 3.8–2 and that

$$G_l^{1/3} \, \Psi_l = t^{1/(6h)} \langle \xi \rangle, \; G_l^{1/3} \, t^{-1} \langle f^{1/(10h)} \, \Psi_l \rangle \le 2 \, t^{1/(6h)-1} \langle \xi \rangle \, ,$$
$$G_l^{1/3} \langle \xi \rangle^{-1} = t^{1/(6h)} \, \Psi_l^{-1} \, ,$$

we have

(3.98) $\partial_t \, \sigma_{lijk} \in t^{1/(6h)-1} \, \Sigma_{\rho_0/2}^{(0,-\rho_0,1)}, \; \partial_x \, \sigma_{lijk} \in t^{1/(6h)} \, \Sigma_{\rho_0/2}^{(0,-\rho_0,1)} \, ,$

$\partial_\xi \, \sigma_{lijk} \in t^{1/(6h)} \, \Sigma_{\rho_0/2}^{(0,1-\rho_0,0)} \, .$

If $d_{l_1} \equiv 0$ which implies $G_{l_1} \le C \langle \xi \rangle^2$, and if $l_2 \ne l_1$, the definitions of $\mathcal{D}^{l_2}$ and $\Gamma^{l,i}$ show that $G_{l_2} \le C' \langle \xi \rangle^2$ on $\cup_{i=1}^3 \Gamma^{l,i}$ and $H_l \le C' \langle \xi \rangle^2$ for $\Gamma^{l,2} \cup \Gamma_l^{,3}$ for any l. Thus, since $\sigma_{lijk} \in \Xi_{\rho_0}^l(G_l^{1/3})$ for any l, we have that

(3.99) $\sigma_{lijk} \in \Sigma^{2/3(0,-\rho_0,1)} \, ,$

(3.100) $|g_l| \le C \langle \xi \rangle^2$ on $\cup_{i=1}^3 \Gamma^{l,i} \, ,$

$|c| \le C \langle \xi \rangle^{-1/3}$ on $\cup_l (\Gamma^{l,2} \cup \Gamma^{l,3}) \, ,$

if either d_1 or d_2 is identically zero.

Moreover, since (3.79), (3.92), Lemma 3.8–1 imply that $c\xi \cdot \sigma_{l1jk} \in \Xi_{\rho_0}^l(\Gamma^{l,1}, G_l^{1/2} H_l^{1/6})$ for $k \ne 1$, and since $G_l^{1/2} H_l^{1/6} \le C \langle \xi \rangle^{3/2}$ on $\Gamma^{l,1}$ if $d_1 d_2 \equiv 0$, by Lemmas 3.11–3, 3.8–2 we have that

(3.101) $c\xi \cdot \sigma_{l1jk} \in \Sigma_{\rho_0/2}^{3/2(0,-\rho_0,1)}$ for $k \ne 1$ if $d_1 d_2 \equiv 0 \, .$

From (3.62) we have that

(3.102) $\varphi_{lij} \equiv 0$ for $i = 1, 3$ if $c \equiv 0 \, .$

From the fact that supp $\varphi_{lij}^* \subset \Gamma^{l,i,j}$, (3.44), (3.39), and the definition of σ_{lijk},

(3.103) $C |\sigma_{l2jk} - \sigma_{l2jk'}| \ge \max_s |\sigma_{l2js}| + G_l^{1/3}$ on supp φ_{lj}^* if $k \ne k' \, .$

End of the proof of Lemma 3.2.   We may assume that the set of suffixes of $\{U_{ij}\}_j$ is $\{1, \cdots, J(i)\}$ for some $J(i) \in N$. We numberize the elements of the set $\Lambda = \{(l, i, j); l = 1, 2, i = 1, 2, 3, j = 1, \cdots, J(i)\}$ so that $\Lambda = \{\nu(s); s = 1, \cdots, s_0\}$, $s_0 = \#(\Lambda)$, and denote φ_{lij} by $\varphi_{\nu(s)}$, σ_{lijk} by $\sigma_{\nu(s),k}$ if $\nu(s) = (l, i, j)$. We define

$$\Phi = \Sigma_s \, \varphi_{\nu(s)} + \Sigma_l \, \tilde{\varphi}_l \, .$$

Then since $\Phi \in \Sigma_{\rho_0/2}^0$, $\Phi \ge 1$, we have that $\Phi \in \Xi(R_{T_0}^3, \Phi, \psi_{\rho_0/2, \alpha})$ so that $\Phi^{-1} \in \Sigma_{\rho_0/2}^0$ by Lemma 3.8–3. We shall define φ_i, σ_{ij}, φ_{0l}, φ_0 in the statement, Case (1). Set $I = \{1, \cdots, s_0\}$ and $\varphi_{0l} = \tilde{\varphi}_l \, \Phi^{-1}$ for $l = 1, 2$, $\varphi_s = \varphi_{\nu(s)} \, \Phi^{-1}$, $\sigma_{sk} = \sigma_{\nu(s)k}$ for $s \in I$.

Case (2). Set $I=\{1, \cdots, s_0\}$, $\Lambda_1=\{(l, i, j)\in\Lambda, i\neq 1\}$, $\Lambda_2=\Lambda\setminus\Lambda_1$, $I_i=\{s\in I;$ $\nu(s)\in\Lambda_i\}$. The definitions of φ_i, σ_{ij}, φ_{0l} are the same as in case (1).

Case (3). Taking account of (3.102), (3.15) we set $\Lambda_3=\{(l, i, j)\in\Lambda; i=2\}$ ($\neq\phi$). With a numbering $\Lambda_3=\{\nu'(s); s=1, \cdots, s_3\}$ and the definition $I=\{1, \cdots, s_3\}$ we set $\varphi_s=\varphi_{\nu'(s)}\,\Phi^{-1}$, $\varphi_s^*=\varphi_{1j}^*$, where $\nu'(s)=(l, 2, j)$, and $\sigma_{sk}=\sigma_{\nu'(s)k}$ for $s\in I$, $\varphi_0=(\tilde{\varphi}_1+\tilde{\varphi}_2)\,\Phi^{-1}$. Note that $\sum_s\varphi_s+\varphi_0\geq 1$ from (3.102).

Then the conclusion of Lemma 3.2 follows from (3.14), (3.15), (3.29), (3.51), (3.56), (3.59), (3.60), (3.62), (3.63), (3.78), (3.93), (3.94), (3.97)$\sim$(3.101), (3.103). The proof is complete.

§4. Pseudodifferential Operators Used in the Proof of Proposition 2.3

Let $S_{\rho,\theta}^m(\boldsymbol{R})$, $1\geq\rho\geq\theta\geq 0$, $\theta<1$ be the usual symbol class i.e. the set of all $a(x, \xi)\in C^\infty(\boldsymbol{R}^2)$ satisfying for any α, β

$$|a_{(\alpha)}^{(\beta)}(x, \xi)|\leq C_{\alpha\beta}\langle\xi\rangle^{m+\theta\alpha-\rho\beta}\quad\text{with some constant } C_{\alpha\beta}$$

where $a_{(\alpha)}^{(\beta)}=\partial_\xi^\beta\,\partial_x^\alpha\,a$ which is a Frechet space with the topology defined by semi-norms

$$|a|_l^{(m)}=\max_{|\alpha|+|\beta|\leq l}\sup_{x,\xi}|a_{(\alpha)}^{(\beta)}|\langle\xi\rangle^{-m-\theta\alpha+\rho\beta}.$$

We recall the definition (2.2) of the function f. We define $^f a\in C^\infty(\boldsymbol{R}^2)$ for $a\in C^\infty(\boldsymbol{R}^2)$ by

$$(4.1)\qquad\qquad {}^f a(x, \xi)=a(x, f(x)\,\xi).$$

We denote $L^2(\boldsymbol{R})$ norm by $\|\ \|$ and the symbol in $S_{\rho,\theta}^{2m}(\boldsymbol{R})$ of $a(x, D)\,b(x, D)$ for $a, b\in S_{\rho,\theta}^m(\boldsymbol{R})$ with some ρ, θ, m by $a\circ b$. Then we prove an interpolation lemma.

Lemma 4.1. (1) *Suppose $m'\geq m$ and let B be a bounded set of $S_{1,0}^m(\boldsymbol{R})$ and $b\in S_{1,0}^{m'}(\boldsymbol{R})$ with $|b|\geq C_0\langle\xi\rangle^{m'}$ when $|\xi|\geq R$ for some $C_0>0$, $R>0$. Then there exists $C>0$ such that*

$$\|{}^f a(x, D)\,u\|\leq C(\varepsilon^{m'-m}\|\,{}^f b(x, D)\,u\|+\varepsilon^{-\max(m,0)}\,\|u\|)\,,$$
$$\text{for}\quad a\in B, u\in\mathcal{S}(\boldsymbol{R}), \varepsilon\in(0, 1]\,.$$

(2) *Let m', m, b as in (1) and $a\in S_{1,\theta}^m(\boldsymbol{R})$ with $\theta>0$. Then there exists $C>0$ such that*

$$\|f^{\max(m,0)}\,a(x, D)\,u\|\leq C(\varepsilon^{m'-m}\|\,\Lambda_{m'}\,u\|+\varepsilon^{-\max(m,0)}\,\|u\|)\,,$$

for $u \in \mathcal{S}(\boldsymbol{R})$, $\varepsilon \in (0, 1]$, *where*

$$\Lambda_s = (\langle f(x) \xi \rangle^s)(x, D) \quad for \quad s \in \boldsymbol{R} .$$

Proof. (1) Let $\chi \in C^\infty(\boldsymbol{R})$ with $\chi = 1$ on $[2R, \infty)$ and $\chi = 0$ on $(-\infty, R]$. For $a \in B$ and b we set $b_\varepsilon = \varepsilon^{m'-m} b\chi$, $c_\varepsilon = \varepsilon^{-\max(m,0)} b/|b| \chi$, $a_1 = a \cdot \chi(2^{-1} \cdot)$, $a_0 = a - a_1$, $s_\varepsilon = a_1/\varphi_\varepsilon$ for $|\xi| \geq 2R$ and $= 0$ for $|\xi| < 2R$ where $\varphi_\varepsilon = \varepsilon^{m'-m} b + \varepsilon^{-\max(m,0)} b/|b|$ for $|\xi| > R$. Then $\{a_1\}_{a \in B}$, $\{a_0\}_{a \in B}$ are bounded in $S_{1,0}^m$ and $S_{1,0}^0$ respectively, and $\varepsilon^{\max(m,0)} c_\varepsilon$ is bounded in $S_{1,0}^0$ in ε by the assumption on b. From the assumption on b it is easy to see that there exists $C_{\alpha\beta} > 0$ for any α, β such that

(4.2) $\quad |s_{\varepsilon(\alpha)}^{(\beta)}| \leq C_{\alpha\beta} \varphi_\varepsilon^{-1} \langle \xi \rangle^{m-\beta}$ for $|\xi| > R$ and any $a \in B$,

$\quad |b_{\varepsilon(\alpha)}^{(\beta)}| \leq C_{\alpha\beta} \varepsilon^{m'-m} |b| \langle \xi \rangle^{-\beta} \leq C_{\alpha\beta} \varphi_\varepsilon \langle \xi \rangle^{-\beta}$ for $|\xi| > R$,

which imply that $\varepsilon^{m-m'} b_\varepsilon$ is bounded in $S_{1,0}^{m'}$ in ε and that s_ε is bounded in $S_{1,0}^0$ in ε from the inequality $\langle \xi \rangle^m \leq C |\varphi_\varepsilon(x, \xi)|$ when $|\xi| > R$. We note that $a = s_\varepsilon b_\varepsilon + s_\varepsilon c_\varepsilon + a_0$. Then we need the followings:

(4.3) There exists $C_{\alpha\beta}$ for any $\theta \in (0, 1)$ such that for any $e \in S_{1,0}^k$

$\quad |(^f e)_{(\alpha)}^{(\beta)}| \leq C_{\alpha\beta} |e|_{|\alpha|+|\beta|}^{(k)} \langle f(x) \xi \rangle^{k-\theta\alpha} \langle \xi \rangle^{\theta\alpha-\beta}$.

(4.4) For any $e \in C^\infty(\boldsymbol{R}^2)$,

$\quad (^f e)_{(\alpha)}^{(\beta)}(x, \xi) = \sum_{(\alpha', \beta') \in D_{\alpha,\beta}} (\delta^2 - x^2)^{-2(\alpha-\alpha')} \psi_{\alpha',\beta'}(x) f^\beta(x) (f(x) \xi)^{(\beta'-\beta)}$

$\quad \times {}^f(e_{(\beta')}^{(\alpha')})(x, \xi), \quad |x| < \delta$

where $D_{\alpha,\beta} = \{(\alpha', \beta') \in \boldsymbol{Z}_+^2; \alpha \geq \alpha', \beta' \geq \beta, \alpha'+\beta'-\beta \leq \alpha, \beta' \geq \min(1, \alpha-\alpha')\}$ and $\psi_{\alpha',\beta'}$ are polynomials depending only on $\alpha, \beta, \alpha', \beta', f$ such that $\psi_{\alpha',\beta'} = 1$ when $\alpha = \alpha'$ (which implies $\beta = \beta'$).

(4.4) is proved by induction on α and (4.3) is a simple consequence of chain rule and inequality that

$$|(f(x) \xi)_{(\alpha)}^{(\beta)}| \leq C_{\alpha\beta\theta}(\langle \xi \rangle / \langle f(x) \xi \rangle)^{\theta\alpha} \langle f(x) \xi \rangle \langle \xi \rangle^{-\beta} .$$

Taking $\theta = 2^{-1}$ in (4.3) we see that

$$^f s_\varepsilon \circ (\varepsilon^{m-m'} {}^f(b_\varepsilon)) = \sum_{|\alpha| \leq [2m']+1} \partial_\xi^\alpha (^f s_\varepsilon) D_x^\alpha (\varepsilon^{m-m'} {}^f(b_\varepsilon))/\alpha! + r_\varepsilon$$

with $r_\varepsilon \in S_{1,1/2}^0$ bounded in ε and $a \in B$ in $S_{1,1/2}^0$.

From (4.4) each term in the summation can be written as $\varepsilon^{m-m'} {}^f a_{\varepsilon,\alpha}$ where $a_{\varepsilon,\alpha} = \sum_{(\alpha',\beta') \in D_{\alpha,0}} g_{\alpha',\beta'} \xi^{\beta'} \partial_x^{\alpha'} \partial_\xi^{\beta'} b_\varepsilon \cdot \partial_\xi^\alpha s_\varepsilon$ with $g_{\alpha',\beta'} \in C_0^\infty(\boldsymbol{R})$ depending only on α, β, α',

β', f so that $a_{\varepsilon,\alpha}$ is bounded in $S_{1,0}^{m-\alpha}$ in ε and $a \in B$ in view of (4.2).
Now we have

(4.5) $^f a = \varepsilon^{m'-m} {}^f S_\varepsilon \circ (^f b - ^f b^f(1-\chi)) - \sum_{\alpha \neq 0} {}^f a_{\varepsilon,\alpha} - \varepsilon^{m'-m} r_\varepsilon + {}^f(S_\varepsilon c_\varepsilon) + {}^f a_0$.

Since the case $m \leq 0$ is trivial from (4.3), one can prove (1) by induction on $[m]$ using (4.3) and applying induction hypothesis to the summation in (4.5). This completes the proof of (1).

(2)   It is easy to check that

(4.6)   there exists $C_{\alpha\beta\theta}$ for any $\theta \in (0, 1)$ such that for any $e \in S_{1,\theta}^k$ with $k > 0$,
$$|(f^k e)_{(\alpha)}^{(\beta)}| \leq C_{\alpha\beta\theta}|a|_{|\alpha|+|\beta|}^{(k)} \langle f\xi \rangle^k \langle \xi \rangle^{\theta\alpha-\beta} .$$

Suppose that $m > 0$. Set $a_0 = f^m a \langle f\xi \rangle^{-m}$. Then,
$$a_0 \circ \langle f\xi \rangle^m = \sum_{|\alpha| \leq [m/(1-\theta)]} \partial_\xi^\alpha a_0 D_x^\alpha (\langle f\xi \rangle^m)/\alpha! + r, \quad r \in S_{1,\theta}^0 .$$

Using (4.3) and Leibniz rule it is easy to see that for $\alpha \neq 0$
$$a_\alpha \equiv i^{-\alpha}(a \langle f\xi \rangle^{-m})^{(\alpha)}(\langle f\xi \rangle^m)_{(\alpha)} \in S_{1,\theta}^{m-(1-\theta)\alpha} .$$

Now when $m > 0$.
$$f^m a = a_0 \circ \langle f\xi \rangle^m - \sum_{0 < |\alpha| \leq [m/(1-\theta)]+1} \alpha!^{-1} f^m a_\alpha - r ,$$

and when $m \leq 0$, (2) is trivial so that one can show (2) by induction on $[m/(1-\theta)]$. Indeed, when $m > 0$, we only have to estimate the first and last term on the right of the above equality by using the boundedness of a_0 and r on L^2 and (1) and apply the induction hypothesis to the summation by noting $[\{m-(1-\theta)\alpha\}/(1-\theta)] = [m/(1-\theta)]-\alpha$ and $\max(m-(1-\theta)\alpha, 0) \leq m$. The proof is complete.

The next lemma is one on the commutation.

Lemma 4.2.   (1)   *Let* $a_i \in S_{1,0}^m(\mathbf{R})$ *with* $s \geq 0$. *Then there exists* $C > 0$ *such that*

(4.7) $\|[^f a(x, D), f^s] u\| \leq C(\varepsilon \|\Lambda_m f^s u\| + \varepsilon^{-\max(m-1,0)}\|u\|), \ u \in \mathcal{S}, \ \varepsilon \in (0, 1]$.

(2)   *Let* $a_i \in S_{1,0}^{m_i}$, $i = 1, 2$. *Then there exists* $C > 0$ *such that with the motation* $m = m_1 + m_2$,

(4.8) $\|(^f(a_1 a_2) - ^f a_1 \circ ^f a_2)(x, D) u\| \leq C(\varepsilon \|\Lambda_m u\| + \varepsilon^{-\max(m-1,0)}\|u\|)$,
$$\text{for} \quad u \in \mathcal{S}, \ \varepsilon \in (0, 1] .$$

Corollary 4.3. (1) *Let $b \in S_{1,0}^m(\boldsymbol{R})$ and assume b satisfies the assumption for b in Lemma 4.1–(1). Then there exists $C > 0$ such that*

$$(4.9) \quad C^{-1} \leq (\|f^s \, {}^f b(x, D) \, u\| + \|u\|) / (\|{}^f b \circ f^s(x, D) \, u\| + \|u\|) \leq C, \, u \in \mathcal{S}, \, u \neq 0 \, .$$

(2) *Let $a_i \in S_{1,0}^{m_i}(\boldsymbol{R})$ and assume that a_i satisfies the assumption for b in Lemma 4.1–(1) for any i. Then there exists $C > 0$ such that*

$$(4.10) \qquad C^{-1} \leq (\|{}^f(a_1 \, a_2)(x, D) \, u\| + \|u\|) / (\|{}^f a_1 \circ {}^f a_2(x, D) \, u\| + \|u\|)$$
$$\leq C, \, u \in \mathcal{S}, \, u \neq 0 \, .$$

Corollary 4.3 follows from Lemma 4.2.

Proof of Lemma 4.2. (1) We prove (1) by induction on $[m]$. It is trivial if $m \leq 0$ and we assume $m > 0$. From (3.64),

(4.11) for any positive k, ε, and $j \in \boldsymbol{N}$,

$$f^{\varepsilon j} \left(\frac{d}{dx} \right)^j (f^k) = \psi_{\varepsilon, j, k} f^k \qquad \text{for some} \quad \psi_{\varepsilon, j, k} \in C_0^\infty(\boldsymbol{R}) \, .$$

From (4.3),

$$^f a \circ f^s = \sum_{\alpha \leq [m]}^f (\partial_\xi^\alpha a) f^\alpha \, D_x^\alpha(f^s) / \alpha! + r \quad \text{with} \quad r \in S_{1,1/2}^0 \, .$$

From (4.11) and induction hypothesis, for $\alpha > 0$,

$$\| [^f(\partial_\xi^\alpha a) f^\alpha \, D_x^\alpha(f^s)](x, D) \, u \| \leq C_\alpha(\|A_{m-1} f^s \, u\| + \varepsilon^{-\max(m-1, 0)} \|u\|) \, ,$$

the right hand side of which is estimated by that of (4.7) from Lemma 4.1–(1). This proves (1).

(2) is proved in the similar way by using the fact that from (4.4), $f^\alpha \, D^\alpha(^f a_2) = {}^f a_{2, \alpha}$ for some $a_{2, \alpha} \in S_{1,0}^{m_2}$. The details are omitted.

Definition 4.4. Let $h \in \boldsymbol{N}$, $T \in (0, 1)$ be given. Assume the notations in Definition 3.1 and set $\tilde{\psi}_1 = t\psi_2$. Let $d \in \boldsymbol{R}^3$ with $d_1 \leq 0$. Then we say that $a(t, x, \xi) \in C^\infty(\boldsymbol{R}_T^3)$ belongs to $\Sigma^d = \Sigma^d(T, h)$ if for any α

$$|\partial^\alpha a(t, x, \xi)| \leq C_\alpha \, \tilde{\psi}_1^{d_1 - \alpha_1} (\langle \xi \rangle^{-1/2})^{d_2 - \alpha_2} \, \psi_{3, 1/2}^{d_3 - \alpha_3}$$

with some constant C_α .

We write $\Sigma = \Sigma(T, h) = \bigcup_d (\bigcup_p \Sigma_p^d \cup \Sigma^d)$.

Definition 4.5. We set for $\rho \in [0, 1/2]$ and $d \in \boldsymbol{R}^3$

$$(4.12) \quad \mu_\rho(d) = -(\rho d_3 + 2^{-1} d_2), \quad \mu(d) = -2^{-1}(d_1 + d_3), \quad \nu(d) = -2^{-1} d_2 + d_3 \, ,$$
$$\sigma_\rho(d) = d_3 + \max(0, \mu_\rho(d) - 2^{-1} d_1), \quad \sigma(d) = \nu(d) + \max(0, \mu(d)) \, .$$

Then,

$$(4.13) \qquad \psi_2^{d_2} \psi_{3,\rho}^{d_3} = \langle t^{e_h} \xi \rangle^{\mu_\rho(d)} \langle \xi \rangle^{d_3},$$

$$\tilde{\psi}_1^{d_1} \langle \xi \rangle^{-2^{-1}d_2} \psi_{3,1/2}{}^{d_3} = t^{d_1} \langle t^{e_h} \xi \rangle^{\mu(d)} \langle \xi \rangle^{d_3},$$

$$\prod_{j=1}^{2} \psi_j^{d_j} \psi_{3,\rho}^{d_3} = t^{d_1} \langle t^{e_h} f^{1/(10h)} \xi \rangle^{-d_1/2} \langle t^{e_h} \xi \rangle^{\mu_\rho(d)} \langle \xi \rangle^{d_3}.$$

$(4.14) \qquad \{t^{-d_1} a_1(t, \cdot)\}_{t \in (0,T)}$ is bounded in $S_{1-\rho,1/2}^{\sigma_\rho(d)}$ for $a \in \Sigma_\rho^d$,

and in $S_{1/2,1/2}^{\sigma(d)}$ for $a \in \Sigma^d$.

A linear mapping on $\boldsymbol{R}^3$,

$$A: d \to (d_1, \mu(d), \nu(d))$$

is one to one so that for any $d \in \boldsymbol{R}^3$ there exist $J_\rho(d)$, $J(d)$, $\mathcal{J}_\rho(d)$, $\mathcal{J}(d) \in \boldsymbol{R}^3$ respectively determined uniquely such that

$$(4.15) \qquad A(J_\rho(d)) = (d_1, \mu_\rho(d) - d_1/2, d_3),$$

$$A(J(d)) = (0, \mu(d), \nu(d)),$$

$$A(\mathcal{J}_\rho(d)) = (d_1, \max(0, \mu_\rho(d) - d_1/2), \max(0, d_3)),$$

$$A(\mathcal{J}(d)) = (d_1, \max(0, \mu(d)), \max(0, \nu(d))).$$

We set for $a \in C^\infty(\boldsymbol{R}_T^3)$ and $E \subset C^\infty(\boldsymbol{R}_T^3)$,

$$^f a(t, x, \xi) = a(t, x, f(x) \xi), \quad ^f E = \{^f a; a \in E\}.$$

Fact 4.6. 1. $ab \in \Sigma^{d+e}$ if $a \in \Sigma^d$, $b \in \Sigma^e$.

2. If $a \in \Sigma_\rho^d$ and $\varepsilon > 0$, $f^\varepsilon \partial^\alpha f a = {}^f a_{\alpha,\varepsilon}$ with $a_{\alpha,\varepsilon} \in \Sigma_\rho^{d-\alpha}$.

3. If $a \in \Sigma^d$ and $\varepsilon > 0$, $f^\varepsilon \partial^\alpha f a = {}^f a_{\alpha,\varepsilon}$ with $a_{\alpha,\varepsilon} \in \Sigma^{d-\alpha}$.

4. $\Sigma_\rho^d \subset \Sigma^{J_\rho(d)}$.

5. $(\langle \xi \rangle^{-1/2})^{J(d)_2} \cdot \psi_{3,1/2}{}^{J(d)_3} = \tilde{\psi}_1^{d_1} \cdot (\langle \xi \rangle^{-1/2})^{d_2} \cdot \psi_{3,1/2}{}^{d_3}$ and $\Sigma^d = t^{d_1} \Sigma^{J(d)}$.

6. $^f \Sigma_\rho^d \subset \Sigma^{\mathcal{J}_\rho(d)}$.

7. $^f \Sigma \subset \Sigma^{\mathcal{J}(d)}$.

5 follows from (4.13) and 2,3,6,7 follow from the formula which follows from (4.4):

$$(4.16) \qquad \partial^\alpha(^f a(t, x, \xi)) = \sum_\beta (\delta^2 - x^2)^{-2(\alpha_2 - \beta_2)} r_\beta(x) f^{\alpha_3}(x) (f(x) \xi)^{\beta_3 - \sigma_3}$$

$$\times {}^f(\partial^\beta a)(t, x, \xi), \quad |x| < \delta,$$

where the summation is taken over all the set $\{\beta = (\alpha_1, l, \alpha_3 + j); l + k \leq \alpha_2, \alpha_3 + j \geq \min(\alpha_2 - l, 1)\}$ and r_β is a polynomial depending only on α, β, f such that $r_\beta = 1$ for $\beta = \alpha$.

We need Lemma 2 in [1] in this section and its easy extention in the next one. We recall the definition of a pair of weight functions in [1]. We say a pair of positive functions Φ, φ on $R_x^n \times R_\xi^n$ form a pair of weight functions if it satisfies the followings:

(4.17)   there exist positive constants C, c, ε such that (i) $c \leq \Phi(x, \xi) \leq C(1+|\xi|)$, $C \geq \varphi(x, \xi) \geq c(1+|\xi|)^{\varepsilon-1}$; (ii) $\varphi\Phi \geq c$; (iii) $\dfrac{\Phi(x, \xi)}{\varphi(x, \xi)} \sim \dfrac{\Phi(y, \eta)}{\varphi(y, \eta)}$ whenever $|\xi| \sim |\eta|$, where $A \sim B$ means that $|A/B|$ and $|B/A|$ are bounded; (iv) $C^{-1}\Phi(y, \eta) \leq \Phi(x, \xi) \leq C\Phi(y, \eta)$, $C^{-1}\varphi(y, \eta) \leq \varphi(x, \xi) \leq C\varphi(y, \eta)$ for all

$$(y, \eta) \in U(x, \xi) = \{(y, \eta); |y-x| < c\varphi(x, \xi), |\eta-\xi| < c\Phi(x, \xi)\} .$$

Then we have the following.

Lemma 4.7.   *Let* $C > 1$, $C_0 > 1$, $c > 0$, $\varepsilon > 0$, *let* k, $l \in N$, *and let* $(M_1, \cdots, M_{k+l}, m_1, \cdots, m_{k+l})$ *be a permutation of* $2(k+l)$ *real numbers. Let* B *be a set of pairs of weight functions* (Φ, φ) *satisfying* (4.17) *with the* C, c, ε *given above and that* $\sup \{ \dfrac{\Phi(x, \xi)}{\varphi(x, \xi)} \cdot \dfrac{\varphi(y, \eta)}{\Phi(y, \eta)}; |\xi| \leq 4$ *and* $|\eta| \leq 4$, *or* $4^{-1}|\eta| \leq |\xi| \leq 4|\eta| \} \leq C_0$ *according to* (iii). *Then, there exists constant* $C' > 0$ *and* $L \in N$ *having the following property: Suppose that* $b(x, y, \xi, \eta)$, $(x, y, \xi, \eta) \in R^{2n} \times R^{2n}$ *is a smooth function in* (y, η) *for each fixed* (x, ξ) *and assume that for some* $(\Phi, \varphi) \in B$,

(4.18) $\displaystyle\sup_{x, y, \xi, \eta} D_y^\alpha D_\eta^\beta b \cdot (a_{\Phi, \varphi}^{|\alpha|, |\beta|})^{-1}(x, y, \xi, \eta) < \infty$ *for any* $\alpha, \beta \in Z_+^n$

where

$$a_{\Phi, \varphi}^{b, q}(x, y, \xi, \eta) = \sum_{\substack{r_1+\cdots+r_k=q \\ r_{k+1}+\cdots+r_{k+l}=p, r_1, \cdots, r_{k+l} \in Z_+}} \prod_{j=1}^{k} \sup_{Q \in K} (\Phi^{M_j-r_j}(Q) \varphi^{m_j}(Q))$$

$$\times \prod_{j=k+1}^{k+l} \sup_{Q \in K} (\Phi^{M_j}(Q) \varphi^{m_j-r_j}(Q))$$

where K *is the covex hull of* (x, ξ), (x, η), (y, ξ), (y, η).
Then,

$$a(x, \xi) = OS - \iint e^{-iy\cdot\eta} b(x, y+x, \xi, \eta+\xi) \, dy d\eta ,$$

where $OS - \iint e^{-iy\cdot\eta} dy d\eta$ *denotes the oscillatory integral and the above one is well-defined by* (4.17)-(i), *satisfies the estimate*

(4.19) $|a(x, \xi)| \leq C'|b|_L^{\Phi, \varphi}(\Phi^{\Sigma M_j} \cdot \varphi^{\Sigma m_j})(x, \xi) ,$

        where $|b|_j^{\Phi, \varphi} = \displaystyle\max_{|\alpha|+|\beta| \leq j} \sup_{x, y, \xi, \eta} |D_y^\alpha D_\eta^\beta b| (a_{\Phi, \varphi}^{|\alpha|, |\beta|})^{-1}(x, y, \xi, \eta) .$

Proof of Lemma 4.7 is parallel to that of Lemma 2 in [1], but we will give a brief discription of proof in the appendix.

Fact 4.8. *Set* $\Phi_t(x, \xi) = \psi_{1,1/2}(t, \xi)$, $\varphi(x, \xi) = \langle \xi \rangle^{-1/2}$, *for* $(x, \xi) \in \boldsymbol{R} \times \boldsymbol{R}$. *It is easy to check that for any* $t \in (0, T)$, (Φ_t, φ) *is a pair of weight functions on* $\boldsymbol{R}_x \times \boldsymbol{R}_\xi$ *with* $C = 10$, $c = 1/10$, $\varepsilon = 1/2$, *and that* $\{(\Phi_t, \varphi) ; t \in (0, T)\}$ *is a set of weight functions on* $\boldsymbol{R}_x \times \boldsymbol{R}_\xi$ *as* B *in Lemma 4.7.*

Lemma 4.9. *Let* $a_1 \in \Sigma^d$, $a_2 \in \Sigma^e$ *and set for* $N \in \boldsymbol{N}$,

(4.20) $$r_N[a_1, a_2] = \int_0^1 r_\theta[a_1, a_2] (1-\theta)^{N-1} d\theta$$

where

$$r_\theta[a_1, a_2] = (2\pi)^{-1} OS - \iint e^{-iy \cdot \eta} a_1(t, x, \xi + \theta\eta) a_2(t, x+y, \xi) \, dy d\eta .$$

Then, $r_N[a_1, a_2] \in \Sigma^{d+e}$.

Lemma 4.10. *Let* $a \in \Sigma^d$ *and* $g(\theta, x, y) \in C^\infty([0, 1] \times \boldsymbol{R}^2)$ *with* $\sup_{\theta, x, y} |\partial^\alpha_{x,y} g| < \infty$ *for all* α. *We set*

(4.21) $$r_\theta(t, x, \xi) = OS - \iint e^{-iy \cdot \eta} g(\theta, x, y) a(t, x+\theta y, \xi+\theta\eta) \, dy d\eta .$$

Then

$$\sup_{\theta, t, x, \xi} |\partial^\alpha_{t,x,\xi} r_\theta| \, \tilde{\psi}_1^{-d_1+\alpha_1} (\langle \xi \rangle^{-1/2})^{-d_2+\alpha_2} \psi_{1,1/2}^{-d_3+\alpha_3} < \infty \quad \text{for any } \alpha .$$

Remark 4.11. *Let* $a_\theta(t, x, y, \xi, \eta) \in C^\infty([0, 1]_\theta \times \boldsymbol{R}^{4n}_{x,y,\xi,\eta})$ *satisfying that with* $\delta \in [0, 1)$ *and* $M \geq 0$

$$|\partial^j_t \partial^\alpha_x \partial^\beta_y \partial^\gamma_\xi \partial^\nu_\eta a_\theta(t, x, y, \xi, \eta)| \leq C_{j\alpha\beta\gamma\nu}(t, x, \xi) (1+|y|)^M (1+|\eta|)^{M+\delta|\beta|}$$

where $C_{j\alpha\beta\gamma\nu}$ is assumed to be locally bounded in (t, x, ξ) and to be independent of θ. Then

(1) for any compact $K \subset \boldsymbol{R}^{2n}$, $OS - \iint e^{-iy \cdot \eta} a_\theta(t, x, y, \xi, \eta) \, dy d\eta$ is continuous in $(\theta, x, \xi) \in [0, 1] \times K$,

(2) one can differentiate the above oscillatory integral in (t, x, ξ) by differentiation under integral sign any times. These are easily verified from the definition of the oscillatory integral.

Proof of Lemma 4.9. We assume the notations in Fact 4.8. From Remark 4.11

(4.22) $$\partial^\alpha r_N[a_1, a_2] = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} r_N[\partial^{\alpha-\beta} a_1, \partial^\beta a_2] .$$

Setting $b_{\beta\theta}(t, x, y, \xi, \eta) = (\partial^{\alpha-\beta} a_1)(t, x, \xi+\theta\eta)(\partial^\beta a_2)(t, x+y, \xi)$, we have in view of Fact 4.6–5 that there exists C_{ij} such that for any $\beta \leq \alpha$,

$$(4.23) \quad |\partial_y^i \partial_\eta^j(b_{\beta\theta}(t, x, y-x, \xi, \eta-\xi))| \leq C_{ij}\, t^{d_1+e_1-\alpha_1} \sup_{Q \in \overline{\xi\eta}} \Phi_t^{m_1} \cdot \varphi^{m_2}(Q)$$
$$\times \sup_{Q \in \overline{\xi\eta}} \Phi_t^{m_1'} \cdot \varphi^{m_2'}(Q),$$

where $\overline{\xi\eta}$ is the segment joining ξ to η and

$$m_1 = J(d-\alpha+\beta)_3-j, \ m_2 = J(d-\alpha+\beta)_2, \ m_1' = J(e-\beta)_3, \ m_2' = J(e-\beta)_2-i.$$

By virtue of Fact 4.8 we get from (4.23) and Lemma 4.7 that there exists C_α such that for any $\beta \leq \alpha$,

$$(4.24) \quad |r_\theta[\partial^{\alpha-\beta} a_1, \partial^\beta a_2]| \leq C_\alpha\, t^{d_1+e_1-\alpha_1}\{\Phi_t^{J(d+e-\alpha)_3} \cdot \varphi^{J(d+e-\alpha_2)}\}(\xi)$$
$$\text{for} \quad \theta \in [0, 1].$$

(4.22) and (4.24) prove Lemma 4.9.

Lemma 4.10 is proved in the similar way.   The proof is omitted.

Corollary 4.12. *In the following, for (ii) and (iv) we suppose that* $a \in \Sigma^d$, $b \in \Sigma^e$ *with* $\nu(d) \geq 0$, $\mu(d) \geq 0$, *and for the others we supose that* $a \in \Sigma_\rho^d$, $b \in \Sigma^e$ *with* $d_3 \geq 0$, $\mu_\rho(d) \geq 0$.

Then we have

(i) $r_N[\partial_\xi^N {}^f a, b] \in f^{N\rho} \Sigma^{J\rho(d)+e+2N(0,1-\rho,0)}$,

(ii) $r_N[\partial_\xi^N {}^f a, b] \in f^{N/2} \Sigma^{d+e+N(0,1,0)}$,

(iii) $r_N[\partial_\xi^N {}^f a, D_x^N b] \in f^{N\rho} \Sigma^{J\rho(d)+e+N(0,1-2\rho,0)}$,

(iv) $r_N[\partial_\xi^N {}^f a, D_x^N b] \in f^{N/2} \Sigma^{d+e}$,

(v) $r_M^\ast[\partial_\xi^N D_x^N {}^f a] \in f^{N\rho/2} \Sigma^{J\rho(d)+N(0,1-2\rho,0)}+t^{d_1} \Sigma^{(0,N,0)}$

where by definition,

$$(4.25) \qquad\qquad r_M^\ast[c] = \int_0^1 r_\theta^\ast[c]\,(1-\theta)^{M-1}\,d\theta$$

where

$$r_\theta^\ast[c] = (2\pi)^{-1}\, OS-\iint e^{-iy\eta}\, c(t, x+\theta y, \xi+\theta\eta)\, dy d\eta.$$

Proof. (i) ${}^f\Sigma_\rho^d \subset \Sigma^{J\rho(d)}$ from Fact 4.6–4 and 7, the assumption on d, and the definition $\mathcal{J}$, J_ρ, and we have

$$(4.26) \qquad\qquad \partial_\xi^N {}^f a = f^{N\rho}\, a_N \quad \text{with}$$
$$a_N = (f\langle\xi\rangle/\langle f\xi\rangle)^{(1-\rho)N}\langle\xi\rangle^{-(1-\rho)N}\langle f\xi\rangle^{(1-\rho)N} {}^f\partial_\xi^N a.$$

Using (4.3) and (4.6), and noting $\langle\xi\rangle^{(1-\rho)N} \partial_\xi^N a \in \Sigma^{J\rho(d)}$ we see that $a_N \in$

$\Sigma^{J_\rho(d)+2N(0,1-\rho,0)}$. Since $b\in\Sigma^e$, Lemma 4.1 implies (i), (ii) is similarly proved. (iii), (iv) are particular cases of (i), (ii) respectively. Next we consider (v). Using (4.26) and Leibniz rule we see that $D_\xi^N \partial_x^N {}^f a = f^{2N\rho/3} \tilde{a}_N$ for some $\tilde{a}_N \in \Sigma^{J_\rho(d)+N(0,1-2\rho,0)}$ so that

$$(4.27) \qquad r_M^{\natural}[\partial_\xi^N D_x^N {}^f a]$$

$$= (2\pi)^{-1}\int_0^1 (1-\theta)^{M-1}(OS - \iint e^{-iy\eta} f^{2N\rho/3}(x+\theta y)$$

$$\cdot \tilde{a}_N(t, x+\theta y, \xi+\theta \eta)\, dy d\eta)\, d\theta\ .$$

Using Taylor expansion at x of order N_1 and integrating by parts in η we see that the right hand side of (4.27) equals to, with some $g(\theta, x, y)\in C^\infty([0,1]\times \boldsymbol{R}^2)$ satisfying $\sup\limits_{\theta,x,y} |\partial_{x,y}^\alpha g| \leq C_{N,N_1,\rho}$ for any α,

$$(2\pi)^{-1}\{\sum_{j<N_1} j!^{-1}\Big(\frac{d}{dx}\Big)^j [f^{2N\rho/3}(x)]$$

$$\times \int_0^1 \theta^{2j}(1-\theta)^{M-1}(OS - \iint e^{-iy\eta} D_\xi^j \tilde{a}_N(t, x+\theta y, \xi+\theta \eta)\, dy d\eta)\, d\theta$$

$$+ \int_0^1 \theta^{2N_1}(1-\theta)^{M-1}(OS - \iint e^{-iy\eta} D_\xi^{N_1} \tilde{a}_N(t, x+\theta y, \xi+\theta \eta)\, g(\theta, x, y)\, dy d\eta)\, d\theta\}\ .$$

By Lemma 4.10 the integral in each term of the summation belong to $\Sigma^{J_\rho(d)+N(0,1-2\rho,0)}$ and the last term does to $\Sigma^{\tilde{d}}$ with $\tilde{d}=J_\rho(d)+N(0, 1-2\rho, 0)-(0, 0, N_1)$. Since N_1 is an arbitrary positive integer, we have proved (v) in view of (3.64) and Lemma 3.8–5. The proof is complete.

We define $a_t \in C^\infty((0, T); C^\infty(\boldsymbol{R}^2))$ for $a\in C^\infty(\boldsymbol{R}_T^3)$ by $a_t(\cdot)=a(t, \cdot)$. Then taking account of (4.14),

$$(4.28) \qquad a_t\circ b_t = \sum_{\alpha<N} \partial_\xi^\alpha a_t\cdot D_x^\alpha b_t/\alpha! + r_N[\partial_\xi^N a, D_x^N b]/(N-1)!\ ,$$

$$a_t^{\natural} = \sum_{\alpha<N} \partial_\xi^\alpha D_x^\alpha \bar{a}_t/\alpha! + r_N^{\natural}[\partial^N D_x^N \bar{a}]/(N-1)!\ ,$$

for $a, b\in\Sigma$ where in general we denote by $c^{\natural}$ the symbol in $S_{\rho,\theta}^m$ of formally adjoint of the operator $c(x, D)$, $c\in S_{\rho,\theta}^m(\boldsymbol{R})$ on $\mathcal{S}(\boldsymbol{R})$.

We also define $a\circ b\in\Sigma$, $a^{\natural}\in C^\infty(\boldsymbol{R}_T^3)$ for $a, b\in\Sigma$ by

$$(4.29) \qquad (a\circ b)_t = a_t\circ b_t, \quad a^{\natural}{}_t = a_t{}^{\natural}\ .$$

Lemma 4.13. $a\circ b\in\Sigma^{d+e}$ if $a\in\Sigma^d$, $b\in\Sigma^e$.

This is an immediate consequence of (4.28), *Lemma 4.9, and Fact 4.6–1.*

Lemma 4.14. (1) *Let $a\in\Sigma_\rho^d$ and suppose that $\mu(d)\geq 0$. Then there exists*

$C>0$ *such that*

$$\|{}^f a_t(x, D)\, u\| \leq C t^{d_1}(t^{e_h(\mu_\rho(d)-(d_1/2))}\|f^{-d_1/20h}\, A_{\sigma_\rho(d)}\, u\|$$
$$+ t^{e_h \mu_\rho(d)}\|A_{\nu_\rho(d)+d_3}\, u\| + \|A_{d_3}\, u\| + \|u\|)\quad for\quad u\in\mathcal{S},\, t\in(0, T)\,.$$

(2)   *Let* $a\in\Sigma^d$ *and suppose that* $\mu(d)\geq 0$. *Then there exists* $C>0$ *such that*

$$\|{}^f a_t(x, D)\, u\| \leq C t^{d_1}(t^{e_h \mu(d)}\|A_{\sigma(d)}\, u\| + \|A_{\nu(d)}\, u\| + \|u\|)\quad for\quad u\in\mathcal{S},\, t\in(0, T)\,.$$

(3)   *Let* $a\in\Sigma^d$ *and suppose that* $\mu(d)\geq 0$, *then there exists* $C>0$ *such that*

$$\|a_t(x, D)\, u\| \leq C t^{d_1}(t^{e_h \mu(d)}\|u\|_{\sigma(d)} + \|u\|_{\nu(d)} + \|u\|)\quad for\quad u\in\mathcal{S},\, t\in(0, T)$$
where $\|u\|_s = \|\langle D\rangle^s\, u\|$.

(4)   *Let* $a\in\Sigma^d$ *and suppose that* $\mu(d)\geq 0$. *Then there exists a constant* $C>0$
such that

$$\|f^{\max(\sigma(d),0)}\, a_t(x, D)\, u\| \leq C t^{d_1}(t^{e_h \mu(d)}\,\|A_{\sigma(d)}\, u\| + \|A_{\nu(d)}\, u\| + \|u\|)$$
for $u\in\mathcal{S},\, t\in(0, T)$.

Fact 4.15.   Set $\Psi_1 = t^s\langle\xi\rangle,\, \Psi = f^{s'}\, t^s\langle\xi\rangle,\, t\in(0, 1),\, \xi\in\mathbf{R},\, |x|<\delta;\, s>0, s'>0$. Then for any $\varepsilon>0$,

$$|\partial^\alpha\,\Psi_1| \leq C_\alpha\,\Psi_1\, t^{-\alpha_1}\langle\xi\rangle^{1-\alpha_3},$$
$$|\partial^\alpha\,\Psi_2| \leq C_{\alpha\varepsilon}\,\Psi_2(\langle t^s\,\xi\rangle\,\Psi_2^{-1})^{\varepsilon\alpha_2}\, t^{-\alpha_1}\langle\xi\rangle^{-\alpha_3}\,.$$

Proof of Lemma 4.14.   (1)   Set

(4.31) $\Psi = t^{d_1}(t^{e_h\mu_\rho(d)}\langle\xi\rangle^{\mu_\rho(d)}+1)\,\langle\xi\rangle^{d_3}(t^{-e_h d_1/2}\, f^{-d_1/(20h)}\langle\xi\rangle^{-d_1/2}+1)\,.$

Then it is easy to see using Fact 4.15 and Lemma 3.8–3 that

(4.32)   for any $\varepsilon>0,\, \alpha\in\mathbf{Z}_+^3,\, |\partial^\alpha\,\Psi| \leq C_{\alpha\varepsilon}\, t^{-\alpha_1}\langle t^{e_h}\xi\rangle^{\varepsilon\alpha_2}\langle\xi\rangle^{-\alpha_3}\,\Psi$.

Using (4.32) and (4.16) it is easy to check that $\partial^\alpha\Psi\in t^{-\alpha_1}\Sigma_\rho^{d^\alpha}$ with $d^\alpha = d+\alpha_3$
$(0, 2\rho, -1)+\varepsilon\alpha_2(0, -2, 0)$ for any $\varepsilon>0$ and $\alpha\in\mathbf{Z}_+^3$, and that the estimate
$|\partial^\alpha\,{}^f\Psi| \leq C_{\alpha\varepsilon}\, t^{-\alpha_1}\langle\xi\rangle^{\varepsilon\alpha_2-\alpha_3}\,{}^f\Psi$ holds for any $\varepsilon>0$ and $\alpha\in\mathbf{Z}_+^3$. In particular,

(4.33) $f^\varepsilon\,\partial_x^k\,{}^f\Psi\in{}^f\Sigma_\rho^{d+k/4(0,-2,0)},\, \partial_x^k\,{}^f\Psi\in\Sigma^{J_\rho(d)+\varepsilon(0,-1,0)}$,
for any $\varepsilon>0$.

Setting $b=a\Psi^{-1}$, and using (4.28), (4.32), (4.33), and Corollary 4.12–(i) we see
that

(4.34) $b\in\Sigma_\rho^0,\,\Psi\in\Sigma_\rho^d,\,{}^f b\circ{}^f\Psi = {}^f a+{}^f a_1+{}^f a_2$ for some
$a_1\in\Sigma^{d'},\, a_2\in t^{d_1}\Sigma^0$ with d' satisfying

$$\sigma_\rho(d') \le \sigma_\rho(d) - 1/4, \ \mu_\rho(d') = \mu_\rho(d), \ d_1' = d_1 \ .$$

If we denote by $S_d(u)$ the expression defined by deleting C from the right hand side of the inequality in (1), we have that

(4.35) $\qquad S_{d'}(u) \le CS_d(u)$ and $\|{}^f\Psi_t(x, D)u\| \le CS_d(u)$,

$\qquad\qquad$ with C independent of $t \in (0, 1]$, $u \in \mathcal{S}$.

Indeed, the latter is clear and the former is easily shown by the properties of d', Corollary 4.3–(1), and Lemma 4.1–(1).

Now, since (1) is clear for d with $\sigma_\rho(d) \le 0$, in particular with $[4\sigma_\rho(d)] \le -1$, (1) can be shown by induction in $[4\sigma_\rho(d)]$ in view of (4.34), (4.35), (4.14), and Fact 4.6.

(2) and (3). They can be proved similarly by induction on $[2\sigma(d)]$ by setting $\Psi = t^{d_1}(t^e{}_h\mu^{(d)} \langle \xi \rangle^{\mu(d)} + 1) \langle \xi \rangle^{\nu(d)}$ instead of the Ψ in the proof of (1) and by using for (3) the standard interpolation in the Sobolev spaces instead of Lemma 4.1–(1). We omit the proofs.

(4) is clear when $[2\sigma(d')] \le -1$, and we prove it by induction on $[2\sigma(d)]$. Assume $\sigma(d) \ge 0$ and (4) is true for d' with $[2\sigma(d')] \le [2\sigma(d)] - 1$. It is easy to see that

(4.36) $\qquad [a_t(x, D), f^{\sigma(d)}] = f^{\max(\sigma(d-(0,0,1)),0)} a_{1t}(x, D) + a_2(x, D)$

$\qquad\qquad$ for some $\quad a_1 \in \Sigma^{d-(0,0,1)}, \ a_2 \in t^{d_1} \Sigma^0$.

Let $S_d(u)$ be the expression defined by removing the right of the inequality of (4). Then,

(4.37) $\qquad S_{d-(0,0,1)}(u) \le CS_d(u)$ and $\|a_t(x, D)f^{\sigma(d)}u\| \le CS_d(u)$ with C

$\qquad\qquad$ independent of $\quad t \in (0, 1], \ u \in \mathcal{S}$.

In fact, the former follows from Lemma 4.1–(1) and the latter does from Lemma 4.1–(1) and the inequality

(4.38) $\qquad$ If $\ s' \ge s > 0$, $\|f^{s'}u\|_s \le C_{ss'}(\|A_s u\| + \|u\|)$, $u \in \mathcal{S}$,

which can be shown by expanding the symbol $\langle \xi \rangle^s \circ f^{s'}$ and using Lemma 4.1–(2), –(1) in this order. (4.36), (4.37), and the induction hypothesis show (4). The proof is complete.

Remark on notations. The letters T and h in the definitions of Σ, $B_{h,T}$ are in common for all notations if we do not write these letters.

Recall the definition of $\mathcal{S}_T$((2.11)). If $a \in \Sigma(T, h)$ and $u \in \mathcal{S}_T$, $a_t(x, D)u_t$

as a function of $(t, x) \in (0, T) \times \boldsymbol{R}$ also belongs to $\mathcal{S}_T$ and $\partial_t(a_t(x, D) u_t) = (\partial_t a)_t(x, D) u_t + a_t(x, D)(\partial_t u)_t$.

Definition 4.16. For $a \in \Sigma(T, h)$ and, we define the linear operator on $\mathcal{S}_T$, which we denote by $Op\ a$ or $a(t, x, D)$, by assigning u to the function $(t, x) \to (a_t(x, D) u_t)(x)$. Furthermore, let $\mathcal{P}_T$ be the set of all $q(t, x, \tau, \xi) \in C^\infty(\boldsymbol{R}_T^4)$ being a polynomial in τ with coefficients in $\Sigma(T_0, h)$ in (t, x, ξ). With the unique expression $q(t, x, \tau, \xi) = \sum_{k \leq m} a_k(t, x, \xi) \tau^k$ we define the operator on $\mathcal{S}_T$, denoted by $Op\ q$ or $q(t, x, D)$, sending u to $\sum_{k \leq m} a_k(t, x, D) D_t^k u$.

q is determined by $Op\ q$ in the above. Indeed, $(Op\ q) u = 0$ for any u means $q = 0$, since for $u = \chi(t; t_0)(t - t_0)^k v(x)$ where $k \leq m$, $\chi \in C_0^\infty((0, T))$ with $\chi = 1$ near $t_0 \in (0, T)$, and $v \in \mathcal{S}$ and q as above, $u \in \mathcal{S}_T$ and $(Op\ q) u(t_0, \cdot) = i^{-k} k!\, a_k(t_0, x, D) v$. We call q the symbol of $Op\ q$.

For $q_j \in \mathcal{P}_T$, $j = 1, 2$ with $q_j = \sum_{k \leq m_j} a_{jk} \tau^k$, $a_{jk} \in \Sigma$, we define $q_1 \circ q_2$ by

(4.39) $q_1 \circ q_2(t, x, \tau, \xi)$

$$= OS - \iint e^{-iy\eta} \sum_{j \leq m_1} (2\pi j!)^{-1} \partial_\tau^j q_1(t, x, \tau, \xi + \eta)\, D_t^j q_2(t, x + y, \xi)\, dy d\eta\ .$$

Then,

(4.40) $q_1 \circ q_2(t, x, D) = q_1(t, x, D)\, q_2(t, x, D)\ ,$

$$q_1 \circ q_2 = \sum_{1 \leq m_1} l!^{-1} \Big(\sum_{s=0}^{N-1} s!^{-1} \partial_\xi^s \partial_\tau^l q_1 \cdot D_x^s D_t^l q_2 \\ + (N_l - 1)!^{-1} r_{N_l}[\partial^N \partial_\tau^l q_1, D_x^N D_t^l q_2] \quad \text{for any} \quad N_l \in N$$

where, by definition,

$$r_N[q_1, q_2] = \sum_{\substack{k \leq m_1 \\ j \leq m_2}} r_N[a_{1k}, a_{2j}] \tau^{k+j}\ .$$

Actually these are true when q_i are monomials in τ so that they are true in general, since all terms are bilinear in q_1 and q_2.

For $E \subset \Sigma(T, h)$, we set

$Op\ E =$ the set of all operators with the symbol in E.

Definition 4.17. For real numbers $k \geq 0$, $l \geq 0$, $N \geq k$ we define $A^{k, l, N}$(resp. $B^{k, l, N}$) = the set of all symbols being a finite sums of symbols of the form:

$$a + f^N b;\ a \in {}^f \Sigma_{1/2}^d\ (\text{resp. } {}^f \Sigma^d),\ b \in \Sigma^e\ \text{with}$$
$$d \in \varDelta^{k, l},\ e \in \varDelta_{k, l}\ (\text{resp. } d, e \in \varDelta_{k, l})$$

where

$$A^{k,l} = \{d \in \mathbf{R}^3;\, 0 \geq d_1 \geq -2l,\, \mu_{1/2}(d) \geq 0,\, d_3 \geq 0,\, \sigma_{1/2}(d) = k\}\,,$$

$$\varDelta_{k,l} = \{d \in \mathbf{R}^3;\, 0 \geq d_1 \geq -2l,\, \tilde{\mu}(d) \geq 0,\, \nu(d) \geq 0,\, \sigma(d) = k\}$$

with the notation

$$\tilde{\mu}(d) = \mu((0, d_2, d_3))\,,$$

and we set

$$\mathcal{A}^{k,l,N} \ (\text{resp. } \mathcal{B}^{k,l,N}) = Op\, A^{k,l,N} \ (\text{resp. } Op\, B^{k,l,N})\,,$$

$$A^{k,N} \ (\text{resp. } B^{k,N}) = A^{k,0,N} \ (\text{resp. } B^{k,0,N})\,,$$

$$\mathcal{A}^{k,N} \ (\text{resp. } \mathcal{B}^{k,N}) = \mathcal{A}^{k,0,N} \ (\text{resp. } \mathcal{B}^{k,0,N})\,.$$

From Fact 4.6–4

$$A^{k,l,N} \subset B^{k,l,N}\,,$$

and clearly,

$$A^{k_1,l_1,N_1} \ (\text{resp. } B^{k_1,l_1,N_1}) \subset A^{k_2,l_2,N_2} \ (\text{resp. } B^{k_2,l_2,N_2})$$

$$\text{if} \quad k_1 \leq k_2,\, l_1 \leq l_2,\, N_1 \geq N_2\,.$$

In general, for a ring R and $E_i \subset R$, $a_i \in R$, $i \in$ finite set I, the set $\sum_{i \in I} a_i E_i$ is defined by $\{\sum_{i \in I} a_i e_i;\, e_i \in E_i\}$. We use this notation for the ring of symbols with the product as functions and the one of their defining operators on $\mathcal{S}_T$ in Definition 4.16.

Lemma 4.18.

(1) $A_1 A_2 \in \mathcal{A}^{k_1+k_2,l_1+l_2,N}$ *(resp. $\mathcal{B}^{k_1+k_2,l_1+l_2,N}$)*
if $A_i \in \mathcal{A}^{k_i,l_i,N}$ *(resp. $\mathcal{B}^{k_i,l_i,N}$) and $N \geq k_1+k_2$.*

(2) *If $b \in C^\infty(\mathbf{R}_T^3)$ satisfies with $d \in \varDelta_{k_1,0}$,*

(4.41) $$|\partial^\alpha b(t, x, \xi)| \leq C_\alpha\, t^{-\alpha_1} (\langle \xi \rangle^{-1/2})^{d_2}\, \psi_{3,1/2}{}^{d_3} \langle \xi \rangle^{-\alpha_3} \quad \text{for all } \alpha\,,$$

$$Op(a \circ {}^f b - a^f b),\, Op({}^f b \circ a - a^f b) \in \mathcal{B}^{\max(k_1+k_2-1/2,0),N}$$

$$\text{for} \quad a \in B^{k_2,N},\, N \geq \max(k_1+k_2, 1)\,.$$

(3) *Suppose that $a \in \Sigma_{1/2}^{(0,-1,1)}$ satisfies*

(4.42) $$\partial_x a \in \Sigma_{1/2}^{(0,-1,1)},\, \partial_\xi a \in \Sigma_{1/2}^0\,.$$

Then we have the followings:

(i) $Op({}^f a \circ b - {}^f a \cdot b),\, Op(b \circ {}^f a - b \cdot {}^f a) \in \mathcal{A}^{k+1/2,l,N}$ *(resp. $\mathcal{B}^{k+1/2,l,N}$) if $b \in A^{k,l,N}$ (resp. $B^{k,l,N}$) with $N \geq 1/2+k$.*

(ii) $Op\ {}^f\bar{a}-(Op\ {}^f a)^*\in\mathcal{A}^{1/2,N}+Op\ \Sigma^{(0,N,0)}$ *for any* t *and* $N\geq 1$, *where for* $q=\sum_k a_k\,\tau^k\in\mathcal{P}_T$ *we denote by* $(OP\ q)^*$ *the operator with the symbol* $q^{\sharp}=\sum_k\sum_{j=0}^k\binom{k}{j}\,D_t^{k-j}\,a_k^{\sharp}\,\tau^j$ *so that*

$$((Op\ q)\,u,\,v)=(u,\,(Op\ q)^*\,v)\quad \textit{for}\quad u,\,v\in\mathcal{S}_T\,,$$

where (,) *denotes the inner product if* $L^2((0,\,T)\times\boldsymbol{R})$.

(iii)   *If* b *satisfies* (4.41), $[Op\ {}^f a,\ Op\ {}^f b]\in Op\ {}^f\Sigma^d+f^N\,Op\ \Sigma^{(0,N,0)}$ *for any* $N\geq 1$.

(iv)   *If* b *satisfies* (4.42), $Op({}^f a\circ{}^f b-{}^f(ab))\in\mathcal{A}^{1,N}$ *for any* $N\geq 1$.

(4)   *For any* $a\in\Sigma_\rho^d$ (*resp.* Σ^d), $s\in[0,\,\infty)$, *and* $N\in\boldsymbol{N}$, *there exists* $b\in\Sigma_\rho^{d-(0,0,1)}$ (*resp.* $\Sigma^{d-(0,0,1)}$) *such that*

$$\text{supp}\ b\subset\text{supp}\ a,\quad f^s\cdot{}^f a-({}^f a+{}^f b)\circ f^s\in f^N\,t^{d_1}\,\Sigma^{(0,N,0)}\,.$$

(4)'   *For any* $a\in\Sigma_\rho^d$ (*resp.* Σ^d), $s\in[0,\,\infty)$, *and* $N\in\boldsymbol{N}$, *there exists* $b\in\Sigma_\rho^{d-(0,0,1)}$ (*resp.* $\Sigma^{d-(0,0,1)}$) *such that*

$$\text{supp}\ b\subset\text{supp}\ a,\quad {}^f a\circ f^s-f^s({}^f a+{}^f b)\in f^N\,t^{d_1}\,\Sigma^{(0,N,0)}\,.$$

(5)   *For any* $a\in\Sigma^d$, $s\in(0,\,\infty)$, *and* $\varepsilon\in(0,\,s)$, *and* $N\in\boldsymbol{N}$ *there exists* $b\in\Sigma^{d-(0,0,1)}$ *such that*

$$f^s\,a-a\circ f^s-b\circ f^{s-\varepsilon}\in t^{d_1}\,\Sigma^{(0,N,0)}\,.$$

Proof. (1)   Let us prove the former case.   We may assume

$$A_i=Op({}^f a_i+f^N\,b_i),\quad \text{with}\ a_i\in\Sigma_{1/2}^{d^i},\ b_i\in\Sigma^{e^i},\ d^i\in\varDelta^{k_i,l_i},\ e^i\in\varDelta_{k_i,l_i}\,.$$

Now,

$$({}^f a_1+f^N\,b_1)\circ({}^f a_2+f^N\,b_2)={}^f a_1\circ{}^f a_2+{}^f a_1\circ(f^N\,b_2)+f^N\,b_1\circ({}^f a_2+f^N\,b_2)\,,$$

and the first term $\in A^{k_1+k_2,l_1+l_2,N}$ from (4.28), Fact 4.6–2 and –4, and Corollary 4.12–(iii), and for the third one this is also clearly true in view of Fact 4.6–4. Since for $\alpha>0$

$$\partial_\xi^\alpha\,{}^f a_1=f\partial_\xi^{\alpha-1}\,{}^f(\partial_\xi\,a_1)\in f\sum{}^{J_{1/2}(d)-(\alpha-1)(0,0,1)}\,,$$

Leibniz rule shows

$$\partial_\xi^\alpha\,{}^f a_1\cdot D_x^\alpha(f^N\,b_2)\in f^N\sum{}^{J_{1/2}(d)+e}\quad \text{for any}\ \alpha\,,$$

which would show that the second term also $\in A^{k_1+k_2,l_1+l_2,N}$ in view of Corollary 4.12–(iii).   This proves the former one, and the latter is similary proved by using Fact 4.6–3 and –7, and Corollary 4.12–(iii) and (iv).

(2) We first note that

$$(4.43) \qquad b \in \cap_\rho \Sigma_\rho^{d^\rho} \quad \text{with } d^\rho \text{ satisfying } J_\rho(d^\rho) = d \,,$$

and that for $k \in N$ we have

$$(4.44) \qquad f^\varepsilon \, \partial_x^k ({}^f b) \in {}^f \Sigma^d \quad \text{for} \quad \varepsilon > 0, \; \partial_x^k ({}^f b) \in \bigcap_{\varepsilon > 0} \Sigma^{d - (0, \varepsilon, 0)} \,,$$

$$\partial_\xi^k ({}^f b) \in \bigcap_{1 > \varepsilon \geq 0} f^\varepsilon \, \Sigma^{d + 2(k - \varepsilon)(0, 1, 0)} \,.$$

(4.43) follows from (4.15), (4.41), and that $d_1 = 0$ which follows from $d \in \Delta_{k_1, 0}$. The first one of (4.44) are easy consequences of (4.41) and (4.16) which imply that

$$(4.45) \qquad |\partial^\alpha \, {}^f b| \leq C_\alpha \langle \xi \rangle^{\varepsilon - \alpha_3} \, t^{-\alpha_1} (\langle f \xi \rangle^{-1/2})^{d_2} \, {}^f \psi_{3, 1/2}{}^{d_3}$$

$$\text{for all } \alpha \text{ and } \varepsilon \in (0, 1) \,,$$

$$(4.46) \qquad \partial_\xi^k \, {}^f b = f^\varepsilon \langle \xi \rangle^{-k + \varepsilon} \, (f \langle \xi \rangle / \langle f \xi \rangle)^{k - \varepsilon} \, {}^f (\langle \xi \rangle^{k - \varepsilon} \, \partial_\xi^k \, b)$$

$$\text{for} \quad \varepsilon \in [0, 1) \text{ and } k \in N \,.$$

The second one follows from (4.45), since $d \in \Delta_{k_1, 0}$ implies that $\nu(d) \geq 0$, $\mu(d) \geq 0$, $d_1 = 0$. The last one follows from (4.46), (4.41), (4.3), (4.6), Fact 4.6–7, and that $d \in \Delta_{k_1, 0}$ as in the proof of Corollary 4.12–(i).

We may assume that $a = {}^f a_1 + f^N \, b_1$ with a_1, b_1 in $\Sigma^{k_2 (0, -2, 0)}$, for proof of (2), because $\langle t^{e_h} \xi \rangle \leq \langle \xi \rangle$ and $d_1 = 0$. (4.41), (4.44), Fact 4.6–2, and Leibniz rule show that for all $k \in N$

$$(4.47) \qquad \partial_\xi^k \, b_1 \cdot f^{1/2} \, D_x^k \, {}^f b \in \Sigma^{(k_1 + k_2 - k/2)(0, -2, 0)} \,,$$

$$\partial_\xi^k ({}^f b) \cdot D_x^k (f^N \, b_1) \in f^N \, \Sigma^{(k_1 + k_2 - k/2)(0, -2, 0)} \,,$$

$$\partial_\xi^k \, {}^f b \cdot D_x^k \, {}^f a_1, \; \partial_\xi^k \, {}^f a_1 \cdot D_x^k ({}^f b) \in {}^f \Sigma^{(k_1 + k_2 - k/2)(0, -2, 0)} \,.$$

An application of Corollary 4.12–(i) with $\rho = 1/4$ and of Corollary 4.12–(ii) and –(iv) respectively show that the reminder term r_M in the asymptotic expansion (4.18) with $N = M$ of ${}^f b \circ a$ and $a \circ {}^f b$ belong to $B^{0, N}$ for any large M, in view of (4.43) and the second one of (4.44). So (4.47) shows (2).

(3) We first note that from (4.42) and (4.16),

$$(4.48) \qquad \partial_x \, {}^f a \in \Sigma^{(0, -2, 0)}, \; {}^f \partial_\xi \, a \in \Sigma^0, \; f^\varepsilon \, \partial_x^k \, {}^f a \in {}^f \Sigma_{1/2}^{(0, -1, 1) + (k - 1)(0, -1, 0)}$$

$$\text{for any } \varepsilon > 0 \text{ and } k \in N \,.$$

To simplify the notation we set

$$\tilde{r}_N[a, b] = r_N[\partial_\xi^N \, a, \, D_x^N \, b] \,,$$

$$I_N[a, b] = \sum_{k=1}^{N-2} \partial_\xi^k a \cdot D_x^k b/k!, \quad \text{for} \quad a, b \in \Sigma .$$

(i)   To show the former one we may assume $b = {}^f a_1 + f^N b_1$, $a_1 \in \Sigma_{1/2}^d$, $b_1 \in \Sigma^e$, $d \in \varDelta^{k,l}$, $e \in \varDelta_{k,l}$.   The second one of (4.48) and Fact 4.6–2 and –3 show that

(4.49) $I_M[{}^f a, {}^f a_1] \in {}^f \Sigma_{1/2}^{d+(0,-1,0)}$, $I_M[{}^f a, f^N b_1] \in f^N \Sigma^{e+(0,-1,0)}$,

$\tilde{r}_M[{}^f a_1, {}^f a] \in f^{(M+1)/2} \sum^{J_{1/2}(d)+(0,-1,0)}$,

$\tilde{r}_M[{}^f a, f^N b_1] \in f^{(M+1)/2} \sum^{e+(0,-1,0)}$,

for any $M > 0$,

and using the first and third one of (4.48) we see that

(4.50) $I_M[{}^f a_1, {}^f a] \in {}^f \Sigma_{1/2}^{d+1/2(0,-1,1)}$, $I_M[b_1, {}^f a] \in \Sigma^{e+(0,-1,0)}$,

$\tilde{r}_M[{}^f a_1, {}^f a] \in f^{M/2} \sum^{J_{1/2}(d)+(0,-1,0)}$, $\tilde{r}_M[b_1, {}^f a] \in \Sigma^{e+(0,-1,0)}$,

for any M.

This shows the former of (i).   The latter is shown in the similar way.
(ii)   The second of (4.42), Fact 4.6–2, and Leibniz rule show that

$$\partial_\xi^\alpha D_x^\alpha {}^f \bar{a} \in {}^f \Sigma_{1/2}^{1/2(0,-1,1)} \quad \text{for any } \alpha \in N ,$$

and that

$$r_N^\sharp[\partial_\xi^N D_x^N {}^f \bar{a}] = r_N^\sharp[\partial_\xi^{N-1} D_x^{N-1} \partial_\xi D_x({}^f \bar{a})] \in f^{(N-1)/4} \Sigma^{(0,-1,0)} + \Sigma^{(0,N-1,0)}$$

from (4.48), Corollary 4.12–(v).   This proves (ii).
(iii)   (4.42)∼(4.44) and (4.48) show that

$$I_N[{}^f a, {}^f b], I_N[{}^f b, {}^f a] \in {}^f \Sigma^d ,$$

$\tilde{r}_N[{}^f a, {}^f b] \in f^{(N+1)/2} \Sigma^{d+(0,N-1-\varepsilon,0)}$ for any ε,

$\tilde{r}_N[{}^f b, {}^f a] \in f^{N/4} \Sigma^{d+(N/2-1)(0,1,0)}$,

as in the proof of (2).   So (iii) is proved.
(iv)   From (4.42) and the last one of (4.48)

$$\partial_\xi^k({}^f a) \cdot D_x^k({}^f b) \in {}^f \Sigma_{1/2}^{(0,-1,1)} \quad \text{for any } k \in N .$$

From (4.42) and the first one of (4.48)

$$\tilde{r}_N[{}^f a, {}^f b] \in f^{(N+1)/2} \Sigma^{(0,-2,0)} .$$

This proves (iv).
(4)   Let us define inductively $a_j \in f^{j/2} \Sigma_\rho^{d-(0,0,j)}$ for $j \in Z_+$ so that

$$a_0 = a,$$

$(*)_j$
$$f^s\ {}^f a_j - {}^f a_j \circ f^s - f^s\ {}^f a_{j+1} \in t^{d_1} f^N\ \Sigma^{(0,N,0)},$$

$$a_j = f^{j/2}\ a'_j \quad \text{for some } a'_j \text{ in } \Sigma_\rho^{d-(0,0,j)}$$

with supp $a'_j \subset$ supp a.

Assume $a_i,\ 0 \le i \le j$ has been chosen. Since $a_j = f^{1/2}\ a'_j$ with $a'_j \in \Sigma_\rho^{d-(0,0,j)}$, and since

$$k!^{-1} f^k\ D_x^k(f^s) = g_k f^{s+1/2} \quad \text{with } g_k \in C_0^\infty(\mathbf{R}) \quad \text{for any } k \in \mathbf{N},$$

setting

$$a'_{j+1} = -\sum_{k=1}^{\tilde{N}} g_k\ \partial_\xi^k\ a'_j \quad \text{and} \quad a_{j+1} = f^{(j+1)/2}\ a'_{j+1}$$

where

$$\tilde{N} = 4(N + \|d\| + 1) \quad \text{with} \quad \|d\| = \Sigma_{j=1}^3\ |d_j|$$

we obtain

$$f^s\ {}^f a_j - {}^f a_j \circ f^s - f^s\ {}^f a_{j+1} = -\tilde{N}!^{-1}\ r_{\tilde{N}+1}[\partial_\xi^{\tilde{N}+1}\ {}^f a_j,\ D_x^{\tilde{N}+1} f^s],$$

$$a'_{j+1} \in \Sigma^{d-(0,0,j+1)} \quad \text{with supp } a'_{j+1} \subset \text{supp } a$$

so that $(*)_j$ holds in view of Corollary 4.12–(ii) and the fact that $a_j \in \Sigma^{\mathcal{S}_\rho(d)} \subset t^{d_1} \Sigma^{\|d\|(0,-2,0)}$ which follows from Lemma 3.8–2 and 5. Since $f^{j/4} = \langle \xi \rangle^{-j/4} (f \langle \xi \rangle / \langle f \xi \rangle)^{j/4} \langle f \xi \rangle^{j/4}$, and since $\langle \xi \rangle^{j/4} \Sigma^{d-(0,0,j)} \subset \Sigma_\rho^d$, we have

$$f^{j/2}\ {}^f\Sigma_\rho^{d-(0,0,j)} \subset f^{j/4}\ t^{d_1}\ \Sigma^{(\|d\|-j/4)(0,-2,0)}$$

in view of (4.3), (4.6), Fact 4.6–6. Now by setting $b = \Sigma_{j=1}^N\ a_j$ and by using $(*)_j$,

$$f^s \cdot {}^f a - a \circ f^s - {}^f b \circ f^s - f^s \cdot {}^f a_{\tilde{N}+1} \in f^N\ t^{d_1}\ \Sigma^{(0,N,0)}$$

which proves the former of (4) in view of

$$\Sigma^{(\|d\|-(\tilde{N}+1)/4)(0,-2,0)} \subset \Sigma^{(0,N,0)}.$$

The latter is similar.

(4)′ By the asymptotic expansion, for any $a \in \Sigma_{1/2}^d$ (resp. Σ^d) and $s \ge 0$,

$$^f a \circ f^s = \sum_{j \le M} {}^f(\partial^j\ a) f^j D_x^j(f^s) / j! + M!^{-1}\ r_{M+1}[\partial_\xi^{M+1}\ {}^f a,\ D_x^{M+1} f^s].$$

The summation can be written as $f^s ({}^f a + {}^f b)$ with $b \in \Sigma_{1/2}^{d-(0,0,1)}$ (resp. $\Sigma^{d-(0,0,1)}$) satisfying supp $b \subset$ supp a as in the proof of (4). Since $\Sigma_{1/2}^d \subset \Sigma^{\mathcal{S}_\rho(d)}$ (resp.

$\Sigma^d \subset \Sigma^{\mathcal{G}}(d)$), the last term is in

$$f^{(M+1)/2} \Sigma\mathcal{G}_{\rho}(d) + (M+1)(0,1,0) \text{ (resp. } f^{(M+1)/2} \Sigma\mathcal{G}(d) + (M+1)(0,1,0))$$

by Corollary 4.12–(ii). By Lemma 3.8–2 and –5 for any N there exists $M \in N$ such that the last term is in

$$f^N t^{d_1} \Sigma^{(0,N,0)} \text{ (resp. } f^N t^{d_1} \Sigma^{(0,N,0)}) .$$

This completes the proof.

(5) The proof is similar to that of (4). One can define inductively a_j, $j = 1, \ldots,$ $\tilde{N} = N + 3(\|d\| + 1) + 1$, in $\Sigma^{d-(0,0,j)}$ so that

$$a_0 = a ,$$
$$f^{s_j} \, Op \, a_j - (Op \, a_j) f^{s_j} - f^{s_{j+1}} \, Op \, a_{j+1} \in t^{d_1} \, Op \, \Sigma^{(0,N,0)}, \, 0 \le j \le \tilde{N} - 1$$
$$\text{where } s_j = s - j\varepsilon/(\tilde{N}+1) .$$

Then setting $b = -\sum_{j=1}^{\tilde{N}-1} a_j \circ f^{s_j - s + \varepsilon}$ we have

$$b \in \Sigma^{d-(0,0,1)} ,$$
$$f^s \, Op \, a - (Op \, a) f^s + (Op \, b) f^{s-\varepsilon} - f^s \tilde{N} \, Op \, a_{\tilde{N}} \in t^{d_1} \, Op \, \Sigma^{(0,N,0)}$$

which show the assertion. The proof is complete.

Lemma 4.19. *Let* $\|\cdot\| = \|\cdot\|_{L^2(R_x)}$.

(1) *Let* $a \in \Sigma^d_{1/2}(T, h)$ *with* $\sigma_{1/2}(d) = k$, $d_1 = -2l$, $l \ge 0$, *and* $\mu_{1/2}(d) \ge 0$. *Then, there exists* $C > 0$ *such that for any* $u \in \mathcal{S}_T$

$$\|((Op \, {}^f a) u)_t\| \le C t^{-2l} (t^{e_h l} \|A_k f^{1/(10h)} u_t\| + \|A_{k-1} u_t\| + \|u_t\|), \, t \in (0, T) .$$

(2) *Let* $a \in \Sigma^d(T, h)$ *with* $\sigma(d) = k$, $d_1 = -2l$, $l \ge 0$, *and* $\tilde{\mu}(d) \ge 0$. *Let* $s \ge 0$. *Then, there exists* $C > 0$ *such that for any* $u \in \mathcal{S}_T$,

$$\|f^{s+\max(\sigma(d),0)}((Op \, a) u)_t\| \le C t^{-2l} (t^{e_h l} \|A_k f^s u_t\| + \|A_{k-l} u_t\| + \|u_t\|) ,$$
$$t \in (0, T) .$$

(3) *Let* $a \in \Sigma^d(T, h)$ *and* $s \ge 0$ *be as in* (2). *Then there exists* $C > 0$ *such that for any* $u \in \mathcal{S}_T$,

$$\|f^s((Op \, {}^f a) u)_t\| \le C t^{-2l} (t^{e_h l} \|A_k f^s u_t\| + \|A_{k-l} u_t\| + \|u_t\|), \, t \in (0, T) .$$

Proof. (1) Since $\mu_{1/2}(d) + d_3 = k - l$ and $\mu_{1/2}(d) \ge 0$, $\langle t^{e_h} \xi \rangle^{\mu_{1/2}(d)} \langle \xi \rangle^{d_3} \le \langle \xi \rangle^{k-l}$ so that $a \in \Sigma^{(-2l,0,0)+(k-l)(0,-1,1)}_{1/2}$. So, by applying Lemma 4.13–(1) with u_t for u and with $(-2l, 0, 0) + (k-l)(0, -1, 1)$ for d and by estimating the term $\|f^{1/(10h)} A_k u_t\|$ by Corollary 4.3–(1) one obtain the desired inequality.

(2)   From Lemma 4.13–(4) and Corollary 4.3–(1) one can easily see that for $A \in f^{\max(k,0)} \, Op \, \Sigma^{(-2l,-2(k-l),0)}$ and $s' \geq s \geq 0$, there exists $C > 0$ such that for $u \in \mathcal{S}_T$,

$$(4.51) \qquad \|(Af^{s'} u)_t\| \leq Ct^{-2l}(t^{\varepsilon_k l}\|A_k f^s u_t\| + \|A_{k-l} u_t\| + \|u_t\|), \, t \in (0, T) \, .$$

As in the proof of (1) we have $a \in \Sigma^{(-2l,-2(k-l),0)}$ from $\tilde{\mu}(d) \geq 0$ and $k - l = \tilde{\mu}(d) + \nu(d)$. When $s = 0$, (4.51) with $s' = s = 0$ implies (2). When $s > 0$, from Lemma 4.18–(5) and the uniqueness of the symbol,

(4.52)   for $\varepsilon \in (0, s)$ there exists $b \in \Sigma^{d-(0,0,1)}$, $r \in t^{-2l} \Sigma^{(0,1,0)}$ satisfying
$$f^s a - a \circ f^s - b \circ f^{s-\varepsilon} = r \, .$$

We consider separately two subcases: (i) $\max(k, 0) \leq 1/2$, (ii) $\max(k, 0) > 1/2$. In (i), we have $b \in t^{-2l} \Sigma^0$ so that from (4.52) one can derive (2) by using (4.51) with $s' = s$ and with $f^{\max(k,0)} \, Op \, a$ for A. In (ii), again Lemma 4.18–(5) implies that

(4.53)   for $\varepsilon \in (0, 1/2)$ there exists $b_1 \in \Sigma^{d-(0,0,2)}$ and $r_1 \in t^{-2l} \Sigma^{(0,1,0)}$
   satisfying $f^{1/2} b - b \circ f^{1/2} - b_1 \circ f^{1/2-\varepsilon_1} = r_1 \, .$

Take ε and ε_1 in (4.52) and (4.53) as $\varepsilon + \varepsilon_1 < 1/2$ so that setting $\varepsilon' = 1/2 - \varepsilon$ and $\varepsilon'' = 1/2 - \varepsilon - \varepsilon_1$, we have

$$f^{k+s} a - f^k a \circ f^s - f^{k-1/2} b \circ f^{s+\varepsilon'} - f^{k-1/2} b_1 \circ f^{s+\varepsilon''} = f^k r + f^{k-1/2} r_1 \, .$$

Since $b, b_1 \in \Sigma^{(-2l,-2(k-1/2-l),0)}$ from $\tilde{\mu}(d) \geq 0$, one would obtain (2) by applying (4.51) with $Op \, f^k a$, $Op \, f^{k-1/2} b$, and $Op \, f^{k-1/2} b_1$ respectively for A and using Lemma 4.1–(1) to estimate the terms $\|A_{k-1/2} f^s u_t\|$ and $\|A_{k-1/2-l} u_t\|$. This completes the proof of (2).

(3)   The proof is similar to that of (2). From Lemma 4.18–(4), there exists $b \in \Sigma^{d-(0,0,1)}$ and $r \in \Sigma^{(0,1,0)}$ such that $f^s \, {}^f a - {}^f a \circ f^s - {}^f b \circ f^s = t^{-2l} r$. From $\tilde{\mu}(d) \geq 0$ and $k - l = \tilde{\mu}(d) + \nu(d)$ we have that $a, b \in \Sigma^{(-2l,-2(k-l),0)}$. Then, Lemma 4.13–(2) and Corollary 4.13–(1) shows (3). The proof is complete.

Now we recall the definition of a norm $\|\cdot\|^{(n)}$ on $\mathcal{S}_T$.

Definition 4.20. For $s = 0, \cdots, 6$ and $n \geq 1$ we define norms on $\mathcal{S}_T$:

$$\|u\|_{(s)}^{(n)} = \sum_{i+2j \leq s} n^{3/2 - i/2 - j} \|A_{i/2} D_t^j u\|^{(n+3-i/2-j)} \, ,$$
$$\|u\|_{(s,n)} = \sum_{\substack{i+2j \leq s \\ i,j \in Z_+}} \|A_{i/2} D_t^j u\|^{(n)} \, .$$

It is easy to see that

(4.54) $\|u\|_{(s)}^{(n)} \leq \|u\|_{(s')}^{(n)}, \ \|u\|_{(s,n)} \leq \|u\|_{(s',n)}, \ \text{if} \ s < s'$,

(4.55) $\|u\|^{(n)} \leq T^{\alpha} \|u\|^{(n+\alpha)}, \ \|u\|_{(s)}^{(n)} \leq \left(\dfrac{1+\alpha}{\alpha}\right)^{-3/2} T^{\alpha} \|u\|_{(s)}^{(n+\alpha)}$,

$\|u\|_{(s,n)} \leq \|u\|_{(s,n+\alpha)}$, for $\alpha > 0$.

Lemma 4.21. (1) *Let* $A \in \mathscr{B}^{k,l,N}$. *Then there exists* $C > 0$ *such that*

$$\|Au\|^{(n)} \leq C(\|A_k u\|^{(n+l(1-1/(10h)))} + \|A_{k-l} u\|^{(n+2l)} + \|u\|^{(n+2l)})\ .$$

(2) *Let* $A \in \mathscr{B}^{k,l,N}$ *with* $s/2 + k \leq N$. *Then there exists* $C > 0$ *such that*

$$\|Au\|_{(s)}^{(n)} \leq C(\|A_k u\|_{(s)}^{(n+l(1-1/(10h)))} + \|A_{k-l} u\|_{(s)}^{(n+2l)} + \|u\|_{(s)}^{(n+2l)})\ ,$$

$$\|Au\|_{(s,n)} \leq C \sum_{i=0}^{s} (\|A_k u\|_{(s-i,n+l(1-1/(10h))+i/2)}$$
$$+ \|A_{k-l} u\|_{(s-i,n+2l+i/2)} + \|u\|_{(s-i,n+2l+i/2)})\ .$$

(3) *Let* $\alpha \in \mathbf{R}$. *Then there exists* $C > 0$ *such that for* $n \geq \max(0, \alpha) + 1$,

$$\|t^{\alpha} u\|_{(s,n)} \leq C \|u\|_{(s,n-\alpha)}\ ,$$
$$\|t^{\alpha} u\|_{(s)}^{(n)} \leq C \|u\|_{(s)}^{(n-\alpha)}\ .$$

(4) *Let* $a \in B_T$. *Then there exists* $C > 0$ *such that*

$$\|au\|_{(s,n)} \leq C \|u\|_{(s,n)}\ .$$

Proof. (1) We may assume that $A = Op({}^f a + f^N b)$, $a \in \Sigma^d$, $b \in \Sigma^e$ with $d, e \in \varDelta_{k,1}$. Since $\sigma(\tilde{d}) = k - l$, $\tilde{\mu}(\tilde{d}) \geq 0$, $t^{-1} \langle t^{e_h} \xi \rangle^{1/2} \geq \langle \xi \rangle^{1/2}$, and $\tilde{d}_1 \geq -2l$ when $\tilde{d} \in \varDelta_{k,1}$, we see $t^{\tilde{d}_1} \langle t^{e_h} \xi \rangle^{-\tilde{d}_1/2+\tilde{\mu}(\tilde{d})} \langle \xi \rangle^{\nu(\tilde{d})} \leq (t^{-1} \langle t^{e_h} \xi \rangle^{1/2})^{2l} \langle \xi \rangle^{k-l}$. So $a, b \in \Sigma^{(-2l,-2(k-l),0)}$. Thus (1) is an immediate consequence of Lemma 4.19–(2), –(3). (2) We may assume $A = Op({}^f a + f^N b)$ with $a, b \in \Sigma^{(0,-2k,0)}$ from the proof of (1). We set

(4.56) $A^{(l)} = Op(D_t^l a)$ if $A = Op\ a$, by definition.

Then by using Lemma 4.18–(1),

$$A_{i/2} D_t^j A = \sum_{m \leq j} A_{i/2} A^{(j-m)} D_t^m, \quad \text{and}$$
$$A_{i/2} A^{(j-l)} \in \mathscr{B}^{k+(i+j-m)/2,(j-m)/2+l,N}\ .$$

Now by using this, (1), and Lemma 4.2–(2), (2) is proved in a straightforward way.
(3) A simple consequence of Leibniz rule and (1). This completes the proof.

Lemma 4.22. *Let* $A \in \mathscr{A}^{k,l,N}$ (*resp.* $\mathscr{B}^{k,l,N}$). *Then if* $N \geq k + j/2$, $A^{(j)} \in \mathscr{A}^{k+j/2,l+j/2,N}$ (*resp.* $\mathscr{B}^{k+j/2,l+j/2,N}$). *In particular* $[D_t, A] \in \mathscr{A}^{k+1/2,l+1/2,N}$ (*resp.*

$\mathcal{B}^{k+1/2, l+1/2, N}$) *when* $N \geq k+1/2$.

This lemma is easy.

§5. The Basic Estimates

Let $T^{(0)} \in (0, 1)$, $a \in B_{h, T}(0)$, and $\tilde{\sigma} \in t^{\varepsilon} \Sigma_{1/2}^{(0, -1, 1)}(T^{(0)}, h)$ with $\varepsilon \in (0, 1)$ satisfying

(5.1) $\qquad \partial_{\xi} \tilde{\sigma} \in t^{\varepsilon} \Sigma_{1/2}^{0}, \partial_{x} \tilde{\sigma} \in t^{\varepsilon} \Sigma_{1/2}^{(0, -1, 1)}, \partial_{t} \tilde{\sigma} \in t^{\varepsilon-1} \Sigma_{1/2}^{(0, -1, 1)}$,

(5.2) $\quad \inf |\operatorname{Im} a| > 0$,

and set

(5.3) $\qquad\qquad L = D_{t} - Op(af\xi + \sigma), \sigma = {}^{f}\tilde{\sigma}$.

Then the following proposition is fundamental for the proof of Proposition 2.3.

Proposition 5.1. *There exists* $C > 0$ *and* $T_{0} \leq T^{(0)}$ *such that for* $T \in (0, T_{0})$ *and* $u \in \mathcal{S}_{T}, n \geq 1$,

$$\|Lu\|^{(n)} \geq C \{n^{-1/2}(\|D_{t} u\|^{(n)} + \|A_{1} u\|^{(n)}) + n^{1/2} \|u\|^{(n+1)}\} .$$

Proof. Let $T \in (0, T^{(0)})$ and $u \in \mathcal{S}_{T}$, and set

(5.4) $\qquad A_{1} = Op (\operatorname{Re} (af\xi + \sigma)), A_{2} = Op (\operatorname{Im} (af\xi + \sigma))$,
$\qquad\qquad L_{1} = D_{t} - A_{1}, L_{2} = A_{2} + nt^{-1}$.

Then, since $t^{-n} Lu = L_{1} v - iL_{2} v$, setting $v = t^{-n} u, n \geq 1$ we have

$$(\|Lu\|^{(n)})^{2} = \|L_{1} v\|^{2} + J(v) + \|L_{2} v\|^{2} ,$$
$$J(v) = -2 \operatorname{Im} (L_{1} v, L_{2} v) ,$$

where $\| \cdot \| = \| \cdot \|_{L^{2}((0, T) \times R)}$ and $(,) = (,)_{L^{2}((0, T) \times R)}$. Let us show that

(5.5) there exists $T_{0} \in (0, T^{(0)})$ such that for any $T \in (0, T_{0})$, $n \geq 1$, $v \in \mathcal{S}_{T}$,
$\quad J(v) \geq n/2 \|t^{-1} v\|^{2} - 1/5 (\|L_{1} v\|^{2} + \|L_{2} v\|^{2})$.

To do so we first note that

(5.6) $2i \operatorname{Im} (L_{1} v, L_{2} v) = (v, [L_{1}, L_{2}] v) + ((L_{1} - L_{1}^{*}) v, L_{2} v) + ((L_{2}^{*} - L_{2}) v, L_{1} v)$.

Next noting

$$L_{1}^{*} - L_{1} = A_{1} - A_{1}^{*}, L_{2}^{*} - L_{2} = A_{2}^{*} - A_{2} ,$$
$$[L_{1}, L_{2}] = [D_{t}, A_{2}] - [A_{1}, A_{2}] + int^{-2} ,$$

and using Lemma 4.18–(ii) and –(iv) and (5.1) we see, in view of $\varepsilon \in (0, 1)$,

(5.7) $L_i^* - L_i \in \mathcal{A}^{1/2,1/2} + Op\,\Sigma^0,\ [L_1, L_2] - \mathrm{int}^{-2} \in t^{\varepsilon-1}\,\mathcal{A}^{1,1}$.

From the latter of (5.7),

(5.8) $S_n \equiv i\,(2\,\mathrm{Im}(L_1\,v,\,L_2\,v) - n\,\|t^{-1}\,v\|^2)$
$= (t^{\varepsilon-1}\,v,\,Av) + ((L_1 - L_1^*)\,v,\,L_2\,v) + ((L_2^* - L_2)\,v,\,L_1\,v)$
        for some $A \in \mathcal{A}^{1,1}$.

So, estimating $\|Av\|$, $\|(L_i - L_i^*)\,v\|$ by (5.7) and Lemma 4.29–(1), –(2), and using $\|v\| \leq \|t^{\varepsilon-1}\,v\|$ we see that there exists $C_1 > 0$ such that for $T \in (0, T^{(0)})$, $n \geq 1$, $v \in \mathcal{S}_T$,

(5.9) $|S_n| \leq C_1(\|A_1\,v\|\,\|t^{\varepsilon-1}\,v\| + \|A_{1/2}\,v\|^2 + \|v\|\,\|t^{\varepsilon-1}\,v\|) + \sum_{i=1}^{2} \|L_i\,v\|^2/10$.

On the other hand, (5.2) and Lemma 4.1–(i) show that there exist C_2, $C_3 > 0$ such that for $T \in (0, T^{(0)})$, $v \in \mathcal{S}_T$, $t \in (0, T)$,

$$\|\mathrm{Im}\,a(t, \cdot)\,f D_x\,v(t, \cdot)\|_0 \geq C_2\,\|A_1\,v(t, \cdot)\|_0 - C_3\,\|v(t, \cdot)\|_0$$
$$\text{where}\quad \|\cdot\|_0 = \|\cdot\|_{L^2(R_x)} \ .$$

Thus, since $\tilde{\sigma} \in t^{\varepsilon}\,\Sigma^{(0,-1,1)}$ with $\varepsilon > 0$ from Lemma 4.29–(1), there exists $T_1 \in (0, T^{(0)})$ and $C_4 > 0$ such that for $T \in (0, T_1)$, $v \in \mathcal{S}_T$, $t \in (0, T)$,

$$\|(A_2\,v)\,(t, \cdot)\|_0 \geq C_0/2\,\|A_1\,v(t, \cdot)\|_0 - C_4\,\|v(t, \cdot)\|_0$$

which means that there exists $C_5 > 0$ such that

(5.10)   for $T \in (0, T_1)$, $v \in \mathcal{S}_T$, $t \in (0, T)$,
$$\|A_1\,v\| \leq C_5(\|A_2\,v\| + \|v\|) \ .$$

Since $\|A_{1/2}\,v\|^2 = (A_{1/2}^*\,A_{1/2}\,v,\,v)$, and since by noting $\langle f\xi \rangle^{1/2} \in S_{1,0}^{1/2}(R)$ from (4.3), the asymptotic expansion the symbol of $A_{1/2}^*$ shows by breaking off after 2 terms, that

$$A_{1/2}^* = {}^f a(x, D) + b(x, D)\quad \text{with}\quad a \in S_{1,0}^{1/2}\quad \text{and}\quad b \in S_{1,1/2}^{-1/2}\,,$$

one can obtain using Lemma 4.2–(ii) and 4.1–(i) that there exists $C_6 > 0$ such that

(5.11) $\|A_{1/2}\,v\|^2 \leq C_6(\|A_1\,v\| + \|v\|)\,\|v\|$.

From (5.9) $\sim$ (5.11) it follows by using $\|v\| \leq \|t^{\varepsilon-1}\,v\|$ that there exists $C_7 > 0$ such that for $T \in (0, T_1)$, $v \in \mathcal{S}_T$, $t \in (0, T)$, $n \geq 1$,

$$|S_n| - \sum_{i=1}^{2} \|L_i \, v\|/10 \le C_7 \, T^{\varepsilon}(\|A_2 \, v\| \, \|t^{-1} \, v\| + \|t^{-1} \, v\|^2) \, .$$

Now using this and

(5.12) $$\|A_2 \, v\| \le \|L_2 \, v\| + n \, \|t^{-1} \, v\|, \, v \in \mathcal{S}_T \, ,$$

we obtain that there exists $T_0 \in (0, T_1)$ such that for $T \in (0, T_0)$, $v \in \mathcal{S}_T$,

(5.13) $$|S_n| \le \sum_{i=1}^{2} \|L_i \, v\|^2/5 + \|t^{-1} \, v\|^2/2 \, .$$

This shows (5.5) which implies that for $T \in (0, T_0)$, $u \in \mathcal{S}_T$, $n \ge 1$,

(5.14) $$(\|Lu\|^{(n)})^2 \ge n \, \|t^{-1} \, v\|^2/2 + 4 \sum_{i=1}^{2} \|L_i \, v\|^2/5 \, .$$

Finally we observe that there exists $C_8 > 0$ such that for $T \in (0, T^{(0)})$, $u \in \mathcal{S}_T$, $n \ge 1$,

(5.15) $$\|D_t \, u\|^{(n)} \le \|L_1 \, v\| + C_8 \, \|A_1 \, v\| + n \, \|t^{-1} \, v\| \, .$$

(5.10), (5.12), (5.14), (5.15) show the desired inequlity. This completes the proof.

Let $L_i = D_t - Op \, (a_i f \xi + \sigma_i)$, $i = 1, 2, 3$ be three operators as in (5.3) for fixed ε. We define

(5.16) $$M'_n(u) = M_n(u, L_i) = \|L_1 \, L_2 \, u\|^{(n)} + \|L_2 \, L_1 \, u\|^{(n)} \, ,$$
$$M_n(u) = M_n(u, L_i) = \sum_{\sigma \in \mathcal{G}_3} \|L_{\sigma(1)} \, L_{\sigma(2)} \, L_{\sigma(3)} \, u\|^{(n)} \, ,$$

for $u \in \mathcal{S}_T$, $n \ge 1$ where $\mathfrak{S}_3$ denotes the symmetric group of degree 3.

Corollary 5.2. *There exists $C > 0$ and $T_0 \in (0, T^{(0)})$ such that for $n \ge 1$, $T \in (0, T_0)$, $u \in \mathcal{S}_T$.*

(i) $$n\|u\|^{(n+2)} + n^{-1/2} \sum_{i=1}^{2} (\|D_t \, L_i \, u\|^{(n)} + \|A_1 \, L_i \, u\|^{(n)}) + n^{-1} \sum_{k=0}^{2} \|A_{2-k} \, D_t^k \, u\|^{(n)}$$
$$\le CM'_n(u),$$

(ii) $$n^{3/2} \|u\|^{(n+3)} + n^{-1/2} \sum_{i \ne j} (\|D_t \, L_i \, L_j \, u\|^{(n)} + \|A_1 \, L_i \, L_j \, u\|^{(n)})$$
$$+ n^{-1} \sum_{i=1}^{3} \sum_{k=0}^{2} \|A_k \, D_t^{2-k} \, L_i \, u\|^{(n)} + n^{-2/3} \sum_{k=0}^{3} \|A_{3-k} \, D_t^k \, u\|^{(n)} \le CM_n(u)$$

Proof. For the sake of simplicity we denote by C or C_i a constant indentdepen of T, n, $u \in \mathcal{S}_T$. We only prove that Proposition 5.1 and (i) imply (ii). the similar way one can prove (i) assuming Proposition 5.1. Exept the inequality

(5.17) $$S \equiv n^{-3/2} \sum_{k=0}^{3} \|A_{3-k} \, D_t^k \, u\|^{(n)} \le CM_n(u) \, ,$$

the validity of the others immediately follows from the assumption. So, it suffices to prove only (5.17) assuming all other inequalities in (ii), and those in (i), Proposition 5.1, for any $T \in (0, T_1)$ with uniform constant C with some $T_1 \in (0, T^{(0)})$. Using the assumption and applying Corollary 4.3–(2) to $A_2 A_1$ we see that

$$(5.18) \qquad S \leq n^{-3/2} \{ \sum_{k=0}^{2} ||A_{2-k} D_t^{k+1} u|| + C_1 (||A_2 A_1 u||^{(n)} + ||u||^{(n)}) \}$$
$$\leq C_2 \{ n^{-1/2} (M_n'(D_t u) + M_n'(A_1 u)) + M_n(u) \} \quad \text{for} \quad T < T_1 .$$

We note that from Lemma 4.18–(3)–(iv) and (5.1)

$$[L_i, D_t] \in t^{\varepsilon-1} \mathcal{A}^{1,2}, [L_i, A_1] \in \mathcal{A}^{1,2} ,$$

and that from Lemmas 4.18–(3)–(i) and 4.22,

$$[L_i, [L_j, D_t]] \in t^{\varepsilon-1} \mathcal{A}^{3/2,1/2,3/2} + t^{\varepsilon-2} \mathcal{A}^{1,2} ,$$
$$[L_i, [L_j, A_1]] \in \mathcal{A}^{3/2,1/2,2} .$$

Therefore, since

$$[L_i L_j, D_t] = [L_i, [L_j, D_t]] + [L_j, D_t] L_i + [L_i, D_t] L_j ,$$
$$[L_i L_j, A_1] = [L_i, [L_j, A_1]] + [L_j, A_1] L_i + [L_i, A_1] L_j ,$$

we see by Lemma 4.21–(1),

$$(5.19) \qquad ||L_i L_j D_t u||^{(n)} \leq C_3 \{ ||A_{3/2} u||^{(n+3/2-\varepsilon)} + ||A_1 u||^{(n+2-\varepsilon)} + ||u||^{(n+2-\varepsilon)}$$
$$+ \sum_{l \in \{i,j\}} (||A_1 L_l u||^{(n+1-\varepsilon)} + ||L_l u||^{(n+1-\varepsilon)}) \} + ||D_t L_i L_j u||^{(n)}$$
$$\text{for} \quad T < T_1 ,$$

$$(5.20) \qquad ||L_i L_j A_1 u||^{(n)} \leq C_3 \{ ||A_{3/2} u||^{(n+1/2)} + ||A_1 u||^{(n+1)} + ||u||^{(n+1)}$$
$$+ \sum_{l \in \{i,j\}} (||A_1 L_l u||^{(n)} + ||L_l u||^{(n)}) \} + ||A_1 L_i L_j u||^{(n)}$$
$$\text{for} \quad T < T_1 .$$

Applying Lemma 4.1–(1) with $\varepsilon = t n^{-1}$ and using the assumption we see that

$$(5.21) \qquad ||A_{3/2} u||^{(n+3/2-\varepsilon)} + ||A_1 u||^{(n+2-\varepsilon)} + ||u||^{(n+2-\varepsilon)}$$
$$\leq C_4 T^\varepsilon (n^{-3/2} ||A_3 u||^{(n)} + M_n(u)) \quad \text{for} \quad T < T_1 .$$

(5.18)~(5.21) and the assumption imply that if $T < T_1$,

$$S \leq C_5 (T^\varepsilon n^{-3/2} ||A_3 u||^{(n)} + M_n(u)) .$$

This proves (5.17) for sufficiently small T. The proof is complete.

§6. Lemmas on the Commutation

We first study commutations involving in operators as in (5.3). We prepare three lemmas.

Lemma 6.1. *Let $a_i \in C^\infty(\boldsymbol{R}_y \times \boldsymbol{R}_\eta)$, $i=1, 2, 3$ such that for some $M>0$ and $\delta \in [0, 1)$,*

$$|\partial_y^\alpha \partial_\eta^\beta a_i(y, \eta)| \leq C_{\alpha\beta} \langle \eta \rangle^{M+\delta|\alpha|} \quad \text{for all } \alpha, \beta .$$

Then, we have the followings.

(i) *If a_2 is constant in η,*

$$Os - \iint e^{-iy\eta} a_1(y, \eta)\, a_2(y)\, dy d\eta$$

$$= \iint e^{-iy\eta}\Big(a_1(y, \eta)\, a_2(0) + D_\eta\, a_1(y, \eta) \int_0^1 (\partial_y a_2)\,(\theta y)\, d\theta\Big)\, dy d\eta .$$

(ii) *If a_1 and a_2 are constant in y,*

$$Os - \iint e^{-iy\eta} a_1(\eta)\, a_2(\eta)\, a_3(y, \eta)\, dy d\eta$$

$$= OS - \iint e^{-iy\eta}\Big(a_1(0)\, a_2(0)\, a_3(y, \eta) + \int_0^1 (\partial_\eta a_1)\,(\theta\eta)\, d\theta \cdot a_2(0)\, D_y\, a_3(y, \eta)$$

$$+ a_1(0) \int_0^1 (\partial_\eta a_2)\,(\theta\eta)\, d\theta \cdot D_y\, a_3(y, \eta) + \prod_{i=1}^{2} \int_0^1 (\partial_\eta a_i)\,(\theta\eta)\, d\theta \cdot D_y^2\, a_3(y, \eta)\Big)\, dy d\eta .$$

This is an immediate consequence of the integration by parts. We omit the proof.

Lemma 6.2. *For $d^1, \cdots, d^k \in \boldsymbol{R}^3$ with $d_1^j \leq 0$ for any j, we denote by S $(d^1, \cdots, d^k)$ the set all families $\mathcal{F} = \{\mathcal{F}_{\alpha\beta}\}_{\alpha \in \boldsymbol{Z}_+^3, \beta \in \boldsymbol{Z}_+^2}$ of subsets $\mathcal{F}_{\alpha\beta} = \{b_j^{\alpha\beta}$ $(t, x, y, \xi, \eta) \in C^\infty(\boldsymbol{R}_T^5), j=1, \cdots, n(\alpha, \beta)\} \cup \{\lambda^{i,j} \in \boldsymbol{Z}_+^3, \mu^{i,j} \in \boldsymbol{Z}_+^2, i=1, \cdots, k,$ $j=1, \cdots, n(\alpha, \beta); \sum_{i=1}^{k} \lambda^{i,j} = \alpha, \sum_{i=1}^{k} \mu^{i,j} = \beta\}$ where $n(\alpha, \beta) \in \boldsymbol{N}$ such that for any $\tau \in \boldsymbol{Z}_+^2$ there exists $C_{\alpha\beta\tau} > 0$ such that for $j=1, \cdots, n(\alpha, \beta)$*

$$(6.1) \qquad |\partial_{y,\eta}^\gamma b_j^{\alpha\beta}(t, x, y, \xi, \eta)| \leq C_{\alpha\beta\gamma}\, t^{\sum_{i=1}^{k} d_1^i - \alpha_1} \cdot \sum_{\sum_{i=1}^{k} \tau^i = \tau} \prod_{i=1}^{k}$$

$$\{\sup_{0 \leq \theta \leq 1} \Phi_t^{J(d^i - \lambda^{i,j})_3 - \mu^{i,j}_2 - \gamma^i_2}(x, \xi + \theta(\eta - \xi))$$

$$\times \sup_{0 \leq \theta \leq 1} \varphi^{J(d^i - \lambda^{i,j})_2 - \mu^{i,j}_1 - \gamma^i_1}(x, \xi + \theta(\eta - \xi))\}$$

where Φ_t, φ are functions in Fact 4.8.

We denote by $|\mathcal{F}(\alpha, \beta)|_\gamma$ the infimum of constants $C_{\alpha\beta\gamma}$ satisfying (6.1) for $\mathcal{F} = \{\mathcal{F}_{\alpha\beta}\}$. We denote by $F(d^1, \cdots, d^k)$ for $d^1, \cdots, d^k$ as above the set of all $b(t, x, y,$

$\xi, \eta) \in C^{\infty}(\mathbf{R}_T^5)$ *such that there exists* $\mathscr{F} \in S(d^1, \cdots, d^k)$ *satisfying with the notations as above*

(6.2) $\partial_{t,x,\xi}^\alpha \partial_{y,\eta}^\beta b(t, x, y, \xi, \eta) = \sum_{j=1}^{n(\alpha, \beta)} b_j^{\alpha\beta}(t, x, y, \xi, \eta)$ *for any* α, β .

Then we have the followings.

(1)   *Let* $b_1 \in F(d^1, \cdots, d^k)$, $b_2 \in F(e^1, \cdots, e^l)$, *and set* $b = b_1 \cdot b_2$. *Let* $\mathscr{F}^{(1)} \in S(d^1, \cdots, d^k)$, $\mathscr{F}^{(2)} \in S(e^1, \cdots, e^l)$ *be families for* b_1, b_2 *as above respectively. Then* $b \in F(d^1, \cdots, d^k, e^1, \cdots, e^l)$ *and one can find a family* $\mathscr{F}$ *for* b *as above such that*

$$|\mathscr{F}(\alpha, \beta)|_\gamma \leq 2^{|\alpha|+|\beta|+|\gamma|} \sup_{\substack{\alpha^1+\alpha^2=\alpha \\ \beta^1+\beta^2=\beta \\ \gamma^1+\gamma^2=\gamma}} |\mathscr{F}^{(1)}(\alpha^1, \beta^1)|_{\gamma^1} \cdot |\mathscr{F}^{(2)}(\alpha^2, \beta^2)|_{\gamma^2} .$$

(2)   *Let* $d^1, \cdots, d^k$ *as above be given. Then there exist* $C_\alpha > 0$ *and* $L_\alpha \in \mathbf{N}$ *for any* $\alpha \in \mathbf{Z}_+^3$ *such that if* $b \in F(d^1, \cdots, d^k)$, *if* $\mathscr{F} \in S(d^1, \cdots, d^k)$ *is a family for* b *as above, and if*

$$b_0(t, x, \xi) = OS - \iint e^{-i(y-x)(\eta-\xi)} b(t, x, y, \xi, \eta) \, dy d\eta ,$$

we have

(6.3) $|\partial^\alpha b_0(t, x, \xi)| \leq C_\alpha (\sup_{\substack{\lambda+(0,\mu)=\alpha \\ |\nu| \leq L_\alpha}} n(\lambda, \mu) |\mathscr{F}_{(\lambda,\mu)}|_\nu) \, t^{\sum_{i=1}^k d_1^i - \alpha_1}$

$\times (\Phi_t^{\sum_{i=1}^k J(d^i)_3 + J(-\alpha)_3}) (x, \xi) \cdot (\varphi^{\sum_{i=1}^k J(d^i)_2 + J(-\alpha)_2}) (x, \xi)$

for any $\alpha \in \mathbf{Z}_+^3$.

Indeed, (1) is a simple concequence of Leibniz rule and (2) follows by noting

$$\partial^\alpha b_0(t, x, \xi) = \sum_{\lambda+(0,\mu)=\alpha} \binom{\alpha}{\lambda} OS - \iint e^{-i(y-x)(\eta-\xi)}$$

$$\times \partial_{t,x,\xi}^\lambda \partial_{y,\eta}^\mu b(t, x, y, \xi, \eta) \, dy d\eta ,$$

by using (6.2) ,and by applying Lemma 4.7 with B in Fact 4.8 and $l = k$.

Let us define the topology in Σ^d by the semi-norms

$$|a|_\alpha^{(d)} = \sup_{t,x,\xi} |\partial^\alpha a| \, \tilde{\psi}_1^{-d_1+\alpha_1} (\langle \xi \rangle^{-1/2})^{-d_2+\alpha_2} \cdot \psi_{3,1/2}^{-d_3+\alpha_3}$$ for $\alpha \in \mathbf{Z}_+^3$.

Then Σ^d becomes a Frechet space.

Lemma 6.3.   *Let* $a_1 \in \Sigma^d$, $a_2 \in \Sigma^e$, *and* $\tau_i \in \Sigma^{(0,-2,0)}$, $i = 1, 2$ *satisfying* $\partial_x \tau_i \in \Sigma^{(0,-2,0)}$ *and* $\partial_\xi \tau_i \in \Sigma^0$. *Set* $\pi_i = \tau - \tau_i(t, x, \xi)$ *and for* $\theta \in [0, 1]$, *define*

$$I_{1,\theta} = I_{1;\theta}^{\pi_1}[a_1, a_2] = OS - \iint e^{-iy\eta} a_1(t, x, \xi+\theta\eta) \pi_1(t, x, \tau, \xi+\theta\eta) a_2(t, x+y, \xi)$$

$dy d\eta ,$

$$I_{2,\theta} = I_{2,\theta}^{\pi_1,\pi_2}[a_1, a_2] = OS - \iint e^{-iy\eta} a_1(t, x, \xi+\theta\eta) (\pi_1 \pi_2) (t, x, \tau, \xi+\theta\eta)$$
$$\times a_2(t, x+y, \xi) \, dy d\eta \,,$$

$$I_{3,\theta} = I_{3,\theta}^{\pi_1}[a_1, a_2] = OS - \iint e^{-iy\eta} a_1(t, x, \xi+\theta\eta) \pi_1(t, x+y, \tau, \xi)$$
$$\times a_2(t, x+y, \xi) \, dy d\eta \,.$$

Then, for $i=1$ or 3 there exists $b_{j,\theta}(t, x, \xi) \in C^\infty([0, 1]_\theta \times \boldsymbol{R}_T^3)$, $j=1, 2$ such that $\{b_{j,\theta}\}_\theta$ and $\{b_{2,\theta}\}_\theta$ are bounded in Σ^{d+e} and $\Sigma^{d+e-(0,1,0)}$ respectively and satisfying $I_{i,\theta}=b_{1,\theta}\,\pi_1+b_{2,\theta}$, and for $i=2$ there exist $b_{j,\theta}(t, x, \xi) \in C^\infty([0, 1]_\theta \times \boldsymbol{R}_T^3)$, $j=1, \cdots, 4$ such that $\{b_{1,\theta}\}_\theta$, $\{b_{2,\theta}\}_\theta \cup \{b_{3,\theta}\}_\theta$, and $\{b_{4,\theta}\}_\theta$ are bounded in Σ^{d+e}, $\Sigma^{d+e-(0,1,0)}$, and $\Sigma^{d+e-(0,2,0)}$ respectively and satisfying

$$I_{2,\theta} = b_{1,\theta}\,\pi_1\,\pi_2 + \sum_{j=1}^2 b_{1+j,\theta}\,\pi_j + b_{4,\theta} \,.$$

Proof. Applying Lemma 6.1—(ii) for $i=1, 2$ and —(i) for $i=3$ we obtain

$$I_{1,\theta} = OS - \iint e^{-iy\eta} a_1(t, x, \xi+\theta\eta) (\pi_1(t, x, \tau, \xi) a_2(t, x+y, \xi)$$
$$-\tilde{\tau}_1(\theta, t, x, \xi, \eta) D_x\, a_2(t, x+y, \xi)) \, dy d\eta \,,$$

$$I_{2,\theta} = OS - \iint e^{-iy\eta} a_1(t, x, \xi+\theta\eta) ((\pi_1 \pi_2) (t, x, \tau, \xi) a_2(t, x+y, \xi)$$
$$- \sum_{\{k,j\}=\{1,2\}} \tilde{\tau}_k(\theta, t, x, \xi, \eta) \pi_j(t, x, \tau, \xi) D_x\, a_2(t, x+y, \xi)$$
$$+(\widetilde{\tau_1 \tau_2}) (\theta, x, \xi, \eta) D_x^2\, a_2(t, x+y, \xi)) \, dy d\eta \,,$$

$$I_{3,\theta} = OS - \iint e^{-iy\eta} (a_1(t, x, \xi+\theta\eta) \pi_1(t, x, \tau, \xi) - \theta\partial_\xi\, a_1(t, x, \xi+\theta\eta)$$
$$\times \tilde{\tilde{\tau}}_1(t, x, y, \xi)) a_2(t, x+y, \xi) \, dy d\eta \,,$$

where, by definition, for $a \in \Sigma$,

$$\tilde{a}(\theta, t, x, \xi, \eta) = \theta \int_0^1 \partial_\xi\, a(t, x, \xi+\theta s\eta) \, ds, \; \tilde{\tilde{a}}(t, x, y, \xi) = \int_0^1 \partial_x\, a(t, x+sy, \xi) \, ds \,.$$

Now the desired results follow from Lemma 6.2 in view of the assumption on a_i, τ_i.

For $q(t, x, \tau, \xi) \in C^\infty(\boldsymbol{R}_T^4)$ we define $^f q \in C^\infty(\boldsymbol{R}_T^4)$ by

$$^f q(t, x, \tau, \xi) = q(t, x, \tau, f(x)\,\xi) \,.$$

Now, we are ready to prove the following two commutation lemmas.

Lemma 6.4. *Let $L_i = Op\, \pi_i$, $i=1, 2, 3$ with $\pi_i = \tau - \tau_i$, $\tau_i = {}^f\tilde{\tau}_i$, $\tilde{\tau}_i \in \Sigma_{1/2}^{(0,-1,1)}$ satisfying with $\varepsilon \in (0, 1)$,*

$$(6.4) \qquad \partial_x\, \tilde{\tau}_i \in \Sigma_{1/2}^{(0,-1,1)}, \; \partial_\xi\, \tilde{\tau}_i \in \Sigma_{1/2}^0, \; \partial_t\, \tilde{\tau}_i \in t^{\varepsilon-1}\Sigma_{1/2}^{(0,-1,1)} \,.$$

Then, for $A = Op\, a,\ a \in \Sigma,$

(i) $AL_i - Op(a\pi_i) \in$
$$\begin{cases} \mathcal{A}^{1,N} & \text{if } a = Op\, \tau_j \text{ for some } j,\ N \geq 1, \\ \mathcal{A}^{k+1/2,l,N} & \text{if } A \in \mathcal{A}^{k,l,N} \text{ and } N \geq k+1/2, \\ \mathcal{B}^{k+1/2,l,N} & \text{if } A \in \mathcal{B}^{k,l,N} \text{ and } N \geq k+1/2, \end{cases}$$

(ii) $L_i L_j - Op(\pi_i \pi_j) \in t^{\varepsilon-1}\, \mathcal{A}^{1,N}$ *for* $N \geq 1,$

(iii) $L_i A - Op(a\pi_i) \in$
$$\begin{cases} \mathcal{A}^{k+1/2,l+1/2,N} & \text{if } A \in \mathcal{A}^{k,l,N} \text{ and } N \geq k+1/2, \\ \mathcal{B}^{k+1/2,l+1/2,N} & \text{if } a \in \mathcal{B}^{k,l,N} \text{ and } N \geq k+1/2, \end{cases}$$

(iv) $A\, Op(\pi_i \pi_j) - Op(a\pi_i \pi_j) = A_i L_i + A_j L_j + A_{ij}$ *with*

(1) $A_i,\ A_j \in f^{1/2}\, \mathcal{A}^{k+1/2,l,N-1/2},\ A_{ij} \in \mathcal{A}^{k+1,l,N}$ *if* $A \in \mathcal{A}^{k,l,N},\ N \geq k+3/2,$

(2) $A_i,\ A_j \in \mathcal{B}^{k+1/2,l,N},\ A_{ij} \in \mathcal{B}^{k+1,l,N}$ *if* $A \in \mathcal{B}^{k,l,N},\ N \geq k+1,$

(3) $A_i,\ A_j \in f^{1/2}\, \mathcal{A}^{1,N},\ A_{ij} \in \mathcal{A}^{3/2,N}$ *if* $A = Op\, \tau_l$ *for some* $l,\ N \geq 3/2,$

(v) $AL_i L_j - Op(a\pi_i \pi_j) = A_i L_i + A_j L_j + A_{ij}$ *with*

(1) $A_i,\ A_j \in f^{1/2}\, \mathcal{A}^{k+1/2,l,N-1/2},\ A_{ij} \in t^{\varepsilon-1}\, \mathcal{A}^{k+1,l,N}$ *if*
$A \in \mathcal{A}^{k,l,N},\ N \geq k+3/2,$

(2) $A_i,\ A_j \in \mathcal{B}^{k+1/2,l,N},\ A_{ij} \in t^{\varepsilon-1}\, \mathcal{B}^{k+1,l,N}$ *if* $\mathcal{A} \in \mathcal{B}^{k,l,N},\ N \geq k+1,$

(vi) $L_i L_j L_k - Op\, \pi = \sum_{k=1}^{3} A_1 L_1 + A_{ijk}$

where $\pi = \prod_{i=1}^{3} \pi_i,$ *with* $A_l \in t^{\varepsilon-1}\, \mathcal{A}^{1,3}$ *and* $A_{ijk} \in t^{\varepsilon-1}\, \mathcal{A}^{3/2,1/2,3}.$

To state the next lemma we make a definition.

Definition 6.5. Let $L_i,\ i = 1, 2, 3,$ be three operators as in Lemma 6.4. We define

$\mathcal{C} = \mathcal{C}(L_i) := $ the set of all operators on $\mathcal{S}_T$ of the form:

$$\sum_{i \neq j} A_{ij} L_i L_j + \sum_{i=1}^{3} A_i L_i \quad \text{with } A_{ij} \in Op\, {}^f \Sigma_{1/2}^{(-1,0,0)} + f^{1/2}\, \mathcal{A}^{1/2,1/2,3},$$
$$A_i \in Op\, {}^f \Sigma_{1/2}^{(-2,0,0)} + f^{1/2}\, \mathcal{A}^{1,1,3} :$$

$F := F(L_i) = $ the set of all symbols q with $Op\, q \in \mathcal{C}(L_i);$

$E := $ the set of symbols of the form

$$\sum_{k=0}^{2} a_k \tau^k \quad \text{with}$$
$$a_2 \in t^{-1}({}^f \Sigma^{(0,1,0)} + f^{3/2}\, \Sigma^{(0,1,0)}),\ a_1 \in t^{-1}\, B^{1/2,1/2,3/2},$$
$$a_0 \in \sum_{k=0}^{\min(2,j)} \sum_{l=0}^{3} \sum_{j=0}^{3-l} t^{-(k/2+3-l-j)}\, B^{(l+j)/2,1/2,3/2};$$
$$\mathcal{C}_0 := Op\, E.$$

Lemma 6.6. *Let* $L_i,\ \tau_i,\ \tilde{\tau}_i,\ \pi_i,\ i = 1, 2, 3$ *be as in Lemma 6.4 and set*
$\pi = \prod_{i=1}^{3} \pi_i$ *and let* $\varphi \in \Sigma_{1/2}^{0}.$

(i) $\pi \circ {}^f\varphi - \pi \cdot {}^f\varphi \in F + E.$

(ii) *Let* $q(t, x, \tau, \xi) = \sum_{k+l \leq 3} a_{kl}(t, x)\, \xi^l \tau^k,\ a_{kl} \in B_{h,T},\ a_{30} = 1.$ *Assume that* π

$(t, x, \tau, \xi) = {}^f q(t, x, \tau, \xi)$ when $(t, x, \xi) \in \text{supp } {}^f \varphi$. Then it follows that $\pi^f \varphi - {}^f \varphi \circ {}^f q \in F + E$.

Before the proof of these lemmas we note

Fact 6.7. (1) If $a \in \Sigma^d_{1/2}$ (resp. Σ^d) with $\mu_{1/2}(d) \geq 0$ and $d_3 \geq 0$ (resp. $\mu(d) \geq 0$ and $\nu(d) \geq 0$), $\partial^j_\xi {}^f a \in f^{j/2} \Sigma^{J_{1/2}(d) + (0,j,0)}$ (resp. $\partial^j_\xi {}^f a \in f^{j/2} \Sigma^{d + (0,j,0)}$). This follows from Fact 4.6—4 and —7, (4.3), and (4.6). (See the proof of Corollary 4.12—(i).)

(2) Let $a \in A^{k,l,N}$ (resp. $B^{k,l,N}$). Then $\partial_\xi {}^j a$ is a sum of the symbols of the form: $a_1 + f^N a_2$; $a_1 \in f^{j/2} \Sigma^{J_{1/2}(d) + (0,j,0)} \cap f^j {}^f \Sigma^{d-}_{1/2}{}^{(0,0,j)}$ (resp. $f^{j/2} \Sigma^{d + (0,j,0)} \cap f^j {}^f \Sigma^{d-(0,0,j)}$), $a_2 \in \Sigma^{e+(0,j,0)}$ with $d \in \Delta^{k,l}$ (resp. $\Delta_{k,l}$), $e \in \Delta_{k,l}$.

(3) Let τ_i be symbols in Lemma 6.4. Then from the proof of Corollary 4. 12—(i),

$$\partial^j_\xi \tau_i \in f^{(j+1)/2} \Sigma^{(0,j-1,0)} \cap f^j {}^f \Sigma^{(1-j)(0,0,1)}_{1/2} \cap f^j {}^f \Sigma^{(1-j)(0,0,1)}$$

for $j \neq 0$. τ_i as well as $\tilde{\tau}_i$ also satisfies (6.4), and

$$f^\theta \partial^j_x \tau_i \in f \Sigma^{(0,-j,1)}_{1/2} \cap f \Sigma^{(0,-j-1,0)}$$

for $j \neq 0$, $\theta > 0$. This follows from (4.16).

Proof of Lemma 6.4. (i) follows from Lemma 4.18—(3)—(i) and —(iv). (ii) follows from (i) and the assumption on τ_i. (iii) follows from Lemmas 4.18— (3)—(i) and 4.22. Let us show (1) of (iv). We have

$$(6.5) \quad a \circ (\pi_i \pi_j) - a\pi_i \pi_j = \sum_{l=1}^{M-1} l!^{-1} \partial^l_\xi a(t, x, \xi) D^l_x(\pi_i \pi_j) (t. x. \tau, \xi)$$

$$+ (2\pi)^{-1}(M-1)!^{-1} \int_0^1 (1-\theta)^{M-1}(OS - \iint e^{-iy\eta} \partial^M_\xi a(t, x, \xi+\theta\eta)$$

$$\times D^M_x(\pi_i \pi_j) (t, x+y, \tau, \xi) \, dy d\eta) \, d\theta .$$

From Fact 6.7—(2) and —(3) the summation in (6.5) takes the form:

$$(6.6) \quad b_i \pi_i + b_j \pi_j + b_{ij} \text{ with } b_i, b_j \in f^{1/2} \mathcal{A}^{k+1/2,l,N-1/2}, b_{ij} \in \mathcal{A}^{k+1,l,N} .$$

Since $b_i = f^{1/2} c_i$ for some $c_i \in \mathcal{A}^{k+1/2,l,N-1/2}$, the summation defines the operator in the desired form by (i). The oscillatory integrals in the integral terms are sums of terms with the notation in Lemma 6.3:

$$r_\theta[\partial^M_\xi a, D^l_x \tau_i D^{M-l}_x \tau_j], 1 \leq l \leq M; I^{\tau_k}_{3,\theta}[\partial^M_\xi a, D^M_x \tau_l], \{k, l\} = \{i, j\} .$$

Thus from Lemma 4.9 and 6.3, the integral term takes the form (6.6) when $M \geq 2N$ so that this term defines the operator in the desired form (i). (2) and (3) can be shown in the similar way.

In view of the equality

$$AL_i\, L_j - Op\,(a\pi_i\, \pi_j) = A\,(L_i\, L_j - Op\,(\pi_i\, \pi_j)) + Op\,(a\circ(\pi_i\, \pi_j - a\pi_i\, \pi_j))\,,$$

(v) follows immediately from (ii), (iv), Lemma 4.18—(1), and the fact that $A^{k,l,N} \subset B^{k,l,N}$ in general.

To see (iv) we note that

$$L_i\, L_j\, L_k - Op\,\pi = (L_j\, L_k - Op\,(\pi_j\, \pi_k))\, L_i + [L_i,\, L_j\, L_k - Op\,(\pi_j\, \pi_k)]$$
$$+ [D_t,\, Op\,(\pi_j\, \pi_k)] - (Op\,\tau_i \cdot Op\,(\pi_j\, \pi_k) - Op\,(\tau_i\, \pi_j\, \pi_k))\,.$$

The first and last terms have the desired forms respectively from (ii) and (iv). From (i) and (ii) we see

(6.7) $[L_i, A] \in \mathcal{A}^{k+1/2, l+1/2, N}$ if $A \in \mathcal{A}^{k,l,N}$ and $N \geq k+1/2$.

So the second term has the desired form in view of (ii) and that in general

(6.8) $t^{-1}\, \mathcal{A}^{k,l,N} \subset \mathcal{A}^{k+1/2, l+1/2, N}$ if $N \geq k+1/2$.

Finally, the symbol of the third term equals to $-D_t\, \tau_j \cdot \pi_k - D_t\, \tau_k \cdot \pi_j$. Hence, one can easily see, from the fact that $\partial_t\, \tau_i \in t^{\varepsilon-1}\, \mathcal{A}^{1,N}$ for $N \geq 1$ and (i), that this term the desired form. This completes the proof.

Remark 6.8. The presence of the factor $f^{1/2}$ for A_i in (v)—(i) in Lemma 6.4 is important.

Proof of Lemma 6.6. (i) We have

(6.9) $\pi \circ {}^f\varphi = \sum_{s=0}^{3} s!^{-1} \sum_{j=0}^{N-1} j!^{-1}\, \partial_\tau^s \partial_\xi^j \pi \cdot D_t^s D_x^j\, {}^f\varphi(t, x, \tau, \xi)$

$+ \sum_{s=0}^{2} s!^{-1}(N-1)!^{-1}(2\pi)^{-1} \int_0^1 (1-\theta)^{N-1}(OS - \iint e^{-iy\eta}\, \partial_\tau^s \partial_\xi^N\, \pi(t, x, \tau, \xi+\theta\eta)$
$\times D_t^s D_x^N\, {}^f\varphi(t, x+y, \xi)\, dy d\eta)\, d\theta\,.$

Since $f^\lambda D_t^s D_x^j\, {}^f\varphi \in {}^f\Sigma_{1/2}^{(-s,-j,0)}$ for any $\lambda > 0$ from Fact 4.6—2, in view of Fact 6.7—(3) the first summation minus $\pi \cdot {}^f\varphi$ takes the form

(6.10) $\sum_{k \neq l} b_{kl}\, \pi_k\, \pi_l + \sum_{k=1}^{3} b_k\, \pi_k + b_0$

with $b_{kl} \in {}^f\Sigma_{1/2}^{(-1,0,0)} + f^{1/2} \bigcap_{N \geq 3/2} A^{1/2,N}$, $b_k \in {}^f\Sigma_{1/2}^{(-2,0,0)} + f^{1/2} \bigcap_{N \geq 3/2} A^{1,1/2,N}$ for $k \neq 0$, $b_0 \in {}^f\Sigma^{(-3,0,0)} + f^{1/2} \bigcap_{N \geq 3/2} A^{3/2,1,N}$.

The symbols of the form (6.10) are in $F+E$ in view of Lemma 6.4—(i) and —(v). On the other hand, the oscillatory integrals in the integral terms are

sums of terms with notations in Lemma 6.3 respectively in s as in the following.

$$s = 0: I_{1,\theta}^{\tau_{p_0}}[\partial_\xi^{N-k}\tau_q\,\partial_\xi^k\tau_r, D_x^N{}^f\varphi], \{p,q,r\} = \{1,2,3\}, 0<k<N;$$
$$I_{2,\theta}^{\tau_{p_i}\pi_q}[\partial_\xi^N\tau_r, D_x^N{}^f\varphi], \{p,q,r\} = \{1,2,3\};$$
$$r_\theta[\prod_{i=1}^3\partial_\xi^{j_i}\tau_i, D_x^N{}^f\varphi], \sum_{i=1}^3 j_i = N, j_i>0.$$
$$s = 1: I_{1,\theta}^{\tau_p}[\partial_\xi^N\tau_q, D_t\,D_x^N{}^f\varphi], p\neq q; r_\theta[\partial_\xi^{N-k}\tau_p\,\partial_\xi^k\tau_q, D_t\,D_x^N{}^f\varphi], 0<k<N.$$
$$s = 2: r_\theta[\partial_\xi^N\tau_p, D_t^2\,D_x^N{}^f\varphi].$$

Thus, by Lemmas 6.3 and 4.9 the summation of the integral terms takes the form

$$(6.11) \quad \sum_{k\neq l} c_{kl}\,\pi_k\,\pi_l + \sum_{k=1}^3 c_k\,\pi_k + c_0$$
$$\text{with } c_{kl}\in f^{(N+1)/2}\,\Sigma^{(0,-1,0)}, c_k\in f^{(N+1)/2}\,\Sigma^{(-1,-1,0)} \quad \text{for} \quad k\neq 0,$$
$$c_0\in f^{(N+1)/2}\,\Sigma^{(-2,-1,0)}.$$

From Lemma 6.4 —(i) and (v) the symbols of the form (6.11) are in $F+E$ when $N\geq 6$. This proves (i).

(ii) We have that

$$^f\varphi\circ{}^fq - {}^f(\varphi q) = \sum_{j=1}^{N-1} j!^{-1}\,\partial_\xi^j{}^f\varphi\cdot D_x^j{}^fq$$
$$+(N-1)!^{-1}\sum_{k<3} r_N[\partial_\xi^N{}^f\varphi, a_{kl}(f\xi)^l]\,\tau^k.$$

Since $\partial_\xi^N{}^f\varphi\in f^{N/2}\,\Sigma^{(0,N,0)}$, the second summation $\in E$ for $N\geq 6$ in view of the definition of E. On the other hand ,from the assumption, for $j\neq 0$.

$$\partial_\xi^j{}^f\varphi\cdot\partial_x^j{}^fq = \sum_{\substack{I\subset\{1,2,3\}\\0<\#(I)\leq 3}}\prod_{i\in I^c}\pi_i\cdot\Big(\sum_{\substack{J=(j_k)\in N^{\#(I)}\\\sum_{k\in I}j_k=j}} C_{I,J}\,\partial_\xi^j{}^f\varphi\cdot\prod_{k\in I}\partial_x^{j_k}\tau_k\Big)$$

where $\prod_{i\in I^c}\pi_i$ means 1 when $I^c=\phi$.

Since $\partial_\xi^j{}^f\varphi\cdot\prod_{k\in I}\partial_x^{j_k}\tau_k\in f^{j-1/2}\,{}^f\Sigma^{(0,-\#(I),0)}$, this is the symbol of the from (6.10) so that it is in $F+E$ from the proof of (i). This completes the proof.

Lemma 6.9. *Let* $\tau_i\in{}^f\Sigma_0^{(0,0,2)}$, $i=1, 2, 3$ *and set* $\pi=\prod_{i=1}^3(\tau-\tau_i)$. *Let* $\varphi\in\Sigma^0$ *and let* $q(t,x,\tau,\xi)=\sum_{k+l\leq 3} a_{kl}(t,x)\,\xi^l\tau^k, a_{kl}\in B_{h,T}$. *Assume that* $\pi(t,x,\tau,\xi)$ $={}^fq(t,x,\tau,\xi)$ *when* $(t,x,\xi)\in\text{supp}\,{}^f\varphi$. *Then,*

$$\pi\circ{}^f\varphi - {}^fq\circ{}^f\varphi\in E.$$

Proof. We have

(6.12) $\pi \circ {}^f \varphi = \sum_{j+k \le 3} (j! \, k!)^{-1} \, D_\tau^j \, D_\xi^k \, \pi \cdot \partial_t^j \, \partial_x^k \, {}^f \varphi + \sum_{j+k=4} (j! \, (k-1)!)^{-1}$

$\times r_k [D_\tau{}^j D_\xi{}^k \pi, \, \partial_t{}^j \partial_x{}^k \, {}^f \varphi] \, .$

From (4.3) and (4.6), it follows that if $\tau_j = {}^f \tilde\tau_j$ with $\tilde\tau_j \in \Sigma_0^{(0,0,2)}$,

$$\partial_\xi^k \tau_j = f^k \, {}^f \partial_\xi^k \, \tilde\tau_j = f^\lambda (f \langle \xi \rangle / \langle f\xi \rangle)^{k-\lambda}$$

$$\times {}^f (\langle \xi \rangle^{k-\lambda} \, \partial_\xi^k \, \tilde\tau_j) \langle \xi \rangle^{-k+\lambda} \in f^\lambda \Sigma^{(k-\lambda)(0,2,0)}$$

for $k \in N$, $\lambda \in R$, $1 \le \lambda < k$.   This implies particularly

$$\partial_\xi^4 \tau_j \in f^{3/2} \Sigma^{5(0,1,0)}, \; \partial_\xi^3 \tau_j \in f \Sigma^{4(0,1,0)} \cap f^{3/2} \Sigma^{3(0,1,0)} ,$$

$$\partial_\xi^2 \tau_j \in f^{3/2} \Sigma^{(0,1,0)} \cap f \Sigma^{2(0,1,0)} \, .$$

Using this and Lemma 4.3, one can easily see that the second summaiton in (6.12) takes the form

$$f^{3/2} (b_2 \tau^2 + b_1 \tau + b_0) \quad \text{with} \quad b_2 \in \Sigma^{(0,1,0)}, \; b_1 \in \Sigma^{(-1,0,0)}, \; b_0 \in \Sigma^{(-2,-1,0)} ,$$

which is in $F+E$.   On the other hand, from the assumption

$$D_\tau^j \, D_\xi^k \, \pi \cdot \partial_t^j \, \partial_x^k \, {}^f \varphi = D_\tau^j \, D_\xi^k \, {}^f q \cdot \partial_t^j \, \partial_x^k \, {}^f \varphi$$

so that the first summation is equals to ${}^f q \circ {}^f \varphi$.   This completes the proof.

Lemma 6.10.   *Let* $L_i = D_t - a_i(t, x) f(x) D_x$, $a_i \in B_{h,T}$, $i = 1, 2, 3$.

(1) $L_1 L_2 = Op \left(\prod_{i=1}^2 (\tau - a_i f\xi) \right) + t^{1/h - 1} \, b f D_x$ *for some* $b \in B_{h,T}$.

(2)   *Assume that* $a_1 - a_2 \in E_{h,T} \setminus \{0\}$.   *Then there exists* $r \in N$ *such that any differential operator of the form* $L = b_1 f D_x + b_2 D_t$, $b_i \in t^r f^r B_{h,T}$ *can be expressed as* $L = c_1 L_1 + c_2 L_2$, $c_i \in B_{h,T}$.

(3)   *Assume that* $a_i - a_j \in E_{h,T} \setminus \{0\}$ *for any distinct* i, j.   *Then there exists* $r \in N$ *such that any differential operator of the form* $R = \sum_{0 < i+j \le 2} b_{ij} f^i D_x^i D_t^j$, $b_{ij} \in t^r f^r B_{h,T}$ *can be expressed as* $R = \sum_{i \ne j} c_{ij} L_i L_j + \sum_i c_i L_i$, $c_{ij}, c_i \in B_{h,T}$.

(4)   *Set* $Q = Op \left(\prod_{i=1}^3 (\tau - a_i f\xi) \right)$.   *Then for any* $\alpha \ge 0$, $[Q, f^\alpha]$ *is expressed as* $\sum_{i \ne j} b_{ij} L_i L_j + \sum_i b_i L_i + t^{1/h - 1} c_1 f D_x + c_0$ *with* $b_{ij}, b_i, c_i \in B_{h,T}$.

(5)   *Let* $R = \sum_{i+j \le m} a_{ij} f^i D_x^i D_t^j$, $a_{ij} \in B_{h,T}$, $m \ge 1$, *and let* $a \in B_T$.   *Then*, $[R, a]$ *can be written in the form* $\sum_{i+j \le m-1} b_{ij} f^i D_x^i D_t^j$, $b_{ij} \in B_{h,T}$.

(1) is trivial, (2) and (3) is a simple consequence of Lagrange's interpolation formula and (1), and (4) is that of Leibniz rule and (1). (5) is proved by a direct calculation.   The details are omitted.

Lemma 6.11. *Let* $L_j = D_t - A_j$, $L'_j = D_t - A'_j$, *with* A_j *and* A'_j *in* $\Sigma^{(0,-2,0)}$, $j=1,2,3$, *and let* $R = D_t^3 + \sum\limits_{j=0}^{2} B_j D_t^j$, $B_j \in Op\, \Sigma^{-2(0,3-j,0)}$. *Then if* $B \in Op\, \Sigma^{(0,6,0)}$, $L_1 L_2 L_3 B - BR$ *can be expressed as* $\sum\limits_{i\neq j} B'_{ij} L'_i L'_j + \sum\limits_i B'_i L'_i + B'_0$ *with* $B'_{ij} \in t^{-1} Op\, \Sigma^0$, $B'_i \in t^{-2} Op\, \Sigma^0$ *for* $i \neq 0$, $B'_0 \in t^{-3} Op\, \Sigma^0$.

Lemma 6.11 is proved in a straightforward way by using Lemma 4.13 and the fact that $[L, A] \in Op\, \Sigma^{(-1,-1,0)+\alpha}$ if $L = D_t - A_1$ with $A_1 \in Op\, \Sigma^{(0,-2,0)}$, and if $A \in Op\, \Sigma^{\alpha}$.

Lemma 6.12. *Let* $\varphi \in {}^f\Sigma^0$ *and* $q = \sum\limits_{i+j\leq 2} a_{ij} f^j \tau^i \xi^j$, $a_{ij} \in B_{h,T}$. *Then*

$$q \circ \varphi - q \cdot \varphi, \varphi \circ q - q \cdot \varphi \in E .$$

Proof. Set $b_i = \sum\limits_{j\leq 2-i} a_{ij} \xi^j$ so that $q = \sum\limits_{i\leq 2} {}^f b_i \tau^i$. Since b_i satisfies (4.41) in Lemma 4.18—(2), we have $\varphi \circ {}^f b_i - \varphi \cdot {}^f b_i$, ${}^f b_i \circ \varphi - \varphi^f b_i \in B^{3/2-i,N}$ for $i < 2$ and any N. $b_2 = a_{20}$ and $\varphi \circ a_{20} - \varphi \cdot a_{20} \in {}^f\Sigma^{(0,0,-1)} + f^{3/2} \Sigma^{(0,1,0)}$, since $\partial_\xi^k \varphi \in f^{k/2} \Sigma^{k(0,1,0)} \cap f^k {}^f\Sigma^{(0,0,-k)}$. Thus, we have proved the second statement and first one is also clear, since moreover, ${}^f b_i \circ (\tau^i \circ \varphi - \tau^i \varphi) \in E$ for $i \neq 0$ in view of Lemma 4.18—(1).

Lemma 6.13. *Let* $\varphi \in {}^f\Sigma^0$ *and* $q \in E$. *Then* $q \circ \varphi$, $\varphi \circ q \in E$.

Proof. Let $q = \sum\limits_{i\leq 2} a_i \tau^i \in E$. By virtue of the fact that ${}^f\Sigma^{(0,1-k,0)} \subset \Sigma^{(0,1-k,0)}$, $k \in N$, one can show in the same way as in the proof of Corollary 4.12—(i) that

$$\varphi \circ a_2,\, a_2 \circ \varphi \in t^{-1}({}^f\Sigma^{(0,1,0)} + f^{3/2} \Sigma^{(0,1,0)}) .$$

It follows from Lemma 4.18—(1) that $\sum\limits_{i\leq 1} \varphi \circ a_i \cdot \tau^i$, $\sum\limits_{i\leq 1} a_i \circ \varphi \cdot \tau^i \in E$, and also $a_i \circ (\tau^i \circ \varphi - \tau^i \varphi) \in E$ for $i \neq 0$ by Leibniz rule. These facts prove the result.

Definition 6.14. *Let* $\varphi \in \Sigma^0_{1/2}(T, h)$. *Then we define the set* Σ^d_φ, $d \in \mathbf{R}^3$ *with* $d_1 \leq 0$ *by*

$$\Sigma^d_\varphi := \{{}^f a;\ a \in \Sigma^d_{1/2},\ \text{supp } a \subset \text{supp } \varphi\} .$$

Definition 6.15. *Let* $\varphi \in \Sigma^0_{1/2}$ *and* $q = \sum\limits_{i+j\leq 3} a_{ij} \tau^i \xi^j$ *with* $a_{ij} \in B_{h,T}$ *and* $a_{30} = 1$ *be given. We define*

$$\varDelta_{\alpha,i} = \{d \in \mathbf{R}^3;\ d_1 = -\alpha_1,\ \sigma_{1/2}(d) = \alpha_1 + \alpha_2 - i/2,\ \mu_{1/2}(d) \geq 0\} \quad \text{for}$$
$\alpha \in \mathbf{Z}^3_+$, $i \in \mathbf{Z}_+$; $F_i =$ the set of all finite sums of symbols in
$\{f^{(i-\alpha_1)/2} a^f \tilde{\partial}^\alpha q;\ a \in \Sigma^d_\varphi, d \in \varDelta_{\alpha,i}, 0 < |\alpha| \leq i\} \quad$ for
$i \in N$, where $\tilde{\partial}^\alpha = \partial_\tau^{\alpha_1} \partial_\xi^{\alpha_2} \partial_{x^3}^{\alpha_3}$.

We also define

$$q_{\alpha,i} = f^{(i-\alpha_1)/2}\,\psi^{d_{\alpha i}}\,{}^f\tilde{\partial}^\alpha q \quad \text{for} \quad i \in N \text{ and } \alpha \in Z_+^3 \text{ with } 0 < |\alpha| \le i,$$

$$q_{0,0} = {}^f q, \quad \text{and}$$

$$\tilde{F} = \{q_{\alpha,i}\,;\, 0 < |\alpha| \le i\} \quad \text{for} \quad i \in N, \text{ where}$$

$$d_{\alpha,i} = (-\alpha_1, 0, 0) + (\alpha_1/2 + \alpha_2 - i/2)\,(0, -1, 1) \quad \text{for} \quad \alpha \in Z_+^3 \text{ and } i \in Z_+\,,$$

$$\psi^d = {}^f\tilde{\psi}^d, \quad \tilde{\psi}^d = \psi_1^{d_1}\,\psi_2^{d_2}\,\psi_{3,1/2}{}^{d_3} \quad \text{for} \quad d \in R^3\,.$$

Fact 6.16. (1) $f^\varepsilon\,\partial^\alpha\,{}^f b \in {}^f\Sigma_{1/2}^{d-\alpha}$ for any $b \in \Sigma_{1/2}^d$ and $\varepsilon > 0$.

(2) $|\partial^\alpha\,\tilde{\psi}^d| \le C_{\alpha\varepsilon}\,t^{-\alpha_1}(\langle t^{\varepsilon_h}f^{1/(10h)}\,\xi\rangle^{-1}\langle t^{\varepsilon_h}\,\xi\rangle)^{\varepsilon\alpha_2/2}\,\langle\xi\rangle^{-\alpha_3}\,\tilde{\psi}^d$ for $\varepsilon > 0$.

(3) $f^{\varepsilon\alpha_2}\,\partial^\alpha\,\psi^d \in t^{-\alpha_1}\,{}^f\Sigma_{1/2}^{d-\alpha_3(0,-1,1)}$ for $\varepsilon > 0$.

(4) If $q = a\tau^i\,\xi^j$ with $a \in B_{h,T}$, $i + j \le 3$, and $a = 1$ when $i = 3$ and $j = 0$, we have

$$q_{\alpha,i} = f^{(i-\alpha_1)/2}\,{}^f(b\psi_1^{-\alpha_1}\,\langle\xi\rangle^{j+\alpha_1/2-i/2})\,\tau^l$$

with $\alpha_1 + j + l \le 3$, $j \in Z_+$, $l \le 2$, and $b \in \Sigma_0^0$ satisfying $|\partial^\alpha b| \le C_\alpha\,t^{-\alpha}\langle\xi\rangle^{-\alpha_3}$.

Proof. (1) Nothing but Fact 4.6—2. (2) A result of the estimates that

$$|\partial^\alpha(t^s f^{s'}\,\xi)| \le C_\alpha\langle t^s f^{s'}\,\xi\rangle\,(\langle t^s f^{s'}\,\xi\rangle^{-1}\langle t^s\,\xi\rangle)^{\varepsilon\alpha_2}\,\langle\xi\rangle^{-\alpha_3}\cdot t^{-\alpha_1}$$

for $\varepsilon > 0$, $s \ge 0$, $s' \ge 0$, and Lemma 3.8—3.

(3) A simple consequence of (4.16) and (2).

(4) This is shown by a direct calculation.

Lemma 6.17. (1) *Let* $q = a\tau^k$ *with* $a \in {}^f\Sigma^d + f^{3/2}\,\Sigma^d$, $\sigma(d) + k \le 3/2$, $-d_1 + k \le 3$, $k \le 2$, $\tilde{\mu}(d) = \mu((0, d_2, d_3)) \ge 0$. *Then,* $q \in E$.

(2) $F_j \subset E$ *if* $j \ge 3$.

(3) $d_{\alpha,i} \in \Delta_{\alpha,i}$, $\Sigma_{1/2}^d \subset \Sigma_{1/2}^{d_{\alpha_2}i}$ *for* $d \in \Delta_{\alpha,i}$.

(4) $a\tilde{\psi}^{-d} \in \Sigma_{1/2}^0$ *if* $a \in \Sigma_{1/2}^d$.

(5) $aq_{\alpha,i} \in F_i$ *for* $a \in \Sigma_\varphi^0$ *and* $i \in N$.

(6) *any* $q \in F_i$ *can be expressed in the form of a sum of symbols of the form* $aq_{\alpha,i}$ *with* $a \in \Sigma_\varphi^0$ *and* $0 < |\alpha| \le i$.

Proof. (1) follows from the inequality that

$$\tilde{\psi}_1^{d_1}\,\langle\xi\rangle^{-d_2/2}\,\psi_{3,1/2}{}^{d_3} \le \tilde{\psi}_1^{-(3-k)}\,\langle\xi\rangle^{-(3-k)/2+\sigma(d)}\,.$$

(2) follows from Fact 4.16—4 and (1). (3) is clear from the definition. (4) follows from Fact 4.16—2. (5) follows from (3) and the definition. (6) follows from (3), (4), and the definition of F_i.

Lemma 6.18. *Let* φ *and* q *as in Definition 6.15 be given and suppose that* $i \le 2$ *and* $a \in \Sigma_\varphi^0$. *Then,*

$$a \circ q_{\alpha,i} - a \cdot q_{\alpha,i}, \quad q_{\alpha,i} \circ a - a \cdot q_{\alpha,i} \in \sum_{i<j\le2} F_j + E .$$

Proof. In view of Fact 6.16—(4) and —(2) for $i \in N$ one can express $q_{\alpha,i}$ as

$$\sum_{1=0}^{2} f^{(i-\alpha_1)/2} \, {}^f b_l \, \tau^l \quad \text{where } b_l = 0 \text{ if } \alpha_1 + l > 3 \text{ and where } b_l$$

satisfies $|\partial^\beta b_l| \le C_{\beta\varepsilon} \psi_1^{-\alpha_1} \langle\xi\rangle^{3-l-\alpha_1/2-i/2-\beta_3+\varepsilon\beta_2} \cdot t^{-\beta_1}$ for $\varepsilon > 0$. Thus we have in view of Corollary 4.12—(i) and Lemma 6.17—(1),

$$a \circ q_{\alpha,i} - \sum_{k=0}^{N} D_\xi^k a \cdot \partial_x^k q_{\alpha,i}/k!, \quad q_{\alpha,i} \circ a - \sum_{s=0}^{3} \sum_{k=0}^{N} (s! \, k!)^{-1} D_\tau^s D_\xi^k q_{\alpha,i} \cdot \partial_t^s \partial_x^k a \in E$$

for sufficiently large N (in fact $N \ge 5$). So, in view of Lemma 6.17—(2), it suffices to show that

(6.13) $D_\xi^s a \cdot \partial_x^s q_{\alpha,i} \in F_{i+s}, \quad D_\tau^k D_\xi^s q_{\alpha,i} \cdot \partial_t^k \partial_x^s a \in F_{i+s+k}$ for $s \in N$.

To do so we note that $a = {}^f \tilde{a}$ with some $\tilde{a} \in \Sigma_{1/2}^0$ satisfying supp $\tilde{a} \subset$ supp φ, from the assumption. Then

(6.14) $D_\xi^s a \cdot \partial_x^s q_{\alpha,i} = \sum_{\substack{k+|\beta|\le s \\ \beta_1=0}} e_{k;\beta}^{\alpha;i}(x) \, {}^f D_\xi^s \tilde{a} \cdot f^{(i+s-\alpha_1)/2} f^{k/4} \cdot \partial_x^k \psi^{d\alpha,i}$
$$\times (f\xi)^{\beta_2} \, {}^f \tilde{\partial}^{\alpha+\beta} q ,$$

where $e_{k;\beta}^{\alpha;i} \in C_0^\infty(\mathbf{R})$ with $e_{k;0}^{\alpha;i} = 0$ if $|\alpha| = i = 0$, and

(6.15) $D_\tau^k D_\xi^s q_{\alpha,i} \cdot \partial_t^k \partial_x^s a = \sum_{\substack{\beta=(k,\beta_2,0) \\ \beta_2\le s}} \binom{s}{\beta_2} f^{(i+s-\alpha_1)/2+s/2}$
$$\times \partial_t^k \partial_x^s a \, {}^f \partial_\xi^{s-\beta_2} \tilde{\psi}^{d\alpha,i} {}^f \partial^{\alpha+\beta} q .$$

By Fact 6.16—(3), ${}^f D_\xi^s \tilde{a} f^{k/4} \partial_x^k \psi^{d\alpha,i}(f\xi)^{\beta_2} \in \Sigma_\varphi^d{}_{\alpha+\beta,i+s}$ so that $D_\xi^s a \cdot \partial_x^s q_{\alpha,i} \in F_{i+s}$, for when $|\alpha| = i = 0$, $\beta \ne 0$ for non trivial terms in the summation of (6.14). Similarly by Fact 4.16—(1) and—(2), $f^{s/2} \partial_t^k \partial_x^s a \cdot {}^f \partial_\xi^{s-\beta_2} \tilde{\psi}^{d\alpha,i} \in \Sigma_\varphi^d{}_{\alpha+\beta,s+k+i}$ so that in view of the fact that when $|\alpha| = i = 0$, $\beta_2 < s$ implies that $\partial_\xi^{s-\beta_2} \tilde{\psi}^{d\alpha,i} = 0$ in the summation of (6.15), we obtain the latter of (6.13). This completes the proof.

Lemma 6.19. *Let φ and q as in Definition 6.15 be given. Set*
$$\Delta = \{(\alpha, i) \in \mathbf{Z}_+^3 \times \mathbf{Z}_+; \; \alpha = 0 \text{ and } i = 0, \text{ or } 0 < |\alpha| \le i \le 2\}, \; \Delta' = \Delta \backslash \{(0,0)\},$$
and $Q_\lambda = Op \, q_{\alpha,i}$ for $\lambda = (\alpha, i) \in \Delta$. Then for any $s \in N$ there exists $l \in N$ such that for any $\mu \in \Delta$ and $A_1, \cdots, A_s \in Op \, \Sigma_\varphi^0$ one can find $A_{\lambda jk}, \lambda \in \Delta', j = l, \cdots, l_s,$ $k = l, \cdots, s, \in Op \, \Sigma_\varphi^0$ such that

$$[Q_\mu, A_1 \cdots A_s] - \sum_{\lambda,j} Q_\lambda A_{\lambda j1} \cdots A_{\lambda js} \in C_0 .$$

Proof. Set $q_\lambda = q_{\alpha,i}$ for $\lambda = (\alpha, i) \in \varDelta$. Let $a \in \Sigma_\varphi^0$ and $\mu = (\alpha, i)$. By Lemma 6.18, $a \circ q_\mu - q_\mu \circ a \in \Sigma_{i < j \le 2} F_j + E$. By Lemma 6.18 and 6.17—(6), for any $q \in F_j$ with $j \le 2$ one can find $a_\lambda \in \Sigma_\varphi^0$ for all $\lambda \in \varDelta_j \equiv \{\lambda \in \varDelta; \lambda = (\alpha, i)$ with $j \le i \le 2\}$ such that $q - \Sigma_{\lambda \in \varDelta_j} q_\lambda \circ a_\lambda \in E$. These two facts shows the Lemma for $s = 1$. Next we assume that $s > 1$ and that the Lemma is true for $1, \cdots, s-1$ so that $1_i \in N, 1 \le i \le s-1$ as in the statement exist. Then for any $\nu \in \varDelta$ there exist $A_{\lambda jk}^\nu, \lambda \in \varDelta', j = 1, \cdots, l_{s-1}, k = 1, \cdots, s-1, \in Op\ \Sigma_\varphi^0$ such that $[Q_\nu, A_1 \cdots A_{s-1}] - \Sigma_{\lambda, j} Q_\lambda A_{\lambda j1}^\nu \cdots A_{\lambda js-1}^\nu \in C_0$, and there exist $A_{\lambda j} \in Op\ \Sigma_\varphi^0, \lambda \in \varDelta', j = 1, \cdots, l_1$ such that $[Q_\mu, A_s] - \Sigma_{\lambda, j} Q_\lambda A_{\lambda j} \in C_0$. Since $[Q_\mu, A_1 \cdots A_s] = [Q_\mu, A_1 \cdots A_{s-1}] A_s + A_1 \cdots A_{s-1}[Q_\mu, A_s]$, the Lemma also holds for s in view of Lemma 6.13.

Next corollary is important for estimation of commutators in §7.

Corollary 6.20. *Let* $\tilde\tau_i, \tau_i, \pi_i, L_i,$ *for* $i = 1, 2, 3$ *be as in Lemma 6.4. Let* φ *and* q *be as in Definition 6.15. Suppose that* $q(t, x, \tau, \xi) = \prod_{i=1}^3 (\tau - \tilde\tau_i(t, x, \xi))$, $(t, x, \xi) \in$ supp φ. *Then one can find* $1_s \in N$ *for any* $s \in N$ *with* $s \ge 2$ *such that for any* $A_1, \cdots, A_s \in Op\ \Sigma_\varphi^0$ *there exist* $A_{lj} \in Op\ \Sigma_\varphi^0$ *and* $Q_l \in C(L_l), l = 1, \cdots, 1_s,$ $j = 1, \cdots, s-1,$ *such that*

$$[Op^f q, A_1 \cdots A_s] - \Sigma_{l=1}^{l_s} Q_l A_{l1} \cdots A_{ls-1} \in C_0.$$

Proof. Assume the notations $\varDelta', q_\lambda$ in the statement and the proof of Lemma 6.19. By Lemma 6.19 one can find $l_s \in N$ such that for any $A_j \in Op\ \Sigma_\varphi^0, j = 1, \cdots, s,$ there exist $A_{\lambda jk} = Op\ a_{\lambda jk}$ with $a_{\lambda jk} \in \Sigma_\varphi^0$ for $\lambda \in \varDelta', j = 1, \cdots, l_s,$ $k = 1, \cdots, s,$ such that

$$[Op^f q, A_1 \cdots A_s] - \Sigma_{\lambda, j} Op(q_\lambda \circ a_{\lambda jk}) A_{\lambda j2} \cdots A_{\lambda js} \in C_0.$$

Since $q_\lambda \circ a_{\lambda j1} \in F_1 + F_2 + E$ by Lemma 6.18, it turns out in view of Lemma 6.13 that the proof is complete if we show that

(6.16) $Op\ p \in C + C_0$ if $p \in F_1 \cup F_2$.

To see this we may assume that $p = f^{(i-\alpha_1)/2} a^f \tilde\partial^\alpha q$ with $a = {}^f \tilde a, \tilde a \in \Sigma_{1/2}^d$ with $d \in \varDelta_{\alpha,i}$ and supp $\tilde a \subset$ supp $\varphi, 0 < |\alpha| \le i \le 2$. We devide the argument into two cases: (1) $\alpha_2 + \alpha_3 > 0$ (2) $\alpha_2 = \alpha_3 = 0$. First we consider (1). From the assumption on $q, \tilde\partial^\alpha q \cdot \tilde a$ is a sum of the following form:

$$\tilde b \prod_{j \in J} (\tau - \tilde\tau_j); 0 < \#(J) < 3, \tilde b = \tilde a \cdot \prod_{j \in I} \partial_\xi^{\beta_{j1}} \partial_x^{\beta_{j2}} \tilde\tau_j, \text{ with } I \ne \phi,$$

$$I \cap J = \phi, \#(I \cup J) = 3 - \alpha_l, \sum_{j \in I} \beta_{j1} = \alpha_{l+1}, |\beta_j| > 0 \text{ for } \beta_j = (\beta_{j1}, \beta_{j2}).$$

From the definition of $d_{\alpha,i}$ and the assumption on $\tau_j, b \in \Sigma_{1/2}^e$ with $e = (-\alpha_1,$

$0, 0)+\#(I)/2$ $(0, -1, 1)$. Thus, in view of $i>\alpha_1$, p can be written in the form:

$$\sum_{k\neq l} b_{kl}\,\pi_k\,\pi_l + \sum_{k=1}^{3} b_k\,\pi_k; \quad b_{kl}\in f^{1/2}\underset{N\geq 1/2}{\cap} A^{1/2,N}, \quad b_k\in f^{1/2}\underset{N\geq 1}{\cap} A^{1,\alpha_1/2,N},$$

which belongs to $F+E$ by Lemma 6.4—(i), (v). Next we pass to (2). $p=a\cdot$ $\displaystyle\sum_{\#(I)=3-\alpha_1} (3-\#(I))!\prod_{i\in I}\pi_i$ and $a\in {}^f\Sigma_{1/2}^{(-\alpha_1,0,0)}$. So, Lemma 6.4—(i) and (v) imply that $p\in F+E$. This completes the proof.

Next we shall study the localization of the operator $Q_{\mu\nu}$ in (2.17).

Definition 6.21. Let $q=\sum_{i+j\leq 3} a_{ij}\,\tau^i\,\xi^j$ with $a_{ij}\in B_{h,T}$ and $a_{30}=1$. We define for $\nu=(\nu_1,\nu_2)$, $\mu=(\mu_1,\mu_2)\in \mathbf{Z}_+^2$,

$$q_{\nu\mu}=f^{\mu_2}\cdot\partial_x^{\mu_2}\cdot {}^f(\partial_\tau^{\nu_1}\partial_{\xi^2}^{\nu}\partial_t^{\mu_1} q)\cdot\langle f\xi\rangle^{(\nu_2-\mu_2)/2}\langle t^{e_h} f\xi\rangle^{(\nu_1-\mu_1)/2}\,t^{-\nu_1+\mu_1}.$$

Lemma 6.22. $\langle f\xi\rangle^{-1/2} q_{\nu\mu}\in E$ when $l\in\mathbf{Z}_+$, $l+|\nu|+|\mu|\geq 3$, $\mu_1\leq 2$, and $|\nu|+|\mu|>0$. Before the proof we note that

(6.17) $\quad$ for $\varepsilon\in\mathbf{R}$, $\alpha\in\mathbf{Z}_+^3$, $|\partial^\alpha\langle t^\varepsilon\,\xi\rangle|\leq C_{\alpha\varepsilon}\langle t^\varepsilon\,\xi\rangle\,t^{-\alpha_1}\langle\xi\rangle^{-\alpha_3}.$

Proof of Lemma 6.22. If $q=a\tau^i\xi^j$, $i\leq 2$, $a\in B_{h,T}$, and if $\nu\leq(i,j)$, we have

(6.18) $\quad q_{\nu\mu}=\sum_{1\leq\mu_2} g_{\mu\nu l}\,\partial_t^{\mu_1}\partial_x^l\,a\cdot(f\xi)^{i-\nu_2}\cdot\langle f\xi\rangle^{(\nu_2-\mu_2)/2}\langle t^{e_h}f\xi\rangle^{(\nu_1-\mu_1)/2}$
$\qquad\qquad \times t^{-\nu_1+\mu_1}\cdot\tau^{i-\nu_1}$ with $g_{\nu\mu l}\in B^\infty(\mathbf{R}_x)$.

If $q=\tau^3$,

(6.19) $\quad q_{\nu\mu}=0$ when $\nu_2+|\mu|>0$, and
$\qquad q_{(\nu_1,0),0}=t^{-\nu_1}\langle f^{e_h} f\xi\rangle^{\nu_1/2}\cdot\dfrac{3!}{(3-\nu_1)!}\cdot\tau^{3-\nu_1}$ when $\nu_1\leq 3$.

Thus using the fact that $\partial_t^{\mu_1}\partial_x^l\,a\cdot t^{\mu_1}=t^{\min(\mu_1,1)/h}\cdot b_l$ for some $b_l\in B_{h,T}$, the inequalities that $\min(\mu_1,1)/h-\mu_1/(20h)\geq 0$ for $2\geq\mu_1\geq 0$ and that $t^{e_h}\langle\xi\rangle\leq\langle t^{e_h}\xi\rangle$, and (6.17) we see that when $l\in\mathbf{Z}_+$, $|\nu|+|\mu|>0$, and $\mu_1\leq 2$, $\langle f\,\xi\rangle^{-1/2} q_{\nu\mu}$ is a sum of symbols of the form:

(6.20) $\quad {}^f a\tau^k; k\leq 2$, $\nu_1+\mu_1/2+k\leq 3$, $a\in t^{-\mu_1/2}\Sigma^d$ with
$\qquad d_1=-\nu_1$, $\sigma(d)=3-k-(|\nu|+|\mu|+l)/2$, $d_3=0$,
$\qquad$ and a satisfies the estimate that
$\qquad |\partial^\alpha a|\leq Ct^{-\mu_1/2-\alpha_1}\cdot\tilde{\psi}_1^{-\nu_1}\cdot\langle\xi\rangle^{\sigma(d)-\nu_1/2-\alpha_3}.$

In view of the definition of E the symbols in (6.20) are in E when $l+|\nu|+|\mu|\geq 3$. So the proof is complete.

We define a similar set of operators as $\mathcal{C}$.

Definition 6.23. Let L_i, $i=1, 2, 3$ as in Lemma 6.4. We define $\mathcal{C}'=\mathcal{C}'(L_i)$ =the set of all operators on $\mathcal{S}_T$ of the form:

$$\sum_{i\neq j} A_{ij}\, L_i\, L_j + \Sigma_i\, A_i\, L_i + A_0$$
$$\text{with}\ \ A_{ij}\in\mathcal{B}^{1/2,1/2,3},\ A_i\in\mathcal{B}^{1,1,3}\ \ \text{for}\ \ i\neq 0,\ A_0\in\mathcal{B}^{3/2,3/2,3/2}\ ;$$

$F'=F'(L_i)$=the set of all symbols q with $Op\ q\in\mathcal{C}'(L_i)$.

We shall show an analogous result of (6.16).

Lemma 6.24. *Let* $\varphi\in\Sigma^0$, *and let* q *be as in Definition* 6.21. *Let* π_i, τ_i, $\tilde{\tau}_i$, L_i,

$\varepsilon\in(0, 1)$, *and* $\pi=\prod_{i=1}^{3}\pi_i$ *be as in Lemma* 6.4. *We suppose that* $\varepsilon\geq 1/(10h)$, *and that*

$$q(t, x, \tau, \xi) = \prod_{i=1}^{3}(\tau-\tilde{\tau}_i(t, x, \xi)),\ (t, x, \xi)\in\text{supp}\ \varphi\ .$$

Then, $q_{\nu\mu}\circ{}^f\varphi\in F'(L_i)+E$ *if* $0<|\nu|+|\mu|<3$.

For the proof of the lemma we make a definition.

Definition 6.25. We recall Definitions 3.7 and 6.15. With $U=R_T^3$, and $M_\alpha^{(1)}=\tilde{\psi}^{-\alpha}$, $M_\alpha^{(2)}=\psi'^{-\alpha}$, for $\alpha\in Z_+^3$ where $\psi'^d=\tilde{\psi}_1^{d_1}\langle\xi\rangle^{-d_2/2}\psi_{3,1/2}{}^{d_3}$ for $d\in R^3$ by definition, we set for a positive function Z on U,

$$\mathcal{B}^{(i)}(Z) = \mathcal{B}(U, Z, M_\alpha^{(i)}),\ i = 1, 2\ .$$

Clearly,

(6.21) $$\Sigma_{1/2}^d = \mathcal{B}^{(1)}(\psi^d),\ \Sigma^d = \mathcal{B}^{(2)}(\psi'^d)\ ,$$

so that

(6.22) $$\Sigma_{1/2}^d\subset t^{d_1}\Sigma^{(0,d_1+d_2,d_3)}\ .$$

Indeed, using $\psi^d=t^{d_1}(t^{-d_1}\psi^d)\leq t^{d_1}\psi^{(0,d_1+d_2,d_3)}$ and Lemma 3.8–5 and 2 we have $\mathcal{B}^{(1)}(\psi^d)=t^{d_1}\mathcal{B}^{(1)}(t^{-d_1}\psi^d)\subset t^{d_1}\mathcal{B}(\psi^{(0,d_1+d_2,d_3)})$ which implies (6.22). We also have

(6.23) $$t^\varepsilon\Sigma^{d+(0,0,1)}\subset\Sigma^{d+(-1,0,0)}\ \text{when}\ \varepsilon\geq 1/(10h)\ ,$$

(6.24) $$t^{\varepsilon-1/2}B^{k,l,N}\subset B^{k,l+1/2,N}\ \text{if}\ k\geq l+1/2\ \text{and}\ \varepsilon\geq 1/(20h)\ .$$

Indeed, $t^\varepsilon\psi_{3,1/2}=t^\varepsilon\langle t^{\varepsilon_h}\xi\rangle^{-1/2}\langle\xi\rangle\leq t^{\varepsilon-\varepsilon_h/2}\langle\xi\rangle^{-1/2}\leq t^{-1}\langle t^{\varepsilon_h}\xi\rangle^{1/2}$ when $\varepsilon\geq 1/(10h)$. So (6.23) holds by (6.21), Lemma 3.8—5 and 2. And if $d\in\Lambda_{k,l}$ with k, l as in

(6.24), $\psi'^d = \tilde{\psi}_{11}^{d_1} \cdot \langle t^{e_h} \xi \rangle^{\tilde{\mu}(d)} \cdot \langle \xi \rangle^{\nu(d)} \le \tilde{\psi}^{d_1} \langle \xi \rangle^{\nu(d)+\tilde{\mu}(d)} \le \tilde{\psi}_1^{-21} \langle \xi \rangle^{k-l}$ from the in-equalities $d_1 \ge -2l$ and $\tilde{\mu}(d) \ge 0$, so that $t^{\varepsilon-1/2} \psi'^d \le \tilde{\psi}_1^{-2l} \langle \xi \rangle^{k-l-1/2} t^{-1/2+1(20h)}$ $\langle \xi \rangle^{1/2} \le \tilde{\psi}_1^{-2l-1} \langle \xi \rangle^{k-l-1/2}$ from $\varepsilon \ge 1/(20h)$. Thus, $t^{\varepsilon-1/2} \Sigma^d \subset \Sigma^e$ with $e = (-2l-1, -(k-l-1/2), 0) \in \varDelta_{k, l+1/2}$, which shows (6.24) in view of the definition of $B^{k,l,N}$.

Proof of Lemma 6.24. We have

$$(6.25) \qquad q_{\nu\mu} \circ \varphi = \sum_{i=0}^{2} i!^{-1} \Big\{ \sum_{j=0}^{N} j!^{-1} \partial_\tau^i \partial_\xi^j q_{\nu\mu} \cdot D_t^i D_x^j {}^f \varphi$$
$$+ r_{N+1} [\partial_\tau^i \partial_\xi^{N+1} q_{\nu\mu}, D_t^i D_x^{N+1} {}^f \varphi] / N! \Big\}.$$

We assume that $0 < |\nu| + |\mu| < 3$. Since from (6.20), $q_{\nu\mu} = \sum_{k=0}^{2} {}^f a_k \tau^k$ with a_k $(t, x, \xi) \in C^\infty(\boldsymbol{R}_T^3)$ satisfying $|\partial^\alpha a_k| \le C_\alpha t^{-(3-k-\alpha_1)} \langle \xi \rangle^{5/2-k-\alpha_3}$ which implies by Lemma 3.8−5 that the notation in Definition 3.7, $a_k \in t^{-(3-k)} \mathcal{B}(\boldsymbol{R}_T^3, \langle \xi \rangle^{5/2-k}, M_\alpha)$ with $M_\alpha = t^{-\alpha_1} \langle \xi \rangle^{-\alpha_3}$, the terms $r_{N+1}[\cdot]$ in (6.25) are in E for large N, for in-deed, by Corollary 4.12−(i) with $\rho = 1/4$ they can be written in the form $\sum_{k=0}^{2} b_k \tau^k$ with $b_k \in t^{-(3-k)} f^3 \Sigma^0$ for $N \ge 12$, and $t^{-2} f^3 \Sigma^0 \subset t^{-1} B^{1/2,1/2,3}$, $t^{-3} f^3 \Sigma^0 \subset B^{3/2,3/2,3}$. A direct calculation shows that

$$(6.26) \quad \partial_\tau^i \partial_\xi^i q_{\nu\mu} = \sum_{\substack{l \le j \\ m \le \mu_2}} C_{lnm} f^{\mu_2-m} \partial_x^{\mu_2-m} (f^{j-l}) f^l \cdot {}^f a_{lm} \cdot t^i q_{(\nu_1+i, \nu_2+j-l)(\mu_1, m)}$$

with $a \in C^\infty(\boldsymbol{R}_T^3)$ satisfying

$$|\partial^\alpha a_{lm}| \le C_\alpha \langle \xi \rangle^{(-\mu_2+m-j-l)/2-\alpha_3} \cdot \langle t^{e_h} \xi \rangle^{-i/2} t^{-\alpha_1}.$$

By (6.26) one can easily see that

$$(6.27) \qquad \partial_\tau^i \partial_\xi^j q_{\nu\mu} \cdot D_t^i D_x^j {}^f \varphi = \sum_{\substack{l \le j \\ m \le \mu_2}} {}^f a_{lm} q_{(\nu_1+i, \nu_2+l)(\mu_1, \mu_2-m)} \text{ with}$$

another $a_{lm} \in \Sigma^{(j-l+m)(0,1,0)}$ such that $a_{0m} = 0$ if $\mu_2 > 0$ and $m \ne 0$, and such that supp $a_{lm} \subset$ supp φ.

For non trivial terms in the summation in (6.27), $|(\nu_1+i, \nu_2+l)+(\mu_1, \mu_2-m)|$ > 0, since when $\mu_2 > 0$, $a_{lm} \ne 0$ implies that $\mu_2 - m + l > 0$. So noting that $a \in {}^f \Sigma^0$ and $q \in E$ implies $aq \in E$, the fact that by $\mu_1 \le 2$, Lemma 6.22, and that $a_{lm} \in \Sigma^0$, we have only to show that

$$(6.28) \quad {}^f a q_{\nu\mu} \in F' \text{ if } a \in \Sigma^0 \text{ with supp } a \subset \text{ supp } \varphi, \text{ and if } 0 < |\nu| + |\mu| < 3.$$

Since $f^{\mu_2} \cdot \partial_x^{\mu_2} {}^f(\partial_\tau^{\nu_1} \partial_\xi^{\nu_2} \partial_t^{\mu_1} q) = \sum_{i+k \le \mu_2} e_{ik} {}^f(\partial_\tau^{\nu_1} \partial_\xi^{\nu_2+k} \partial_t^{\mu_1} \partial_x^i q) (f\xi)^k$ with $e_{ik} \in B^\infty(R_x)$, and since supp $a \subset$ supp φ, ${}^f a q_{\nu\mu}$ with $0 < |\nu| + |\mu| < 3$ can be expressed

as a sum of the following symbols:

(1) The case $\nu_2 + |\mu| > 0$.

$$^f b \cdot \prod_{s \in I} (\tau - \tau_s);\ 0 < \#(I) < 3,\ b = a_1 a_2 \prod_{s \in J} \partial^{\gamma^s} \tau_s \ \text{with}$$

$$J \neq \phi,\ I \cap J = \phi,\ \#(I \cup J) = 3 - \nu_1,\ \sum_{s \in J} r^s = (\mu_1, i, \nu_2 + k) \ \text{with}$$

$$i + k \le \mu_2,\ r^s \neq \phi,\ a_1 = ag \tilde{\psi}^{-\nu_1} \ \text{with} \ g \in B^\infty(\boldsymbol{R}_x),\ a_2 = \tilde{\psi}_1^{\mu_1} \langle \xi \rangle^{(\nu_2 - \mu_2)/2} \xi^k .$$

(2) The case $\nu_2 = |\mu| = 0$. ($\nu_1 = |\mu| + |\nu| > 0$ then.)

$$^f b \prod_{s \in I} (\tau - \tau_s);\ \#(I) = 3 - \nu_1,\ b = at^{-\nu_1} \langle t^{\varepsilon_h} \xi \rangle^{\nu_1/2} .$$

We consider the case (1) first. We set $b_1 = a_2 \prod_{s \in J} \partial^{\gamma^s} \tau_s$ and $b_2 = \prod_{s \in J} b_{2s}$ with $b_{2s} = \tilde{\psi}_1^{\gamma_1^s} \cdot \langle \xi \rangle^{(\gamma_3^s - \gamma_2^s)/2} \partial^{\gamma^s} \tau_s$. From (6.17) we have $\langle t^{\varepsilon_h} \xi \rangle^{-\gamma_1^s/2} \langle \xi \rangle^{(\gamma_3^s - \gamma_2^s)/2} \in \Sigma_{1/2}^{\gamma_1^s(0,1,0) + (\gamma_3^s - \gamma_2^s)/2(0,-1,1)}$. So if we set $J_1 = \{s \in J;\ r_1^s > 0\}$ and $J_2 = J \backslash J_1$, $b_{2s} \in t^\varepsilon \Sigma_{1/2}^{(0,0,1)}$ for $s \in J_l$ and $b_{2s} \in \Sigma_{1/2}^{(0,-1,1)/2}$ for $s \in J_2$ in view of (6.22) and Lemma 3. $8-2$. Hence, by (6.23) and Lemma 3.8-2, $b_{2s} \in \Sigma^{(-1,0,0)}$ for any $s \in J$ which implies $b_2 \in \Sigma^{\#(J)(-1,0,0)}$. Since $a_2 = b_0 b_2$ with $b_0 = \xi^k \langle \xi \rangle^{-k} \langle \xi \rangle^{(i+k-\mu_2)/2} \in \Sigma^0$, we have $a_2 \in \Sigma^{\#(J)(-1,0,0)}$, and therefore $b \in \Sigma^{(3-\#(I))(-1,0,0)}$. We have shown that $^f aq_{\nu\mu}$ takes the following form in the case (1):

$$(6.29) \qquad \sum_{k \neq l} b_{kl} \pi_k \pi_l + \sum_{k=1}^{3} b_k \pi_k;\ b_{kl} \in {}^f \Sigma^{(-1,0,0)},\ b_k \in {}^f \Sigma^{(-2,0,0)} .$$

It is clear from (6.17) that in the case (2), $^f aq_{\nu\mu}$ also takes the form (6.29). Now, by Lemma 6.4$-$(i) and (v), $^f aq_{\nu\mu}$ with $0 < |\nu| + |\mu| < 3$ defines an operator of the form of the right hand side of (v) in the same lemma with $k = l = 1/2$ and $N = 3$, and from (6.24), $t^{\varepsilon-1} \mathscr{B}^{3/2,1/2,3} \subset \mathscr{B}^{3/2,3/2,3}$ which completes the proof.

We shall use for proof of Proposition 2.3 the localization operator of the form Φ^n, $n \in N$ with $\Phi = Op\ \varphi$, $\varphi \in \Sigma_{1/2}^0$, and φ^n being an element of partitions of unity of $\boldsymbol{R}_T^3$ which is contained in $\Sigma_{1/2}^0$ and any element ψ of which satisfies supp $d\psi \subset \{t^\varepsilon \langle \xi \rangle \ge c\}$ for some $c > 0$ and $0 < \varepsilon \le e_h$. Such partitions of unities have been constructed in Lemma 3.2. So, we need to know that $\Phi^n - Op\ \varphi^n$ is small in norm.

Lemma 6.26. *Let $0 < \varepsilon \le e_h$ and $c > 0$. Let $a, b \in \Sigma_{1/2}^0(T, h)$ whose gradients supported in $D_\varepsilon \equiv \{t^\varepsilon \langle \xi \rangle \ge c\}$. Then, if T is small, $Op(^f a \circ {}^f b) - Op(^f a \cdot {}^f b) \in t^\varepsilon \mathscr{A}^{0,N}$ for any $N > 0$.*

Proof. Since $\langle t^{\varepsilon_h} \xi \rangle \le Ct^\varepsilon \langle \xi \rangle$ in D_ε, for $\tilde{a} \in \Sigma_{1/2}^0$ supported in D_ε

$$(6.30) \qquad |\partial^\alpha \, \partial_x^k \, \partial_\xi^l \, \tilde{a}| \le C_{\alpha k l} \, t^{\varepsilon(k+l)/2} \, \langle \xi \rangle^{(k-l)/2} \cdot \tilde{\gamma}^{-\alpha} \,.$$

If T is small, $|\xi| \ge 1$ on D_ε so that using (4.16) and Lemma 3.8—5 one can show that for $\tilde{a}$ as above,

$$f^\varepsilon \, \partial^\alpha \, {}^f\tilde{a} \in t^{\varepsilon(\alpha_2 + \alpha_3)/2} \, f \Sigma_{1/2}^{(-\alpha_1, 0, 0) + (\alpha_2 - \alpha_3)/2(0, -1, 1)} \quad \text{for} \quad \varepsilon > 0 \,,$$

and that ${}^f a$ also satisfies (6.30) if $k + l > 0$, which imply that $\partial_x^k \, {}^f a \cdot \partial_\xi^k \, {}^f b \in t^{\varepsilon} {}^f \Sigma_{1/2}^0$ for $k > 0$ and that $r_N[\partial_\xi^N \, {}^f a, \, t^{-N\varepsilon/2} \, D_x^N \, {}^f b] \in f^{N/2} \, \Sigma^0$ in view of Lemma 3.8—5 and Corollary 4.12—(i). This completes the proof.

Corollary 6.27. *Let a be as in the above lemma. Then, if T is small, for any $n \ge 2$ we have*

$$(Op \, {}^f a)^n - Op(({}^f a)^n) \in t^\varepsilon \, \mathcal{A}^{0, N} \quad \text{for} \quad N > 0 \,.$$

Next lemma is the one to handle the reminder term in Corollary 6.27 when the localization of $q_{\nu\mu}$ is carried out.

Lemma 6.28. *Let q be as in Definition 6.21. There exists $N_0 > 0$ such that if $0 < |\nu| + |\mu| < 3$, for any $a \in t^\varepsilon \, \mathcal{A}^{0, N}$ with $N > N_0$ and $\varepsilon > 0$, we have*

$$q_{\nu\mu} \circ a - a \circ q_{\nu\mu} - \sum_{\substack{0 < |\alpha| + |\beta| < 3 \\ \beta_1 = \mu_1}} a_{\alpha\beta} \circ q_{\alpha\beta} \in E$$

with $a_{\alpha\beta} \in t^\varepsilon \, \Sigma^0$.

Proof. Let $a \in t^\varepsilon({}^f \Sigma_{1/2}^0 + f^N \Sigma^0)$ with $\varepsilon > 0$ and let $\nu, \mu \in \mathbf{Z}_+^2$ with $0 < |\nu| + |\mu| < 3$. Using (6.26), the fact that $\partial^\alpha \, b \in t^\varepsilon \, \Sigma^{d-\alpha}$ (resp. $f^{\varepsilon'} \, \partial^\alpha \, b \in t^\varepsilon \, {}^f \Sigma_{1/2}^{d-\alpha}$ for $\varepsilon' > 0$) if $b \in t^\varepsilon \, \Sigma^d$ (resp. $b \in t^\varepsilon \, {}^f \Sigma_{1/2}^d$) which follow from (6.21), Lemma 3.8—5, and (4.16), and the one that $f^s \langle t^{\varepsilon_h} \xi \rangle^s \langle t^{\varepsilon_h} f\xi \rangle^{-s} \in \Sigma_{1/2}^0$ for $s > 0$, we have that when $i, j \in \mathbf{Z}_+$ and $N > \dfrac{i+j}{2} > 0$,

$$(6.31) \qquad \partial_\tau^i \, \partial^i \, q_{\nu\mu} \, \partial_t^i \, \partial_x^j \, a = \sum_{\substack{l \le j \\ m \le \mu_2}} b_{lm} \, q_{(\nu_1 + i, \, \nu_2 + j - l)(\mu_1, m)}$$

$$\text{with} \quad b_{lm} \in t^\varepsilon({}^f \Sigma_{1/2}^{(-\mu_2 + m - l)/2(0, -1, 1)} + f^{(N - (i+j)/2)} \, \Sigma^{(\mu_2 - m + l)(0, 1, 0)})$$

$$\text{such that} \quad b_{jm} = 0 \text{ if } \mu_2 > 0 \text{ and } \mu_2 > m \,.$$

We note that from the fact that

$$\text{if } a \in B^{k, l, N} \text{ and } b \in B^{k', l', N} \text{ with } N \ge k + k', \, ab \in B^{k + k', l + l', N}$$

which follows from the fact that $d + e \in \Delta_{k + k', l + l'}$ if $d \in \Delta_{k, l}$ and $e \in \Delta_{k', l'}$, that $\mathcal{G}(d) = d$ if $d \in \Delta_{k, l}$, and Fact 4.6—7, and we also note that $f^{1/2} \, a \in \Sigma^{(0, 1, 0)}$ if

$a \in {}^f \Sigma^{(0,1,0)}$, which follows from (4.3) and (4.6). Using these facts we see

(6.32)   if $a \in B^{0,2}$ and $q \in E$, $aq \in E$.

Thus, noting that $\langle f\xi \rangle^{\alpha/2} a \in t^\varepsilon \mathcal{A}^{0,N}$ for $a \in {}^f \Sigma_{1/2}^{\alpha/2(0,1,-1)} + f^N \Sigma^{(0,\alpha,0)}$ and $\alpha \geq 0$, and that $A^{0,N} \subset B^{0,N}$, and using (6.32) and Lemma 6.22 we see that

(6.33) $aq_{\nu\mu} \in E$ for $a \in {}^f \Sigma_{1/2}^{j/2(0,1,-1)} + f^2 \Sigma^{(0,j,0)}$ with $j \in \mathbf{Z}_+$ and

$\nu, \mu \in \mathbf{Z}_+^2$ with $|\nu| + |\mu| > 0$, $j + |\nu| + |\mu| \geq 3$, and $\mu_1 \leq 2$.

Since it follows from (6.20) that for $\nu, \mu \in \mathbf{Z}_+^2$ with $0 < |\nu| + |\mu| < 3$, $q_{\nu\mu} = \sum_{k=0}^{2} {}^f a_k \tau^k$ with $a_k(t, x, \xi) \in C^\infty(\mathbf{R}_T^3)$ satisfying $|\partial^\alpha a_k| \leq C_\alpha t^{-(3-k)-\alpha_1} \langle \xi \rangle^{5/2-k-\alpha_3}$ which implies by (4.16) that $|\partial^\alpha {}^f a_k| \leq C_\varepsilon t^{-(3-k)-\alpha_1} \langle f\xi \rangle^{5/2-k} \langle \xi \rangle^{-\alpha_3 + \varepsilon\alpha_2}$ for any $\varepsilon \in (0, 1)$, there exists $M_1 \in N$ such that for any $M \geq M_1$ and $i \leq 2$,

(6.34) $r_M[\partial_\xi^M a, D_x^M q_{\nu\mu}], r_M[\partial_\tau^i \partial_\xi^M q_{\nu\mu}, D_t^i D_x^M a] \in E$,

by using the fact that $D_x^M a_k \in t^{-(3-k)} \Sigma^{(5-M/2)(0,-1,0)}$,

Corollary 4.12—(i), and Lemma 4.19 for the former, and by using the fact that $a_k \in t^{-(3-k)} \Sigma_{1/4}^{5(0,-1/2,1)}$ and Corollary 4.12—(i) for the latter. In (6.31) we have that $i + j + |\nu| + |\mu| = \mu_2 - m + l + |(\nu_1 + i, \nu_2 + j - l) + (\mu_1, m)|$, that $|(\nu_1 + i, \nu_2 + j - l) + (\mu_1, m)| > 0$ if $b_{lm} \neq 0$, and that $\mu_1 \leq 2$. So, from (6.33) we have in (6.31) that $\partial_\tau^i \partial_\xi^i q_{\nu\mu} \cdot \partial_t^i \partial_x^i a \in E$ if $i + j + |\nu| + |\mu| \geq 3$ and $N \geq \frac{i+j}{2} + 2$. Thus, for $a \in t^\varepsilon A^{0,N}$ with $\varepsilon > 0$ it follows that

(6.35) $q_{\nu\mu} \circ a - \sum_{i+j \leq 2} (i! \, j!)^{-1} D_\tau^i D_\xi^j q_{\nu\mu} \cdot \partial_t^i \partial_x^i a \in E$

if $0 < |\nu| + |\mu| < 3$ and $N \geq (M_1 + 2)/2 + 2$.

Now we claim that for any $d, e \in \mathbf{R}^3$ with $d_1 = e_1 = 0$, $\sigma_{1/2}(d) \leq 0$, $\sigma(e) \leq 0$, there exists $N_0 > 0$ such that

(6.36) for $N > N_0$, $a \in {}^f \Sigma_{1/2}^d + f^N \Sigma^e$, and $\alpha, \beta \in \mathbf{Z}_+^2$ with $0 < |\alpha| + |\beta| < 3$,

there exist $a_{\lambda\gamma} \in \Sigma$ for pairs $(\lambda, \gamma) \in \mathbf{Z}_+^2 \times \mathbf{Z}_+^2$ with $\lambda \geq \alpha$, $\gamma \geq \beta$ and with $\gamma_1 = \beta_1$ and $|\lambda| + |\gamma| < 3$, such that $a_{\alpha\beta} \in {}^f \Sigma_{1/2}^{1/2(0,1,-1)+d} + f^{N/2} \Sigma^{(0,1,0)+e}$, $a_{\lambda\gamma} \in {}^f \Sigma_{1/2}^d + f^{N/2} \Sigma^e$ for $(\lambda, \gamma) \neq (\alpha, \beta)$, satisfying that

$$a \circ q_{\alpha\beta} - a \cdot q_{\alpha\beta} - \sum_{(\lambda,\gamma)} a_{\lambda\gamma} q_{\lambda\gamma} \in E.$$

It is not difficult to show Lemma 6.28 using (6.31), (6.33), (6.35), (6.36). The details are omitted and we shall prove (6.36).

Since for any $\varepsilon > 0$ and any $b \in C^\infty(\mathbf{R}_T^3)$ satisfying with $d \in \mathbf{R}^3$ that $|\partial^\alpha b$

$(t, x, \xi)| \leq C_\alpha\, t^{-\alpha}{}_1 \langle\xi\rangle^{-\alpha}{}_3 \cdot \psi'^d$ for any $\alpha\in Z_+^3$ (see Definition 6.25 for the defini-
tion of ψ'^d),we have by (4.16) that $f^\varepsilon\,\partial^\alpha\,{}^f b = {}^f b_\alpha$ with $b_\alpha(t, x, \xi)\in C^\infty(R_T^3)$ satisfy-
ing $|\partial^\beta\, b_\alpha|\leq C_\beta\, t^{\alpha_1-\beta_1}\langle\xi\rangle^{-\alpha_3-\beta_3}\cdot\psi'^d$, and since $|\partial^\alpha\,\psi'^d|\leq C_\alpha\,\psi'^d\,t^{-\alpha}{}_1\langle\xi\rangle^{-\alpha_3}$
by (6.17), for any $i\in N$

$$(6.37) \qquad f^i\,\partial_x^i\,q_{\alpha\beta} = \sum_{l\leq i} f^{l+\beta_2}\cdot\partial_x^{l+\beta_2}f(\partial_T^{\alpha_1}\,\partial_\xi^{\alpha_2}\,\partial_t^{\beta_1}\,q)\langle f\xi\rangle^{(\alpha_2-\beta_2-l)/2}$$
$$\times\langle t^{\varepsilon_h}f\xi\rangle^{(\alpha_1-\beta_1)/2}\,t^{-\alpha}{}_1{}^{+\beta_1}\cdot {}^f b_l$$

with $b_l\in C^\infty(R_T^3)$ satisfying $|\partial^\alpha\,b_l(t, x, \xi)|\leq C_\alpha\,t^{-\alpha}{}_1\langle\xi\rangle^{l/2-\alpha_3}$. We may assume
that $a={}^f a_1+f^N\,a_2$ with $a_1\in\Sigma_{1/2}^d$, $a_2\in\Sigma^e$ for d, e as in (6.36). Since $\partial_\xi^i\,a_1$
$\langle\xi\rangle^{l/2}\in\Sigma^{d-1/2(i-l)(0,-1,1)}$ and $\partial_\xi^i\,a_2\langle\xi\rangle^{l/2}\in\Sigma^{e-(i-l)(0,-1,0)}$ for $i, l\in Z_+$ with $i\geq l$,
we have by (6.37) that when $N > M_1$ and $i\in N$ with $i\leq M_1$ where M_1 is the one
in (6.34),

$$(6.38) \qquad \partial_\xi^i\,aD_x^i\,q_{\alpha\beta} = \sum_{l\leq i} q_{\alpha\beta+(0,l)}\,b_l \quad \text{for some} \quad b_l\in {}^f\Sigma^{d-1/2(i-l)(0,-1,1)}$$
$$+f^{N-i}\,\Sigma^{e+(i-l)(0,1,0)}\,.$$

We take $N_0=2M_1+2$ so that $N-i\geq N/2\geq 2$ if $N > N_0$. From (6.33) and the
assumption on d, e, we have that

$$(6.39) \qquad\qquad \text{if } |\alpha|+|\beta|+l\geq 3, \, q_{\alpha\beta+(0,l)}\,b_l\in E\,.$$

And we also have in (6.38) that

$$(6.40) \qquad\qquad b_0\in {}^f\Sigma_{1/2}^{d-1/2(0,-1,1)}+f^{N/2}\,\Sigma^{e+(0,1,0)}\,,$$

$$(6.41) \qquad\qquad b_l\in {}^f\Sigma_{1/2}^d+f^{N/2}\,\Sigma^e \quad \text{for all } l\,.$$

In view of (6.34), (6.38)~(6.41) prove (6.36) with N_0 as in the above by the
asymptotic expansion of the symbol $a\circ q_{\alpha\beta}$ with breaking off it after M_1 terms.

Next we shall study effects of localization by the partitions of unities in
Lemma 3.2.

Definition 6.29. Let X be a subset of R_T^3 and let $a\in\Sigma$. We say a be-
longs to $\Sigma_{\rho,X}^d$, $d\in R^3$ with $d_1\leq 0$, $\rho\in[0, 1/2]$ (resp. Σ_X^d, $d\in R^3$ with $d_1\leq 0$) if
a satisfies

$$\sup_{(t,x,\xi)\in X}|\partial^\alpha\,a|\,\psi_1^{-d_1}\cdot\psi_2^{-d_2}\cdot\psi_{3,\rho}^{-d_3} < \infty$$
$$(\text{resp. }\sup_{(t,x,\xi)\in X}|\partial^\alpha\,a|\,\tilde\psi_1^{-d_1}\langle\xi\rangle^{d_2/2}\,\psi_{3,1/2}^{-d_3} < \infty)\,.$$

It is clear that

(6.42) $a\in\Sigma^d_{\rho,X}$ (resp. Σ^d_X) and supp $a\subset X$ implies $a\in\Sigma^d_\rho$ (resp. Σ^d),

   $a\in\Sigma^d_{\rho,X}$ (resp. Σ^d_X) implies $\partial^\alpha a\in\Sigma^{d-\alpha}_{\rho,X}$ (resp. $\Sigma^{d-\alpha}_X$)

   $a\in\Sigma^d_{\rho,X}$ (resp. $\Sigma^d_{\rho,X}$) and $b\in\Sigma^e_{\rho,X}$ (resp. Σ^e_X) implies $ab\in\Sigma^{d+e}_\rho$,
   (resp. Σ^{d+e}_X),

   $a,b\in\Sigma^d_{\rho,X}$ (resp. Σ^d_X) implies $a+b\in\Sigma^d_{\rho,X}$ (resp. Σ^d_X).

We set

$$\Sigma_X = \bigcup_d \left(\bigcup_\rho \Sigma^d_{\rho,X}\cup\Sigma^d_X\right).$$

Lemma 6.30. *Let* $a\in C^\infty(\boldsymbol{R}^3_T)$ *satisfying with some* $k, l\geq 0$ *that* $|\partial^\alpha a(t, x, \xi)|\leq C_\alpha\, t^{-k-\alpha_1}\langle\xi\rangle^{l-\alpha_3}$ *for any* $\alpha\in\boldsymbol{Z}^3_+$, *and let* $b\in\Sigma^d_{1/2}$ (*resp.* Σ^d). *Then, for any* $N\in\boldsymbol{N}$ *there exists* $M\in\boldsymbol{N}$ *such that*

$$^fa\circ{}^fb-\sum_{j\leq M}\partial^{jf}_\xi a\cdot D^{jf}_x b/j\,!,\ ^fb\circ{}^fa-\sum_{j\leq M}\partial^{jf}_\xi b\cdot D^{jf}_x a/j\,!\in t^{d_1-k}\,f^N\,\Sigma^{(0,N,0)}.$$

Proof. Since $\Sigma^d_{1/2}\subset\Sigma^{J}{}_{1/2}{}^{(d)}$, it suffices to consider the latter case that $b\in\Sigma^d$. Since $a\in t^{-k}\Sigma^l_{1/4}{}^{(0,-1/2,1)}$, Fact 4.6$-$7 and Corollary 4.2$-$(i) show the first one. Next, from (4.16), we have for any $\varepsilon>0$ that $|\partial^\alpha\, {}^fa(t, x, \xi)|\leq C_{\alpha\varepsilon}\, t^{-k-\alpha_1}\langle\xi\rangle^{l-\alpha_3+\varepsilon\alpha_2}$ so that $D^{Mf}_x a\in t^{-k}\Sigma^{(l+\varepsilon)(0,-2,0)}$. Thus, Corollary 4.12$-$(ii) shows the second one. The proof is complete.

The following lemma is an easy consequence of (4.16).

Lemma 6.31. (1) *Let* $a\in\Sigma^d_{\rho,X}$ (*resp.* Σ^d_X). *Then, for any* $\varepsilon>0$ *and* $\alpha\in\boldsymbol{Z}^3_+$, *there exists* $a_{\alpha\varepsilon}\in\Sigma^{d-\alpha}_{\rho,X}$ (*resp.* $\Sigma^{d-\alpha}_X$) *such that* $f^{\varepsilon\alpha_2}\cdot\partial^\alpha\, {}^fa={}^fa_{\alpha\varepsilon}$ *with* supp $a_{\alpha\varepsilon}\subset$ supp a.

(2) *Let* $a\in C^\infty(\boldsymbol{R}^3_T)$ *satisfying the estimates of the* a *in Lemma 6.30. Then, for any* $\varepsilon>0$ *and* $\alpha\in\boldsymbol{Z}^3_+$, *there exists* $a_{\alpha\varepsilon}\in C^\infty(\boldsymbol{R}^3_T)$ *such that* $f^{\varepsilon\alpha_2}\cdot\partial^\alpha\, {}^fa={}^fa_{\alpha\varepsilon}$ *with the estimates that* $|\partial^\beta a_{\alpha\varepsilon}(t, x, \xi)|\leq C_\beta\, t^{-k-\alpha_1-\beta_1}\langle\xi\rangle^{l-\alpha_3-\beta_3}$.

Lemma 6.32. *Let* $D_t-Op^f a$ *with* $a\in C^\infty(\boldsymbol{R}^3_T)$ *satisfying the estimates for* a *in Lemma 6.30. Let* $b\in\Sigma^d_{1/2}$ (*resp.* Σ^d). *Then, for any* $N\in\boldsymbol{N}$ *there exists* $c\in t^{-k}\Sigma^{d+(l-1/2)(0,-1,1)}_{1/2}+\Sigma^{d+(-1,0,0)}_{1/2}$ (*resp.* $t^{-k}\Sigma^{d+(l-1/2)(0,-2,0)}+\Sigma^{d+(-1,0,0)}$) *with* supp $c\subset$ supp b *such that* $[L, Op\,^fb]-Op\,^fc\in f^N\, t^{d_1-k}\Sigma^{(0,N,0)}$.

Proof. This follows from Lemma 6.30 and 6.31 in view of the equality, $[L, Op\,^fb]=Op\, D_t\,^fb-[Op\,^fa, Op\,^fb]$.

Lemma 6.33. *Let* $g(t, x, \xi)=d\xi^3+f^{1/2} a\xi^2$ *with* $d\in E_{h,T}$ *and* $a\in B_{h,T}$. *Let* X *be a subset of* $\boldsymbol{R}^3_T$ *and assume that* $|g|\leq Ct^{-\kappa}\langle\xi\rangle^{3/2}$ *with* $\kappa=3/2-1/(3h)$ (*resp.*

$|g| \leq C \langle \xi \rangle^2)$ on X. Then,

$$|\partial^\alpha g| \leq C_\alpha \, t^{-\kappa-\alpha_1} \langle \xi \rangle^{3/2-\alpha_3} \cdot \langle t^{e_h} f^{1/(10h)} \, \xi \rangle^{\min(|\alpha|,1)/2} \text{ on } X$$

$$(\textit{resp. } |\partial^\alpha g| \leq C_\alpha \, t^{-\alpha_1} \langle \xi \rangle^{2+\min(1,\alpha_2)-\alpha_3} \text{ on } X)$$

Proof.   We only prove the former case and the latter one is proved similarly.   When $\langle \xi \rangle \leq 2$, the inequality clearly holds.   To consider the opposite case, we note that

$$(6.43) \qquad\qquad t^{e_h/2} f^{1/(20h)} \langle \xi \rangle^{1/2} \leq \langle t^{e_h} f^{1/(10h)} \, \xi \rangle^{1/2} .$$

When $\langle \xi \rangle \geq 2$, using $d\xi^3 = g - f^{1/2} a\xi^2$, $|g|^{1/3} \leq C(|tf|^{1/3h}|\xi| + f^{1/6}|\xi|)$, and the assumption, we see for $\alpha \neq 0$ and $(t, x, \xi) \in X$ that

$$|\partial^\alpha g(t, x, \xi)| \leq C_\alpha \, t^{-\kappa-\alpha_3} \langle \xi \rangle^{3/2-\alpha_3} (t^{\kappa/3+1/(3h)} f^{1/(4h)} \langle \xi \rangle^{1/2}$$
$$+ t^{\kappa/3} f^{1/7} \langle \xi \rangle^{1/6} + t^\kappa f^{1/3} \langle \xi \rangle^{1/2}) .$$

Using $\kappa/3 + 1/(3h) > e_h/2$ and (6.43) show that the first term in the parenthesis on the right can be estimated by $f^{1/(4h)-1/(20h)} \langle t^{e_h} f^{1/(10h)} \, \xi \rangle^{1/2}$, and using $2\kappa/3 > e_h/2$ and (6.43) show that the second one and the last one are estimated respectively by $f^{1/7-1/(60h)} \langle t^{e_h} f^{1/(10h)} \, \xi \rangle^{1/6}$ and $f^{1/3-1/(20h)} \langle t^{e_h} f^{1/(10h)} \, \xi \rangle^{1/2}$.   Thus, we get the desired inequality for $\alpha \neq 0$ and for $\alpha = 0$ nothing but the assumption.   The proof is complete.

Lemma 6.34.   *Let g and X be as in Lemma 6.33 and let $b \in \Sigma_{1/2,X}^e$ (resp. Σ_X^e).   Then, $db \in t^{-\kappa} \Sigma_{1/2,X}^{-3/2(0,-1,1)+\varepsilon} + f^{1/2} \Sigma_{1/2,X}^{-(0,-1,1)+\varepsilon}$ (resp. $db \in \Sigma_X^{-(0,-2,0)+\varepsilon}$).*

Proof.   Let $\chi(\xi) \in C^\infty(\mathbf{R})$ with $\chi = 1$ when $|\xi| \geq 1$ and $\chi = 0$ when $|\xi| \leq 1/2$. Break up db as $\chi db + (1-\chi) \, db$.   The second term $\in t^{e_1} \Sigma_{1/2,X}^{N(0,1,-1)}$ (resp. $t^{e_1} \Sigma_X^{(0,N,0)}$) for any $N \in \mathbf{N}$, and expressing the first term as $\xi^{-3} b\chi g - f^{1/2} ab\xi^{-1}\chi$, and using the fact that $g \in t^{-\kappa} \Sigma_{1/2,X}^{3/2(0,-1,1)}$ (resp. $\Sigma_X^{2\,(0,-2,0)}$) by Lemma 6.33, we have in view of (6.34) that $\chi db \in t^{-\kappa} \Sigma_{1/2,X}^{-3/2(0,-1,1)} + f^{1/2} \Sigma_{1/2,X}^{-(0,-1,1)+\varepsilon}$ (resp. $\Sigma_X^{-(0,-2,0)+\varepsilon}$).   This completes the proof.

Lemma 6.35.   *Let $L_i = D_t - a_i(t, x) f(x) D_x$ with $a_i \in B_{h,T}$, $i = 1, 2, 3$, let $b \in {}^f\Sigma_{1/2}^0$, and let with $c_i, c_{ij} \in B_{h,T}$,*

$$R = f^{1/2}(x) \left(\sum_{i=1}^3 c_i(t, x) f(x) D_x \cdot L_i \right.$$
$$\left. + \sum_{i \neq j} c_{ij}(t, x) L_i L_j + t^{1/h-1} c_0(t, x) f(x) D_x \right) .$$

Then,

$$[Op \, b, R] \in \mathcal{C}_0, \quad Op \, b \cdot R \in \mathcal{C}(L_i) + \mathcal{C}_0 .$$

Proof. It is easy to see that $b \circ f^{1/2} \in f^{1/2} \, {}^f\Sigma^0_{1/2} + f^N \, \Sigma^{(0,1,0)}$ for all $N \in \mathbf{N}$. So, the second one follows from Lemma 4.18—(1). Next, we set $B = Op \, b$, $C_i = f^{1/2} c_i fD_x$ for $i \neq 0$, $C_{ij} = f^{1/2} c_{ij}$, $C_0 = t^{1/h-1} c_0 f^{3/2} D_x$. Then,

$$(6.44) \quad [B, R] = \sum_{i=1}^{3} ([B, C_i] L_i + C_i [B, L_i]) + \sum_{i \mp j} ([B, C_{ij}] L_i L_j + C_{ij}[B, L_i] L_j$$
$$+ C_{ij}[L_i, [B, L_j]] + C_{ij}[B, L_j] L_i) + [B, C_0],$$

$$(6.45) \quad [B, C_i] \in \bigcap_{N \geq 1/2} \mathcal{A}^{1/2,N} \quad \text{for} \quad i \neq 0 \quad \text{and} \quad [B, L_i] \in \bigcap_{N \geq 1/2} \mathcal{A}^{1/2,1/2,N}$$

by Lemmas 4.18—(3)—(i) and 4.22,

$$[B, C_{ij}] \in Op \, {}^f\Sigma^{1/2(0,-1,1)}_{1/2} + f^N \, Op \, \Sigma^{(0,N,0)} \quad \text{for} \quad N > 0$$

by Lemma 4.18—(4),

$$[L_i, [B, L_j]] \in \bigcap_{N \geq 1} \mathcal{A}^{1,1,N}$$

by Lemmas 4.18—(3)—(i) and 4.22,

$$[B, C_0] \in \bigcap_{N \geq 1/2} t^{1/h-1} \mathcal{A}^{1/2,N}$$

by Lemma 4.18—(1).
Thus, from Lemma 4.18—(1) and that $\mathcal{A}^{k,l,N} \subset \mathcal{B}^{k,l,N}$, all terms are in (6.44) except the first ones in the each one of the second summation. Next, we note

$$(6.46) \quad L_i L_j = D_t^2 + d_1(t, x) f(x) D_x D_t + d_2(t, x) f(x)^2 D_x^2$$
$$+ t^{1/h-1} d_3(t, x) f(x) D_x,$$

and we let $a \in C^\infty(\mathbf{R}^3_T)$ satisfying the estimate for a in Lemma 6.30. From Lemmas 6.30 and 6.31, for $b_1 \in {}^f\Sigma^{1/2(0,1,-1)}_{1/2}$ we have $b_1 \circ {}^f a \in t^{-k} \, {}^f\Sigma^{(l-1/2)(0,-2,0)} + f^M t^{-k} \Sigma^{(0,M,0)}$ for any $M \in \mathbf{N}$. So, (6.45) and (6.46) show that $[b, C_{ij}] L_i L_j \in C_0$. This completes the proof.

Lemma 6.36. *Let* $b \in \Sigma^d_{1/2}$ *(resp.* Σ^d*) and let* $a \in C^\infty(\mathbf{R}^3_T)$ *satisfying* $|\partial^\alpha a| \leq C_\alpha t^{-\alpha_1-k} \langle \xi \rangle^{l-\alpha_3}$. *Then, there exists* c_1, c_2, *belonging to* $t^{-k} \Sigma^{d+(l-1/2)(0,-1,1)}_{1/2}$ *(resp.* $t^{-k} \Sigma^{d+(l-1/2)(0,-2,0)}$*) with* $\mathrm{supp} \, c_i \subset \mathrm{supp} \, a \cap \mathrm{supp} \, b$ *satisfying* ${}^f a \circ {}^f b - {}^f(a \cdot b) - {}^f c_1$ *and* ${}^f b \circ {}^f a - {}^f(a \cdot b) - {}^f c_2$ *are in* $f^N t^{d_1-k} \Sigma^{(0,N,0)}$.

This is an immediate consequence of Lemma 6.30 and 6.32.

Lemma 6.37. *Let* $b = {}^f\tilde{b}$ *with* $\tilde{b} \in \Sigma^0_{1/2}$. *Let* $L = D_t - f(x) a(t, x) D_x$ *with* $a \in B_{h,T}$. *Then,*

$$[L, Op \, b] \in Op(\Sigma^{(-1,0,0)}_{\tilde{b}} + \Sigma^{1/2(0,-1,1)}_{\tilde{b}} + f^N \Sigma^{(0,N,0)}) \quad \text{for} \quad N \in \mathbf{N}.$$

Lemma 6.38. *Let* $g=d\xi^3+f^{1/2}\,a\xi^2$ *with* $d\in E_{h,T}$ *and* $a\in B_{h,T}$, *and let* $\varphi\in\Sigma^0_{1/2}$. *Assume that* $|g|\leq Ct^{-\kappa}\langle\xi\rangle^{3/2}$ *on* supp φ *with* κ *in Lemma 6.33. Then, for any* $b\in B_{h,T}$,

$$^f\varphi\circ^f(bg)\in t^{-\kappa}\,\mathscr{A}^{3/2,3}\,.$$

Lemma 6.37, 6.38 are immediate consequences of Lemmas 6.30, 6.31, 6.33.

Lemma 6.39. *Let* $g=d\xi^3+f^{1/2}\,a\xi^2$ *with* $d\in E_{h,T}$ *and* $a\in B_{h,T}$, *and let* $\varphi\in\Sigma^e_{1/2}$ (*resp.* Σ^e). *Assume that* $|g|\leq Ct^{-\kappa}\langle\xi\rangle^{3/2}$ (*resp.* $|g|\leq C\langle\xi\rangle^2$) *on* supp φ. *Then,*

(1) $d(f\xi)^l\circ^f\varphi-d(f\xi)^l\cdot^f\varphi\in t^{-\kappa}\,{}^f\Sigma^{(l-2)(0,-1,1)+e}_{1/2}+f^{1/2}\,{}^f\Sigma^{(l-3/2)(0,-1,1)}_{1/2}$
   (*resp.* $^f\Sigma^{(l-3/2)(0,-2,0)+e}$) *for any* $l\in\mathbf{Z}_+$,

(2) $d(f\xi)^l\circ^f\varphi\in t^{-\kappa}\,{}^f\Sigma^{(l-3/2)(0,-1,1)+e}_{1/2}+f^{1/2}\,\Sigma^{(l-1)(0,-1,1)+e}_{1/2}$
   (*resp.* $^f\Sigma^{(l-1)(0,-2,0)+e}$) *for any* $l\in\mathbf{Z}_+$,

(3) $^f\varphi\circ(bd(f\xi)^l)-bd(f\xi)^l\cdot^f\varphi\in t^{-\kappa}\,{}^f\Sigma^{(l-2)(0,-1,1)+e}_{1/2}+f^{1/2}\,{}^f\Sigma^{(l-3/2)(0,-1,1)+e}$
   $\qquad+t^{e_1}f^N\,\Sigma^{(0,N,0)}$
   (*resp.* $^f\Sigma^{(l-3/2)(0,-2,0)+e}+t^{e_1}f^N\,\Sigma^{(0,N,0)}$) *for any* $b\in B_{h,T},l\in\mathbf{Z}_+$,
   and $N\in\mathbf{N}$,

(4) $^f\varphi\circ(bd\,(f\xi)^l)\in t^{-\kappa}\,{}^f\Sigma^{(l-3/2)(0,-1,1)}+f^{1/2}\,{}^f\Sigma^{(l-1)(0,-1,1)+e}_{1/2}+t^{e_1}f^N\,\Sigma^{(0,N,0)}$
   (*resp.* $^f\Sigma^{(l-1)(0,-2,0)}+f^N\,t^{e_1}\,\Sigma^{(0,N,0)}$) *for any* $b\in B_{h,T}$, $l\in\mathbf{Z}_+$, *and* $N\in\mathbf{N}$.

Proof. (1) and (2) follow from Lemmas 6.30, 31–(1), 34. (3) and (4) follow from Lemmas 6.30, 31, 34, and that $d\in E_{h,T}$. Indeed, for any $\varepsilon>0$ and $j\in\mathbf{N}$ we have for some $b_{j\varepsilon}\in B_{h,T}$ that $f^\varepsilon\,\partial^j_x(bd(f\xi)^l)=b_{j\varepsilon}\,d(f\xi)^l$. This completes the proof.

Lemma 6.40. *Let* $g=d\xi^3+f^{1/2}\,a\xi^2$ *with* $d\in E_{h,T}$ *and* $a\in B_{h,T}$, *let* $\varphi\in\Sigma^0_{1/2}$, *and let* $L=D_t-f(x)\,\lambda(t,x)\,D_x$ *with* $\lambda\in B_{h,T}$. *Assume that* $|g|\leq Ct^{-\kappa}\langle\xi\rangle^{3/2}$ *on* snpp φ.

(1) *For any* $b\in B_{h,T}$ *and any* $l\in\mathbf{Z}_+$, *there exists* c *belonging to*
   $t^{-\kappa}\,{}^f\Sigma^{(l-3/2)(0,-1,1)}_{1/2}+f^{1/2}\,{}^f\Sigma^{(l-1)(0,-1,1)}_{1/2}+f^N\,\Sigma^{(0,N,0)}$ *such that*
   $Op\,{}^f\varphi\cdot bdf^lD^l_x\cdot L=Op\,c\cdot L.$

(2) *For any* $l\in\mathbf{Z}_+$ *and any* $N\in\mathbf{N}$, *there exist* c_1, c_2 *belonging respectively to*
   $t^{-\kappa}\,{}^f\Sigma^{(l-3/2)(0,-1,1)}_{1/2}+f^{1/2}\,{}^f\Sigma^{(l-1)(0,-1,1)}_{1/2},\ t^{-\kappa}\,{}^f\Sigma^{(l-3/2)(0,-1,1)}_{1/2}$
   $+f^{1/2}\,{}^f\Sigma^{(l-1)(0,-1,1)+(-1,0,0)}_{1/2}+t^{-\kappa}\,{}^f\Sigma^{(l-1)(0,-1,1)}_{1/2}+f^{1/2}\,{}^f\Sigma^{(l-1/2)(0,-1,1)}_{1/2}$
   $+f^N\,\Sigma^{(0,N,0)}$ *such that*

   $$df^lD^l_x\cdot L\cdot Op\,{}^f\varphi = Op\,c_1\cdot L+Op\,c_2\,.$$

(3) *For any* $b\in B_{h,T}$, *any* $l\in\mathbf{Z}_+$, *and any* $N\in\mathbf{N}$, *there exist* c_1,c_2 *being res-*

pectively in $t^{-\kappa} f \Sigma_{1/2}^{(l-2)(0,-1,1)} + f^{1/2} f \Sigma^{(l-3/2)(0,-1,1)} + f^N \Sigma^{(0,N,0)}$,

$t^{-\kappa} (f \Sigma_{1/2}^{(l-3/2)(0,-1,1)+(-1,0,0)} + f \Sigma_{1/2}^{(l-1)(0,-1,1)}$

$+ f^{1/2} (f \Sigma_{1/2}^{(l-1)(0,-1,1)+(-1,0,0)} + f \Sigma_{1/2}^{(l-1/2)(0,-1,1)}) + f^N \Sigma^{(0,N,0)})$ *such that*

$$[bdf^l D_x^l \cdot L, \, Op \, {}^f\varphi] = Op \, c_1 \cdot L + Op \, c_2 \, .$$

Proof. (1) follows from Lemma 6.39–(4). Next, as for (2) and (3), setting $A = df^l D_x^l$, $B = Op \, {}^f\varphi$ we have

$$ALB = A[L, B] + ABL \quad \text{and} \quad b[AL, B] = bA[L, B] + [bA, B] \, L \, .$$

So, (2) and (3) follow from Lemmas 6.36, 37, 39–(1)~(3), and the fact that if $r \in \Sigma^{(0,N,0)}$ with $N \in \mathbb{N}$, $f^l \, \partial_x^l (f^N r) \in f^N \Sigma^{(0,N-l,0)}$ for any $l \in \mathbb{N}$. This completes the proof.

§7. Lemmas on the Estimates

Remark on notations. Constants denoted by C or C_i in the inequalities involving the norms $|| \; ||^{(n)}$, $|| \; ||_{(n,s)}$, $|| \; |_{\{s\}}^{\{n\}}$ are *always independent of n, T, $u \in \mathcal{S}_T$. We also remark that u always denotes an element of $\mathcal{S}_T$.*

Lemma 7.1. (1) *For any $k, l \in \mathbb{N}$ with $1 \leq k, l \leq 6$, $n \geq 1$, $T \in (0, 1)$,*

$$||A_{k/2} u||^{(n+(l-k)/2)} \leq C(n^{(k-l)/2} ||A_{l/2} u||^{(n)} + n^{k/2} ||u||^{(n+l/2)}) \, .$$

(2) $||D_t u||^{(n)} \geq 2^{1/2} n ||u||^{(n+1)}$ *for $n \geq 1$, $T \in (0, 1)$.*

(3) *Let L_i, $i = 1, 2, 3$ be three operators as in Corollary 5.2. Then for small T, $n \geq 1$,*

$$||u||_{\{6\}}^{\{n\}} \leq C M_n(u) \, .$$

(4) *Let L_i be as in (3). Then, for small T, $n \geq 1$,*

$$\sum_{i \neq j} \sum_{k=0}^{2} n^{(l-k)/2} ||A_{k/2} L_i L_j u||^{(n+1-k/2)}$$
$$+ \sum_{i=1}^{3} \sum_{k=0}^{4} n^{l-k/2} ||A_{k/2} L_i u||^{(n+2-k/2)} \leq C M_n(u) \, .$$

(5) *Let Q be an operator on $\mathcal{S}_{T_0}$ with T_0 in the statement of Corollary 5.2, and L_i as in (3) of the form,*

$$Q = \sum_{i \neq j} A_{ij} L_i L_j + \sum_{i=1}^{3} A_i L_i + A_0 \quad \text{with}$$
$$A_{ij} \in t^{-1} \mathcal{B}^{0,0}, \, A_i \in t^{-1} \mathcal{B}^{1/2,1/2,1/2} \quad \text{for} \quad i \neq 0 \, ,$$
$$A_0 \in \sum_{k=0}^{\min(2,j)} \sum_{l=0}^{3} \sum_{j=0}^{3-l} t^{-(k/2+3-l-j)} \mathcal{B}^{(l+j)/2,1/2,3/2} \, .$$

Then, for small T, $n \geq 2$,

$$\|Qu\|^{(n)} \leq C(n^{-1/2} M_n(u) + \|u\|^{(n-1/(20h))}_{(3)}) .$$

(6) $n^{3/2-s/2} \|u\|_{(s,n)} \leq \|u\|^{(n-3+s/2)}_{(s)}, \ n \geq 4-s/2.$

(7) $\sum_{i+j \leq s} \|f^i D^i_x D^j_t u\|^{(n)} \leq C\|u\|_{(2s,n)}, \ s = 1, 2, 3.$

(8) $\|u\|_{(2s,n)} \leq C \sum_{i=0}^{s} \|f^{s-i} D^{s-i}_x D^i_t u\|^{(n)}, \ s = 1, 2, 3.$

Proof. To see (1) one only have to apply Lemma 4.1–(i) with $\varepsilon = tn^{-1}$. (2) is a direct consequence of a simple integration by parts. (3) and (4) immediately follow from Corollary 5.2 and (1), (2) in this lemma. (5) is proved by straightforward but tedious applications of Lemmas 4.21–(1) and –(3), 7.1–(4), and 4.18–(1). We omit the details. (6) follows from (4.55) and the definition. (7) follows from Lemma 4.1–(1). (8) follows from Lemma 4.1–(1), and (2) in this lemma.

Lemma 7.2. *Let L_i be three operators as in Corollary 5.2.*

(1) *Let $Q \in C'(L_i)$. Then, for small T, $n \geq 2$,*

$$\|Qu\|^{(n)} \leq C(M_{n-1/(20h)}(u) + n^{-1/2} M_n(u)) .$$

(2) *Let $Q \in C(L_i)$. Then, for small T, $n \geq 2$,*

$$\|Qu\|^{(n)} \leq C(M_{n-1/(20h)}(f^{1/(20h)} u) + n^{-1/2} M_n(u)) .$$

(3) *Let $q \in C_0$ Then, for small T, $n \geq 2$,*

$$\|Qu\|^{(n)} \leq C(n^{-1/2} M_n(u) + \|u\|^{(n-1/(20h))}_{(3)}) .$$

Proof. (1) is easily proved by Lemmas 4.21, and 7.1–(3), –(4). (2) is proved by Lemmas 4.19, 7.1–(3) and –(4) for the estimation, and by Lemmas 4.18–(3), 4.22 for the commutation involving $f^{1/(20h)}$ and L_i. Next, to see (3) we let $Q = Op \ q$ with $q = \sum_{k=0}^{2} u_k \tau^k$ with $a_k \in \Sigma$ satisfying the condition for a_k in the definition of E. Then, it is easy to see using Lemma 4.18–(1) and –(3)–(iv), and Lemma 4.22 that $Q - Op \ a_2 \cdot L_1 L_2$ is an operator as in Lemma 7.1–(5) which proves (3) in view of Lemma 7.1–(4). This completes the proof.

Lemma 7.3. *Let L_i be three operators as in Corollary 5.2. Set $\pi = \prod_{i=1}^{3} (\tau - a_i$ $(t, x) f(x) \xi - \sigma_i(t, x, \xi))$ with the notations just above (5.16). Then we have the followings.*

(1) *For small T, $n \geq 1$,*

$$C^{-1} M_n(u) \leq \|Op \ \pi \ u\|^{(n)} \leq CM_n(u) .$$

(2)   *Let φ, q be as in Lemma 6.6 with $T=T_0$ being as in Lemma 7.1–(5) and we assume that the assumption in (ii) of that Lemma holds for π, φ, q here.   Then for small $T, n \geq 2$,*

$$\|Op\ \pi \circ {}^f \varphi\ u\|^{(n)} \leq C(M_{n-1/(20h)}\ (f^{1/(20h)}\ u) + n^{-1/2}\ M_n(u) + \|Op\ {}^f q\ u\|^{(n)}$$
$$+ \|u\|_{(3)}^{(n-1/(20h))})\ .$$

   Proof.   (1) is proved by Lemmas 6.4–(vi), 4.22–(1), and 7.1–(4).   Next, from (5.1) and that $a_i \in B_{h,T_0}$, $a_i\ \xi + \tilde{\sigma}_i$ being in $\Sigma_{1/2}^{(0,-1,1)}(T_0, h)$ satisfies (6.4).   Thus, by the assumption one can apply Lemma 6.6–(ii).   Now, the desired inequailty follows from Lemmas 7.2–(2), –(3), 4.21–(1).   This completes the proof.

   Lemma 7.4.   *Let $\varphi \in \Sigma_{1/2}^0(T_0, h)$, $q = \sum\limits_{i+j\leq 3} a_{ij}(t, x)\ \tau^i\ \xi^j$ with $a_{30}=1$ and $a_{ij} \in B_{h,T_0}$, and q_3 be the homogeneous part of q of degree 3 in (τ, ξ).   Suppose that $q_3 = \prod\limits_{j=1}^{3}(\tau - \lambda_j\ \xi)$ with $\lambda_j \in B_{h,T_0}$ satisfying $\inf\limits_{t,x}|\operatorname{Im}\lambda_j| > 0$.   Then, we have the followings.*
   (1)   *For small T and $n \geq 2$,*

$$n^{-1/2}\ M_{0,n}(u) \leq C(n^{-1/2}\ \|Op\ {}^f q\ u\|^{(n)} + \|u\|_{(4)}^{(n-1)})\ ,$$

where by definition,

(7.1) $$M_{0,n}(\cdot) = M_n(\cdot, L_{0,i}), L_{0,i} = D_t - \lambda_i(t, x)\ f(x)\ D_x\ .$$

(2)   *For small T and $n \geq 2$,*

$$M_{0,n}(Op\ {}^f \varphi\ u) \leq C(\|Op\ {}^f q\ u\|^{(n)} + \|u\|_{(3)}^{(n-1/(20h))} + \|u\|_{(4,n)})\ .$$

   Proof.   (1)   We just have to apply Lemmas 7.3–(1), 7.1–(7) and –(6) in this order to the left hand side in view of the assumption on q_3.
   (2)   Noting the assumption on q_3 we apply Lemma 7.3–(2) for φ with φ here and $f^{1/(20h)}$ to get that

$$\|Op\ {}^f q_3 \circ {}^f \varphi\ u\|^{(n)} + \|Op\ {}^f q_3 f^{1/(20h)}\ u\|^{(n)}$$
$$\leq C(M_{0,n-1/(20h)}\ (f^{1/(20h)}\ u) + n^{-1/2}\ M_{0,n}(u) + \|Op\ {}^f q_3\ u\|^{(n)}$$
$$+ \|u\|_{(3)}^{(n-1/(20h))})\ .$$

The desired inequality can be obtained by estimating the left hand side from below by Lemma 7.3, the term involving q_3 on the right by Lemma 7.1–(7), and the second term on the right by (1).   The proof is complete.

   Definition 7.5.   We define for $\varphi \in \Sigma_{1/2}^0(T, h)$,

$\Sigma_{k,\varphi}: = \{Op\ \varphi;\ \varphi = \varphi_1\circ\cdots\circ\varphi_k\ $ for some $\ \varphi_i\in\Sigma_\varphi^0, i = 1, \cdots, k\}\ $ for $\ k\in N$,

$\Sigma_{0,\varphi}: = \{$identity on $\mathcal{S}_T\}$,

where we define $\varphi_1\circ\cdots\circ\varphi_k=\varphi_1\circ(\varphi_2\circ\cdots\circ\varphi_k)$ for $k\geq 2$ inductively.

Lemma 7.6. *Let* $\varphi\in\Sigma_{1/2}^0(T_0, h)$ *and* $q = \sum_{i+j\leq 3} a_{ij}\ \tau^i\ \xi^j, a_{ij}\in B_{h,T_0}, a_{30}=1$.
Let L_i, π *be as in Lemma 7.3 and let* $M_n(\cdot)=M_n(\cdot, L_i)$. *We suppose that the assumption on* q *in Lemma 7.4 holds for* q *here, and we let* $M_{0,n}(\cdot)$ *be defined by* (7.1). *Finally we let* $R=D_t^3+\sum_{k=0}^2 Op\ b_k\ D_t^k$ *with* $b_k\in\Sigma^{-(3-k)(0,2,0)}$. *Then, we have the followings.*

(1) *Let* $A\in\Sigma_{k,\varphi}$ *with* $k\geq 1$ *and let* $\alpha\geq 0$. *Then, one can find* $A'\in\Sigma_{k,\varphi}$ *such that for small* T *and* $n\geq 2$,

$$|M_n(f^\alpha Au-A'f^\alpha u)|\leq C(n^{-1/2}||Op\ ^f q\ u||^{(n)}+||Ru||^{(n)}+||u||_{(4)}^{(n-1)})\ .$$

(2) *Let* k, A, φ *be as in* (1). *Then, one can find* $A'\in\Sigma_{k,\varphi}$ *such that for small* T *and* $n\geq 2$,

$$|M_n(Af^\alpha u-f^\alpha A'u)|\leq C(n^{-1/2}||Op\ ^f q\ u||^{(n)}+||Ru||^{(n)}+||u||_{(4)}^{(n-1)})\ .$$

(3) *Assume in addition that there exists* $\tilde{q}=\sum_{i+j\leq 3}\tilde{a}_{ij}\ \tau^i\ \xi^j, \tilde{a}_{ij}\in B_{h,T_0}, \tilde{a}_{30}=1$ *such that the assumption on* π, q, φ *in Lemma 6.6-(ii) holds for* $\pi, \tilde{q}, \varphi$ *here. Then, for any* $k\in N$ *there exists* $l\in N$ *such that for any* $A\in\Sigma_{k,\varphi}$ *one can find* $A_i\in\Sigma_{k-1,\varphi}$, $i=1, \cdots, l$ *such that for small* T *and* $n\geq 2+1/(20h)$,

$$||[Op\ ^f\tilde{q}, A]\ u||^{(n)}\leq C\{\sum_{i=1}^l (M_{n-1/(20h)}\ (A_i\ f^{1/(20h)}\ u)+n^{-1/2}\ M_n(A_i\ u))$$
$$+n^{-1/2}\ ||Op\ ^f q\ u||^{(n)}+||Ru||^{(n-1/(20h))}+||u||_{(4)}^{(n-1/(20h))}\}\ .$$

Proof. We shall prove (1) first. Successive appliaction of Lemma 4.18–(4) shows that one can find $A'\in\Sigma_{k,\varphi}$ with $f^\alpha A-A'f^\alpha\in\Sigma^{(0,10,0)}$. Then, using Lemmas 7.3–(1) and 6.11 with $L_{0,i}$ for L_i' and with L_i, R given here for L_i, R respectively, and applying Lemma 7.1–(3) and –(4) we obtain that for small T and $n\geq 1$,

$$M_n((f^\alpha A-A'f^\alpha)\ u)\leq C(n^{-1/2}\ M_{0,n}(u)+||Ru||^{(n)})\ .$$

Now, (1) follows from Lemma 7.4–(1). (2) is proved in the same way using Lemma 4.18–(4)'. We pass to (3). From Corollary 6.20 and Lemma 7.2–(2) and –(3) one can find $l\in N$ for any $k\in N$ such that for any $A\in\Sigma_{k,\varphi}$ one can find $A_l, \cdots, A_1\in\Sigma_{k-1,\varphi}$ such that for small T and $n\geq 2$,

$$\|[Op\ ^f\bar{q},\ A]\ u\|^{(n)} \leq C\ \{ \sum_{i=1}^{l} (M_{n-1/(20h)}\ (f^{1/(20h)}\ A_i\ u) + n^{-1/2}\ M_n(A_i\ u))$$

$$+ n^{-1/2}\ M_{0,n}(u) + \|u\|_{(n)}^{(n-1/(20h))}\ .$$

Now, the desired inequality is clear in view of (1), Lemma 7.4–(1), and (4.55). This completes the proof.

Corollary 7.7. *Let φ, q, L_i, π be as in Lemma 7.6 and assume that the assumption on π, q, φ in Lemma 6.6–(ii) holds for π, φ, q here. Let k, $k' \in N$. Then, one can find $l \in N$ such that for any $A \in \Sigma_{2k+k',\varphi}$, one can find $A_{ij} \in \Sigma_{k',\varphi}$, $i = 0, \cdots, k'$, $j = 1, \cdots, l$ such that for small T and $n \geq k+3$,*

$$M_n(Au) \leq C(\|Op\ ^f q\ u\|^{(n)} + \sum_{i=0}^{k} \sum_{j=1}^{l} n^{-(k-i)/2}\ M_{n-i/(20h)}\ (A_{ij}\ f^{i/(20h)}\ u)$$

$$+ \|f^{k/(20h)}\ u\|_{(4,\,n-k/(20h))} + \|u\|_{(4)}^{(n-1/(20h))})\ .$$

Before the proof we prepare a lemma.

Lemma 7.8. *Let q be as in Lemma 7.4 and let $\alpha \geq 0$. Then, for small T and $n \geq 2$,*

$$\|[Op\ ^f q,\ f^\alpha]\ u\|^{(n)} \leq C(n^{-1/2}\ \|Op\ ^f q\ u\|^{(n)} + \|u\|_{(4)}^{(n-1)})\ .$$

Proof. By the assumption on q_3 one can apply Lemma 6.10–(4) to $Q = Op\ ^f q_3$ and α. Then, using Lemmas 7.1–(4) and –(5) shows that the left hand side of the desired inequality can be dominated by $n^{-1/2}\ M_{0,n}(u) + \|u\|_{(2,n+1)}$ when T is small and $n \geq 1$. Now, the desired inequality is clear from Lemmas 7.4–(1) and 7.1–(6). The proof is complete.

Proof of Corollary 7.7. We claim that for any $l \in N$ with $l \geq 2$ there exists $s \in N$ such that for any $A \in \Sigma_{l,\varphi}$ one can find $A_j \in \Sigma_{l-2,\varphi}$, $j = 1, \cdots, s$ satisfying that with the notation $Q = Op\ ^f q$, for small T and $n \geq 4$,

$$(7.2) \qquad M_n(Au) + \|QAu\|^{(n)} \leq C\ \{\|Qu\|^{(n)} + \|u\|_{(4)}^{(n-1/(20h))}$$

$$+ \sum_{j=1}^{s} (\|AQ_j\ f^{1/(20h)}\ u\|^{(n-1/(20h))} + n^{-1/2}\ \|QA_j\ u\|^{(n)}$$

$$+ M_{n-1/(20h)}\ (A_j\ f^{1/(20h)}\ u) + n^{-1/2}\ M_n(A_j\ u))\}\ .$$

Proof of (7.2). Let $l \in N$. Writing $A = A' \cdot A''$ with $A' \in \Sigma_{1,\varphi}$, $A'' \in \Sigma_{l-1,\varphi}$ and applying Lemma 7.3–(1), –(2) to $M_n(Au)$ in this order shows that $M_n(Au)$ is dominated by the expression made from the right hand side of Lemma 7.3–(2) substituting $A''u$ for u when T is small and $n \geq 2$. Then applying Lemma 7.6–(1) to the first term of this expression and Lemma 4.21–(2) to the last term of it, one finds that there exists B_1, $B_2 \in \Sigma_{l-1,\varphi}$ such that for small T and $n \geq 3$,

$$(7.3) \qquad M_n(Au) \leq C(||Qu||^{(n-1/(20h))} + ||u||_{(4)}^{(n-1/(20h))}$$
$$+ \sum_{j=1}^{2} (||QB_j u||^{(n)} + M_{n-1/(20h)} (B_j f^{1/(20h)} u) + n^{-1/2} M_n(B_j u)) .$$

Lemma 7.6–(3) shows that when $l \geq 2$, thee there exists $s' \in N$ and $B'_j \in \Sigma_{l-1,\varphi}$, $B''_j \in \Sigma_{l-2,\varphi}$, $j = 1, \cdots, s'$ satisfying that for small T and $n \geq 2$,

$$(7.4) \qquad ||QAu||^{(n)} + \sum_{j=1}^{2} ||QB_j u||^{(n)} \leq C \Big[\sum_{j=1}^{s'} \{ M_{n-1/(20h)} (B'_j f^{1/(20h)} u)$$
$$+ M_{n-1/(20h)} (B''_j f^{1/(20h)} u) + n^{-1/2}(M_n(B'_j u) + M_n(B''_j u)) \}$$
$$+ ||Qu||^{(n)} + ||u||_{(4)}^{(n-1/(20h))} \Big] .$$

(7.2) follows by first applying (7.3), (7.4) in this order to the left hand side of (7.2), and again to terms involving B_j, B'_j, and finally using Lemmas 7.6–(2), 7.8, 4.21–(2) for estimating a term of the form $M_{n-1/(10h)}(Bf^{1/(10h)} u)$ with $B \in \Sigma_{l-2,\varphi}$ by a term of the form $M_{n-1/(20h)}(B' f^{1/(20h)} u) + ||Qu||^{(n)} + ||u||_{(4)}^{(n-1/(20h))}$ with $B' \in \Sigma_{l-2,\varphi}$. This proves (7.2).

Using (7.2) iteratively with the aid of Lemmas 7.8 and 4.21–(2) we get that for any $k, k' \in N$ there exists $s \in N$ satisfying that for small T and $n \geq k+3$,

$$(7.5) \qquad M_n(Au) \leq C(||Qu||^{(n)} + ||u||_{(4)}^{(n-1/((20h))}$$
$$+ \sum_{i=0}^{k} \sum_{j=1}^{s} n^{-i/2}(||QA_{ij} f^{(k-i)/(20h)} u||^{(n-(k-i)/(20h))}$$
$$+ M_{n-(k-i)/(20h)}(A_{ij} f^{(k-i)/(20h)} u))) .$$

On the other hand for any $A \in \Sigma_{l,\varphi}$, $l \geq 1$, it follows from Lemmas 7.4–(2), 7.1–(6) and –(7), and 4.21–(2) that one can find $A' \in \Sigma_{l-1,\varphi}$ satisfying that for small T and $n \geq 2$,

$$||QAu||^{(n)} \leq C(||QA' u||^{(n)} + ||u||_{(4)}^{(n-1/(20h))} + ||u||_{(4,n)}) .$$

So, iteration of this process yields that for any $A \in \Sigma_{l,\varphi}$, $l \geq 1$ we have that for small T and $n \geq 2$,

$$(7.6) \qquad ||AQu||^{(n)} \leq C(||Qu||^{(n)} + ||u||_{(4)}^{(n-1/(20h))} + ||u||_{(4,n)}) .$$

Applying this and Lemma 7.8 to terms in the summation in (7.5) we get that

$$\sum_{i=0}^{k} \sum_{j=1}^{s} n^{i-/2} ||QA_{ij} f^{(k-i)/(20h)} u||^{(n-(k-i)/(20h))}$$
$$\leq C_1 (\sum_{i=0}^{k} n^{-i/2}(||Qf^{(k-i)/(20h)} u||^{(n-(k-i)/(20h))}$$
$$+ ||f^{(k-i)/(20h)} u||_{(4,n-(k-i)/(20h))} + ||u||_{(4)}^{(n-1/(20h))})$$
$$\leq C_2(||Qu||^{(n)} + ||u||_{(4)}^{(n-1/(20h))} + ||f^{h/(20h)} u||_{(4,n-k/(20h))}) .$$

This proves Corollary 7.7.

Throughout the following discussions in this section we assume the notations and the assumptions in §3 and that p_3 there satisfies the assumption for q_3 in Lemma 7.4. Thus, $p_3(t, x, 1, \zeta(t, x))^{-1} q$ with in (3.6) also satisfies the same assumption, and we assume the notation (7.1) with it. *We fix some $T_0 \in (0, 1)$ as in Lemma 3.2 and we remark that the symbols of operators always are in $\mathscr{P}_{T_0}$ and u always $\in \mathscr{S}_T$, $T \in (0, T_0)$.*

Note that

$$(7.7) \qquad (1 - \mu_l \zeta)^3 (1 + A_{0l}) = p_3(t, x, 1, \zeta).$$

We set noting (3.17)

$$(7.8) \qquad \tilde{L}_l = Op\left((1 - \mu_l \zeta)^{-1} {}^f L_l\right),$$
$$L'_l = \tilde{L}_l + Op\left((1 + A_{0l})^{-1}(1 + \mu_l \zeta)^{-1} ((-1)^l cb_l + d_l A_{2l}) f\xi\right).$$

Recall three cases (1)~(3) in (3.13). Using Lemma 3.2 and the notation in it we set in case (1),

$$(7.9) \qquad L_{ij} = \tilde{L}_{l(i)} - Op\, {}^f\tilde{\sigma}_{ij}, \; i \in I,$$

where

$$(7.10) \qquad \tilde{\sigma}_{ij} = (1 - \mu_{l(i)} \zeta)^{-1} \sigma_{ij}.$$

In case (2) we set that

$$(7.11) \quad L_{ij} = \tilde{L}_{l(i)} - Op\, {}^f\tilde{\sigma}_{ij} \quad \text{for} \quad i \in I_1, \text{ or } i \in I_2 \text{ and } j \neq 1, \text{ and that}$$
$$L_{i1} = \tilde{L}_{l(i)} - (1 - \mu_{l(i)} \zeta)^{-1} c_l f D_x - Op\, {}^f\tilde{\sigma}_{ij} \quad \text{for} \quad i \in I_2,$$

where $\tilde{\sigma}_{ij}$ are defined by (7.10). Here, we note that $c_l = (1 + A_{0l})^{-1} (-1)^{l+1} c_l$. See (3.59). In case (3) we set

$$(7.12) \qquad L_{ij} = \tilde{L} - Op\, {}^f\tilde{\sigma}_{ij} \quad \text{for} \quad i \in I,$$

where $\tilde{\sigma}_{ij}$ are defined by (7.10) and $\tilde{L}$ are done, noting $L_1 = L_2$ in case (3), by

$$(7.13) \qquad \tilde{L} = \tilde{L}_1.$$

Note that

$$\text{Im}\,(\mu_l/(1 - \mu_l \zeta)) = \text{Im}\,\mu_l/|1 - \zeta\mu_l|^2$$

so that $\tilde{L}_l, L'_l, L_{ij}$ are operators as in (5.3). We set

$$(7.14) \qquad \varphi_{0l} = 2^{-1} \varphi_0 \quad \text{for} \quad l = 1, 2 \text{ in case (3)},$$

and we also set for $N \in \mathbb{N}$ with $N \geq 2$,

$$(7.15) \qquad \varphi_N = (\sum_{i \in I} \varphi_i^N + \sum_{l=1}^{2} \varphi_{0l}^N)^{-1/N} ,$$

$$\varphi_{iN} = \varphi_i \cdot \varphi_N, \; \varphi_{0lN} = \varphi_{0l} \cdot \varphi_N .$$

Then, $\varphi_{iN}, \varphi_{0lN} \in \Sigma_{1/2}^0$ in cases (1) and (2), $\in \Sigma_0^0$ in case (3), by Lemma 3.8–(3), and

$$(7.16) \qquad \sum_{i \in I} \varphi_{iN}^N + \sum_{l=1}^{2} \varphi_{0lN}^N = 1 .$$

Lemma 7.9. *We assume the above notations and assumptions. Let $Q_{\mu\nu}$ be as in (2.17) with q in (3.6) for $\tilde{Q}$. Then, for small T and $n \geq 2$,*

$$\sum_{0 < |\nu|+|\mu| \leq 2} \|Q_{\mu\nu} u\|^{(n)} \leq C (\sum_{i \in I} \tilde{M}_{i,n}(\Phi_{iN}^{N-1} u) + \sum_{l=1}^{2} \tilde{M}_{0l,n} (\Phi_{0lN}^{N-1} u)$$
$$+ n^{-1/2} \|Op^f q\, u\|^{(n)} + \|u\|_{(4)}^{(n-1/(20h))}) ,$$

where

$$(7.17) \quad \Phi_{iN} = Op^f \varphi_{iN}, \; \Phi_{0lN} = Op^f \varphi_{0lN} ,$$
$$M_{i,n}(\cdot) = M_n(\cdot, L_{ij}), M_{0l,n}(\cdot) = M_n(\cdot, L_1 = L_2 = \tilde{L}_l, L_3 = L_l') ,$$
$$\tilde{M}_{*,n} = M_{*,n-1/(20h)} + n^{-1/2} M_{*,n}, \; n \geq 1 + 1/(20h) .$$

Proof. By Lemma 3.2,

$$\inf t^{2\kappa/3} \langle \xi \rangle \gg 0 \text{ on } \bigcup_{i \in I} \text{supp } \varphi_i \cup \bigcup_{l=1}^{2} \text{supp } |d\varphi_{0l}| .$$

So, since $0 < 2\kappa/3 \leq e_h$, it follows from Lemmas 6.24, 6.28, Corollary 6.27, and Lemma 7.2–(1) and –(3) that for small T and $n \geq 2$,

$$(7.18) \quad \sum_{0 < |\mu|+|\nu| \leq 2} \|Q_{\mu\nu} u\|^{(n)} \leq C \{ \Sigma_i \tilde{M}_{i,n}(\Phi_{iN}^{N-1} u)$$
$$+ \sum_{0 < |\nu|+|\mu| \leq 2} \|Q_{\mu\nu} \Phi_{0lN}^N u\|^{(n)} + n^{-1/2} M_{0,n}(u) + \|u\|_{(3)}^{(n-1/(20h))} \} .$$

To handle the second summation we set

$$q^{(l)} = (1 - \zeta \mu_l)^{-3} L_l^2 (L_l + (1 + A_{0l})^{-1}((-1)^l cb_l + d_l A_{2l}) \xi) .$$

Then,

$$q = p_3(t, x, 1, \zeta(t, x)) q^{(l)} + d_l A_{1l} L_l \xi^2 + g_1 + f^{1/2}(p_2 - a_{0l} \xi^2) .$$

If we extend the definition of $q_{\nu\mu}$ in Definition 6.21 for q of the form with in it with $a_{ij} \in B_{h,T_0}$ and $a_{30} = \text{const.}$ by the same equality,

$$Q_{\mu\nu} = Op(q^{(l)}{}_{\mu\nu} + (p_3(t, x, 1, \zeta(t, x)))^{-1}$$

$$\times (d_l\, A_{1l}\, L_l\, \xi^2 + g_l + f^{1/2}(p_2 - a_{0l}\, \xi^2)))\,_{\mu\nu})\,.$$

By Lemma 6.4,

(7.19) $Op(q^{(l)}{}_{\mu\nu})\, \varPhi_{0lN} \in \mathcal{C}(L_1 = L_2 = \tilde{L}_l,\, L_3 = L_l') + \mathcal{C}_0,\ 0 < |\mu| + |\nu| \le 2\,.$

Noting that $\langle t^{e_h}\xi\rangle^{-1}\, t^{e_h}\langle\xi\rangle \le 1$ when $t \in (0,1)$, (6.17), (3.65), and the fact that $d_l \in E_{h,T}$, we see that when $1 \le |\nu| + |\mu| \le 2$,

$$(p_3(t, x, 1, \zeta(t, x))^{-1}\, d_l\, A_{1l}\, L_l\, \xi^2)_{\mu\nu} = d_l\,(a_1\,{}^f L_l + a_2)$$

with

$$a_1 \in t^{-1/2 - 1/(20h)}\, {}^f\Sigma_0^{3/2(0,0,1)},\ a_2 \in t^{-1/2 - 1/(20h)}\, {}^f\Sigma_0^{5/2(0,0,1)} + t^{-1}\, {}^f\Sigma_0^{(0,-1,2)}\,,$$

and

(7.20) $(p_3(t, x, 1, \zeta(t, x))^{-1} f^{1/2}(p_2 - a_{0l}\, \xi^2))_{\mu\nu} \in E\,.$

Using the equality

$$
\begin{aligned}
d_l(a_1\,{}^f L_l + a_2)\circ{}^f\varphi_{0lN} \\
= d_l((a_1(1 - \mu_l\,\zeta))\circ{}^f D_t\varphi_{0lN}) + d_l((a_1(1 - \mu_l\,\zeta))\circ{}^f\varphi_{0lN})\,\tau \\
- d_l((a_1\,\mu_l\,f\xi)\circ{}^f\varphi_{0lN}) + d_l(a_2\circ{}^f\varphi_{0lN})\,,
\end{aligned}
$$

the fact that (3.19) holds, Lemmas 6.31–(1), 6.33, 6.34, and Corollary 4.12–(i) with $\rho = 1/4$, we see by asymptotic expansion that

(7.21) $d_l(a_1\,{}^f L_l + a_2)\circ{}^f\varphi_{0lN} \in E\,.$

Similarly, using (3.19), Lemma 6.33, and Leibniz rule we see $g_l \in t^{-\kappa}\, \Sigma_{\mathrm{supp}\,\varphi_{0l}}^{3/2(0,-2,0)}$ so that Lemma 6.31–(1) and Leibniz rule show that $f^{\nu_2}\cdot \partial_{t,x}^\nu f(\partial_{\xi^2}^\mu g_l) = {}^f g_l^{\mu\nu}$ with $g_l^{\mu\nu} \in t^{-\kappa}\, \Sigma_{\mathrm{supp}\,\varphi_{0l}}^{3/2(0,-2,0) - (\nu_1,\nu_2,\mu_2)}$ for $\nu,\mu \in \mathbf{Z}_+^2$. So, (6.17) and Leibniz rule show that $(g_l)_{\mu\nu} \in t^{-\kappa}\, {}^f\Sigma_{\mathrm{supp}\,\varphi_{0l}}^{3/2(0,-2,0)}$. Thus, since by (6.17) and Lemma 6.31–(2) $(g_l)_{\mu\nu} = {}^f\tilde{g}_l$ with $|\partial^\alpha \tilde{g}_l| \le C_\alpha\langle\xi\rangle^{3-\alpha_3}$, we have $(g_l)_{\mu\nu}\circ{}^f\varphi_{0lN} \in t^{-\kappa}\, {}^f\Sigma^{3/2(0,-2,0)} + f^{3/2}\, \Sigma^0$ by Lemmas 6.30, 6.31. Since $t^{-1/2 + 1/(20h)}\langle\xi\rangle^{1/2} \le t^{-1}\langle t^{e_h}\xi\rangle^{1/2}$ when $t \le 1$, by Lemmas 3.8–2, 3.8–5, and (6.21) we have $t^{-\kappa}\, {}^f\Sigma^{3/2(0,-2,0)} \subset B^{3/2,3/2,3/2}$. Hence,

(7.22) $(g_l)_{\mu\nu}\circ{}^f\varphi_{0lN} \in E\,.$

Therefore, by (7.19)~(7.22), Lemmas 6.13, and 7.1–(1) and –(3) it follows that for small T, $n \ge 2$,

(7.23) $\displaystyle\sum_{1 \le |\nu| + |\mu| \le 2} ||Q_{\mu\nu}\, \varPhi_{0lN}\,{}^N u||^{(n)} \le C(\tilde{M}_{0l,n}(\varPhi_{0lN}\,{}^{N-1}u) + n^{-1/2}\, M_{0,n}(u)$

$$+ ||u||_{(3)}^{(n-1/(20h))})\,.$$

Substituting (7.23) into (7.18), estimating $n^{-1/2} M_{0,n}(u)$ by Lemma 7.4–(1), and using (4.55) we get the desired inequality. The proof is complete.

§8. The Derivation of (2.13) and (2.14)

§§1. The Derivation of (2.13-1) and (2.14-1)

(2.13–1) is nothing but Proposition 5.1. On the other hand, noting that $\langle t^{e_h}\xi\rangle^{-1} t^{e_h}\langle\xi\rangle \leq 1$ for $t \in (0, 1)$, and (6.17), we see that

$$Q_{\mu\nu} \in Op\ {}^f\Sigma^{(-1,0,0)} \quad \text{when} \quad |\nu| + |\mu| = 1 .$$

Thus, (2.14–1) follows from Lemma 4.21 and Proposition 5.1.

§§2. The Derivation of (2.13-2) and (2.14-2)

From the assumption of Proposition 2.3,

$$p_2 = \prod_{i=1}^{2} (\tau - \lambda_i \xi) \quad \text{with} \quad \lambda_i \in B_{h,T_0} \text{ satisfying inf } |\operatorname{Im} \lambda_i| > 0 .$$

(2.13–2) is an immediate consequence of Corollary 5.2 and Lemma 6.10–(1), and similarly as §§1,

$$Q_{\mu\nu} = \sum_{j=1}^{2} Op\,(a_j(\tau - (1 - \lambda_j \zeta)^{-1} \lambda_j f\xi)) \bmod Op\ {}^f\Sigma^{(-1,0,0)}$$
$$\text{with } a_j \in {}^f\Sigma^{(-1,0,0)} \text{ when } |\mu| + |\nu| = 1 .$$

(2.14–2) is an easy consequence of Lemmas 4.18–(3), 4.21–(1), and Corollary 5.2–(i).

§§3. The Derivation of (2.13-3) and (2.14-3)

p_3 and p_2 in Proposition 2.3 satisfy the assumptions for those in §3. So, identifying the latters with formers respectively, we use the notations, the assumptions, the conventions, and the results in §3 and in §7 after Corollary 7.7.

Here, again, we remark that all symbols of operators are in $\mathscr{P}_{T_0}$, where $T_0 \in (0, 1)$ is as in Lemma 3.2, that u denotes an element in $\mathcal{S}_T$ with $T \in (0, T_0)$.

The proof of (2.13) and (2.14) is devided into three cases (1)~(3) in (3.13). These cases are considered in order from (1).

Remark on notation. In §8, §9, the constants denoted by C or C_i in the estimates are always independet of T, n, $u \in \mathcal{S}_T$ as in §7.

Proposition 8.1. Assume that (1) of (3.13) holds. Then, there exists r_0, $n_0 \in N$ with $n_0 \geq r_0 + 1$ such that for small T and $n \geq n_0$,

$$\|f^{r_0} u\|_{(4,n-r_0)} + \|u\|_{(6)}^{(n)} \leq C \|Qu\|^{(n)},$$

$$\|Q_{\mu\nu} u\|^{(n)} \leq C(\|Qu\|^{(n-1/(20h))} + n^{-1/2} \|Qu\|^{(n)}) \quad \text{when} \quad 1 \leq |\nu| + |\mu| \leq 2.$$

From the assumptions (2.7), (2.8), in all cases (1)~(3),

(8.1) $p_3(t, x, \tau, \zeta(t,x)\tau+\xi) = p_3(t, x, 1, \zeta(t,x)) \prod_{i=1}^{3} (\tau - \lambda_i(t,x)\xi)$ with

   $\lambda_i \in B_{h,T_0}$ satisfying that

(8.2) $\inf_{i,x} |\operatorname{Im} \lambda_i| > 0$ and $\lambda_i - \lambda_j \in E_{h,T_0}$ for any i, j.

   Proof of Proposition 8.1. It suffices to prove that two inequalities in the statement replacing Q on the right hand side by $p_3(t, x, 1, \zeta(t,x))^{-1} \, Op^f q$ with q in (3.6). In the following we denote this operator by Q. In case (1),

$$\lambda_i - \lambda_j \in E_{h,T_0} \setminus \{0\} \quad \text{if} \quad i \neq j.$$

So, from Lemma 6.10–(3) one can find $r_0 \in N$ such that

(8.3)   for any $0 \leq i \leq 2$ there exist $c_{kl}, c_k \in B_{h,T_0}$, $0 \leq k \leq 3$, $1 \leq l \leq 3$ satisfying

$$t^{r_0} \cdot f^i D_x^i D_t^{2-i} f^{r_0} = \sum_{l \neq k} c_{kl} L_{0,l} L_{0,k} + \sum_{k=1}^{3} c_k L_{0,k} + c_0.$$

So, by Lemma 7.1–(4), for small T and $n \geq r_0 + 1$,

$$\sum_{i=0}^{2} \|f^i D_x^i D_t^{2-i} f^{r_0} u\|^{(n-r_0)} \leq C_0 \, n^{-1/2} \, M_{0,n}(u),$$

where by our convention in the beginning of §§3 we set

$$M_{0,n}(\cdot) = M_n(\cdot, L_{0,i}), \; L_{0,i} = D_t - \lambda_i f D_x.$$

Thus, from Lemma 7.1–(8), for small T and $n \geq r_0 + 1$,

(8.4) $\|f^{r_0} u\|_{(4,n-r_0)} \leq C_1 n^{-1/2} M_{0,n}(u) \leq C_2(n^{-1/2} \|Qu\|^{(n)} + \|u\|_{(4)}^{(n-1)}),$

where the right inequality follows from Lemma 7.4–(1). Next, from (3.20), Corollary 6.27, Lemma 4.21, (4.55), Lemma 7.1–(3) we have that for small T and $n \geq 2$,

(8.5) $\|u\|_{(6)}^{(n)} \leq C_3(\sum_{i \in I} M_{i,n}(\Phi_{iN}{}^N u) + \sum_{l=1}^{2} M_{0l,n}(\Phi_{0lN}{}^N u) + T^{2\kappa/3} \|u\|_{(6)}^{(n)}),$

       where $\Phi_{iN} = Op^f \varphi_{iN}$, $\Phi_{0lN} = Op^f \varphi_{0lN}$.

We set

$$N_0 = 20h(2r_0 + 4).$$

Lemma 8.2. *Let* $N_j \in N$, $j = 1, 2$ *with* $N_1 = 2N_2 + 1 \geq 2N_0 + 1$, *and let* $\Phi \in$

Σ_{1,φ_i} for $i \in I$. Set $J = J(u) = M_{i,n}(\Phi^{N_1} u)$ for $u \in \mathcal{S}_T$. Then, for small T and $n \geq N_2 + 3$,

$$J \leq C(\|Qu\|^{(n)} + \|u\|_{(4)}^{(n-1/(20h))} + \|f^{r_0} u\|_{(4,n-r_0-3)}) .$$

Proof. We apply Corollary 7.7 for J taking $k = N_2$ and $k' = 1$ to obtain that there exist $l_0 \in N$ and $B_j \in \Sigma_{1,\varphi_i}$, $j = 1, \cdots, l_0$ such that for small T and $n \geq N_2 + 3$,

$$(8.6) \qquad J \leq C_1(\|Qu\|^{(n)} + \sum_{r=0}^{N_2} \sum_{j=1}^{l_0} n^{-(N_2-r)/2} M_{i,n-r/(20h)}(B_j f^{r/(20h)} u)$$
$$+ \|f^{2r_0+4} u\|_{(4,n-2r_0-4)} + \|u\|_{(4)}^{(n-1/(20h))}) .$$

From (3.21), (3.22), (3.24) of Lemma 3.2, L_{ij} can be expressed as $L_{ij} = Op$ $(\tau - f \tilde{\tau}_{ij})$ with some $\tilde{\tau}_{ij} \in \Sigma_{1/2}^{(0,-1,1)}$ satisfying the condition for $\tilde{\tau}_j$ of Lemma 6.4 with $\varepsilon = 1/(6h)$ and that $q = \prod_{j=1}^{3} (\tau - \tilde{\tau}_{ij}(t, x, \xi))$ for $(t, x, \xi) \in \text{supp } \varphi_i$. So, it follows from Lemma 6.6, (5.1), Lemma 4.18–(1) by setting $\pi^{(i)} = \prod_{j=1}^{3} (\tau - f \tilde{\tau}_{ij})$ for each $i \in I$ that

$$(8.7) \qquad Op\, \pi^{(i)} \cdot B_j - B_j Q - \sum_{k=0}^{2} A_{ijk} D_t^k \in \mathcal{C}_0$$

with some $A_{ij2} \in \mathscr{A}^{1/2,1/2,3}$, $A_{ij1} \in \mathscr{A}^{3/2,1/2,3} + \mathscr{A}^{1,1,3}$,

$$A_{ij0} \in \mathscr{A}^{5/2,1/2,3} + \mathscr{A}^{2,1,3} + t^{1/(6h)-1} \mathscr{A}^{3/2,1/2,3} .$$

By Lemmas 7.2–(3) and 7.4–(1), for any $R \in \mathcal{C}_0$ there exists $C > 0$ such that for small T and $n \geq 2$,

$$(8.8) \qquad \|Ru\|^{(n)} \leq C(n^{-1/2} \|Qu\|^{(n)} + \|u\|_{(4)}^{(n-1/(20h))}) .$$

The summation in (8.7) can be setimated by using Lemma 4.2–(1), –(2), and (4.55) as

$$(8.9) \qquad \sum_{k=0}^{2} \|A_{ijk} D_t^k u\|^{(n)} \leq C_2(\sum_{k=0}^{2} \|A_{5/2-k} D_t^k u\|^{(n+1/2)} + \|D_t^k u\|^{(n+3-k)})$$

for small T and $n \geq 2$.

From (8.7)~(8.9) we get that for small T and $n \geq 2$,

$$(8.10) \qquad \|(Op\, \pi^{(i)} \cdot B_j - B_j Q) u\|^{(n)} \leq C_3(n^{-1/2} \|Qu\|^{(n)} + \|u\|_{(4)}^{(n-1/(20h))}$$
$$+ \sum_{k=0}^{2} (\|A_{5/2-k} D_t^k u\|^{(n+1/2)} + \|D_t^k u\|^{(n+3-k)})) .$$

Since for $0 \leq r \leq N_2$ and $n \geq N_2 + 3$,

$$n - r/(20h) \geq N_2 + 3 - N_2/(20h) \geq (20h-1)(2r_0+4) + 3 ,$$

we have that for $0 \leq r \leq N_2$, small T, and $n \geq N_2 + 3$,

$$
(8.11) \quad \|Op\ \pi^{(i)}\ B_j\ u\|^{(n_r)} \leq C_4(\|Qu\|^{(n_r)} + n_r^{-1}\ \|Qu\|^{(n_r)} + \|u\|_{(4)}^{(n_r+1)}
$$
$$
+ \sum_{k=0}^{2} (\|A_{5/2-k}\ D_t^k\ u\|^{(n_r+1/2)} + \|D_t^k\ u\|^{(n_r+3-k)}))
$$

with the notation that

$$
n_r = n - r/(20h) \quad \text{for} \quad r \in \mathbf{Z}_+ \text{ and } n \geq 1 + r/(20h)\ .
$$

Applying Lemma 7.3–(1) to $M_{i,n_r}(B_j\ f^{r/(20h)})$ in (8.6), using (8.11), and estimating $\|Qf^{r/(20h)}\ u\|^{(n_r)}$ by Lemma 7.8 and $\|f^{r/(20h)}\ u\|_{(4)}^{(n_r+1)}$ by Lemma 4.21–(2) one obtain that

$$
(8.12) \quad M_{i,n_r}(B_j\ f^{r/(20h)}\ u) \leq C_5(\|Qu\|^{(n_r)} + \|u\|_{(4)}^{(n_r+1)}
$$
$$
+ \sum_{k=0}^{2} (\|A_{5/2-k}\ D_t^k\ f^{r/(20h)}\ u\|^{(n_r+1/2)} + \|D_t^k\ f^{r/(20h)}\ u\|^{(n_r+3-k)})
$$
$$
\text{for} \quad 0 \leq r \leq N_2, \text{ small } T, n \geq N_2 + 3\ .
$$

We use the inequalities that

$$
(8.13) \quad \|A_{5/2-k}\ u\|^{(n)} \leq C(\|A_{1/2}\ f^{2-k}\ D_x^{2-k}\ u\|^{(n)} + \|u\|^{(n)})
$$
$$
\text{for} \quad 0 \leq k \leq 2, n \geq 1\ ,
$$

and that

$$
(8.14) \quad \|A_{1/2}\ f^{2-k}\ D_x^{2-k}\ D_t^k\ f^{r_0}\ u\|^{(n-r_0)} \leq CM_{0,n-1/2}(u)\ ,
$$
$$
\|D_t^2\ f^{r_0}\ u\|^{(n-r_0)} \leq CM_{0,n-1}(u)\ ,
$$
$$
\text{for} \quad 0 \leq k \leq 2, \text{ small}, T, n \geq r_0 + 1\ ,
$$

where (8.13) follows from Corollary 4.13–(ii) and (8.14) is shown by using (8.3), Lemmas 7.1–(3) and –(4), 4.18, 4.21–(1), and (4.55).

When $N_2 \geq r \geq N_2 - 2$, we have $r/(20h) \geq 2r_0 + 3$ and we can apply (8.13), (8.14), and Lemma 7.1–(3) to get that

$$
(8.15) \quad M_{i,n_r}(B_j\ f^{r/(20h)}\ u) \leq C_6(\|Qu\|^{(n_r)} + \|u\|_{(4)}^{(n_r+1)}
$$
$$
+ M_{0,n_r+r_0}(f^{r/(20h)-r_0}\ u))
$$
$$
\text{for} \quad N_2 \geq r \geq N_2 - 2, \text{ small } T, n \geq N_2 + 3\ .
$$

By Lemmas 7.4–(2), 7.8, 4.21–(1), the last term on the right can be dominated by $\|Qu\|^{(n_r+r_0)} + \|u\|_{(4)}^{(n_r+r_0-1)} + \|u\|_{(3)}^{(n_r+1+r_0)} + \|f^{r/(20h)-r_0}\ u\|_{(4,n_r+r_0)}$. Thus, noting $r/(20h) \geq 2r_0 + 3$ in (8.15) and using Lemma 4.21–(4) to estimate $\|f^{r/(20h)-r_0}\ u\|_{(4,n_r+r_0)}$ by $\|f^{r_0}\ u\|_{(4,n-r_0-3)}$ we get from (8.15) that

$$
(8.16) \quad M_{i,n_r}(B_j\ f^{r/(20h)}\ u) \leq C_7(\|Qu\|^{(n-r_0-3)} + \|u\|_{(4)}^{(n-r_0-3)} + \|f^{r_0}\ u\|_{(4,n-r_0-3)})
$$
$$
\text{for} \quad N_2 \geq r \geq N_2 - 2, \text{ small } T, n \geq N_2 + 3\ .
$$

On the other hand, when $0 \leq r \leq N_2 - 3$, we have applying Lemmas 4.21–(2), 7.1–(3), (4.55), and Lemma 7.4–(1) to (8.12) that for small T and $n \geq N_2 + 3$,

$$n^{-(N_2-r)/2} M_{i,n_r}(B_j f^{r/(20h)} u) \leq C_8 \, n^{-3/2} M_{i,n}(B_j f^{r/(20h)} u) \leq C_9 (n^{-3/2}$$
$$\times ||Qu||^{(n)} + n^{-3/2} ||u||_{(4)}^{(n-1/(20h))} + n^{-1/2} ||u||_{(4)}^{(n)}) \leq C_{10} (n^{-3/2} ||Qu||^{(n)}$$
$$+ n^{-3/2} M_{0,n-1/(20h)}(u) + n^{-1/2} M_{0,n}(u)) \leq C_{11} (n^{-1/2} ||Qu||^{(n)} + ||u||_{(4)}^{(n-1)}) \, .$$

So,

$$(8.17) \qquad n^{-(N_2-r)/2} M_{i,n_r}(B_j f^{r/(20h)} u) \leq C_{11}(n^{-1/2} ||Qu||^{(n)} + ||u||_{(4)}^{(n-1)})$$
$$\text{for} \quad 0 \leq r \leq N_2 - 3, \text{ small } T, \text{ and } n \geq N_2 + 3 \, .$$

Now, the desired inequality follows from (8.6), (8.16), (8.17), (4.55), and Lemma 4.21–(4).

By Lemma 8.2 it follows that the term involving Φ_{iN} in (8.5) can be estimated for large odd N, and to estimate the term involving Φ_{0lN} we need the following two lemmas.

Lemma 8.3.   *Let* $m \in N$ *with* $m \geq 2$. *Then, one can find* $s \in N$ *such that for any* $A \in \Sigma_{m,\varphi_{0l}}$ *there exists* $A_j \in \Sigma_{m-1,\varphi_{0l}}$, $j = 1, \cdots, s$ *satsifying that for small* T *and* $n \geq 3$,

$$M_{0l,n}(Au) \leq C \left\{ \sum_{j=1}^{s} (M_{0l,n-1/(20h)}(A_j f^{1/(20h)} u) + n^{-1/2} M_{0l,n}(A_j u)) + ||Qu||^{(n)} + ||u||_{(4)}^{(n-1/(20h))} \right\} \, .$$

Proof.   By Lemma 7.3 we have that for small T and $n \geq 1$,

$$(8.18) \qquad M_{0l,n}(Au) \leq C_1 (|| [Op \, {}^h q^{(l)}, A] u ||^{(n)} + ||A \cdot Op \, {}^f q^{(1)} u||^{(n)}) \, .$$

We set

$$(8.19) \quad R = Op \, \{ p_3(t, x, 1, \zeta(t, x))^{-1} \, {}^f (d_l \, A_{1l} \, L_l \, \xi^2 + f^{1/2} p_2 - a_{0l} f^{1/2} \xi^2) \}$$

so that

$$(8.20) \qquad Q = Op \, {}^f q^{(1)} + R + Op \, (p_3(t, x, 1, \zeta(t, x))^{-1} \, {}^f g_l) \, .$$

Noting (3.19), we have for any $\varphi \in \Sigma_{\varphi_{0l}}^0$ that

$$(8.21) \qquad \varphi \circ {}^f (p_3(t, x, 1, \zeta(t, x))^{-1} g_l) \in E \quad \text{from Lemma 6.38} \, ,$$
$$[Op \, \varphi, R] \in \mathcal{C}_0 \quad \text{from Lemmas 6.35, 40–(3), and (6.24)} \, ,$$
$$Op \, \varphi \cdot R \in \mathcal{C} + \mathcal{C}_0 \quad \text{with} \quad L_1 = L_2 = \tilde{L}_l, \, L_3 = L_l'$$
$$\text{from Lemmas 6.35, 40–(1), and (6.24)} \, .$$

Writing $A = A^{(1)} \cdots A^{(m)}$ with $A^{(i)} \in \Sigma_{1, \varphi_{0l}}$, we have from (8.20) that

(8.22) $A \cdot Op\, {}^f q^{(l)} = A(Q - R - Op\,(p_3(t, x, 1, \zeta(t, x))^{-1} {}^f g_l))$,

$$AR = A^{(1)} RA^{(2)} \cdots A^{(m)} + A^{(1)} \sum_{j=2}^{m} A^{(2)} \cdots [A^{(j)}, R] \cdots A^{(m)}\,.$$

Thus, noting (8.21), (8.22), Lemma 6.13, and using Lemma 7.2–(2) and –(3) we obtain setting $A_1 = A^{(2)} \cdots A^{(m)}$ that for small T, $n \geq 2$,

(8.23) $\|A \cdot Op\, {}^f q^{(l)} u\|^{(n)} \leq C_2 (\|Qu\|^{(n)} + M_{0l, n-1/(20h)} (f^{1/(20h)} A_1 u)$
              $+ n^{-1/2} M_{0, n}(u) + \|u\|_{(3)}^{(n-1/(20h))})$.

From Lemma 7.6–(1) there exists $A_2 \in \Sigma_{m-1, \varphi_{0l}}$ such that for small T, $n \geq 3$,

$$M_{0l, n-1/(20h)} (f^{1/(20h)} A_1 u) \leq M_{0l, n-1/(20h)} (A_2 f^{1/(20h)} u)$$
$$+ C_3 (\|Qu\|^{(n-1/(20h))} + \|u\|_{(4)}^{(n-1/(20h)-1)})\,.$$

Substituting this into (8.23) and estimating the forth term in (8.23) by Lemma 7.4–(1) we see that

(8.24) $\|A \cdot Op\, {}^f q^{(l)} u\|^{(n)} \leq C_3 (\|Qu\|^{(n)} + M_{0l, n-1/(20h)} (A_2 f^{1/(20h)} u)$
              $+ n^{-1/2} M_{0l, n} (A_1 u) + \|u\|_{(4)}^{(n-1/(20h))})$.

Now, applying Lemma 7.6–(3) to $\|[Op\, {}^f q^{(l)}, A] u\|^{(n)}$ taking $\bar{q} = q^{(l)}$ and $R = Q$, and using (8.18) and (8.24) we see that the desired conclusion holds. This completes the proof of Lemma 8.3.

Lemma 8.4. *Let $m \in N$ with $m \geq 2$ and $A \in \Sigma_{m, \varphi_{0l}}$. Then, for small T and $n \geq m+1$,*

$$M_{0l, n}(Au) \leq C(\|Qu\|^{(n)} + \|u\|_{(4)}^{(n-1/(20h))}$$
$$+ \|f^{(m-1)/(20h)} u\|_{(4, n-(m-1)/(20h))})\,.$$

Proof. Successive use of Lemmas 8.2, 7.8, 4.21–(2) shows that there exist $s \in N$ and $A_1, \cdots, A_s \in \Sigma_{l, \varphi_{0l}}$ such that for small T and $n \geq m+1$,

(8.25) $M_{0l, n}(Au)$
$$\leq C_1 \left(\sum_{i=1}^{s} \sum_{j=0}^{m-1} n^{-(m-j-1)/2} M_{0l, n-j/(20h)} (A_i f^{j/(20h)} u) \right.$$
$$\left. + \|Qu\|^{(n)} + \|u\|_{(4)}^{(n-1/(20h))} \right)\,.$$

From Lemma 6.39–(2), and (6.24) there exist $B_i \in Op\, {}^f \Sigma_{1/2}^{2(0, -1, 1)}$ for $i = 1, \cdots, s$ such that

(8.26) $Op\,(p_3(t, x, 1, \zeta(t, x))^{-1} d_l (f\xi)^3) \cdot A_i - B_i \in \mathcal{C}_0$.

On the other hand, since we can write $Op({}^fL_l(f\xi)^2)=(1-\mu_l\,\zeta)f^2\,D_x^2\cdot\tilde{L}_l+$ $\sum_{k=1}^{2}a_k\,f^k\,D_x^k$ with $a_k\in B_{h,T_0}$, it follows from (3.19), Lemmas 6.40–(2), 6.39–(2), and (6.24) that there exist $B_i'\in Op\ {}^f\Sigma_{1/2}^{(0,-1,1)}$, $i=1,\cdots,s$ such that

$$(8.27)\qquad Op(p_3(t,x,1,\zeta(t,x))^{-1}\,d_l\,A_{1l}\ {}^fL_l(f\xi)^2)\cdot A_i-B_i'\,\tilde{L}_l\in\mathcal{C}_0\,.$$

Thus, since

$$(8.28)\qquad p_3(t,x,1,\zeta(t,x))^{-1}\,{}^fq_3={}^fq^{(l)}+p_3(t,x,1,\zeta(t,x))^{-1}(d_l(f\xi)^3$$
$$+d_l\,A_{1l}\ {}^fL_l(f\xi)^2)\,,$$

it follows from (8.26), (8.27), Lemmas 7.3–(1), 7.2–(3), 7.4–(1), 4.21–(1), 4.18–(i) that for small T and $n\geq m+1$,

$$M_{0l,n-j/(20h)}\,(A_i\,u)\leq C_2\,||Op\ {}^fq^{(l)}\,A_i\,u||^{(n-j/(20h))}$$
$$\leq C_3(||p_3(t,x,1,\zeta(t,x))^{-1}\,Op\ {}^fq_3\,u||^{(n-j/(20h))}$$
$$+n^{-1/2}\,M_{0,n-j/(20h)}\,(u)+||u||_{(3)}^{(n-(j+1)/(20h))}+||B_i\,u||^{(n-1/(20h))}$$
$$+||B_i'(\tilde{L}_l-D_t)\,u||^{(n-j/(20h))}+||B_i'\,D_t\,u||^{(n-j/(20h))})$$
$$\leq C_4(||Qu||^{(n-j/(20h))}+||u||_{(4)}^{(n-(j+1)/(20h))}+||u||_{(4,n-j/(20h))})\,.$$

Combining this and (8.25) and using Lemmas 4.21–(2), 7.8, (4.55), and Lemma 7.1–(6) we have that for small T, $n\geq m+1$,

$$M_{0l,n}(Au)\leq C_5\{\sum_{j=0}^{m-1}n^{-(m-j-1)/2}(||Q\,f^{j/(20h)}\,u||^{(n-j/(20h))}$$
$$+||f^{j/(20h)}\,u||_{(4)}^{(n-1(j+1)/(20h))}+||f^{j/(20h)}\,u||_{(4,n-j/(20h))})$$
$$+||Qu||^{(n)}+||u||_{(4)}^{(n-1/(20h))}\}$$
$$\leq C_6(\sum_{j=0}^{m-1}n^{-(m-j-1)/2}\,||f^{j/(20h)}\,u||_{(4,n-j/(20h))}+||Qu||^{(n)}+||u||_{(4)}^{(n-1/(20h))})$$
$$\leq C_7(||f^{(m-1)/(20h)}\,u||_{(4,n-(m-1)/(20h))}+||Qu||^{(n)}+||u||_{(4)}^{(n-1/(20h))})\,.$$

This completes the proof of Lemma 8.4.

Remark 8.5.   In the proof of Lemmas 8.3 and 8.4 we do not use (1) of (3.13).   So these two lemmas hold in all cases of (3.13).

Taking $N=2N_0+1$ in (8.5) and estimating the first (resp. second) summation by Lemma 8.2 (resp. 8.5) we obtain that for small T and $n\geq 2N_0+2$,

$$||u||_{(6)}^{(n)}\leq C_4(||Qu||^{(n)}+||u||_{(4)}^{(n-1/(20h))}+||f^{r_0}\,u||_{(4,n-r_0-3)}$$
$$+||f^{N_0/(10h)}\,u||_{(4,n-N_0/(10h))})\,.$$

Since $N_0/(10h)=2(2r_0+4)>r_0+3$, the last term on the right can be dominated

by $||f^{r_0} u||_{(4, n-r_0-3)}$ in view of Lemma 4.21–(4) and (4.55). So combining this inequality and (8.4), and using (4.55) we get that for small T and $n \geq N_0 + 3$,

$$(8.29) \qquad\qquad ||u||_{\{6\}}^{(n)} + ||f^{r_0} u||_{(4, n-r_0)} \leq C_5 ||Qu||^{(n)} .$$

Taking $N = 2N_0 + 2$ in the inequality of Lemma 7.9 we see by the similar argument that for small T and $n \geq 2N_0 + 3$,

$$(8.30) \qquad \sum_{1 \leq |\mu| + |\nu| \leq 2} ||Q_{\mu\nu} u||^{(n)} \leq C_6 (||Qu||^{(n-1/(20h))} + n^{-1/2} ||Qu||^{(n)}$$
$$+ ||u||_{\{4\}}^{(n-1/(20h))} + ||f^{r_0} u||_{(4, n-r_0-3)}) .$$

Applying (8.29) to the latter two terms on the right we get the second desired inequality. The proof is complete.

Next we consider the case (2).

Proposition 8.6. *Assume that* (2) *of* (3.13) *holds. Then there exists* r_0, $n_0 \in \mathbb{N}$ *with* $n_0 \geq r_0 + 1$ *such that for small* T *and* $n \geq n_0$,

$$||f^{r_0} u||_{(4, n-r_0)} + ||u||_{\{6\}}^{(n)} \leq C ||Qu||^{(n)} ,$$
$$||Q_{\mu\nu} u||^{(n)} \leq C (||Qu||^{(n-1/(20h))} + n^{-1/2} ||Qu||^{(n)}) \quad when \quad 1 \leq |\mu| + |\nu| \leq 2 .$$

Proof. As in the proof of Proposition 8.1 we may assume $Q = p_3 (t, x, 1, \zeta (t, x))^{-1} Op^f q$. Set $\Phi_{iN} = Op^f \varphi_{iN}$, $\Phi_{0lN} = Op^f \varphi_{0lN}$ for $N \geq 2$. Then using Corollary 6.27, Lemma 4.18–(4) and (5), Lemmas 4.21, 7.2–(3), and the fact that $\sum_{i=0}^{2} A_i D_t^i f^{3/2} A \in \mathcal{C}_0$ for $A \in Op \Sigma^{(0, M, 0)}$, $A_i \in Op^f \Sigma^{M'(0, -2, 0)}$, $i = 0, 1, 2$ with $M, M' \geq 0$ and $M' - 2^{-1} M \leq -1$ which can be shown by using Fact 4.6–7, (4.3), (4.6), and Corollary 4.12–(iv) as in the proof of Corollary 4.12–(i), it is not hard to see that for $r \in \mathbb{N}$ we have that

$$(8.31) \quad ||f^r u||_{(4, n-r)} \leq C_1 (n^{-1/2} ||Qu||^{(n-r-2\kappa/3)}$$
$$+ ||u||_{\{3\}}^{(n-2\kappa/3 - r - 1/(20h))} + \sum_{i \in I} ||f^r \Phi_{iN}^N u||_{(4, n-r)} + \sum_{l=1}^{2} ||f^r \Phi_{0lN}^N u||_{(4, n-r)})$$
$$\text{for small } T \text{ and } n \geq 2\kappa/3 + r + 2 .$$

In the same way as in the proof of Proposition 8.1 we have from Corollary 6.27, Lemma 4.21, (4.55),

$$(8.32) \qquad\qquad ||u||_{\{6\}}^{(n)} \leq C_2 (\sum_{i \in I} ||\Phi_{iN}^N u||_{\{6\}}^{(n)} + \sum_{l=1}^{2} ||\Phi_{0lN}^N u||_{\{6\}}^{(n)})$$
$$\text{for small } T \text{ and } n \geq 2\kappa/3 + 1 .$$

In case (2) of (3.13) we have

$$c \in E_{h, T_0} \backslash \{0\} .$$

So setting, keeping (7.11) and a remark just after it in mind, that

$$(8.33) \qquad \widetilde{\widetilde{L}}_l := \widetilde{L}_l - (1 - \mu_l \zeta)^{-1} c_l f D_x$$
$$= \widetilde{L}_l - (1 + A_{0l})^{-1} (-1)^{l+1} (1 - \mu_l \zeta)^{-1} c b_l f D_x,$$

and using Lemma 6.10–(2), and Lemmas 4.21, 7.1–(8), one can show that there exists $r_1 \in N$ such that for any $r \geq r_1$, $r \in N$ we have that $t^r f^r D_t$ and $t^r f^{r+1} D_x$ can be expressed as a linear combinations of $\widetilde{L}_l$ and $\widetilde{\widetilde{L}}_l$ with coefficients in B_{h,T_0}, and that

$$(8.34) \qquad \|f^r \Phi_{iN}{}^N u\|_{(4, n-r)} \leq C_3 (\|f^{r-r_1} \widetilde{L}_{l(i)} \Phi_{iN}{}^N u\|_{(2, n-r+r_1)}$$
$$+ \|f^{r-r_1} \widetilde{\widetilde{L}}_{l(i)} \Phi_{iN}{}^N u\|_{(2, n-r+r_1)} + n^{-1/2} \|u\|_{(2)}^{(n-1)}),$$

$$(8.35) \qquad \|f^r \Phi_{0lN}{}^N u\|_{(4-r)} \leq C_3 (\|f^{r-r_1} \widetilde{L}_l \Phi_{0lN}{}^N u\|_{(2, n-r+r_1)}$$
$$+ \|f^{r-r_1} \widetilde{\widetilde{L}}_l \Phi_{0lN}{}^N u\|_{(2, n-r+r_1)} + n^{-1/2} \|u\|_{(2)}^{(n-1)}),$$

for $n \geq r+1$, where $l(i)$, $i \in I$ is as in Lemma 3.2. (3.26) and the fact that $c \in E_{h,T_0}$ shows

$$c\xi \in \Sigma_{\text{supp } \varphi_i}^{2/3(0, -2, 0)} \quad \text{for} \quad i \in I_1.$$

So from (8.33), Lemmas 6.30, 6.31–(1), and (6.42)

$$(8.36) \qquad \widetilde{\widetilde{L}}_{l(i)} \Phi_{iN} - \widetilde{L}_{l(i)} \Phi_{iN} \in Op^f \Sigma^{2/3(0, -2, 0)}.$$

Using (8.34), (8.36), Lemmas 4.18, 4.21, and the inequality that

$$(8.37) \qquad \|A_\alpha u\|_{(s, n)} \leq C \|u\|_{(s + [2\alpha] + 1, n)}$$
$$\text{for} \quad s \in N \text{ with } 0 \leq s \leq 5 \text{ and } \alpha > 0 \text{ with } s + 2\alpha \leq 6, \, 2\alpha \notin N,$$

we have for any $r \in N$ with $r \geq r_1$ that for $n \geq r+1$ and $i \in I_1$,

$$(8.38) \qquad \|f^r \Phi_{iN}{}^N u\|_{(4, n-r)} \leq C_4 (\|f^{r-r_1} \widetilde{L}_{l(i)} \Phi_{iN}{}^N u\|_{(2, n-r+r_1)}$$
$$+ \|A_{2/3} f^{r-r_1} \Phi_{iN}{}^{N-1} u\|_{(2, n-r+r_1)} + \|u\|_{(3)}^{(n-1)}).$$

Similarly using the fact that by (7.11) and (3.23),

$$(8.39) \qquad \widetilde{\widetilde{L}}_{l(1)} - L_{i1} \in Op^f \Sigma_{1/2}^{2/3(0, -1, 1)},$$

we have for any $r \in N$ with $r \geq r_1$ that for $n \geq r+1$ and $i \in I_2$,

$$(8.40) \qquad \|f^r \Phi_{iN}{}^N u\|_{(4, n-r)} \leq C_5 (\|f^{r-r_1} \widetilde{L}_{l(i)} \Phi_{iN}{}^N u\|_{(2, n-r+r_1)}$$
$$+ \|f^{r-r_1} L_{i1} \Phi_{iN}{}^N u\|_{(2, n-r+r_1)} + \|u\|_{(3)}^{(n-1)}).$$

Noting (7.11) and applying the similar argument as above to the first terms on the right of (8.38) and (8.40) we obtain for any $r \in N$ with $r \geq r_1$ that for $n \geq r+1$

and $i \in I$,

(8.41) $\|f^r \Phi_{iN}{}^N u\|_{(4, n-r)} \leq C_6 \left(\sum_{j=1}^{2} \|f^{r-r_1} L_{ij} \Phi_{iN}{}^N u\|_{(2, n-r+r_1)} \right.$

$\left. + \|A_{2/3} f^{r-r_1} \Phi_{iN}{}^{N-1} u\|_{(2, n-r+r_1)} + \|u\|_{(3)}^{(n-1)} \right) .$

Since by using Lemmas 7.1–(8), 4.12–(1), and Cor. 4.3–(ii) the second term on the right of (8.41) is dominated by $\|A_{2/3} D_t f^{r-r_1} \Phi_{iN}{}^{N-1} u\|^{(n-r+r_1)} + \|A_{2/3} f^{r-r_1+1}$ $D_x \Phi_{iN}{}^{N-1} u\|^{(n-r+r_1)} + n^{-1/2} \|u\|_{(2)}^{(n-2)}$, repeating the similar argument as above once again we get for any $r \in N$ with $r \geq 2r_1$ that for $n \geq r+1$ and $i \in I$,

(8.42) $\|f^r \Phi_{iN}{}^N u\|_{(4, n-r)} \leq C_7 \left(\sum_{j=1}^{2} \|L_{ij} \Phi_{iN}{}^N u\|_{(2, n-r+r_1)} \right.$

$\left. + \|A_{2/3} L_{ij} \Phi_{iN}{}^{N-1} u\|^{(n-r+2r_1)} + \|u\|_{(3)}^{(n-r+2r_1-1)} \right) .$

On the other hand, using the fact that from (3.19) and Lemma 6.39–(2),

(8.43) $d_l f D_x \cdot Op\, \varphi \in t^{-\kappa} \mathcal{A}^{0,3}$ for $\varphi \in \Sigma_{\varphi_{0l}}^0 ,$

in the similar way as above we have for $r \in N$, $r \geq r_1$ that for $n \geq r+1$

(8.44) $\|f^r \Phi_{0lN}{}^N u\|_{(4, n-r)} \leq C_8 \left(\|f^{r-r_1} \tilde{L}_l \Phi_{0lN}{}^N u\|_{(2, n-r+r_1)} \right.$

$\left. + \|f^{r-r_1} L_l' \Phi_{0lN}{}^N u\|_{(2, n-r+r_1)} + n^{-1/2} \|u\|_{(2)}^{(n-r+r_1+\kappa-2)} \right) .$

Therefore, taking

(8.45) $r = r_0 := r_1 + \max(r_1, \kappa) + 1 ,$

and Lemma 7.1–(1) and Corollary 5.2 we obtain from (8.42) and (8.44) that

(8.46) $\|f^{r_0} u\|_{(4, n-r_0)} \leq C_9 \left\{ n^{-1/2} \|Qu\|^{(n-r_0-1)} + \|u\|_{(3)}^{(n-2)} \right.$

$+ \sum_{i \in I} (M_{i, n-3}(\Phi_{iN}{}^N u) + M_{i, n-3}(\Phi_{iN}{}^{N-1} u)) + \sum_{l=1}^{2} (M_{0l, n-3}(\Phi_{0lN}{}^N u)$

$\left. + M_{0l, n-3}(\Phi_{0lN}{}^{N-1} u)) \right\} ,$

for small T and $n \geq r_0 + 3 .$

Estimating the right of (8.32) by Lemma 7.1–(3) we have for small T and $n \geq 2$,

(8.47) $\|u\|_{(6)}^{(n)} \leq C \left(\sum_{i \in I} M_{i, n}(\Phi_{iN}{}^N u) + \sum_{l=1}^{2} M_{0l, n}(\Phi_{0lN}{}^N u) \right) .$

We set

(8.48) $N_0 := 20h(2r_0 + 1) .$

Lemma 8.7. *Let $N_1, N_2 \in N$ with $N_2 \geq N_0$ and $N_1 = 2N_2 + 3$. Let $A \in \Sigma_{N_1, \varphi_i}$, $A_2 \in \Sigma_{N_1, \varphi_i}$ for $i \in I$. Then for small T and $n \geq N_2 + 3$,*

$$\sum_{j=1}^{2} M_{i,n}(A_j\, u) \le C(\|Qu\|^{(n)} + \|u\|_{(4)}^{(n-1/(20h))} + \|f^{r_0+r_1}\, u\|_{(4,n-r_0-r_1)})\,.$$

Proof. Using Corollary 7.7 with $k = N_2$ and $k' = 3$ to $M_{i,n}(A_1\, u)$, and next estimating the terms of the form $n^{-(N_2-r)/2} M_{i,n-1/(20h)}(A f^{r/(20h)}\, u)$, $A \in \Sigma_{3,\varphi_i}$, $0 \le r \le N_2$ on the right of the resulting inequality by using (7.3), (7.6), Lemmas 7.8, 4.21–(1), 7.1–(6) we get for some $B_1, \cdots, B_s \in \Sigma_{2,\varphi_i}$ that for small T and $n \ge N_2+3$,

$$(8.49) \qquad M_{i,n}(A_1\, u) \le C_1(\|Qu\|^{(n)} \\ + \sum_{r=0}^{N_2+1} \sum_{j=1}^{s} n^{-(N_2+1-r)/2} M_{i,n-r/(20h)}(B_j\, f^{r/(20h)}\, u) \\ + \|f^{N_2/(20h)}\, u\|_{(4,n-N_2/(20h))} + \|u\|_{(4)}^{(n-1/(20h))})\,.$$

Using Corollary 7.7 with $k = N_2$ and $k' = 2$ we have for some $B_{s+1}, \cdots, B_{s+s'} \in \Sigma_{2,\varphi_i}$ that for small T and $n \ge N_2+3$,

$$(8.50) \qquad M_{i,n}(A_2\, u) \le C_2(\|Qu\|^{(n)} \\ + \sum_{r=0}^{N_2} \sum_{j=s+1}^{s+s'} n^{-(N_2-r)/2} M_{i,n-r/(20h)}(B_j\, f^{r/(20h)}\, u) \\ + \|f^{N_2/(20h)}\, u\|_{(4,n-N_2/(20h))} + \|u\|_{(4)}^{(n-1/(20h))})\,.$$

Let us estimate terms involving B_j in (8.49) and (8.50). To do so we let $\Phi \in \Sigma_{2,\varphi_i}$ and we write $\Phi = \Phi_1 \Phi_2$, $\Phi_j \in \Sigma_{1,\varphi_i}$. From Lemma 6.6, $Op\, \mu \cdot \Phi_1 - \Phi_1 Q \in \mathcal{C} + \mathcal{C}_0$ with $L_j = L_{ij}$ and π there. So using (3.23) and the last fact of (3.26) we have

$$(8.51) \qquad Op\, \pi \cdot \Phi_1 - \Phi Q - \sum_{k=0}^{2} \tilde{A}_k\, \tilde{L}_{I(i)}^{\,k} \in \mathcal{C}_0 \\ \text{with } \tilde{A}_2 \in \mathcal{A}^{1/2,1/2,3},\ \tilde{A}_1 \in \mathcal{A}^{3/2,1/2,3}, \\ \tilde{A}_0 \in \mathcal{A}^{3/2,3/2,3} + \mathcal{A}^{2,1,3} + t^{-1+1/(6h)}\, \mathcal{A}^{3/2,1/2,3}\,.$$

Then for any $\alpha \ge 0$ we have that for small T and $n \ge 3$,

$$(8.52) \quad M_{i,n}(\Phi f^\alpha\, u) \le C_1'\, \|Op\, \pi \cdot \Phi f^\alpha\, u\|^{(n)} \quad \text{(Lemma 7.3–(1))} \\ \le C_2'(\|Q\Phi_2 f^\alpha\, u\|^{(n)} + \sum_{k=0}^{2} \|\tilde{A}_k\, \tilde{L}_{I(i)}\, \Phi_2 f^\alpha\, u\|^{(n)} + n^{-1/2}\, \|Qu\|^{(n)} \\ + \|u\|_{(4)}^{(n-1/(20h))}) \quad \text{((8.51), Lemmas 6.13, 7.2–(3), 7.4–(1), 4.21–(2))} \\ \le C_3'(\|Qu\|^{(n)} + M_{0I(i),n}(\Phi_2 f^\alpha\, u) + \|f^\alpha\, u\|_{(4,n+1-1/(20h))} + \|u\|_{(4)}^{(n-1/(20h))} \\ + \|A_{3/2}\, \tilde{L}_{I(i)}\, \Phi_2 f^\alpha\, u\|^{(n+1/2(1-1)/(10h))}) \quad \text{((7.6), Lemmas} \\ \text{7.8, 4.18–(1), 4.21–(1) and (2)).}$$

By Corollary 4.3 we have that for small T and $n \ge 2$,

$$(8.53) \qquad \|A_{3/2}\, \tilde{L}_{I(i)}\, \Phi_2 f^\alpha\, u\|^{(n)} \le C_8'(n^{-1/2} M_{0I(i),n-1}(\Phi_2 f^\alpha\, u) \\ + \|A_{1/2} f D_x\, \tilde{L}_{I(i)}\, \Phi_2 f^\alpha\, u\|^{(n)})\,.$$

It is not hard to check using Lemma 4.18–(4) that for any $r \in N$,

$$(8.54) \qquad fD_x \tilde{L}_{l(i)}[\Phi_2, f^r] = f^r \tilde{B}_1 D_t + f^r \tilde{B}_2$$
$$\text{with} \quad \tilde{B}_1 \in \mathcal{A}^{1/2,3}, \quad \tilde{B}_2 \in \mathcal{A}^{3/2,3} + \mathcal{A}^{1,1/2,3} \ .$$

From (3.26) and Lemma 6.39–(2)

$$(8.55) \qquad (d_{l(i)} f\xi) \circ \varphi \in {}^f \Sigma^0 \quad \text{for} \quad \varphi \in \Sigma_{\varphi_i} \ .$$

Using (8.54), (8.55), (7.8), and noting the manner of choice of r_1 (cf. (8.34)), one can show in a straightforward way by taking care of commutation that for $n \geq r_1 + 1$

$$(8.56) \qquad ||A_{1/2} fD_x \tilde{L}_{l(i)} \Phi_2 f^{r_1} u||^{(n-r_1)} \leq C_5'(||u||_{(4,n-r_1)} + ||u||_{(3)}^{(n-1)}$$
$$+ ||\tilde{L}_{l(i)}{}^2 \Phi_2 u||_{(1,n)} + ||\tilde{L}_{l(i)} L'_{l(i)} \Phi_2 u||_{(1,n)}) \ .$$

From (8.52), (8.53), (8.56), (4.55), Lemmas 7.1–(4), 4.21–(2) and (4) we have for any $r \in N$ with $r \geq N_2 - 2$ which implies $r/(20h) \geq 2r_0 + 1$,

$$(8.57) \qquad M_{i, n-r/(20h)}(\Phi f^{r/(20h)} u) \leq C_6'(||Qu||^{(n-1/(20h))}$$
$$+ ||u||_{(4)}^{(n-(r+1)/(20h)+r_1)} + M_{0l(i), n-r/(20h)}(\Phi_2 f^{r/(20h)} u)$$
$$+ M_{0l(i), n-r/(20h)+r_1}(\Phi_2 f^{r/(20h)-r_1} u)$$
$$+ ||f^{r/(20h)-r_1} u||_{(4, n+r_1-(r+1)/(20h)+1/2)}) \ .$$

From (8.52), (4.55), Lemmas 7.1–(4) and (6), 4.21–(2) we have for any $r \in N$ with $0 \leq r \leq N_2 - 3$ that for small T and $n \geq N_2 + 3$,

$$(8.58) \qquad n^{-(N_2-r)/2} M_{i, n-r/(20h)}(\Phi f^{r/(20h)} u) \leq C_7'(n^{-3/2} ||Qu||^{(n)}$$
$$+ n^{-1} ||u||_{(4)}^{(n-1/(20h))} + n^{-1} M_{0l(i), n}(\Phi_2 f^{r/(20h)} u)) \ .$$

Now from (3.26) and Lemma 6.39–(2) for $\varphi \in \Sigma_{\varphi_i}^0$,

$$(8.59) \qquad (d_{l(i)} (f\xi)^3 + d_{l(i)} (f\xi)^2 A_{1l(i)} {}^f L_{l(i)}) \circ \varphi = b_1 \tau + b_2$$
$$\text{with some} \quad b_1 \in {}^f \Sigma^{(0,-2,0)}, \ b_2 \in {}^f \Sigma^{(0,4,0)} + {}^f \Sigma^{(-1,-2,0)} \ .$$

So from (8.59) we have for any $A \in \Sigma_{1,\varphi_i}$ that for small T and $n \geq 2$,

$$(8.60) \qquad M_{0l(i), n}(Au) \leq C(M_{0, n}(Au) + ||u||_{(4, n)} + ||u||_{(3)}^{(n-1)})$$
$$\text{(by Lemmas 7.3–(1), 4.21–(1))}$$
$$\leq C'(||Qu||^{(n)} + ||u||_{(4, n)} + ||u||_{(3)}^{(n-1/(20h))})$$
$$\text{(by Lemma 7.4–(2))}.$$

Substituting $f^\alpha u$, $\alpha \geq 0$ for u in (8.60) and using Lemmas 7.8, 4.21 we have for any $A \in \Sigma_{1,\varphi_i}$, $\alpha \geq 0$ that for small T and $n \geq 2$,

$$M_{0l(i),n} (Af^\alpha u) \leq C(||Qu||^{(n)} + ||f^\alpha u||_{(4,n)} + ||u||_{(4)}^{(n-1/(20h))}) .$$

Applying this inequality to (8.57) and (8.58) we get using Lemmas 4.21–(4) and 7.1–(6) that when $r \geq N_2 - 2$, for small T and $n \geq N_2 + 3$,

$$(8.61) \qquad M_{i,n-r/(20h)} (\Phi f^{r/(20h)} u) \leq C_8'(||Qu||^{(n-r/(20h))} + ||u||_{(4)}^{(n-r_0-r_1)}$$
$$+ ||f^{r_0+r_1} u||_{(4,n-r_0-r_1)}) ,$$

and that when $0 \leq r \leq N_2 - 3$, for small T and $n \geq N_2 + 3$,

$$(8.62) \qquad n^{-(N_2-r)/2} M_{i,n-r/(20h)} (\Phi f^{r/(20h)} u) \leq C_9'(n^{-1} ||Qu||^{(n)}$$
$$+ n^{-1/2} ||u||_{(4)}^{(n-1/(20h))}) .$$

Applying (8.61) and (8.62) to the terms involving B_j in (8.49) and (8.50), noting $N_2 \geq 2r_0 + 1$, and using Lemma 4.21–(4) and (4.55) we get the desired inequality. The proof of Lemma 8.7 is complete.

We set

$$(8.63) \qquad\qquad\qquad N_1 = 2N_0 + 3 .$$

Then by Lemma 8.7 we have that for $i \in I$, small T, and $n \geq N_0 + 3$,

$$(8.64) \qquad M_{i,n}(\Phi_{iN_1}{}^{N_1}u) + M_{in}(\Phi_{iN_1}{}^{N_1-1} u) \leq C_{11}(||Qu||^{(n)} + ||u||_{(4)}^{(n-1/(20h))}$$
$$+ ||f^{r_0+r_1} u||_{(4,n-r_0-r_1)}) .$$

By Remark 8.5 we can use Lemma 8.4 to conclude that for any l, small T, and $n \geq N_1 + 1$

$$(8.65) \qquad M_{0l,n}(\Phi_{0lN_1}{}^{N_1} u) + M_{0l,n}(\Phi_{0lN_1}{}^{N_1-1} u) \leq C_{12}(||Qu||^{(n)}$$
$$+ ||u||_{(4)}^{(n-1/(20h))} + ||f^{(N_1-1)/(20h)} u||_{(4,n-(N_1-1)/20h))}$$
$$+ ||f^{(N_1-2)/(20h)} u||_{(4,n-(N_1-2)/(20h))}) .$$

Since $(N_1-2)/(20h) \geq 2(2r_0+1)$, the rest of proof of Proposition 8.6 is now straightforward by using (8.46), (8.47), (8.64), (8.65), and by taking $N=N_1$ in Lemma 7.9 as in the proof of Proposition 8.1. Q.E.D.

Finally we consider the case (3).

Proposition 8.8. *Assume that* (3) *of* (3.13) *holds. Set for* $s=1, 2, n \geq 1$, *and* $u \in \mathcal{S}_T$,

$$||u||^{(s,n)} = \sum_{k=0}^{s} n^{k/2} ||A_{(s-k)/2} u||^{(n+(s+k)/2)} .$$

Then there exists $n_0 \in \mathbf{N}$ *such that for small T and $n \geq n_0$,*

$$||u||_{(6)}^{(n)} + \sum_{k=1}^{2} ||\tilde{L}^k u||^{(3-k,n)} \leq C ||Qu||^{(n)} ,$$

$$\sum_{1 \leq |\mu| + |\nu| \leq 2} ||Q_{\mu\nu} u||^{(n)} \leq C(||Qu||^{(n-1/(20h))} + n^{-1/2} ||Qu||^{(n)}) .$$

Proof. Let $\varphi_i^* \in \Sigma_0^0$, $i \in I$ be in Lemma 3.2 and define for $s, j, k \in \{1, 2, 3\}$ with $s \neq j$,

$$b_{sjk} = \left[\begin{array}{ll} \sigma_{ik}/(\sigma_{is} - \sigma_{ij}) \, \varphi_i^* & \text{on supp } \varphi_i^* \\ 0 & \text{on } (\text{supp } \varphi_i^*)^c . \end{array} \right.$$

Then b_{ijk} satisfies with the notations in Definitions 3.1 and 4.4

(8.66) $$|\partial^\alpha b_{sjk}| \leq C_\alpha \, \tilde{\psi}_1^{-\alpha_1} \cdot \psi_2^{-\alpha_2} \cdot \psi_{3,1/6}^{-\alpha_3} .$$

Proof of (8.66). It suffices to consider this on supp φ_i^*. First from (3.28),

(8.67) $$|b_{sjk}| \leq C .$$

Next from (3.22)′ and (3.28) we have that for α with $\alpha_1 > 0$

(8.68) $$|\partial^\alpha \sigma_{ik}| \leq C_\alpha \, t^{-1+1/(6h)} \, \tilde{\psi}_1^{-\alpha_1+1} \, \psi_2^{-\alpha_2} \cdot \psi_{3,0}^{1-\alpha_3}$$
$$\leq C_\alpha' \, t^{-1+1/(6h)+\kappa/3} \langle \xi \rangle^{1/2} \, \tilde{\psi}_1^{-\alpha_1+1} \, \psi_2^{-\alpha_2} \cdot \psi_{3,0}^{-\alpha_3} \cdot |\sigma_{is} - \sigma_{ij}|$$
$$\leq C_\alpha' \, \tilde{\psi}_1^{-\alpha_1} \, \psi_2^{-\alpha_2} \cdot \psi_{3,0}^{-\alpha_3} |\sigma_{is} - \sigma_{ij}| ,$$

that for α with $\alpha_1 = 0$ and $\alpha_2 > 0$,

(8.69) $$|\partial^\alpha \sigma_{ik}| \leq C_\alpha \, t^{1/(6h)} \psi_2^{-\alpha_2+1} \cdot \psi_{3,0}^{1-\alpha_3} \leq C_\alpha' \, t^{1/(6h)+\kappa/3} \langle \xi \rangle^{1/2}$$
$$\times \psi_2^{-\alpha_2+1} \, \psi_{3,0}^{-\alpha_3} \cdot |\sigma_{is} - \sigma_{ij}| \leq C_\alpha' \cdot \psi_2^{-\alpha_2} \cdot \psi_{3,0}^{-\alpha_3} \cdot |\sigma_{is} - \sigma_{ij}| ,$$

and that for α with $\alpha_3 > 0$ and $\alpha_1 = \alpha_2 = 0$,

(8.70) $$|\partial^\alpha \sigma_{ik}| \leq C_\alpha \, \psi_{3,0}^{2/3-\alpha_3} \leq C_\alpha' \, t^{\kappa/3} \langle \xi \rangle^{1/6} \, \psi_{3,0}^{-\alpha_3} \cdot |\sigma_{is} - \sigma_{ij}|$$
$$\leq C_\alpha' \langle t^{\varepsilon_h} \xi \rangle^{1/6} \, \psi_{3,0}^{-\alpha_3} \cdot |\sigma_{is} - \sigma_{ij}| .$$

(8.67)∼(8.70) and Leibniz rule show (8.66).

The fact that $\varphi_i^* = 1$ on supp φ_i and Lagrange's interpolation formula show

(8.71) $$L\varphi_{i2} = b_{121}(L - \sigma_{i2}) \, \varphi_{i2} + b_{212}(L - \sigma_{i1}) \, \psi_{i2} ,$$
$$L^2 \varphi_{i2} = \sum_{k,j,s : \text{distinct}} b_{ksk} \, b_{kjk}(L - \sigma_{is}) \, (L - \sigma_{ij}) \, \varphi_{i2} ,$$

where we adopt the convention (3.16) on L and φ_{i2} is defined by (7.15). (8.66) implies

(8.72) $$b_{sjk} \in \Sigma^0 .$$

We set

(8.73) $$B_{sjk} = Op\ ^fb_{sjk}\ .$$

Now we recall the definition (7.17) of Φ_{iN}, Φ_{0lN}. Using for $\Phi = \Phi_{i2}$ identities that

$$L_{im}\ \Phi = [L_{im},\ \Phi] + \Phi L_{im}\ ,$$
$$L_{il}\ L_{ij}\ \Phi = [L_{il},\ [L_{,j},\ \Phi]] + [L_{,j},\ \Phi]\ L_{il} + [L_{il},\ \Phi]\ L_{ij} + \Phi L_{il}\ L_{ij}\ ,$$
$$\tilde{L}^2\ \Phi = [\tilde{L},\ [\tilde{L},\ \Phi]] + \Phi\tilde{L}^2 + 2[\tilde{L},\ \Phi]\ \tilde{L}\ ,$$

and using (8.71), (8.72), Lemmas 4.18, 4.22, 6.4, and (6.24) show

(8.74) $$\tilde{L}\Phi_{i2} = B_{121}\ L_{i1}\ \Phi_{i2} + B_{212}\ L_{i2}\ \Phi_{i2} + \sum_{j=1}^{2} A_j\ L_{ij} + A_0$$

$$\text{with } A_j \in \mathscr{B}^{0,3}\quad \text{for}\quad j \neq 0,\ A_0 \in \mathscr{B}^{1/2,1/2,3}\ ,$$

(8.75) $$\tilde{L}^2\ \Phi_{i2} = \sum_{k,j,s\,:\,\text{distinct}} B_{ksk}\ B_{kjk}\ L_{is}\ L_{ij} + \sum_{s\neq j} B_{sj}\ L_{is}\ L_{ij}$$

$$+ \sum_{s=1}^{3} B_s\ L_{is} + \tilde{B}\tilde{L} + B_0$$

$$\text{with } B_{sj} \in \mathscr{B}^{0,3},\ B_s \in \mathscr{B}^{1/2,1/2,3}\ \text{for}\ s \neq 0,\ \tilde{B} \in \mathscr{B}^{1/2,1/2,3},\ B_0 \in \mathscr{B}^{1,1,3}\ .$$

Using (8.74), (8.75), and using Lemma 4.21–(1), (4.55) show that for $n \geq 2$,

(8.76) $$\sum_{k=1}^{2} ||\tilde{L}^k\ \Phi_{i2}\ ^2u\ ||^{(3-k,n)} \leq C_l\ (\sum_{s\neq j}\ \sum_{k=1}^{2} ||L_{is}\ L_{ij}\ \Phi_{i2}\ ^ku||^{(1,n)}$$
$$+ \sum_{s=1}^{3}\ \sum_{k=1}^{2} ||L_{is}\ \Phi_{i2}\ ^ku||^{(2,n)} + ||\tilde{L}\Phi_{i2}\ u||^{(2,n-1/(10h))} + ||u||_{(3)}^{(u-1/(20h))}$$
$$+ n^{-1}\ ||u||_{(2)}^{(n)})\ .$$

If $L_0 = D_t - A$ with $A = Op\ ^fa$ and $a \in \Sigma_{1/2}^{(0,-1,1)}$ satisfying (4.42), Lemmas 4. 18–(3)–(i), 4.22, and the fact that $t^{-1}\ \mathscr{A}^{k,l,N} \subset \mathscr{A}^{k+1/2,l+1/2,N}$ when $N \geq k+1/2$ show

$$[L_0,\ B] \in t^\lambda\ \mathscr{A}^{k+1/2,l+1/2,N}\quad \text{if}\quad B \in t^\lambda\ \mathscr{A}^{k,l,N}\ \text{with}\ n \geq k+1/2\ .$$

So if $L_{0j}, j=1, 2$ are such operators as L_0 and $\Phi \in t^\lambda\ \mathscr{A}^{0,3}$ with $\lambda \geq 0$, using Lemma 4.21–(1), (4.55) show that for $n \geq 2+\lambda$,

(8.77) $$||L_{01}\ \Phi u||^{(2,n)} \leq C(||L_{01}\ u||^{(2,n-\lambda)} + ||u||_{(3)}^{(n-1/(20h)-\lambda)} + n^{-1/2}\ ||u||_{(2)}^{(n-\lambda)})\ ,$$
$$||L_{01}\ L_{02}\ \Phi u||^{(1,n)} \leq C'(||L_{01}\ L_{02}\ u||^{(1,n-\lambda)}$$
$$+ \sum_{j=1}^{2} (||L_{0j}\ u||^{(2,n-\lambda-1/(20h))} + n^{-1/2}\ ||L_{0j}\ u||^{(2,n-\lambda)})$$
$$+ ||u||_{(3)}^{(n-1)/(10h)-\lambda)} + n^{-1}\ ||u||_{(1)}^{(n-\lambda)})\ .$$

Applying (8.77) to each term with $k=2$ of summations on the right and to the third term, and using Lemma 4.21–(2) we obtain

(8.78) $\displaystyle\sum_{k=1}^{2} ||\tilde{L}^k \Phi_{i2}{}^2 u||^{(3-k,n)} \leq C_2 (\sum_{s\neq j} ||L_{is} L_{ij} \Phi_{i2} u||^{(1,n)}$

$\qquad\qquad + \displaystyle\sum_{s=1}^{3} ||L_{is} \Phi_{i2} u||^{(2,n)} + ||\tilde{L}u||^{(2,n-1/(20h))} + ||u||_{(3)}^{(n-1/(20h))} + n^{-1} ||u||_{(2)}^{(n)}) \ .$

Similarly we have that for $n \geq 2$,

(8.79) $\displaystyle\sum_{k=1}^{3} ||\tilde{L}^k \Phi_{0l2}{}^2 u||^{(3-k,n)} \leq C_3 (\sum_{k=1}^{2} ||\tilde{L}^k \Phi_{0l2} u||^{(3-k,n)} + ||u||_{(3)}^{(n-1/(20h))}$

$\qquad\qquad + n^{-1/2} ||u||_{(2)}^{(n)}) \ ,$

and setting $\Phi_i = \Phi_{i2}{}^2 - Op^f(\varphi_{i2}{}^2)$, $\Phi_{0l} = \Phi_{0l2}{}^2 - Op^f(\varphi_{0l2}{}^2)$, and noting $\Phi_i, \Phi_{0l} \in t^{2\kappa/3} \mathcal{A}^{0,3}$ from (3.20) and Lemma 6.26 we have that

(8.80) $\displaystyle\sum_{k=1}^{2} (||\tilde{L}^k \Phi_i u||^{(3-k,n)} + ||\tilde{L}^k \Phi_{0l} u||^{(3-k,n)}) \leq C_4 (\sum_{k=1}^{2} ||\tilde{L}^k u||^{(3-k,n-2\kappa/3)}$

$\qquad\qquad + ||u||_{(3)}^{(n-2\kappa/3)}), \ n \geq 3 \ .$

Note that by Lemma 4.21, $||\Phi_i u||_{(6)}^{(n)} + ||\Phi_{0l} u||_{(6)}^{(n)} \leq C_5 ||u||_{(6)}^{(n-2\kappa/3)}$ for $n \geq 2$. Now using (8.78)~(8.80), (4.55), Lemmas 4.21–(1), 7.1–(3) and (4) show that there exists $n_1 \in N$ and $T_1 \in (0, T_0)$ such that for $T \in (0, T_1)$, $n \geq n_1$,

(8.81) $||u||_{(6)}^{(n)} + \displaystyle\sum_{k=1}^{2} ||\tilde{L}^k u||^{(3-k,n)} \leq C_6 (\sum_{i\in I} ||\Phi_{i2} u||_{(6)}^{(n)} + \sum_{l=1}^{2} ||\Phi_{0l2} u||_{(6)}^{(n)}$

$\qquad\qquad + \displaystyle\sum_{i\neq j} ||L_{is} L_{ij} \Phi_{i2} u||^{(1,n)} + \sum_{s=1}^{3} ||L_{is} \Phi_{i2} u||^{(2,n)}$

$\qquad\qquad + \displaystyle\sum_{l=1}^{2} \sum_{k=1}^{2} ||\tilde{L}^k \Phi_{0l2} u||^{(3-k,n)}) \leq C_7 (\sum_{i\in I} M_{i,n}(\Phi_{i2} u) + \sum_{l=1}^{2} M_{0,n}(\Phi_{0l2} u))$

where we use that $\tilde{L} = L_{0,i}$ for any i from the assumption (3.13)–(3). As in the proof of Proposition 8.1 we may assume that $Q = p_3(t, x, 1, \zeta(t, x))^{-1} Op^f q$. Let $\Phi \in \Sigma_{1,\varphi_i}$ and $\pi = \prod_{j=1}^{3} (\tau - \tilde{\mu} f\xi - {}^f\tilde{\sigma}_{ij})$ where $\tilde{\mu}(t, x) = \mu(t, x)/(1 - (\zeta \cdot \mu)(t, x))$ with the convention (3.16), and note that $p_3(t, x, 1, \zeta(t, x))^{-1} {}^f q = \pi$ on supp φ_i. From Lemmas 6.9, 6.12

(8.82) $\qquad\qquad\qquad\qquad Op \ \pi \cdot \Phi - Q \cdot \Phi \in \mathcal{C}_0 \ ,$

(8.83) $\qquad\qquad [Q_3, \Phi] \in \mathcal{C} + \mathcal{C}_0 \ \text{with} \ \mathcal{C} = \mathcal{C}(L_1 = L_2 = L_3 = \tilde{L}) \ ,$

$\qquad\qquad [Q_2, \Phi] \in \mathcal{C}_0 \ ,$

where $Q_j = p_3(t, x, 1, \zeta(t, x))^{-1} Op^f q_j$ with the homogeneous part q_j of degree j in (τ, ξ) of q.

Using (8.82), (8.83), Lemmas 7.3–(1), 7.4–(1), 4.21 one can easily deduce that for small T and $n \geq 2$,

(8.84) $M_{i,n}(\Phi u) \leq C_8 (||Qu||^{(n)} + \displaystyle\sum_{k=1}^{2} (||A_{(3-k)/2} \tilde{L}^k u||^{(n+((3-k)/2)(1-(1/10h))}$

$$+||\tilde{L}^k u||^{(n+3-k)}+||u||^{(n-(1/20h))}_{(4)}) \, .$$

Next we let $\Phi_0 \in \Sigma_{1,\varphi_{0l}}$. In case (3) we have $g_l = f^{1/2} a_{0l} \xi^2$ so that

$$f^{1/2} p_2 = g_l + f^{1/2} \sum_{j=1}^{2} a_{jl} \xi^{2-j} L^j \, .$$

So from Lemma 6.38,

$$\Phi_0 \cdot Q_2 = \sum_{k=0}^{2} A_k \tilde{L}^k \quad \text{with} \quad A_k \in \mathcal{A}^{(3-k)/2,3} \quad \text{for} \quad k = 1, 2, A_0 \in t^{-\kappa} \mathcal{A}^{3/2,3} \, .$$

Using this and (8.83) which holds for $\Phi \in {}^f \Sigma_{1/2}^0$ we obtain similarly as (8.84) that for small T and $n \geq 2$,

$$(8.85) \qquad M_{0,n}(\Phi_0 \, u) \leq C_9 (||Qu||^{(n)} + \sum_{k=1}^{2} (||A_{(3-k)/2} \, \tilde{L}^k u||^{(n+((3-k)/2)(1-(1/10h)))}$$

$$+||\tilde{L}^k u||^{(n+3-k)}+||u||^{(n-1/(20h))}_{(4)}) \, .$$

Now the rest of proof is straightforward by using (8.81), (8.84), (8.85), and Lemma 7.9. We omit the details. Propositions 8.1, 8.6, 8.8 mean (2.13), (2.14) respectively in cases (1), (2), (3).

§9. Proof of Proposition 2.3

We just have to (2.10) for proof of Proposition 2.3, because the standard argument gives the desired concluion. When $m=1$, or when $m=2$ and $k_2=1$, (2.13) contains (2.10). And the remaining cases are contained in the following Proposotion.

Proposition 9.1. *Suppose the assumptions and notations in Proposition 2.3. Suppose that in (2.6), the number of paires of (i, j) with $j \neq 0$ is greater than or equal to 2. Set for $s = 0, \cdots, 2m$ and $n \geq 1$,*

$$||u||^{[s,n]} = \sum_{i+j \leq 2s} n^{m-(3/2)-(i/2)-j} ||A_{i/2} D_t^j u||^{(n+m-(j/2)-j)}, \quad u \in \mathcal{S}_T \, .$$

Then, there exists $C > 0$, $n_0 \geq 1$, and $T \in (0, T_0)$ such that for $T \in (0, T_0)$, $n \geq n_0$, and $u \in \mathcal{S}_T$

$$(9.1) \qquad\qquad\qquad ||u||^{[2m,n]} \leq C ||Qu||^{(n)} \, .$$

Proof. We note that $m \geq 2$ from the assumption. In view of (2.7) the partial fraction decomposition, or rather the theory of resultants shows that one can write

$$p_m + f^{1/2} p'_{m-1} = \prod_i \prod_{j \neq 0} (p_{ij} + f^{1/2} p'_{ij}) + q_{m-2}$$

with the notation $p'_{m-1}(t, x, \tau, \xi) = p_{m-1}(t, x, \tau, -\zeta(t, x)\tau+\xi)$ where p'_{ij}, q_{m-2} are in $C^\infty(\mathbf{R}^4_{T_0})$ being polynomials in (τ, ξ) with their coeficients in B_{T_0} respectively of degrees $i-1$, $m-2$. We may neglect perturbations of terms of order $\leq m-2$ in $\tilde{Q}$ for proof of (9.1) in view of (9.1) and Lemma 4.21–(1) so that we may assume

$$\tilde{Q} = p_m(t, x, 1, \zeta(t, x))^{-1}(p_m+f^{1/2}p_{m-1}-q_{m-2})(t, x, \tau, \zeta(t, x)\tau+\xi).$$

Set for $j \neq 0$,

$$(9.2)\qquad \tilde{Q}_{ij} = p_{ij}(t, x, 1, \zeta(t, x))^{-1}(p_{ij}+f^{1/2}p'_{ij})(t, x, \tau, \zeta(t, x)\tau+\xi),$$
$$Q_{ij} = {}^f\tilde{Q}_{ij}, \quad Q^{ij} = \prod_{(1,s)\neq(i,j)} Q_{ls}.$$

Then

$$(9.3)\qquad\qquad Q(t, x, \tau, \xi) = \Pi_{i,j} Q_{ij}(t, x, \tau, \xi).$$

Similarly as above we can write for each $1 \leq k \leq m$,

$$(9.4)\qquad\qquad \xi^k \tau^{m-k} = \sum_{i,j} r_{ijk} \tilde{Q}^{ij} + r_k$$

where r_{ijk}, r_k are in $C^\infty(\mathbf{R}^4_{T_0})$ which are polynomials in (τ, ξ) with coeficients in B_{T_0} such that deg $r_{ijk}=i$, degree in τ of $r_{ijk}=i-1$, deg $r_k=m-1$, degree in τ of $r_k=m-2$.

Using (9.4), (2.13), Lemmas 7.1–(2), 4.21–(1), and the inequality $\|D^m_t u\|^{(n)} \leq C(\|Qu\|^{(n)}+\sum_{k=1}^m \sum_{l=0}^{m-k} \|f^k D^k_x D^l_t u\|^{(n)})$, one finds that for small T and large n,

$$(9.5)\quad \|u\|^{[2m,n]}-C_1 n^{-3/2}\|Qu\|^{(n)} \leq C_2(n^{-3/2}\sum_{k=1}^m \|f^k D^k_x D^{m-k}_t u\|^{(n)}+n^{-1}\|u\|^{[2m-2,n-1]})$$

$$\leq C_3(\sum_{i,j} \|Q_{ij}\circ Q^{ij}(t, x, D)u\|^{(n)}+n^{-1}\|u\|^{[2m-2,n-1]}).$$

Let $e_1=(1, 0)$, $e_2=(0, 1)\in\mathbf{R}^2$. Then, symbolic calculus shows that by extending Definition 6.21 to q of the form in it with $a_{30}=$cosnt.,

$$(9.6)\qquad (\tilde{Q}_{ij})_{0,e_l} = \sum_{k=0}^{i-1} a_{kl}\tau^k \text{ with}$$
$$a_{k1}\in t^{1/2-1/(20h)}f \sum^{(i-k-1/2)(0,-2,0)}, \quad a_{k2}\in {}^f\sum^{(i-k-1/2)(0,-2,0)};$$

$$(9.7)\qquad (\tilde{Q}_{ij})_{e_1,0}\circ(\tilde{Q}_{i'j'})_{0,e_1}-(\tilde{Q}_{ij})_{e_1,0}\cdot(\tilde{Q}_{i'j'})_{0,e_1} = \sum_{k=0}^{i+i'-2} a_k\tau^k$$
$$\text{with } a_k\in\sum_{0<l\leq i+i'-k-2} t^{-1+1/2-1/(20h)} B^{i+i'-k-l-1,1/2,m}$$
$$+t^{-1/2-1/(20h)} B^{i+i'-k-2,m};$$

$$(9.8)\qquad (\tilde{Q}_{ij})_{e_2,0}\circ(\tilde{Q}_{i'j'})_{0,e_2}-(\tilde{Q}_{ij})_{e_2,0}\cdot(\tilde{Q}_{i'j'})_{0,e_2} = \sum_{k=0}^{i+i'-2} b_k\tau^k$$

with $b_k \in B^{i+i'-k-2,m}$;

$$(9.9) \qquad [Op\, Q_{i'j'}, Op(\tilde{Q}_{ij})_{0,e_1}] = \sum_{k=0}^{i+i'-2} A_{kl} D_t^k$$

with $A_{kl} \in \sum_{0 < s \le i+i'-k-2} t^{-s+1/2-1/(20h)} \mathcal{B}^{i+i'-k-s-1/2,m}$
$+ t^{1/2-1/(20h)} \mathcal{B}^{i+i'-3/2-k,m}$, $A_{k2} \in \mathcal{B}^{i+i'-k-3/2,m}$.

From (9.3), (9.7), (9.8), and the equalities

$$(\tilde{Q}_{ij})_{e_1,0} \cdot (\tilde{Q}_{i'j'})_{0,e_1} = \partial_\tau Q_{ij} \cdot \partial_t Q_{i'j'},$$
$$(\tilde{Q}_{ij})_{e_2,0} \cdot (\tilde{Q}_{i'j'})_{0,e_2} = {}^f(\partial_\xi \tilde{Q}_{ij}) f D_x Q_{i'j'},$$
$$(\tilde{Q}_{ij})_{e_l,0} \circ (\tilde{Q}_{i'j'})_{0,e_l} \circ Q^{ij,i'j'} - ((\tilde{Q}_{ij})_{e_l,0} \cdot (\tilde{Q}_{i'j'})_{0,e_l}) \cdot Q^{ij,i'j'}$$
$$= ((\tilde{Q}_{ij})_{e_l,0} \circ (\tilde{Q}_{i'j'})_{0,e_l} - (\tilde{Q}_{ij})_{e_l,0} \cdot (\tilde{Q}_{i'j'})_{0,e_l}) \circ Q^{ij,i'j'}$$
$$+ ((\tilde{Q}_{ij})_{e_l,0} \cdot (\tilde{Q}_{ij})_{0,e_l}) \circ Q^{ij,i'j'} - ((\tilde{Q}_{ij})_{e_l,0} \cdot (\tilde{Q}_{i'j'})_{0,e_l}) \cdot Q^{ij,i'j'}$$
$$\text{where } Q^{ij,i'j'} = \prod_{(l,s) \ne (i,j),(i',j')} Q_{ls},$$

we see that

$$Q_{ij} \circ Q^{ij} = Q + \sum_{l=1}^{2} \sum_{(i,j) \ne (i',j')} (\tilde{Q}_{ij})_{e_l,0} \circ (\tilde{Q}_{i'j'})_{0,e_l} \circ Q^{ij,i'j'} + \sum_{k \le m-2} \tilde{a}_k \tau^k$$

with $\tilde{a}_k \in \sum_{0 < s \le m-2} (t^{-s+1/2-1/(20h)} B^{m-k-s-1,1/2,m} + t^{-1/2-1/(20h)} B^{m-k-2,m})$.

This and (2.13) imply that for small T and large n,

$$\sum_{i,j} ||Q_{ij} \circ Q^{ij}(t, x, D)u||^{(n)}$$
$$\le C_4 (||Qu||^{(n)} + \sum_{l=1}^{2} \sum_{(i,j) \ne (i',j')} ||Q_{ij} \circ (\tilde{Q}_{i'j'})_{0,e_l} \circ Q^{ij,i'j'}(t, x, D)u||^{(n)}$$
$$+ ||u||^{[2m-3,n-1/2]}) .$$

By (9.9), (9.6), (2.14), a term in the summation on the right is estimated for small T and large n by

$$||(\tilde{Q}_{i'j'})_{0,e_l} \circ Q_{ij} \circ Q^{ij,i'j'}(t, x, D)u||^{(n)} + ||u||^{[2m-3,n-1/2]}$$
$$\le C_5 (||(\tilde{Q}_{i'j'})_{0,e_l} \circ Q^{i',j'}(t, x, D)u||^{(n)} + ||u||^{[2m-3,n-1/2]})$$
$$\le C_6 (||Q_{i'j'} \circ Q^{i'j'}(t, x, D)u||^{(n-1)/(20h)}$$
$$+ n^{-1/2} ||Q_{i'j'} \circ Q^{i'j'}(t, x, D)u||^{(n)} + ||u||^{[2m-3,n-1/2]}) .$$

Therefore we obtain that for small T and large n,

$$(9.10) \qquad \sum_{i,j} ||Q_{ij} \circ Q^{ij}(t, x, D)u||^{(n)} \le C_7 (||Qu||^{(n)} + ||u||^{[2m-3,n-1/2]}) .$$

The desired conclusion follows from (9.5) and (9.10).   Q.E.D.

§ 10. Appendix. Proof of Lemma 4.7

Our proof is parallel to that of Lemma 2 in [1]. So we make our discription brief. We denote the set of all b as in the statement satisfying (4.18) for $(\Phi, \varphi) \in B$ by $\mathcal{U}^{\Phi, \varphi}$, and we set $\mathcal{U} = \bigcup_{(\Phi, \varphi) \in B} \mathcal{U}^{\Phi, \varphi}$. We denote $a(x, \xi)$ as in the above defined for $b \in \mathcal{U}$ by $I(b)(x, \xi)$. The important fact is that constants in proof are uniform on B and $\mathcal{U}$, but we omit to mention this fact for simplicity.

First we assume that the lemma has proved for $b \in \mathcal{U}$ compactly supported in (y, η) for each fixed (x, ξ). Let $b \in \mathcal{U}^{\Phi, \varphi}$ and set $b_m = \chi_m b$ for $m \in N$ where $\chi_m(x, y, \xi, \eta) = \chi(y/m, \eta/m)$ with $\chi \in C_0^\infty(\mathbf{R}^{2n})$ with $\chi(0) = 1$. Then

$$|D_y^\alpha D_\eta^\beta b_m(x, y, \xi, \eta)| \leq C_{\alpha\beta} |b|_{|\alpha|+|b|}^{\Phi, \varphi} \cdot \sum_{i \leq |\beta|} \sup_{Q \in K} \Phi^{-p}(Q) \cdot a_{|\alpha|, |\beta|-p}^{\Phi, \varphi}$$

with $C_{\alpha\beta}$ independent of m, and $I(b_m)$ converges to $I(b)$ at any point. So the lemma for general b follows from the above assumed special case by replacing k by $k+1$ and $(M_1, \cdots, M_{k+l}, m_1, \cdots, m_{k+l})$ by $(0, M_1, \cdots, M_{k+l}, 0, m_1, \cdots, m_{k+l})$. Thus, we just have to prove (4.19) with C', L independent of $b \in \mathcal{U}^{\Phi, \varphi}$ compactly usported in (y, η) for each fixed (x, ξ), and of (Φ, φ).

We note that $I(b) = \iint e^{-i(y-x) \cdot (\eta - \xi)} b(x, y, \xi, \eta) dy \, d\eta$ then. Let $\chi_1 \in C^\infty(\mathbf{R})$ with $\chi_1(z) = 1$ when $|z| \leq 1/3$, $\chi_1(z) = 0$ when $|z| \leq 1/2$, and write $b = b_1 + b_2$ where $b_1(x, y, \xi, \eta) = b(x, y, \xi, \eta) \chi_1(|\eta - \xi| / (1 + |\xi|))$. Then $b_i \in {}^{\Phi, \varphi}$ and $|b_i|_j^{\Phi, \varphi} \leq C_{1j} |b|_j^{\Phi, \varphi}$ so that $I(b) = I(b_1) + I(b_2)$. We have

$$b_1 = 0 \quad \text{when} \quad |\xi - \eta| \geq (1 + |\xi|)/2 \, ,$$
$$b_2 = 0 \quad \text{when} \quad |\xi - \eta| \leq (1 + |\xi|)/3 \, .$$

The integration by parts gives that $I(b_2) = I(b_2')$ with $b_2' \in \mathcal{U}^{\Phi, \varphi}$ given by $b_2'(x, y, \xi, \eta) = |\eta - \xi|^{-2N} (-\Delta_y)^N [(1 + |x-y|^{2P})^{-1} (1 + (-\Delta_\eta)^P) b_2]$. Taking $P > n/2$ and N large we get $|I(b_2)| \leq |I(b_2')| \leq C_2 |b|_{2(P+N)}^{\Phi, \varphi} \Phi^{\Sigma M_j} \cdot \varphi^{\Sigma m_j}$.

Next choose $\chi_2 \in C^\infty(\mathbf{R})$ with $\chi_2(z) = 1$ when $|z| \leq 1/9$, $\chi_2(z) = 0$ when $|z| \geq 1/2$, and write $b_1 = b_{11} + b_{12}$ where $b_{11} = b_1(x, y, \xi, \eta) \chi_2(c^{-2}(\varphi^{-2}(x, \xi) |y - x|^2 + \Phi^{-2}(x, \xi) |\xi - \eta|^2))$. Again $b_{1i} \in \mathcal{U}^{\Phi, \varphi}$ and $|b_{1i}|_j^{\Phi, \varphi} \leq C_{3j} |b|_j^{\Phi, \varphi}$ so that $I(b_2) = I(b_{21}) + I(b_{22})$. We have

$$b_{11} = 0 \quad \text{when} \quad \varphi^{-1}(x, \xi) |y-x| + \Phi^{-1}(x, \xi) |\eta - \xi| \geq c/2 \, ,$$
$$b_{12} = 0 \quad \text{when} \quad \varphi^{-1}(x, \xi) |y-x| + \Phi^{-1}(x, \xi) |\eta - \xi| \leq c/3 \, .$$

Setting $\Phi_0 = \Phi(x, \xi)$, $\varphi_0 = \varphi(x, \xi)$ we have that when $(x, y, \xi, \eta) \in \text{supp } b_{11}$, $C^{-1} \leq |\Phi(Q)/\Phi_0| \leq C$, $C^{-1} \leq |\varphi(Q)/\varphi_0| \leq C$ for $Q \in K$. Then the integration by parts gives $I(b_{11}) = I(b_{11}')$ with $b_{11}' \in \mathcal{U}$ given by $b_{11}' = (1 + \Phi_0^{2P} |x-y|^{2P})^{-1} \times$

$\times (1+\Phi_0^{2P}(-\varDelta_y)^P) b_{11}$, and $|I(b_{11})| = |I(b'_{11})| \le C_4 |b|_{2P}^{\Phi,\varphi} \Phi_0^{\Sigma M_j} \cdot \varphi_0^{\Sigma m_j}$ by taking $P > n/2$. Finally we consider b_{12}. As in the same way of [1] setting $\Phi_0 = \Phi(x, \xi)$, $\varphi_0 = \varphi(x, \xi)$, and $R = \Phi_0/\varphi_0$ we have that for $(x, y, \xi, \eta) \in \text{supp } b_{12}$ and $Q \in K$, $C_0^{-1} R \le \Phi(Q)/\varphi(Q) \le C_0 R$, $\Phi_0/\Phi(Q) \le (C_0 c^{-1})^{1/2} (\Phi_0 \varphi_0)^{1/2}$, $\Phi(Q) \le 2 \cdot \max(c^{-1} C_0, 3Cc^{-1}) (|\eta - \xi| + R|y-x|)$. By using the first inequality the next two means respectively

$$\varphi_0/\varphi(Q) \le C_5 (\Phi_0 \varphi_0)^{1/2}, \quad \varphi(Q) \le C_6 (R^{-1} |\eta - \xi| + |y-x|).$$

Set $b'_{21}(x, y, \xi, \eta) = (\varphi_0^{-2N}(-\varDelta_\eta)^N + \Phi_0^{-2N}(-\varDelta_y)) \{ b_{21}(x, y, \xi, \eta)/(\varphi_0^{-2N} |y-x|^{2N} + \Phi_0^{-2N} |\eta - \xi|^{2N}) \}$. Then $b'_{21} \in \mathcal{U}^{\Phi,\varphi}$ and $I(b_{21}) = I(b'_{21})$. Leibniz rule shows that

$$|b'_{21}(x, y, \xi, \eta)|$$
$$\le C_{7N} \varphi_0^{-2N} \Phi_0^{-2N} \left\{ \frac{\Sigma_{|\beta| \le 2N} |D_\eta^\beta b_{21}(t, y, \xi, \eta)| |\Phi_0|^{|\beta|}}{(\varphi_0^{-1} |y-x| + \Phi_0^{-1} |\eta - \xi|)^{2N+(2N-|\beta|)}} + \right.$$
$$\left. + \frac{\Sigma_{|\alpha| \le 2N} |D_y^\alpha b_{21}(t, x, \xi, \eta)| |\varphi_0|^{|\alpha|}}{(\varphi_0^{-1} |y-x| + \Phi_0^{-1} |\eta - \xi|)^{2N+(2N-|\alpha|)}} \right\}$$
$$= C_{7N} \varphi_0^{-2N} \Phi_0^{-2N} \{I + II\}.$$

We take $P > \sum_{j=1}^{k+1} (|M_j| + |m_j|)$ and set $\tilde{M}_j = M_j + P$, $\tilde{m}_j = m_j + P$. Then $\tilde{M}_j \ge 0$, $\tilde{m}_j \ge 0$. By virtue of $\Phi \varphi \ge c$ a term of the numerator in I is dominated by $c^{-P(k+1)} |b|_{2N}^{\Phi,\varphi}$ times

$$\sum_{r_1 + \cdots + r_k = |\beta|} \Phi_0^{|\beta|} \prod_{j=1}^{k} \sup_{Q \in K} \Phi^{\tilde{M}_j - r_j}(Q) \varphi^{\tilde{m}_j}(Q) \prod_{j=k+1}^{k+l} \sup_{Q \in K} \Phi^{\tilde{M}_j}(Q) \varphi^{\tilde{m}_j}(Q)$$

$$\le C_8 \prod_{j=1}^{k+1} (\sup_{Q \in K} \Phi^{\tilde{M}_j}(Q) \varphi^{\tilde{m}_j}(Q)) \sup_{Q \in K} (\Phi_0/\Phi(Q))^{|\beta|}$$

$$\le C_9 (\Phi_0 \varphi_0)^N (|\eta - \xi| + R|y-x|)^{\Sigma \tilde{M}_j} \cdot (R^{-1} |\eta - \eta| + |y-x|)^{\Sigma \tilde{m}_j}.$$

Thus $I \le C_{10} |b|_{2N}^{\Phi,\varphi} \Phi_0^{\Sigma \tilde{M}_j - N} \varphi_0^{\Sigma \tilde{m}_j - N} (\varphi_0^{-1} |y-x| + \Phi_0^{-1} |\eta - \xi|)^{\Sigma(\tilde{M}_j + \tilde{m}_j) - 2N}$. II is dominated by the same expression. Therefore,

$$|I(b_{21})| \le C_{11} |b|_{2N}^{\Phi,\varphi} \cdot \Phi_0^{\Sigma \tilde{M}_j + n - N} \cdot \varphi_0^{\Sigma \tilde{m}_j + n - N} \quad \text{if } 2N > \sum_{i=1}^{k+1} (\tilde{M}_j + \tilde{m}_j) + n.$$

This completes the proof.

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