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$\mathscr{D}$-Modules on Analytic Spaces

By

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Introduction

In this note, we give a formalism of '$\mathscr{D}$-Modules' on complex analytic spaces, generalizing some results in the theory of $\mathscr{D}$-Modules on complex manifolds to the singular case. Our definition of $\mathscr{D}$-Modules, which is inspired by that of mixed Hodge Modules on singular varieties [15], uses the local embeddings of analytic spaces into complex manifolds, see 1.5, where the well-definedness follows from Kashiwara's equivalence [6]. The advantage of this definition is that locally we can apply immediately the theory of $\mathscr{D}$-Modules on complex manifolds, and we get some global results, e.g. (5.8.2)(6.1.2–3), etc., once the functors and the canonical morphisms are defined globally. Using these, we can prove, for example, the base change property 4.8, the adjunction formula 4.9, the duality for proper morphism 5.7, and the Riemann-Hilbert correspondence 6.2 in the singular case.

We define the direct image $f_!$, the de Rham functor DR (see 3.1) and the dual **D** (see 5.2) using the Čech covering and reducing essentially to the smooth case by an argument similar to the case of mixed Hodge Modules [15]. For the pull-back $f^!$ we use the theory of algebraic local cohomology [6] in the closed embedding case (see §2) and the Riemann-Hilbert correspondence [7] [10] in the projection case (see 4.2). For the proof of the duality theorem for direct image by a proper morphism, we extend the notion of induced $\mathscr{D}$-Module and good coherent $\mathscr{D}$-Module to the singular case, see 3.5, and relate the analytic dualizing complex in [11] [12] [13] with the dualizing complex for $\mathscr{D}$-Modules in the singular case, see 5.3. We also introduce the notion of inductively good which is stable by the pull-back, see 4.11. The proof of adjunction formula 4.9 is essentially the same as [15] and uses Kashiwara's diagram (4.9.2). Using

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these we can generalize some results in [19] [20], see 4.15. The proof of the Riemann-Hilbert correspondence is a natural generalization of the proof in [16, §4] to the singular case using the diagonal pairing, see 6.2.

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§1. Definition

In this note the analytic spaces are always assumed separated, paracompact, and globally finite dimensional (i.e. the diemension is globally bounded). We use mainly right $\mathcal{D}$-Modules on complex manifolds to simplify the definition of direct image and dual.

1.1. We first review the definition of direct image of $\mathcal{D}$-Modules with proper support for a morphism of complex manifolds. For a complex manifold X we denote by $M(\mathcal{D}_X)$ (resp. $M(\mathcal{O}_X)$) the abelian category of (right) $\mathcal{D}_X$-Modules (resp. $\mathcal{O}_X$-Modules), and $D^b(\mathcal{D}_X)$ (resp. $D^b(\mathcal{O}_X)$) its bounded derived category [17].

Let $f: X \to Y$ be a morphism of complex manifolds, and $M \in D^b(\mathcal{D}_X)$. The direct image with proper support of M is defined by

$$(1.1.1) \qquad f_! M = \boldsymbol{R} f_!(M \overset{L}{\otimes}_{\mathcal{D}_X} \mathcal{D}_{X\to Y}) \in D^b(\mathcal{D}_Y)$$

where $\boldsymbol{R}f_!$ is the sheaf theoretic direct image with proper support defined by taking a canonical c-soft resolution, and $\mathcal{D}_{X\to Y} = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}_Y$. If f is proper, we denote $f_!$ by f_*.

1.2. Let X be a complex manifold, Z a closed analytic subvariety of X. We define

$$(1.2.1) \qquad \Gamma_{[Z]} M = \varinjlim_{\mathcal{I}} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{I}, M), \quad \Gamma_{[X|Z]} M = \varinjlim_{\mathcal{I}} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}, M)$$

for an $\mathcal{O}_X$-Module M, where $\mathcal{I}$ runs over coherent Ideals of $\mathcal{O}_X$ such that $\mathrm{supp}\, \mathcal{O}_X/\mathcal{I} \subset |Z|$ with $|Z|$ the underlying set of Z, cf. [6] [7] [10], etc. For a coherent Ideal $\mathcal{I}$ such that $\mathrm{supp}\, \mathcal{O}_X/\mathcal{I} = |Z|$, $\Gamma_{[Z]} M$ has an exhaustive filtration defined by $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{I}^n, M)$ (i.e. the subsheaf of M annihilated $\mathcal{I}^n$). Although the filtration depends on $\mathcal{I}$, they are all cofinal. By definition we have natural morphisms $\Gamma_{[Z]} M \to M$ and $M \to \Gamma_{[X|Z]} M$. The cohomology sheaves of the derived functors $\boldsymbol{R}\Gamma_{[Z]} M$, $\boldsymbol{R}\Gamma_{[X|Z]} M$ (defined by taking an injective resolution of M) are denoted by $\mathcal{H}^j_{[Z]}M$ and $\mathcal{H}^j_{[X|Z]}M$. Then we have a long exact sequence:

$$(1.2.2) \qquad \to \mathcal{H}^j_{[Z]}M \to \mathcal{H}^j M \to \mathcal{H}^j_{[X|Z]}M \to \mathcal{H}^{j+1}_{[Z]}M \to$$

for $M \in D^b(\mathcal{O}_X)$. Note that they have natural structures of $\mathcal{D}$-Modules if M is a $\mathcal{D}$-Module (or a complex of $\mathcal{D}$-Modules), cf. [loc. cit.]. We denote by $M_Z(\mathcal{D}_X)$ the full subcategory of $M(\mathcal{D}_X)$ defined by the condition

$$(1.2.3) \qquad \Gamma_{[Z]}M \xrightarrow{\sim} M\,.$$

This is equivalent to the condition that for any local section m of M, there exists locally a coherent Ideal $\mathcal{J}$ of $\mathcal{O}_X$ such that $\operatorname{supp}\mathcal{O}_X/\mathcal{J} \subset |Z|$ and $m\mathcal{J}=0$. In particular

$$(1.2.4) \qquad M_Z(\mathcal{D}_X) \subset M_{Z'}(\mathcal{D}_X)$$

for $|Z| \subset |Z'|$. Note that $\Gamma_{[Z]}$: $M(\mathcal{D}_X) \to M_Z(\mathcal{D}_X)$ is a right adjoint functor of the natural functor $M_Z(\mathcal{D}_X) \to M(\mathcal{D}_X)$, i.e. we have a canonical isomorphism

$$(1.2.5) \qquad \operatorname{Hom}_{\mathcal{D}_X}(M, N) = \operatorname{Hom}_{\mathcal{D}_X}(M, \Gamma_{[Z]}N) \quad \text{for} \quad M \in M_Z(\mathcal{D}_X),\ N \in M(\mathcal{D}_X)\,.$$

In particular we get

(1.2.6) $\Gamma_{[Z]}M$ is injective in $M_Z(\mathcal{D}_X)$ if M is an injective $\mathcal{D}_X$-Module.

Let Z_1, Z_2 be closed subvarieties of X. Then

(1.2.7) $\Gamma_{[Z_2]}M$ is $\Gamma_{[Z_1]}$-acyclic if M is injective

by the theory of local cohomology on $\operatorname{Spec}(\mathcal{O}_{X,x})$, cf. [3] (in fact M_x and $(\Gamma_{[Z_2]}M)_x$ correspond to flasque sheaves on $\operatorname{Spec}(\mathcal{O}_{X,x})$, because injective $\mathcal{D}$-Modules are injective $\mathcal{O}$-Modules, and the injectivity is stable by restriction to open subsets). This implies

$$(1.2.8) \qquad \boldsymbol{R}\Gamma_{[Z_1 \cap Z_2]}M = \boldsymbol{R}\Gamma_{[Z_2]}\boldsymbol{R}\Gamma_{[Z_1]}M \qquad \text{for} \quad M \in D^b(\mathcal{D}_X)$$

as in [6]. Let Z_i be divisors on X for $1 \le i \le n$, and $Z = \cap_i Z_i$. Then

$$(1.2.9) \qquad \boldsymbol{R}\Gamma_{[Z]}M = \boldsymbol{R}\Gamma_{[Z_n]} \cdots \boldsymbol{R}\Gamma_{[Z_1]}M \quad \text{with} \quad \boldsymbol{R}\Gamma_{[Z_i]}M = [M \to \Gamma_{[X|Z_i]}M]$$

for $M \in D^b(\mathcal{D}_X)$, where $[M \to \Gamma_{[X|Z_i]}M]$ is a double complex such that the bidegrees of M^j, $\Gamma_{[X|Z_i]}M^j$ are $(0, j)$ $(1, j)$. Here $\Gamma_{[X|Z_i]}$ is the localization by a defining equation of Z_i, and is an exact functor, so that $\Gamma_{[X|Z_i]}M$ is well-defined. This means that $\boldsymbol{R}\Gamma_{[Z]}M$ is represented by a double complex whose (p, q) component is

$$\bigoplus_{|I|=p} \Gamma_{[X|Z_I]}M^q \tag{1.2.10}$$

where $Z_I = \bigcup_{i\in I} Z_i$ and $\Gamma_{[X|\emptyset]} = id$. As a corollary we get

$$\boldsymbol{R}\Gamma_{[Z]}M \xrightarrow{\sim} M \qquad \text{for} \quad M \in M_Z(\mathcal{D}_X)\,. \tag{1.2.11}$$

We denote by $D^b(M_Z(\mathcal{D}_X))$ the bounded derived category of the abelian category $M_Z(\mathcal{D}_X)$, and $D^b_Z(\mathcal{D}_X)$ the full subcategory of $D^b(\mathcal{D}_X)$ defined by the cohomological condition: $\mathcal{H}^j M \in M_Z(\mathcal{D}_X)$. Then the natural functor $D^b(M_Z(\mathcal{D}_X)) \to D^b_Z(\mathcal{D}_X)$ is an equivalence of categories with quasi-inverse $\boldsymbol{R}\Gamma_{[Z]}$ by (1.2.11).

1.3. Let $f\colon X \to Y$ be a morphism of complex manifolds, and Z, Z' closed subvarieties of X, Y such that $Z' \supset f(Z)$. Then the direct image in (1.1.1) induces an functor of derived categories

$$f_!\colon D^b(M_Z(\mathcal{D}_X)) \to D^b(M_{Z'}(\mathcal{D}_Y))\,. \tag{1.3.1}$$

In fact, using the graph of f, we can reduce to the case f projection, where the direct image is defined by the relative de Rham functor, cf. (3.1.2) below. Then we can use the filtration of $\Gamma_{[Z]}M$ in the remark after (1.2.1). More precisely, we replace $\mathrm{DR}_{X\times Y/Y}(M)$ by the inductive limit of (canonical) c-soft resolution of its subsheaves annihilated by coherent Ideals of $\mathcal{O}_Y$ whose quotients are supported in Z', cf. also 3.2. Note that the inductive limit commutes with the direct image with proper support so that c-soft sheaves are stable by inductive limit.

Assume Z smooth, and let $i\colon Z \to X$ be the natural inclusion. Then Kashiwara showed the equivalence of categories (cf. [loc. cit.]):

$$i_*\colon M(\mathcal{D}_Z) \xrightarrow{\sim} M_Z(\mathcal{D}_X) \tag{1.3.2}$$

In fact the assertion is local and we may assume Z is a hypersurface defined by a coordinate function x. Then $M \in M_Z(\mathcal{D}_X)$ has the decomposition $M = \bigoplus_{i\geq 0} M_i$ with $M_i = \mathrm{Ker}\,(x\partial_x - i\colon M \to M)$, and $M = i_* M_0$. This equivalence is generalized to

$$f_!\colon M_Z(\mathcal{D}_X) \xrightarrow{\sim} M_{Z'}(\mathcal{D}_Y) \tag{1.3.3}$$

for any morphism of complex manifolds with closed subspaces $f\colon (X, Z) \to (Y, Z')$ inducing an isomorphism $Z_{\mathrm{red}} \to Z'_{\mathrm{red}}$. In fact, we may assume that

Z, Z' are reduced by definition, and $Z \to Y$ is a minimal embedding locally (i.e. the dimension of Y is equal to the dimension of the Zariski tangent space of Z at a gvien point of Z) by replacing X, Y with their closed submanifolds using (1.3.2), because the assertion is local on Y. Then f has locally a section, and the assertion follows from (1.3.2). Here note that the functor $f_!$ depends only on the restriction of f to Z. In fact we have more generally

(1.3.4) $$f_! = g_!: D^b M_Z(\mathcal{D}_X) \to D^b M_{Z'}(\mathcal{D}_Y)$$

if f, g: $(X, Z) \to (Y, Z')$ coincide on Z. Using the natural factorization of f, g the assertion is reduced to the case of closed embeddings i_f, i_g: $X \to X \times Y$ defined by the graph of f, g, where Z is replaced by $i_f(Z) = i_g(Z)$. Then the assertion follows from (1.3.3) applied to pr_1: $X \times Y \to X$, because the compositions $\mathrm{pr}_1 i_f$, $\mathrm{pr}_1 i_g$ are the identity on X.

1.4. Let X be a complex analytic space. We consider the category $\mathcal{C}(X)$ whose objects are the closed embeddings $U \to V$ such that U are open subsets of X and V are smooth, where the morphisms are the morphisms of V such that their restrictions to U are the canonical open embeddings as open subsets of X. Note that we have a canonical morphism

(1.4.1) $$\mathcal{C}(X) \to \mathcal{C}(X_{\mathrm{red}})$$

by assigning the composition $U_{\mathrm{red}} \to U \to V$ to $U \to V$, where U_{red}, X_{red} denote the associated reduced spaces.

Let $W_i = \{U_i \to V_i\}$ $(i=1, 2)$ be objects of $\mathcal{C}(X)$, and f: $V_1 \to V_2$ be a morphism of $\mathcal{C}(X)$. Put $Z = U_2 \setminus f(U_1)$, $V_2' = V_2 \setminus Z$, and $V_1' = f^{-1}(V_2')$. Then Z is a closed subset of V_2 and V_i' is an open subset of V_i $(i=1, 2)$. We have a canonical factorization $f|_{V_1'} = j_2 f'$, where f': $V_1' \to V_2'$ is the restriction of f, and j_2: $V_2' \to V_2$ is the natural inclusion. Note that f' induces the identity on U_1, and U_1 is a closed subvariety of V_2' by f'.

Let $M \in M_{U_1}(\mathcal{D}_{V_1})$, cf. (1.2.3). Then $f_! M$ (cf. 1.1) is the zero extension of $f'_!(M|_{V_1'})$, and is a $\mathcal{D}_{V_2}$-Module. In fact the first assertion is clear by definition of direct image, and the second is reduced to the case $U_1 = U_2$ and follows from (1.3.3). By (1.3.4) we have

(1.4.2) $f_! M$ depends only on M, W_1, W_2, and is independent of f.

1.5. With the above notation we define $M(X, \mathcal{D})$ the category of '$\mathcal{D}$-Modules' on X as follows: The objects of $M(X, \mathcal{D})$ are $\{M_W\}_{W \in \mathcal{C}(X)}$, where $M_W \in M_U(\mathcal{D}_V)$ with $W = \{U \to V\}$, and M_W are given morphisms

$$u_{W_2W_1}: f_! M_{W_1} \to M_{W_2} \tag{1.5.1}$$

inducing isomorphisms on $V_2\backslash(U_2\backslash U_1)$ for any morphism $f\colon W_1 \to W_2$ in $\mathcal{C}(X)$, cf. (1.4.2), which satisfy the relation

$$u_{W_3W_1} = u_{W_3W_2} \circ g_!(u_{W_2W_1})\colon (gf)_! M_{W_1} \to M_{W_3} \tag{1.5.2}$$

for any W_1, W_2, $W_3 \in \mathcal{C}(X)$, where g is any morphism of W_2 to W_3. The morphisms of $M(X, \mathcal{D})$ are morphisms of M_W compatible with the morphisms (1.5.1). We call M_W the *representative* of M on W.

By definition $M(X, \mathcal{D})$ is an abelian category. We denoted by $D^b(X, \mathcal{D})$ the bounded derived category $D^b(M(X, \mathcal{D}))$ (cf. [17]) of $M(X, \mathcal{D})$ (same for $D(X, \mathcal{D})$, $D^+(X, \mathcal{D})$, etc.). We say that $M = \{M_W\} \in M(X, \mathcal{D})$ is *coherent* (resp. *holonomic*, resp. *regular holonomic*) if so are M_W for any $W \in \mathcal{C}(X)$, and $M \in D^b(X, \mathcal{D})$ is *coherent* (resp. *holonomic*. resp. *regular holonomic*) if so are $\mathcal{H}^j M$ for any j, where $\mathcal{H}^j\colon D(X, \mathcal{D}) \to M(X, \mathcal{D})$ is the natural cohomology functor. We denote by $D^b_{\mathrm{coh}}(X, \mathcal{D})$ (resp. $D^b_{\mathrm{hol}}(X, \mathcal{D})$, resp. $D^b_{rh}(X, \mathcal{D})$ the full subcategory of $D^b(X, \mathcal{D})$ consisting of such objects.

If X is globally embedded into a complex manifold V', $W' = \{X \to V'\}$ belongs to $\mathcal{C}(X)$, and we get a natural functor $M(X, \mathcal{D}) \to M_X(\mathcal{D}_{V'})$. We can check that this functor induces an equivalence of categories

$$M(X, \mathcal{D}) \xrightarrow{\sim} M_X(\mathcal{D}_{V'}) \tag{1.5.3}$$

In fact we can construct a $\mathcal{D}_V$-Module M_W for any $W = \{U \to V\} \in \mathcal{C}(X)$ by applying (1.3.3) to the projections of $V \times V'$ to V, V'.

Remark. Our $\mathcal{D}$-Module on analytic spaces is not a sheaf in the classical sense or that of Grothendieck (although its definition in this paper is largely influenced by the definition of Crystalline cohomology). In fact, it is a 'sheaf' like a perverse sheaf, i.e., to each $\{U \to V\} \in \mathcal{C}(X)$, we associate an abelian category instead of an abelian group (or an object of a category), and for gluing we need a theory of direct image $f_!$ for a morphism f of $\mathcal{C}(X)$.

1.6. Let $\boldsymbol{W} = \{W_i\}_{i \in \Lambda}$ be a family of $W_i = \{U_i \to V_i\} \in \mathcal{C}(X)$ indexed by Λ. We say that $\boldsymbol{W}$ is a covering of $\mathcal{C}(X)$ if $\cup_i U_i = X$. For a covering $\boldsymbol{W}$, put $U_I = \cap_{i \in I} U_I$, $V_I = \prod_{i \in I} V_i$ so that we have a natural closed embedding $U_I \to V_I$ and $W_I = \{U_I \to V_I\}$ belongs to $\mathcal{C}(X)$ (and also to $\mathcal{C}(U_i)$ for $i \in I$). By (1.5.3) $M_i \in M_{U_i}(\mathcal{D}_{V_i})$ determines an object of $M(U_i, \mathcal{D})$, and we denote its representative on W_I by $M_{i,I} \in M_{U_I}(\mathcal{D}_{V_I})$ for $i \in I$. Then by the same argument as above the category $M(X, \mathcal{D})$ is equivalent to the category $M(\boldsymbol{W}, \mathcal{D})$ defined as follows: The objects of $M(W, \mathscr{D})$ are $\{M_i\}$ with $M_i \in M_{U_i}(\mathscr{D}_{U_i})$ such that they are given isomorphisms

$$u_{ij}\colon M_{j,\{i,j\}} \xrightarrow{\sim} M_{i,\{i,j\}} \tag{1.6.1}$$

satisfying the gluing condition

$$u_{ik} = u_{ij} \circ u_{jk}\colon M_{k,\{i,j,k\}} \xrightarrow{\sim} M_{i,\{i,j,k\}} \tag{1.6.2}$$

where u_{ij} denotes also the induced isomorphism $M_{j,\{i,j,k\}} \xrightarrow{\sim} M_{i,\{i,j,k\}}$.

We can also show that $M(W, \mathscr{D})$ is equivalent to $M(W, \mathscr{D})'$ defined as follows: Let $\mathrm{pr}_{IJ}\colon V_J \to V_I$ be the natural projection for $J \supset I$. The objects of $M(W, \mathscr{D})'$ are $\{M_I\}_{I \subset \Lambda}$ with $M_I \in M_{U_I}(\mathscr{D}_{V_I})$, and they are given morphisms of $\mathscr{D}_{V_I}$-Modules

$$v_{IJ}\colon (\mathrm{pr}_{IJ})_! M_J \to M_I \tag{1.6.3}$$

inducing an isomorphism on the complement of $U_I \setminus U_J$, and satisfying

$$v_{IK} = v_{IJ} \circ (\mathrm{pr}_{IJ})_! v_{JK} \text{ on the complement of } U_I \setminus U_K\,. \tag{1.6.4}$$

As a corollary, we get an equivalent of categories

$$M(X_{\mathrm{red}}, \mathscr{D}) \xrightarrow{\sim} M(X, \mathscr{D}) \text{ (same for } D^b(X, \mathscr{D}),\ D^b_{\mathrm{coh}}(X, \mathscr{D}), \text{ etc.)} \tag{1.6.5}$$

induced by (1.4.1). In fact the inverse functor is constructed by using the above definitions of $M(X, \mathscr{D})$. So we can assume that the analytic spaces are reduced in most cases, except for the case we consider induced $\mathscr{D}$-Modules which depend on the non reduced structure of X, cf. 3.5.

Remark. The second definition in 1.6 is essentially the same as that in the filtered case in [14, 2.1.20]. The arguments in 1.5–6 can be applied to the case of filtered $\mathscr{D}$-Module and mixed Hodge Module.

1.7. Let Y be a complex analytic space, and X an open subset with $j\colon X \to Y$ the natural inclusion. Then we have the canonical pull-back functor $j^{-1}\colon M(Y, \mathscr{D}) \to M(X, \mathscr{D})$ by definition. The left adjoint functor

$$j_!\colon M(X, \mathscr{D}) \to M(Y, \mathscr{D}) \tag{1.7.1}$$

of j^{-1} is given by the usual zero extension. Here we use the fact that for $W = \{U \to V\} \in \mathcal{C}(Y)$, $W' := \{U \setminus Z \to V \setminus Z\}$ belongs to $\mathcal{C}(X)$, where $Z = Y \setminus X$. The existence of the right adjoint functor

$$j_*\colon M(X, \mathscr{D}) \to M(Y, \mathscr{D}) \tag{1.7.2}$$

is less trivial. For W, U, V, Z as above we denote by j_W: $V\backslash Z \to V$ the natural inclusion. Then for $M \in \mathcal{C}(X)$ the representative of $j_* M$ on W is defined by $\Gamma_{[U]}(j_W)_* M_{W'}$, where $M_{W'}$ is the representative of M on W' and $(j_W)_*$ denotes the sheaf theoretic direct image. We check that $\Gamma_{[U]}(j_W)_* M_{W'}$ is essentially independent of V under the equivalence (1.3.3) by reducing to the closed embedding case (1.3.2) as in the proof of (1.3.3), and the well-definedness follows. Note that

(1.7.3) $j_* M$ is injective in $M(Y, \mathcal{D})$ if M is injective in $M(X, \mathcal{D})$

by adjunction. Then we have

(1.7.4) The abelian category $M(X, \mathcal{D})$ has enough injectives.

In fact the assertion is reduced to the case X is a closed subvariety of a complex manifold by (1.7.3), and follows from (1.2.6).

1.8. Let X, Y be complex analytic spaces, and put $Z = X \times Y$. Then we have a bifunctor

(1.8.1) $$\boxtimes : M(X, \mathcal{D}) \times M(Y, \mathcal{D}) \to M(Z, \mathcal{D}).$$

In fact it is well-known if X, Y are smooth [7], [10] (cf. also [16, §4]), where $M \boxtimes N$ is defined by

(1.8.2) $$\begin{aligned} &\mathcal{O}_{X\times Y} \otimes_{(\mathrm{pr}_1^{-1}\mathcal{O}_X \otimes_{\boldsymbol{C}} \mathrm{pr}^{-1}\mathcal{O}_Y)} (\mathrm{pr}_1^{-1} M \otimes_{\boldsymbol{C}} \mathrm{pr}_2^{-1} N) \\ &= (\mathcal{O}_{X\times Y} \otimes_{\mathrm{pr}_1^{-1}\mathcal{O}_X} \mathrm{pr}_1^{-1} M) \otimes_{\mathrm{pr}_2^{-1}\mathcal{O}_Y} \mathrm{pr}_2^{-1} N. \end{aligned}$$

The general case is reduced to this case by 1.6. The functor $\boxtimes$ is exact for both factors, cf. [loc. cit.]. This implies a canonical isomorphism

(1.8.3) $$\mathcal{H}^n(M \boxtimes N) = \oplus_{i+j=n} \mathcal{H}^i M \boxtimes \mathcal{H}^j N$$

for complexes M, N, which is induced by $\mathrm{Ker}\, d^i \boxtimes \mathrm{Ker}\, d^j \to \mathrm{Ker}(d^i \boxtimes id + (-1)^i id \boxtimes d^j)$. (In fact, this morphism implies also the degeneration of the spectral sequence associated with a double complex.) If $M \in M(X, \mathcal{D})$ and $N \in M(Y, \mathcal{D})$ have filtrations F, G such that $F_i M = 0$, $G_j N = 0$ for $i, j \ll 0$, we define the filtration $H = F \boxtimes G$ by $H_k(M \boxtimes N) = \sum_{i+j=k} F_i M \boxtimes G_j N$ so that

(1.8.4) $$\mathrm{Gr}_k^H(M \boxtimes N) = \oplus_{i+j=k} \mathrm{Gr}_i^F M \boxtimes \mathrm{Gr}_j^G N.$$

In the case X, Y smooth, the external product $\boxtimes$ commutes with algebraic local cohomology, i.e. for a closed subspace Z of X, we have a canonical isomorphism

(1.8.5) $$(\boldsymbol{R}\Gamma_{[Z]} M) \boxtimes N \xrightarrow{\sim} \boldsymbol{R}\Gamma_{[Z\times Y]}(M \boxtimes N),$$

using free resolutions of $\mathcal{O}_X/\mathcal{J}$ for coherent Ideals $\mathcal{J}$ in (1.2.1).

§2. Local Cohomology

2.1. Let $i\colon X \to Y$ be a closed embedding of analytic spaces. We have a natural functor $\mathcal{C}(Y) \to \mathcal{C}(X)$ by assigning $\{U \cap X \to V\}$ to $\{U \to V\} \in \mathcal{C}(Y)$, and this implies the direct image functor

$$(2.1.1) \qquad i_*\colon M(X, \mathcal{D}) \to M(Y, \mathcal{D}),$$

because $M_{U\cap X}(\mathcal{D}_V) \subset M_U(\mathcal{D}_V)$, cf. (1.2.4). By definition i_* is exact and fully faithful, and the essential image is $M_X(Y, \mathcal{D})$ defined by the condition:

$$(2.1.2) \qquad M_W \in M_{U\cap X}(V, \mathcal{D}) \qquad \text{for any} \quad W = \{U \to V\} \in \mathcal{C}(Y),$$

where M_W is the representative of $M \in M(Y, \mathcal{D})$ on W. By exactness i_* induces

$$(2.1.3) \qquad i_*\colon D^b(X, \mathcal{D}) \to D^b(Y, \mathcal{D}).$$

We define a full subcategory $D^b_X(Y, \mathcal{D})$ of $D^b(Y, \mathcal{D})$ by the condition:

$$(2.1.4) \qquad \mathcal{H}^j M \in M_X(Y, \mathcal{D}) \qquad \text{for any} \quad j.$$

For $M = \{M_W\} \in M(Y, \mathcal{D})$, we define $\Gamma_{[X]}M = \{(\Gamma_{[X]}M)_W\} \in M_X(Y, \mathcal{D})$ with a natural morphism $\Gamma_{[X]}M \to M$ by

$$(2.1.5) \qquad (\Gamma_{[X]}M)_W := \Gamma_{[U\cap X]}M_W \to M_W \qquad \text{for} \quad W = \{U \to V\},$$

cf. (1.2.1). This is well-defined, because we can check that $\Gamma_{[U\cap X]}M_W \to M_W$ is independent of V under the equivalence of categories (1.3.3) by reducing to the closed embedding case (1.3.2) as in the proof of (1.3.3). We denote by $\boldsymbol{R}\Gamma_{[X]}M$ the derived functor defined by taking an injective resolution, cf. (1.7.4), and $\mathcal{H}^j_{[X]}M$ its cohomology sheaves. We have a canonical morphism

$$(2.1.6) \qquad \boldsymbol{R}\Gamma_{[X]}M \to M$$

by definition. Note that

$$(2.1.7) \qquad (\mathcal{H}^j_{[X]}M)_W = \mathcal{H}^j_{[U\cap X]}M_W$$

by (1.2.6), and $\mathcal{H}^j_{[X]}M = 0$ for $j \gg 0$, because V is smooth. We have $\mathcal{H}^0_{[X]}M = \Gamma_{[X]}M$, because $\Gamma_{[X]}M$ is left exact, and

$$(2.1.8) \qquad \boldsymbol{R}\Gamma_{[X]}M \xrightarrow{\sim} M \qquad \text{for} \quad M \in M_X(Y, \mathcal{D}),$$

i.e. $\mathcal{H}^j_{[X]}M = 0$ for $j > 0$ by (1.2.11).

By definition $\mathcal{H}^j_{[X]}M$ belongs to the essential image of i_* in (2.1.1), and we define $\mathcal{H}^j i^! M \in M(X, \mathcal{D})$ by $\mathcal{H}^j_{[X]}M = i_*\mathcal{H}^j i^! M$. More generally we have a pull-back functor

(2.1.9) $$i^!\colon D^b(Y, \mathcal{D}) \to D^b(X, \mathcal{D})$$

by taking injective resolution (cf. (1.7.4)) and then applying $\mathcal{H}^0 i^!$. Note that

(2.1.10) $$\mathbf{R}\Gamma_{[X]}M = i_* i^! M$$

by definition. In the definition of (2.1.9) we used also the equivalence of categories

(2.1.11) $$D^b(X, \mathcal{D}) \xrightarrow{\sim} D^+(X, \mathcal{D})^b,$$

where the right is the full subcategory of $D^+(X, \mathcal{D})$ whose objects have bounded cohomologies, and (2.1.11) follows from the existence of canonical filtration τ, cf. [17]. We have the adjoint relation

(2.1.12) $$\mathrm{Hom}_{D^b(X,\mathcal{D})}(M, i^!N) \xrightarrow{\sim} \mathrm{Hom}_{D^b(Y,\mathcal{D})}(i_*M, N)$$

for $M \in D^b(X, \mathcal{D})$, $N \in D^b(Y, \mathcal{D})$, induced by the canonical morphism $i_*i^!N \to N$, cf. (2.1.6) (2.1.10). In fact (2.1.12) holds for $M \in M(X, \mathcal{D})$, $N \in M(Y, \mathcal{D})$ when N is injective by (1.2.5), and this shows the stability of injective objects by the functor $i^!$. Then (2.1.12) is clear.

2.2. Proposition. *We have an equivalence of categories*

(2.2.1) $$i_*\colon D^b(X, \mathcal{D}) \to D^b_X(Y, \mathcal{D})$$

with quasi-inverse $i^!$.

Proof. It is enough to show the quasi-isomorphism

(2.2.2) $$\mathbf{R}\Gamma_{[X]}M \xrightarrow{\sim} M \qquad \text{for} \quad M \in D^b_X(Y, \mathcal{D}).$$

But it is clear by (2.1.8), because we may assume $M \in M_X(Y, \mathcal{D})$.

2.3. With the notation of 2.1, we define $\Gamma_{[Y|X]}M = \{(\Gamma_{[Y|X]}M)_W\} \in M(Y, \mathcal{D})$ with a natural morphism $M \to \Gamma_{[Y|X]}M$ by

(2.3.1) $$(\Gamma_{[Y|X]}M)_W := M_W \to \Gamma_{[V|U\cap X]}M_W \qquad \text{for} \quad W = \{U \to V\} \in \mathcal{C}(Y),$$

where $M = \{M_W\}$. This is also essentially independent of V under (1.3.3), and well-defined. Let $\mathbf{R}\Gamma_{[Y|X]}M$ be the derived functor defined by taking injective resolution. Then we have a distinguished triangle

$$(2.3.2) \qquad \to \boldsymbol{R}\Gamma_{[X]}M \to M \to \boldsymbol{R}\Gamma_{[Y|X]}M \to .$$

In fact we have a short exact sequence

$$(2.3.3) \qquad 0 \to \Gamma_{[X]}M \to M \to \Gamma_{[Y|X]}M \to 0$$

if M is an injective object, because the injectivity is preserved by restriction to open subsets by (1.7.1), and (2.3.2) is true for $M_U(\mathcal{D}_V)$ by (1.2.6).

§3. Direct Image

3.1. Let X, Y be complex analytic spaces. If X, Y are smooth, the relative de Rham functor $\mathrm{DR}_{X\times Y/Y}$ is defined by

$$(3.1.1) \qquad [M\otimes \Lambda^{\dim X}\Theta_X \to \cdots \to M] \qquad \text{for} \quad M\in M(\mathcal{D}_{X\times Y}),$$

where Θ_X is the sheaf of holomorphic vector fields on X, and M is put in degree zero. Here the sheaf theoretic pull-backs by the first projection are omitted to simplify the notation. By choosing a local coordinate system $(x_1, \cdots, x_n)$ of X, (3.1.1) is identified with the Koszul complex $K(M;\, \partial_1, \cdots, \partial_n)$ shifted to the left by $n=\dim X$, where Θ_X is trivialized by the vector fields $\partial_i=\partial/\partial x_i$, cf. [7] for intrinsic definition of the differential of (3.1.1). By definition, cf. [loc. cit.], we have

$$(3.1.2) \qquad p_! M = \boldsymbol{R}p_!\mathrm{DR}_{X\times Y/Y}(M),$$

where $p=\mathrm{pr}_2$ the second projection, and $\boldsymbol{R}p_!$ is the sheaf theoretic direct image with proper support defined by taking some canonical c-soft resolution.

In general, we take a covering family $\boldsymbol{W}=\{W_i\}_{i\in\Lambda}$ of $\mathcal{C}(X)$ in 1.6 such that $\boldsymbol{W}$ is locally finite, i.e. $\{U_i\}$ is a locally finite covering of X. For any $W'=\{U'\to V'\}\in\mathcal{C}(Y)$, we define a covering family $\boldsymbol{W}'=\boldsymbol{W}\times W'=\{W'_i\}_{i\in\Lambda}$ of $X\times U'$ by $W'_i=W_i\times W'=\{U_i\times U'\to V_i\times V'\}\in\mathcal{C}(X\times U')$, where $W_i=\{U_i\to V_i\}$. For $M\in M(X\times Y, \mathcal{D})$, we have natural morphisms

$$(3.1.3) \qquad (\mathrm{pr}_{IJ})_! M_J \to M_I$$

inducing isomorphisms on the complement of $U_I\times U'\backslash U_J\times U'$ by (1.6.3), where M_I is the representative of M on $W'_I=W_I\times W'$, and $W_I=\{U_I\to V_I\}$ is as in 1.6. By definition of direct image $(\mathrm{pr}_{IJ})_!$, cf. (3.1.2), they induce morphisms of complexes of $\mathrm{pr}_2^{-1}\mathcal{D}_{V'}$-Modules on $X\times V'$:

$$(3.1.4) \qquad (j_J)_!\mathrm{DR}_{V_J\times V'/V'}(M_J) \to (j_I)_!\mathrm{DR}_{V_I\times V'/V'}(M_I)$$

inducing quasi-isomorphisms on $U_J\times V'$, where $j_I\colon U_I\times V'\to X\times V'$ is the natural inclusion, and $\mathrm{DR}_{V_I\times V'/V'}(M_I)$ are viewed as complexes of $p^{-1}\mathscr{D}_{V'}$-Modules on $U_I\times V'$. By the compatibility condition of the morphisms (3.1.3), cf. 1.6, the morphisms (3.1.4) are compatible with composition for $I\subset J\subset K$, and we get a co-Čech complex $\mathrm{DR}_{W}(M)_{W'}$ by the double complex whose (p, q)-component is

$$(3.1.5)\qquad \bigoplus_{|I|=1-p}(j_I)_!\mathrm{DR}_{V_I\times V'/V'}(M_I)^q$$

This construction is compatible with the morphisms of $M\in M(X\times Y, \mathscr{D})$ and the morphisms of $W'\in \mathcal{C}(Y)$, and independent of the choice of $\boldsymbol{W}$ up to quasi-isomorphism, i.e. for two locally finite coverings $\boldsymbol{W}_a=\{W_i\}_{i\in\Lambda_a}$ $(a=1, 2)$ such that $\Lambda_1\subset\Lambda_2$, the natural morphisms of complexes of $p^{-1}\mathscr{D}_{V'}$-Modules

$$(3.1.6)\qquad \mathrm{DR}_{W_1}(M)_{W'}\to \mathrm{DR}_{W_2}(M)_{W'}$$

is a quasi-isomorphism (this can be checked by reducing to the case $|\Lambda_2\backslash\Lambda_1|=1$).

If $Y=pt$, we define

$$(3.1.7)\qquad \mathrm{DR}_X(M)=\mathrm{DR}_W(M)=\mathrm{DR}_W(M)_{pt}\in D^b(\boldsymbol{C}_X)$$

for $M\in D^b(X, \mathscr{D})$, where $\boldsymbol{pt}=\{pt\to pt\}\in\mathcal{C}(pt)$.

In general, we define $p_!$: $D^b(X\times Y, \mathscr{D})\to D^b(Y, \mathscr{D})$ by

$$(3.1.8)\qquad p_!(M)_{W'}=\boldsymbol{R}p_!\mathrm{DR}_W(M)_{W'}\qquad \text{for}\quad W'\in\mathcal{C}(Y),$$

where $p_!(M)_{W'}$ is the representative of $p_!(M)$ on W', and $\boldsymbol{R}p_!$ is defined by taking some canonical c-soft resolution, and we can use the lemma below in this case. Here the cohomological dimension of X is finite by assumption on X, and we can truncate the resolution, or use (2.1.11) after taking the direct image.

By definition we have a canonical isomorphism

$$(3.1.9)\qquad \mathrm{DR}_Y(p_!M)=\boldsymbol{R}p_!\mathrm{DR}_{X\times Y}(M),$$

because $\boldsymbol{W}\times\boldsymbol{W}'$ is a covering family of $\mathcal{C}(X\times Y)$ if $\boldsymbol{W}'$ is that of $\mathcal{C}(Y)$. Here $\boldsymbol{R}p_!$ is the sheaf theoretic direct image with proper supports.

3.2. Lemma. *Let X be an analytic space, and $M\in M(X, \mathscr{D})$. Then there exists canonically $M'\in M(X, \mathscr{D})$ with injection $M\to M'$ such that $M'_W\otimes_{\mathcal{O}_V}L$ is c-soft for any $W=\{U\to V\}\in\mathcal{C}(X)$ and locally free $\mathcal{O}_V$-Module L.*

Proof. Let $N_{\mathcal{J}}$ be the subsheaf of M_W annihilated by a coherent Ideal $\mathcal{J}$ of $\mathcal{O}_V$ such that supp $\mathcal{O}_V/\mathcal{J}\subset U$, and $N'_{\mathcal{J}}$ be the sheaf of discontinuous sections of $N_{\mathcal{J}}$. The functor $N_{\mathcal{J}}\to N_{\mathcal{J}}$ is exact and commutes with the tensor product of locally free sheaves. Then M'_W is obtained by the inductive limit of $N'_{\mathcal{J}}$. This construction is compatible with the morphisms of $\mathcal{C}(X)$, because it is true in the closed embedding case. So we get M' with the injection $M \to M'$.

3.3. Let $f: X \to Y$ be a morphism of analytic spaces, $i_f: X \to X \times Y$ the embedding by graph, and $p = \mathrm{pr}_2: X \times Y \to Y$ the second projection. We define $f_!: D^b(X, \mathcal{D}) \to D^b(Y, \mathcal{D})$ by

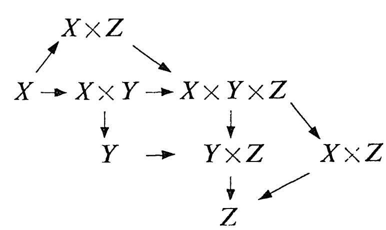
$$f_! = p_!(i_f)_* , \tag{3.3.1}$$

cf. (2.1.3) (3.1.8). If f is proper, we define $f_* = f_!$. We can check

$$(gf)_! = g_! f_! \tag{3.3.2}$$

for $f: X \to Y$, $g: Y \to Z$, using the diagram

(3.3.3)



because the sheaf theoretic direct image with proper support commutes with inductive limit. We have also

$$\mathrm{DR}_Y(f_! M) = \boldsymbol{R} f_! \mathrm{DR}_X(M) \tag{3.3.4}$$

by (3.1.9), because the closed embedding case follows from the well-definedness of DR_X.

3.4. Proposition. *If $f: X \to Y$ is a finite morphism, the direct image $f_*: M(X, \mathcal{D}) \to M(Y, \mathcal{D})$ is faithful and exact.*

Proof. The assertion is local on Y. We may assume Y smooth and X is a closed subspace Z of the product of Y with a complex manifold which will be denoted by X. Then the assertion is reduced to the exactness and faithfulness of the functor

$$f_!: M_Z(\mathcal{D}_{X \times Y}) \to M(\mathcal{D}_Y) , \tag{3.4.1}$$

where $f = \mathrm{pr}_2$, and Z is finite over Y. Taking factorization of f, we may assume $\dim X = 1$. Let x be a local coordinate of X, and $y = (y_1, \cdots, y_m)$ a local coordinate system of Y. Then for $M \in M_Z(\mathcal{D}_{X \times Y})$

$$\mathrm{DR}_{X \times Y/Y}(M) = \mathrm{Cone}(\partial_x: M \to M) . \tag{3.4.2}$$

For the exactness of $f_!$ it is enough to show the injectivity of ∂_x. Let $m \in M$ such that $m\partial_x = 0$. By definition, cf. (1.2.3), we have a holomorphic function g such that $mg = 0$, where we may assume g is a Weierstrass polynomial $P(x, y)$ of x with coefficient in holomorphic functions of y. Let d be the degree of $P(x, y)$. Since the coefficient of x^d is 1, $m\partial_x = 0$ and $mP(x, y)\partial_x^d = 0$ imply $m = 0$. For the proof of faithfulness, it is enough to show $f_!M = 0$ iff $M = 0$, because the functor commutes with Im by exactness. The assertion is stalkwise, and we may assume the stalk of M at $x \in X \times Y$ is finitely generated, and M is coherent (by restricting X, and replacing M by a coherent Module with same stalk at x), because the inductive limit of an injective system whose morphisms are injective and non zero, is not zero. Then the assertion is checked by restricting to a subsapce of Y on which supp M is locally biholomorphic, because $f_!M$ is coherent.

3.5. Let X be a complex analytic space, and L an $\mathcal{O}_X$-Module. With the notation of 1.5, we define $M = \{M_W\} \in M(X, \mathcal{D})$ by

$$M_W = L|_U \otimes_{\mathcal{O}_V} \mathcal{D}_V \qquad \text{for} \quad W = \{U \to V\} \in \mathcal{C}(X)\,, \tag{3.5.1}$$

where the morphisms $u_{W_1W_2}$ in (1.5.1) are naturally defined by the compatibility of the passage to the associated induced Modules with direct images, cf. [16, 3.3]. We call M the $\mathcal{D}$-Module *induced* by L, and denote it by $L \otimes_{\mathcal{O}_X} \mathcal{D}$. A $\mathcal{D}$-Module which is isomorphic to a $\mathcal{D}$-Module induced by an $\mathcal{O}_X$-Module is called an *induced* $\mathcal{D}$-Module. Let $M_i(X, \mathcal{D})$ be the full subcategory of $M(X, \mathcal{D})$ consisting of induced $\mathcal{D}$-Modules. For M, L as above, we have

(3.5.2) M is coherent, iff L is a coherent $\mathcal{O}_X$-Module.

In fact, if M is coherent, $L|_U$ is a subsheaf of a quasi-coherent $\mathcal{O}_V$-Module M, and it is enough to show that L is locally finitely generated. Let $\sum_{|\nu| \le r} m_{i,\nu} \otimes \partial^\nu$ $(1 \le i \le k)$ be local generators of M_W over $\mathcal{D}_V$, where we take a local coordinate system $(x_1, \cdots, x_m)$ and put $\partial^\nu = \prod_j \partial_j^{\nu_j}$ with $\partial_j = \partial/\partial x_j$. Then $m_{i,\nu} \otimes 1$ generate M over $\mathcal{D}_V$. This implies that $m_{i,\nu}$ generate L over $\mathcal{O}_V$, because $\mathcal{D}_V$ is faithfully flat over $\mathcal{O}_V$.

We say that $M \in M(X, \mathcal{D})$ is a *quotient coherent induced* $\mathcal{D}$-Module, if M is coherent and there is a surjective morphism of a coherent induced $\mathcal{D}$-Module M' onto M. We say that a $\mathcal{D}$-Module M on X is *good coherent*, if for any relatively compact open subset X' of X, there exists a finite filtration G of $M|_{X'}$ such that $\mathrm{Gr}_i^G M|_{X'}$ are quotient coherent induced. We say that a $\mathcal{D}$-Module is *inductively n-good*, if for any relatively compact open subset X' of X, there exist increasing exhaustive filtrations G^k of $\mathrm{Gr}_{i_{k-1}}^{G^{k-1}}\cdots\mathrm{Gr}_{i_0}^{G^0}M|_{X'}$ for $i_j\in N$ and $0\leq k\leq n$ inductively such that G^0 is a finite filtration, G^k are bounded below, i.e. $G_i^k\mathrm{Gr}_{i_{k-1}}^{G^{k-1}}\cdots\mathrm{Gr}_{i_0}^{G^0}M|_{X'}=0$ for $i\ll 0$, and $\mathrm{Gr}_{i_n}^{G^n}\cdots\mathrm{Gr}_{i_0}^{G^0}M|_{X'}$ are quotient coherent induced. In particular, M is inductively 0-good iff it is good coherent. By definition, M is inductively n-good, iff $M|_{X'}$ has a finite filtration G and each $\mathrm{Gr}_i^G M|_{X'}$ has an increasing exhaustive filtration G' such that $\mathrm{Gr}_j^{G'}\mathrm{Gr}_i^G M$ are inductively $(n-1)$-good, and zero for $j\ll 0$. We say that M is *inductively good* if it is inductively n-good for some n. Then

(3.5.3) a $\mathcal{D}$-Module is good coherent, if it is coherent and inductively good,

by the noetherian property of $\mathcal{D}$-Module, cf. [7]:

(3.5.4) an increasing sequence of coherent sub-Modules of a coherent $\mathcal{D}$-Module is locally stationary.

We say that a complex of $\mathcal{D}$-Modules is good coherent (resp. inductively n-good, resp. inductively good) if so are its cohomologies. We denoted by $M_{g,\mathrm{coh}}(X, \mathcal{D})$ (resp. $M_{ig}(X, \mathcal{D})$) the full subcategory of $M(X, \mathcal{D})$ consisting of good coherent (resp. inductively good) $\mathcal{D}$-Modules, and by $D^b_{g,\mathrm{coh}}(X, \mathcal{D})$ the full subcategory of $D^b(X, \mathcal{D})$ consisting of good coherent bounded complexes (same for $D^b_{ig}(X, \mathcal{D})$). By (1.8.3–4) we have

(3.5.5) inductively n-good bounded complexes are stable by external product $\boxtimes$.

In fact, it is enough to show that the coherent induced Modules are stable by the external product. But it is clear.

Assume X smooth, and let U be a relatively compact open polydisc in X with $\bar{U}$ its closure. Then, for $M\in M_{ig}(X, \mathcal{D})$ $(\subset M(\mathcal{D}_X))$, we have

$$H^i(\bar{U}, M)=0 \qquad \text{for} \quad i>0\,. \tag{3.5.6}$$

In fact, $\Gamma(\bar{U}, *)$ and $H^i(\bar{U}, *)$ commute with inductive limit and are the inductive limit cf $\Gamma(U', *)$ and $H^i(U', *)$ for $U'\supset\bar{U}$. Then the assertion is reduced to the case of quotient coherent induced $\mathcal{D}$-Modules, and we can proceed by decreasing induction on i using 3.6 below, because the assertion follows from Cartan's theorem B in the case of coherent induced Modules, and $H^i(\bar{U}, M)=0$ for $i\gg 0$. We have also

(3.5.7) $M|_{\bar{U}}$ is generated by $\Gamma(\bar{U}, M)$ over $\mathcal{D}_X$, if $M\in M_{ig}(X, \mathcal{D})$

by Cartan's theorem A and (3.5.6).

We say that a $\mathcal{D}_X$-Module (or an $\mathcal{O}_X$-Module) is quasi-coherent, if it is locally isomorphic to the cokernel of a morphism of free Modules (of infinite rank in general). Here quasi-coherence over $\mathcal{D}_X$ implies that over $\mathcal{O}_X$, because $\mathcal{D}_X$ is locally free over $\mathcal{O}_X$. A quasi-coherent Module M is locally the union of its coherent $\mathcal{D}_X$-sub-Modules (or $\mathcal{O}_X$-sub-Modules). In fact, if $M = \mathrm{Coker}(L' \to L)$ with L', L free, consider the intersections of a finite free sub-Module of L with the images of finite free sub-Modules of L', which are coherent, and use the Noetherian property. In particular, the image of a morphism of a locally finitely generated Module to a quasi-coherent Module is coherent. On the other hand, if M is the union of its coherent $\mathcal{O}_X$-sub-Modules, $M|_U$ is generated by global sections over $\mathcal{D}_X$ (or $\mathcal{O}_X$) by Cartan's theorem A, and we get a surjection $M' \to M|_U$ with M' a free Module. Then M' is the union of its coherent sub-Modules, and so is the kernel of the surjection, because the image of a coherent sub-Module of M' is locally contained in a coherent sub Module of M. Therefore we get

(3.5.8) M is quasi-coherent iff it is locally the union of its coherent sub-Modules.

Actually we showed that quasi-coherence over $\mathcal{D}_X$ is equivalent to that over $\mathcal{O}_X$, because we used only coherent $\mathcal{O}_X$-sub-Modules (and the locally freeness of $\mathcal{D}_X$ over $\mathcal{O}_X$) for the construction of presentation of M as the cokernel of a morphism of free Modules. By a similar argument, we get

(3.5.9) $\mathrm{Ker}(M' \to M)$ is locally a quotient of free Module, and $\mathrm{Im}(M' \to M)$ is quasi-coherent, if M' is locally a quotient of free Module and M is quasi-coherent,

because the assertion is reduced to the case M' free by taking a surjection of a free Module to M'. Note that (3.5.6–7) hold for quasi-coherent Modules, if U is sufficiently small so that we have a presentation of M as the cokernel as above on a neighborhood of $\bar{U}$.

Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence. Then we have

(3.5.10) if two of M', M, M'' are quasi-coherent, so is the remaining one.

In fact the case M quasi-coherent follows from (3.5.8). In the remaining case we can lift locally the presentations of M', M'' to that of M using the assertion in the other cases and (3.5.6) in the quasi-coherent case, cf. the remark after (3.5.9). By (3.5.9) (3.5.10) we have

(3.5.11) quasi-coherent Modules are stable by Ker, Coker, Im.

Using (3.5.7) (3.5.8) we get

(3.5.12) inductively good $\mathcal{D}$-Modules are quasi-coherent.

In fact, we have a canonical surjective morphism $\Gamma(\bar{U}, M)\otimes_{\mathbf{C}}\mathcal{D}_U \to M|_U$ for an inductively good $\mathcal{D}$-Module M, and its kernel is a union of coherent $\mathcal{O}$-Module, because the image of a morphism of a finite free $\mathcal{O}$-Module L to an inductively good $\mathcal{D}$-Module M is a coherent $\mathcal{O}$-Module. In fact, consider the image of the induced morphism $L\to \mathrm{Gr}_{i_n}^{G^n}\cdots \mathrm{Gr}_{i_0}^{G^0}M$ for $i_j \gg 0$ defined on a compact subset of X and replace L by its kernel so that i_j decreases inductively. By the same argument we can show that an inductive limit of inductively good $\mathcal{D}$-Module is quasi-coherent.

3.6. Lemma. *Let*

$$0 \to M' \to M \to M'' \to 0 \tag{3.6.1}$$

be a short exact sequence of quasi-coherent $\mathcal{D}$-Modules on X. Then M is inductively n-good iff M', M'' are inductively n-good. In particular, M' is good coherent, if M is coherent induced.

Proof. The assertion is clear if M', M'' are inductively n-good. Assume M is inductively n-good. Then, restricting X to a relatively compact open subset, we have inductively defined filtrations whose successive graduations are quotient coherent induced by definition. We take inductively the graduations of induced and quotient filtrations on M', M'', and the assertion is reduced to the case M is a quotient coherent induced $\mathcal{D}$-Module, and M', M'' are coherent. Then the assertion on M'' is trivial. For M' we have to use essentially the filtered theory, cf. [14, §2]. Assume first M is induced by L. We define a filtration F on M_W for $W=\{U\to V\}\in\mathcal{C}(X)$ by

$$F_pM_W = L\otimes F_p\mathcal{D}_V\,. \tag{3.6.2}$$

Then $\mathrm{Gr}_p^F M_W$ is an $\mathcal{O}_U$-Module, F is compatible with the morphisms of $W\in\mathcal{C}(X)$, and $\{(M_W, F)\}$ defines an object of $\mathrm{MF}(X, \mathcal{D})$ (denoted by $\mathrm{MF}(\mathcal{D}_X)$ in [loc. cit.]). We define F on M', M'' by induced and quotient filtrations. Then $\{(M'_W, F)\}$, $\{(M''_W, F)\}\in \mathrm{MF}(X, \mathcal{D})$. Let $p(M')=\min\{p\colon F_pM'_W\neq 0$ for some $W\}$ (≥ 0). Then $F_{p(M')}M'_W$ are compatible with the morphisms of W, and defines globally a coherent $\mathcal{O}_X$-Module $L_{p(M')}$. Let $M'_{p(M')}$ be the $\mathcal{D}$-Module induced by L_p. Then we have a morphism $M'_{p(M')}\to M'$ induced by the natural inclusion $F_{p(M')}M'_W \to M'_W$. Let $G_{p(M')}M'$ be its image. Since $p(M'/G_{p(M')}M') > p(M')$, we can proceed by induction, and get a filtration G of M' such that $\mathrm{Gr}_p^G M'$ are quotient coherent induced.

We now show the case M quotient coherent induced. Let N be a coherent induced $\mathcal{D}$-Module with a surjection $N \to M$. Let N', N'' be the kernel of $N \to M$ and $N \to M''$. Then N', N'' are good coherent by the above argument. We have a short exact sequence $0 \to N' \to N'' \to M' \to 0$ by snake lemma, and the assertion follows.

Remark. The above argument shows that the underlying $\mathcal{D}$-Module of a coherent filtered $\mathcal{D}$-Module in the sense of [14, 2.1.15 and 2.1.20] is good coherent. In the case X smooth, this implies that a coherent $\mathcal{D}_X$-Module M having a good filtration F (i.e. $\mathrm{Gr}^F M$ is coherent over $\mathrm{Gr}^F \mathcal{D}_X$) is good coherent. In particular

(3.6.3) a coherent $\mathcal{D}$-Module is locally good coherent.

3.7. Theorem. *Let $f: X \to Y$ be a proper morphism of complex analytic spaces. Then $f_*: D^b(X. \mathcal{D}) \to D^b(Y, \mathcal{D})$ preserves inductively n-good bounded complexes, and induces*

$$f_*: D^b_{g,\mathrm{coh}}(X, \mathcal{D}) \to D^b_{g,\mathrm{coh}}(Y, \mathcal{D}), \text{ etc}. \tag{3.7.1}$$

Proof. Let $p: X \times Y \to Y$ be the second projection. Using 3.6 and the spectral sequence whose E_2-term is $\mathcal{H}^i p_!(\mathcal{H}^j M)$, it is enough to show

(3.7.2) $\mathcal{H}^i p_!(M)$ is good coherent (resp. inductively n-good)

for $M \in M(X \times Y, \mathcal{D})$ such that M is good coherent (resp. inductively n-good) and supp M is proper over Y, because the direct image by closed embedding is an exact functor and preserves coherent induced $\mathcal{D}$-Modules. If M is induced by an $\mathcal{O}_{X\times Y}$-Module L, we have a natural quasi-isomorphism as complexes of $p^{-1}\mathcal{D}_{V'}$-Modules:

$$\mathrm{DR}_W(M)_{W'} \xrightarrow{\sim} L \otimes_{p^{-1}\mathcal{O}_{V'}} p^{-1}\mathcal{D}_{V'} \tag{3.7.3}$$

for $W' = \{U' \to V'\} \in \mathcal{C}(Y)$ compatible with the morphisms of W'. This implies

$$(p_! M)_{W'} \xrightarrow{\sim} \mathbb{R} p_! L \otimes_{\mathcal{O}_{V'}} \mathcal{D}_{V'}, \tag{3.7.4}$$

and the assertion follows from Grauert's coherence theorem for $\mathcal{O}$-Modules.

In general, we may assume M good coherent, by using inductively the spectral sequences

(3.7.5) $$E_1^{i,j} = \mathcal{H}^{i+j}p_!\mathrm{Gr}^G_{-i}M \Rightarrow \mathcal{H}^{i+j}p_! M$$

associated with the filtrations G in the definition of inductively good. Here the differential $d_r^{i,j}$ is zero for $r \gg -i$, and the kernel of the projection $E_r^{i,j} \to E_\infty^{i,j}$ ($r \gg -i$) is quasi-coherent by (3.5.11–12) and 3.6, because the kernel is inductively n-good by the structure of spectral sequence and 3.6 if $E_1^{i,j}$ are inductively (n–1)-good. For $i \in \boldsymbol{Z}$, a similar argument shows

(3.7.6) $\mathcal{H}^i p_! M$ are locally finitely generated and $\mathcal{H}^j p_! M$ ($j > i$) are good coherent for good coherent $\mathcal{D}$-Modules M if this holds for quotient coherent induced $\mathcal{D}$-Modules,

where we use (3.5.9) in the coherent case (i.e. the kernel of a morphism of a locally finitely generated Module to a coherent Module is locally finitely generated) and the fact that $\mathcal{H}^i p_! M$ is locally finitely generated if so are $\mathrm{Gr}^G_k \mathcal{H}^i p_! M$.

We show the assumption of (3.7.6) for i–1 follows from its assertion for i, so that we can proceed by induction, where $\mathcal{H}^i p_! M = 0$ for $i \gg 0$, because the cohomological dimension of X is bounded by hypothesis. We consider a short exact sequence

(3.7.7) $$0 \to M' \to M \to M'' \to 0$$

with M coherent induced and M'' quotient coherent induced so that M' is good coherent by 3.6. We have the associated long exact sequence

(3.7 8) $$\to \mathcal{H}^i p_! M' \to \mathcal{H}^i p_! M \to \mathcal{H}^i p_! M'' \to \mathcal{H}^{i+1} p_! M' \to \mathcal{H}^{i+1} p_! M \to$$

where $\mathcal{H}^i p_! M$ are coherent induced for any i by (3.7.4). Assume the conclusion of (3.7.6) for i. Then $\mathcal{H}^i p_! M''$ is good coherent by 3.6, and $\mathcal{H}^{i-1} p_! M''$ is locally finitely generated using (3.5.9) in the coherent case. So the hypothesis of (3.7.6) is proved for i–1.

3.8. Proposition. *Let f: $X \to Y$ be as in* 3.7, *and Z be a complex analytic space. Then for $M \in D^b_{g,\mathrm{coh}}(X, \mathcal{D})$ and $N \in D^b(Z, \mathcal{D})$, we have a canonical isomorphism*

(3.8.1) $$(f_* M) \boxtimes N \xrightarrow{\sim} (f \times id)_* (M \boxtimes N) \quad \text{in} \quad D^b(Y \times Z, \mathcal{D}).$$

Proof. By definition of direct image and external product, we have a canonical morphism (3.8.1), and the assertion is local and stalkwise on $Y \times Z$, because the stalks of (3.8.1) depends only on M and the stalks of N In particular we may assume Y, Z smooth and N a $\mathcal{D}_Z$-Module. Here we use only the $\mathcal{O}_Z$-Module structure of N in the definition of (3.8.1). Since the two functors commute with inductive limit, we may assume N is a coherent $\mathcal{O}_Z$-Module by taking coherent extension of finitely generated $\mathcal{O}_{Z,z}$-submodules of N_z for $z\in Z$, and then N is a free $\mathcal{O}_Z$-Module by taking resolution.

Therefore the assertion is reduced to the case Z smooth and $N=\omega_Z$. Here we may assume $\dim Z=1$ by factorizing the projection $Y\times Z\to Y$. For $z\in Z$, let i_z: $Y=Y\times\{z\}\to Y\times Z$ be the natural inclusion. By 3.10 below, $i_z^!$ of (3.8.1) is a quasi-isomorphism for any z. Taking the mapping cone, the assertion is reduced to

(3.8.2) $M\in D^b_{\mathrm{coh}}(\mathcal{D}_{Y\times Z})$ is zero, if $i_z^!M=0$ for any z.

The hypothesis is equivalent to the quasi-isomorphism

(3.8.3) $$M\to\Gamma_{[Y\times Z|Y\times\{z\}]}M\,,$$

and we may assume $M\in M_{\mathrm{coh}}(\mathcal{D}_{Y\times Z})$ by taking cohomology, because $\Gamma_{[Y\times Z|Y\times\{z\}]}$ is exact. Then the assertion is further reduced to

(3.8.4) $M\in M_{\mathrm{coh}}(\mathcal{D}_X)$ is zero, if $i_x^!M=0$ for any $x\in X$,

where X denotes the above $Y\times Z$, and i_x: $\{x\}\to X$ is the natural inclusion. Restricting X to a smooth Zariski-open subset of $\mathrm{supp}\,M$, we may assume $X=\mathrm{supp}\,M$. Let F be a good filtration of M. If $Ch(M)\neq T^*X$, take a generic smooth hypersurface i: $Y\to X$ such that T^*_YX is not contained in $Ch(M)$ and the local equation t of Y is a non zero divisor of Gr^FM (by restricting X to a Zariski-open subset if necessary). Then $i^!M=0$ iff $M=0$ on a neighborhood of Y, and we may replace M by $\mathcal{H}^1i^!M$ $(=M/Mt$ locally). So the assertion is reduced to the case of $Ch(M)=T^*X$. Let $u_1,\cdots,u_k$ be local generators of M, and M_j the $\mathcal{D}_X$-sub-Module generated by u_j. Then $Ch(M_j)=T^*X$ for some j, and this means $M_j=\mathcal{D}_X$. Applying the same to M/M_j inductively, we get locally an injective morphism of a finite free $\mathcal{D}_X$-Module M' into M whose cokernel M'' has smaller characteristic variety. Consider a long exact sequence

$$\to\mathcal{H}^{n-1}i_x^!M'\to\mathcal{H}^{n-1}i_x^!M\to\mathcal{H}^{n-1}i_x^!M''\to\mathcal{H}^ni_x^!M'\to\mathcal{H}^ni_x^!M\to\mathcal{H}^ni_x^!M''\to$$

where $n=\dim X$. Since $\mathcal{H}^ni_x^!M'\neq0$ for any $x\in X$, the assertion is reduced to

(3.8.5) $\{x\in X\colon\mathcal{H}^{n-1}i_x^!M=0\}$ is dense in X for $M\in M_{\mathrm{coh}}(\mathcal{D}_X)$,

which we apply to M''. For the proof of (3.8.5), we may assume also $Ch(M)=T^*X$ by the same argument as above. Then the assertion follows from the long exact sequence using induction on $\dim Ch(M)$, because

$\mathcal{H}^{n-1}i_x^!(\mathcal{D}_X)=0$.

3.9. Corollary. *Let $f\colon X\to Y$ and $f'\colon X'\to Y'$ be proper morphisms of complex analytic spaces. Then we have a canonical isomorphism*

$$(f\times f')_*(M\boxtimes M')=f_*M\boxtimes f'_*M' \tag{3.9.1}$$

for $M\in D^b_{g,\mathrm{coh}}(X,\mathcal{D})$, $M'\in D^b_{g,\mathrm{coh}}(X',\mathcal{D})$.

Proof. This follows from 3.8 and $(f\times f')_*=(f\times id)_*(id\times f')_*$, cf. (3.3.2).

3.10. Proposition. *Let $f\colon X\to Y$ be a morphism of complex analytic spaces, and Z be a closed subspace of Y. Put $Z'=f^{-1}(Z)$. Then, for $M\in D^b(X,\mathcal{D})$, we have canonical isomorphisms*

$$\boldsymbol{R}\Gamma_{[Z]}f_!M=f_!\boldsymbol{R}\Gamma_{[Z']}M \tag{3.10.1}$$

$$\boldsymbol{R}\Gamma_{[Y|Z]}f_!M=f_!\boldsymbol{R}\Gamma_{[X|Z']}M\,. \tag{3.10.2}$$

Proof. It is enough to show the acyclicity of $\boldsymbol{R}\Gamma_{[Z]}f_!\boldsymbol{R}\Gamma_{[X|Z']}M$ and $\boldsymbol{R}\Gamma_{[Y|Z]}f_!\boldsymbol{R}\Gamma_{[Z']}M$ by the commutative diagram

$$\begin{array}{ccccccc}
 & \downarrow & & \downarrow & & \downarrow & \\
\to & \boldsymbol{R}\Gamma_{[Z]}f_!\boldsymbol{R}\Gamma_{[Z']}M & \xrightarrow{\sim} & \boldsymbol{R}\Gamma_{[Z]}f_!M & \to & \boldsymbol{R}\Gamma_{[Z]}f_!\boldsymbol{R}\Gamma_{[X|Z']}M & \to \\
 & \downarrow\cong & & \downarrow & & \downarrow & \\
\to & f_!\boldsymbol{R}\Gamma_{[Z']}M & \to & f_!M & \to & f_!\boldsymbol{R}\Gamma_{[X|Z']}M & \to \\
 & \downarrow & & \downarrow & & \downarrow\cong & \\
\to & \boldsymbol{R}\Gamma_{[Y|Z]}f_!\boldsymbol{R}\Gamma_{[Z']}M & \to & \boldsymbol{R}\Gamma_{[Y|Z]}f_!M & \xrightarrow{\sim} & \boldsymbol{R}\Gamma_{[Y|Z]}f_!\boldsymbol{R}\Gamma_{[X|Z']}M & \to \\
 & \downarrow & & \downarrow & & \downarrow &
\end{array} \tag{3.10.3}$$

In particular the assertion is local on Y, and we may assume Y smooth. By (1.2.9–10) it is enough to show that the direct image commutes with the localization by a function on Y. But it is clear by definition, because inductive limit commutes with the sheaf theoretic direct image with proper support.

3.11. *Remark.* With the notation of 3.5, let $L\otimes_{\mathcal{O}_X}\mathcal{D}$ and $L'\otimes_{\mathcal{O}_X}\mathcal{D}$ be the $\mathcal{D}$-Modules on X induced by $\mathcal{O}_X$-Modules L, L'. We define the (filtered) group of differential morphisms of L to L' by

$$\mathcal{H}om_{\mathrm{Diff}}(L,L')|_U=\mathcal{H}om_{\mathcal{D}_V}((L|_U)\otimes_{\mathcal{O}_U}\mathcal{D}_V,(L'|_U)\otimes_{\mathcal{O}_U}\mathcal{D}_V) \tag{3.11.1}$$

for $W=\{U\to V\}\in\mathcal{C}(X)$, where it has a filtration F defined by

$$F_p\mathcal{H}om_{\mathrm{Diff}}(L,L')|_U=\mathcal{H}om_{\mathcal{O}_V}(L|_U,(L'|_U)\otimes_{\mathcal{O}_U}F_p\mathcal{D}_V) \tag{3.11.2}$$

cf. [14, §2] [16]. In fact, it is independent of the closed embeddings of U into manifolds V, and is globally well-defined. Note that this definition coincides with Grothendieck's one [21] by [16], because the latter is also invariant by closed embeddings. In fact, let $i: X \to Y$ be a closed embedding of complex analytic spaces. With the notation of [loc. cit.] (see also [16, 1.20]) we have

$$\mathcal{H}om_{\mathcal{O}_Y}(P_Y^\flat \otimes_{\mathcal{O}_Y} i_*L, i_*L') = \mathcal{H}om_{\mathcal{O}_X}((\mathcal{O}_X \otimes_{\mathcal{O}_Y} P_Y^\flat \otimes_{\mathcal{O}_Y} \mathcal{O}_X) \otimes_{\mathcal{O}_X} L, L')$$

with $\mathcal{O}_X \otimes_{\mathcal{O}_Y} P_Y^\flat \otimes_{\mathcal{O}_Y} \mathcal{O}_X = \mathcal{O}_{X\times X} \otimes_{\mathcal{O}_{Y\times Y}} P_Y^\flat = P_X^\flat$ using the right exactness of tensor, where X, Y may be singular and nonreduced.

We define the ring of differential operators on X by

$$\mathcal{D}_X = \mathcal{H}om_{\mathrm{Diff}}(\mathcal{O}_X, \mathcal{O}_X) . \tag{3.11.3}$$

Let $W = \{U \to V\} \in \mathcal{C}(X)$, and $\mathcal{D}_{U\to V} = \mathcal{O}_U \otimes_{\mathcal{O}_V} \mathcal{D}_V$. Then $\mathcal{D}_{U\to V}$ has commuting structures of left $\mathcal{D}_U$-Module and right $\mathcal{D}_V$-Module by definition (3.11.1). Let M be a (right) $\mathcal{D}_X$-Module. We define

$$M_W^\prime = (M|_U) \otimes_{\mathcal{D}_U} \mathcal{D}_{U\to V} . \tag{3.11.4}$$

Then $M_W^\prime$ are compatible with the morphisms of $\mathcal{C}(X)$, and define a $\mathcal{D}$-Module on X, which we denote by $M \otimes_{\mathcal{D}_X} \mathcal{D}$ and call the $\mathcal{D}$-Module on X induced by a $\mathcal{D}_X$-Module M. So we get a functor $\otimes_{\mathcal{D}_X} \mathcal{D}$ of $M(\mathcal{D}_X)$ the category of $\mathcal{D}_X$-Modules to $M(X, \mathcal{D})$. We say that a $\mathcal{D}_X$-Module M is induced by an $\mathcal{O}_X$-Module L if $M = L \otimes_{\mathcal{O}_X} \mathcal{D}_X$. Then $(L \otimes_{\mathcal{O}_X} \mathcal{D}_X) \otimes_{\mathcal{D}_X} \mathcal{D} = L \otimes_{\mathcal{O}_X} \mathcal{D}$. For the moment the relation between $M(\mathcal{D}_X)$ and $M(X, \mathcal{D})$ in general is not clear, cf. [20] for the one dimensional case.

§4. Pull-Backs

4.1. Let $f: X \to Y$ be a morphism of complex analytic spaces. We define the pull-back $f^!: D^b(Y, \mathcal{D}) \to D^b(X, \mathcal{D})$ by the composition:

$$f^! = i_f^! p^! , \tag{4.1.1}$$

where i_f is the embedding by graph of f, and $p: X \times Y \to Y$ is the second projection. Here $i_f^!$ is defined in (2.1.12). For $p^!$, we will first show in 4.2 the existence of $M \in D_{rh}^b(X, \mathcal{D})$ (cf. 1.5) such that

$$\mathrm{DR}_X(M) \simeq a_X^! \mathbf{C} \in D_c^b(\mathbf{C}_X) , \tag{4.1.2}$$

where $a_X: X \to pt$ is the natural morphism and $a_X^! \mathbf{C}$ is the topological dualizing complex, cf. [18]. The uniqueness of M with isomorphism (4.1.2) will be shown in 4.10, and we denote M by $a_X^! \omega_{pt}$. (This notation must be distinguished with the topological dualizing sheaf $a_X^! \mathbf{C}$, although $\omega_{pt} \simeq \mathbf{C}$.) We have the trace morphism

$$\mathrm{Tr}_{a_X}\colon (a_X)_! a_X^! \omega_{pt} \to \omega_{pt} \tag{4.1.3}$$

by the topological trace morphism $\mathrm{Tr}_{a_X}\colon (a_X)_! a_X^! \boldsymbol{C} \to \boldsymbol{C}$, cf. [18], because $(a_X)_! a_X^! \omega_{pt} \cong (a_X)_! a_X^! \boldsymbol{C}$ by (4.1.2). We define

$$p^! N = a_X^! \omega_{pt} \boxtimes N \qquad \text{for} \quad N \in D^b(Y, \mathcal{D}), \tag{4.1.4}$$

cf. 1.8 for $\boxtimes$. Then we have a canonical isomorphism

$$a_{X\times Y}^! \, \omega_{pt} = a_X^! \, \omega_{pt} \boxtimes a_Y^! \, \omega_{pt} \tag{4.1.5}$$

using the uniqueness of $a_{X\times Y}^! \, \omega_{pt}$ and the commutativity of DR with $\boxtimes$, cf. [7][10][1], etc. (see also the remark after 4.4). Then the functoriality

$$(fg)^! = g^! f^! \tag{4.1.6}$$

can be checked using the diagram (3.3.3). In fact, using the isomorphism (4.7.1) below, it is enough to show the commutativity of the pull-back by projection with local cohomology, i.e.

$$a_X^! \, \omega_{pt} \boxtimes \boldsymbol{R}\Gamma_{[Z]} N \xrightarrow{\sim} \boldsymbol{R}\Gamma_{[X\times Z]}(a_X^! \, \omega_{pt} \boxtimes N), \tag{4.1.7}$$

and it follows from (1.8.5). This definition of pull-back $f^!$ is compatible with the usual definition

$$f^! M = f^{-1} M \otimes_{f^{-1}\mathcal{D}_Y} \mathcal{D}_{Y\leftarrow X}[\dim X - \dim Y] \tag{4.1.8}$$

in the case X, Y smooth, using (2.1.10) in the closed embedding case and the construction in 4.2 below in the smooth projection case, where $\mathcal{D}_{Y\leftarrow X} = \omega_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}(\mathcal{D}_Y \otimes_{\mathcal{O}_Y} (\omega_Y)^{-1})$, cf. [1][7][10], etc. We have also

$$f^! M \in D^b_{\mathrm{hol}}(X, \mathcal{D}) \qquad \text{for} \quad M \in D^b_{\mathrm{hol}}(Y, \mathcal{D}) \text{ (same for } D^b_{rh}(X, \mathcal{D})). \tag{4.1.9}$$

by [6] [7] [10].

4.2. Lemma. *With the above notation, the dualizing complex $a_X^! \omega_{pt}$ exists.*

Proof. If X is a closed subvariety of a manifold V, the assertion follows from [7][10] by taking $\boldsymbol{R}\Gamma_{[Z]}\omega_V[\dim V]$, where ω_V is the analytic dualizing sheaf (i.e. the top degree differential forms) of V. Moreover $a_X^! \, \omega_{pt}$ with isomorphism (4.1.2) is unique by the Riemann-Hilbert correspondence in this case, cf. [loc. cit.]. In the general case this implies the local existence and uniqueness of $a_X^! \, \omega_{pt}$. Let $\boldsymbol{W} = \{W_i\}_{i\in\Lambda}$ be a locally finite covering family of X such that U_i are relatively compact in X, where $W_i = \{U_i \to V_i\} \in \mathcal{C}(X)$. We may assume X connected, and hence Λ is countable, i.e. $\Lambda=\boldsymbol{N}$. (In fact X is covered by $X_j=\bigcup_{U_i\cap X_{j-1}\neq\emptyset} U_i$ $(j>0)$ with $X_0=\{x\}$.) Then the assertion is reduced to the following

4.3. Lemma. *If X is covered by two open subsets U_1, U_2 such that $a^!_{U_\alpha}\omega_{pt}$ exists for $\alpha=1, 2, 3$ (where $U_3=U_1\cap U_2$) and $a^!_{U_3}\omega_{pt}$ with isomorphism* (4.1.2) *is unique, then $a^!_X\omega_{pt}$ exists.*

Proof. Let j_α: $U_\alpha\to X$ be the natural inclusions. Then we have isomorphisms $a^!_{U_\alpha}\omega_{pt}|_{U_3}\cong a^!_{U_3}\omega_{pt}$ $(\alpha=1, 2)$ by the uniqueness of $a^!_{U_3}\omega_{pt}$ This induces morphisms $(j_3)_!a^!_{U_3}\omega_{pt}\to(j_\alpha)_!a^!_{U_\alpha}\omega_{pt}$ in $D^b(X, \mathcal{D})$, and $a^!_X\omega_{pt}$ is given by

(4.3.1) $$\operatorname{Cone}((j_3)_!a^!_{U_3}\omega_{pt}\to(j_1)_!a^!_{U_1}\omega_{pt}\oplus(j_2)_!a^!_{U_2}\omega_{pt}),$$

because we have a distinguished triangle

(4.3.2) $$\to(j_3)_!a^!_{U_3}\boldsymbol{C}\to(j_1)_!a^!_{U_1}\boldsymbol{C}\oplus(j_2)a^!_{U_2}\boldsymbol{C}\to a^!_X\boldsymbol{C}\to$$

4.4. Lemma. *With the notation of* (3.1.5) (3.1.7) *and* 1.8, *we have natural quasi-isomorphisms for $N\in M(Y, \mathcal{D})$:*

(4.4.1) $$\mathrm{pr}_1^{-1}\mathrm{DR}_W(a^!_X\omega_{pt})\otimes_{\boldsymbol{C}}\mathrm{pr}_2^{-1}N_{W'}\to\mathrm{DR}_W(a^!_X\omega_{pt}\boxtimes N)_{W'}$$

compatible with the morphisms of $W'\in\mathcal{C}(Y)$ and $N\in M(Y, \mathcal{D})$.

Proof. We have the natural morphisms (4.4.1) by definition, cf. 1.8 and 3.1. To show the quasi-isomorphism, we may assume Y is smooth and X is a closed subspace Z of a complex manifold which will be denoted by X. By definition we have a canonical morphism

(4.4.2) $$\mathrm{pr}_1^{-1}\mathrm{DR}_X(M)\otimes_{\boldsymbol{C}}\mathrm{pr}_2^{-1}N\to\mathrm{DR}_{X\times Y/Y}(M\boxtimes N)$$

for a $\mathcal{D}_X$-Module M and an $\mathcal{O}_Y$-Module N, where only the $\mathcal{O}_Y$-Module structure of N is used in (4.4.1). Then it is enough to show that (4.4.2) is a quasi-isomorphism in the case M holonomic, because $\boldsymbol{R}\Gamma_{[Z]}\omega_V$ has holonomic cohomologies. Since the assertion is stalkwise on $X\times Y$, and compatible with inductive limit of N, we may assume N coherent, and then free, i.e. $N=\omega_Y$, by taking a resolution of N. In this case the assertion is more or less well-known. In fact, let $x\in X$, and B_ε denote the ε-ball in X with center x (defined by taking local coordinates). Then, if ε is sufficiently small, the direct image as $\mathcal{D}$-Module $(\mathrm{pr}_2)_*(M\boxtimes\omega_Y|B_\varepsilon\times Y)$ is independent of ε, and its cohomology sheaves are free $\mathcal{O}_Y$-Modules of finite rank, cf. [4][8]. This means that the restriction of $\mathcal{H}^j(\mathrm{DR}_{X\times Y/Y}(M\boxtimes\omega_Y))$ to $\{x\}\times Y$ are free $\mathcal{O}_Y$-Modules of finite rank. So it is enough to show the isomorphism (4.4.2) after taking DR_Y. Then the assertion is reduced to the case $M=\omega_X$, by exchanging X and Y, and we may assume $N=\omega_Y$ by the same argument as above. Then the assertion is clear.

Remark. This argument is essentially same as the proof of the commutativity of the de Rham functor DR with the external product $\boxtimes$. cf. for example [1].

4.5. Corollary. *With the notation of* 4.1, *we have a canonical isomorphism*

$$(a_X)_! a_X^! \omega_{pt} \boxtimes N \xrightarrow{\sim} p_! p^! N . \tag{4.5.1}$$

Proof. This is obtained by taking the direct image of (4.4.1) by $\boldsymbol{R}p_!$, because $\boldsymbol{R}p_!$ commutes with inductive limits (and hence with tensor product over $\boldsymbol{C}$).

4.6. With the notation of 4.1 we get the trace morphism

$$\mathrm{Tr}_p\colon\ p_! p^! N \to N . \tag{4.6.1}$$

by 4.5 and (4.1.3). For the closed embedding i_f, we have the trace morphism

$$\mathrm{Tr}_{i_f}\colon\ (i_f)_! i_f^! M = \boldsymbol{R}\Gamma_{[X]} M \to M \tag{4.6.2}$$

by (2.1.6.) (2.1.10), and we get the trace morphism

$$\mathrm{Tr}_f\colon\ f_! f^! N = p_!(i_f)_!\, i_f^! p^! N \to p_! p^! N \to N \tag{4.6.3}$$

by (4.6.1–2).

4.7. Lemma. *Let i: $Y \to X \times Y$ be a section of the projection p: $X \times Y \to Y$, i.e. $pi=id$. Then the composition*

$$i^! p^! M \cong p_! i_! i^! p^! M \to p_! p^! M \to M \tag{4.7.1}$$

is an isomorphism for $M \in D^b(X, \mathcal{D})$, where the morphisms are induced by (3.3.2), (2.1.6) *and* (4.6.1) *respectively.*

Proof. Since the assertion is local, we may assume X, Y are closed subspaces of complex manifolds X', Y', and i is extended to a section i': $Y' \to X' \times Y'$ of the projection p': $X' \times Y' \to Y'$. Let Z, Z' be the image of i, i'. For $M \in M_Y(\mathcal{D}_{Y'}) = M(Y, \mathcal{D})$, we have

(4.7.2) $\boldsymbol{R}\Gamma_{[X \times Y']} p'^! M$ represents $p^! M$ on $X' \times Y'$,

by (1.8.5) and definition of $p'^! M$, $p^! M$. Since the trace morphism $(a_X)_! a_X^! \boldsymbol{C} \to \boldsymbol{C}$ is the composition of the natural morphism

$$(a_X)_! a_X^! \boldsymbol{C} = (a_X)_! \boldsymbol{R\Gamma}_X a_{X'}^! \boldsymbol{C} \to (a_{X'})_! a_{X'}^! \boldsymbol{C} \tag{4.7.3}$$

with the trace morphism $(a_{X'})_! a_{X'}^! \boldsymbol{C} \to \boldsymbol{C}$, and $\boldsymbol{R\Gamma}_{[Z]} p'^! M = \boldsymbol{R\Gamma}_{[Z']} p'^! M$ for $M \in M_Y(\mathcal{D}_{Y'})$ by (1.2.8)(2.1.8), it is enough to show that the composition

$$p'_! \boldsymbol{R\Gamma}_{[Z']} p'^! M \to p'_! p'^! M \to M \tag{4.7.4}$$

is an isomorphism for $M \in M_Y(\mathcal{D}_{Y'})$, and the assertion is reduced to the case X, Y smooth. Then we may assume that the section is constant by replacing the direct decomposition $X \times Y$, and the assertion is clear.

4.8. Theorem. *Let*

$$\begin{array}{ccc} X & \xleftarrow{\;g'\;} & X' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ Y & \xleftarrow{\;g\;} & Y' \end{array} \tag{4.8.1}$$

be a cartesian diagram of complex analytic spaces. Then we have a canonical isomorphism

$$g^! f_! M = f'_! g'^! M \quad \textit{in} \quad D^b(Y', \mathcal{D}) \tag{4.8.2}$$

for $M \in D^b(X, \mathcal{D})$ in the following cases: *i) g is a closed embedding, ii) f is a finite morphism, iii) M is holonomic, iv) f is proper and $M \in D^b_{g,\mathrm{coh}}(X, \mathcal{D})$.*

Proof. The case f is a closed embedding is clear by definition, and we may assume f is the second projection $X \times Y \to Y$. Since the case i) follows from 3.10, we may also assume g is the second projection $Y' \times Y \to Y$, and the case iv) follows from 3.8. In the remaining cases, we have a canonical morphism

$$g^! f_! M \to f'_! g'^! M \quad \text{in} \quad D^b(Y', \mathcal{D}) \tag{4.8.3}$$

cf. also (3.8.1). So the assertion is local, and we may assume Y, Y' smooth by the case i). In the case iii) the assertion follows from (4.4.2). In the case ii), i.e. supp M is finite over Y, (4.4.2) is also a quasi-isomorphism. In fact, we may assume X smooth, and it is enough to show the stalkwise isomorphism

$$N \otimes_{\mathcal{O}_Y} \mathcal{O}_{Y \times Y'} \xrightarrow{\sim} N \otimes_{\mathcal{O}_{X \times Y}} \mathcal{O}_{X \times Y \times Y'} \tag{4.8.4}$$

for $N = M \otimes \wedge^i \Theta_X$, where the sheaf theoretic pull-backs are omitted. Taking graduation of $M \otimes \wedge^i \Theta_X$ by the filtration induced by that on M, cf. (1.2.1), (4.8.4) is reduced to the case N is an $\mathcal{O}_Z$-Module with $Z = \mathrm{supp}\, M$, and then N coherent, because the assertion is stalkwise and (4.8.4) commutes with inductive limit of N. Then it is clear by the base change property of projective morphism by smooth morphism which can be easily checked using free resolutions.

4.9. Theorem. *Let $f\colon X\to Y$ be a morphism of complex analytic spaces. Then we have the adjunction formula*

$$\mathrm{Hom}_{D^b(X,\mathcal{D})}(M, f^!N) \xrightarrow{\sim} \mathrm{Hom}_{D^b(Y,\mathcal{D})}(f_!M, N) \tag{4.9.1}$$

for $M\in D^b(X,\mathcal{D})$, $N\in D^b(Y,\mathcal{D})$, induced by the trace morphism $\mathrm{Tr}_f\colon f_!f^!N\to N$, *cf.* (4.6.3), *if the assertion of* 4.7 *holds for M, f and $g=\mathrm{pr}_2\colon X\times Y\to Y$.*

Proof. By (2.1.12) (4.6.3) we may assume f is the second projection $X\times Y\to Y$. Consider the commutative diagram

$$\begin{array}{ccccc} X\times Y & \xleftarrow{q_1} & X\times X\times Y & \xleftarrow{i} & X\times Y \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle q_2} & & \\ Y & \xleftarrow{p_2} & X\times Y & & \end{array} \tag{4.9.2}$$

where q_1, q_2 are induced by the projections to the first and second factors of $X\times X$, $f=p_1=p_2$ and $q_1i=q_2i=id$. We define $\beta\colon M\to f^!f_!M$ by

$$M\simeq i^!q_1^!M\simeq (q_2)_!i_*i^!q_1^!M\to (q_2)_!q_1^!M\simeq p_2^!(p_1)_!M\,, \tag{4.9.3}$$

where the morphism are induced by (4.7.1), (3.3.2), (2.1.6) and 4.8 respectively. Note that (4.9.3) is defined also for $M=f^!N$, because (4.8.2) holds for $M=f^!N$ using $(q_1)_!q_1^!p_1^!N=(q_1)_!q_2^!p_2^!N$ and 4.4. Then it is enough to show that the compositions

$$f_!M\xrightarrow{f_!\beta} f_!f^!f_!M\xrightarrow{\alpha} f_!M\,,\quad f^!N\xrightarrow{\beta} f^!f_!f^!N\xrightarrow{f^!\alpha} f^!N \tag{4.9.4}$$

induced by $\alpha:=\mathrm{Tr}_f$ and β are isomorphisms. The first is same as $(p_1)_!$ of the composition

$$M\simeq (q_1)_!i_*i^!q_1^!M\to (q_1)_!q_1^!M\xrightarrow{\mathrm{Tr}} M\,, \tag{4.9.5}$$

because Tr is compatible with direct image, i.e. we have a commutative diagram

$$\begin{array}{ccc} (p_1)_!((q_1)_!q_1^!)M & \xrightarrow{(p_1)_!\mathrm{Tr}} & (p_1)_!M \\ \| & & \| \\ ((p_2)_!p_2^!)(p_1)_!M & \xrightarrow{\mathrm{Tr}} & (p_1)_!M \end{array} \tag{4.9.6}$$

by definition of Tr, cf. (4.6.1), since the direct image with proper support commutes with inductive limit. Here the first isomorphism of (4.9.5) is induced by (4.7.1), and hence the composition (4.9.5) is actually the identity. The second is same as

$$f^!N \cong i^! q_2^! p_2^! N \cong (q_2)_! i_* i^! q_2^! p_2^! N \to (q_2)_! q_2^! p_2^! N \xrightarrow{\mathrm{Tr}} p_2^! N \tag{4.9.7}$$

by the symmetry of the diagram (4.9.2) and the compatibility of Tr with pull-backs:

$$\begin{array}{ccc} ((q_2)_! q_2^!) p_2^! N & \xrightarrow{\mathrm{Tr}} & p_2^! N \\ \| & & \| \\ p_2^!((p_1)_! p_1^!) N & \xrightarrow{p_2^! \mathrm{Tr}} & p_2^! N\,. \end{array} \tag{4.9.8}$$

Here we can use either the projection q_1 or q_2 as we like for the first isomorphism of (4.9.7) which is induced by (4.7.1), because it is independent of the choice of the projection by the symmetry of the diagram (4.9.2). Then the composition (4.9.7) is an isomorphism. This completes the proof of 4.9.

4.10. Proposition. *The dualizing complex* $a_X^! \omega_{pt}$ *in* 4.1 *is unique up to a canonical isomorphism.*

Proof. This follows from 4.9 by the standard argument on representable functor, where we can apply 4.9 to $M = a_X^!$, $N = \omega_{pt}$ and a_X: $X \to pt$, because M is holonomic.

Remarks. i) In the proof of 4.9, the functors $f^!$ might depend on the choice of $a_X^! \omega_{pt}$. In the proof of 4.10, we apply 4.9 to the functor $a_X^!$ associated with any dualizing complex $a_X^! \omega_{pt}$. Note that the global uniqueness of $a_X^! \omega_{pt}$ is not used in the previous arguments.

ii) The notation $a_X^! \omega_{pt}$ is justified also by 4.10, because it is the pull-back of $\omega_{pt} \in D^b_{g,rh}(pt, \mathcal{D})$ by $a_X^!$. In particular we have

$$f^!(a_Y^! \omega_{pt}) = a_X^! \omega_{pt} \tag{4.10.1}$$

for f: $X \to Y$ by (4.1.6).

4.11. Theorem. *If* $M \in D^b(X, \mathcal{D})$ *has inductively n-good cohomologies, cf.* 3.5, *then* $f^!M$ *has inductively* $(n+1)$*-good cohomologies. In particular,* $D^b_{g,\mathrm{hol}}(X, \mathcal{D}) = D^b_{g,\mathrm{coh}}(X, \mathcal{D}) \cap D^b_{\mathrm{hol}}(X, \mathcal{D})$, $D^b_{g,rh}(X, \mathcal{D}) = D^b_{g,\mathrm{coh}}(X, \mathcal{D}) \cap D^b_{rh}(X, \mathcal{D})$ *are stable by* $f^!$.

Proof. By (3.5.3) (4.1.9) it is enough to show the first assertion. By the same argument as (3.7.5), we may assume M quotient coherent induced. If f is a projection, the assertion follows from (3.5.5) and 4.12 below. So we may assume f is a closed embedding. If M is coherent induced, the assertion follows from 4.13 below. We consider a short exact sequence

$$0 \to M' \to M \to M'' \to 0 \tag{4.11.1}$$

as in (3.7.7), i.e. M is coherent induced, etc. Then the remaining argument is similar to that after (3.7.6), where $\mathscr{H}^i p_!$ is replaced by $\mathscr{H}^i f^!$, the locally finitely generated condition by locally quotient of free Module, coherence by quasi-coherence, and 0-good (i.e. good coherent) by 1-good. The difference is that the condition: M is locally isomorphic to a quotient of a free Module, might be unstable by extension. So we use

(4.11.2) $\mathrm{Ker}(M' \to M)$ is locally a finite extension of quotients of free Modules, and $\mathrm{Im}(M' \to M)$ is quasi-coherent, if M' is locally a finite extension of quotients of free Modules and M is quasi-coherent,

(4.11.3) finite extensions of quotients of free Modules are stable by quotients,

for the assertion corresponding to (3.7.6), where M' is called a finite extension of quotients of free Modules, if there is a finite filtration on M' whose graded pieces are quotients of free Modules. In fact, (4.11.3) is clear by taking the quotient filtration on the quotient, and (4.11.2) follows from (3.5.9) by applying it inductively to a sub-Module of M' and replacing M', M by their quotients inductively. Then we get the assertion as in the proof of 3.7.

4.12. Lemma. *With the notation of* 4.1 *and* 4.11, *we have*

$$a_X^! \omega_{pt} \in D^b_{g,rh}(X, \mathscr{D}) . \tag{4.12.1}$$

Proof. The assertion is clear if X smooth, because $a_X^! \omega_{pt} = \omega_X [\dim X]$ by definition (using [7] [10]). The general case is reduced to this case by induction on $\dim X$ using Hironaka's desingularization. Let U be the maximal smooth open subset of X with pure dimension n, where n is the (maximal) dimension of X. Then $\bar{U}$ is a closed analytic subspace of X. Let $\pi'\colon X' \to \bar{U}$ be a resolution of singularity, and $\pi\colon X' \to X$ its composition with the natural inclusion $\bar{U} \to X$. Then we have a canonical morphism

$$\mathrm{Tr}\colon \pi_! a_{X'}^! \omega_{pt} \to a_X^! \omega_{pt} \tag{4.12.2}$$

by (4.10.1) and (4.6.3). Let $Z = X \setminus U$ and $Z' = \pi^{-1}(Z)$. We have a distinguished triangle

(4.12.3) $$\to \boldsymbol{R}\Gamma_{[Z]}a_X^! \omega_{pt} \to a_X^! \omega_{pt} \to \boldsymbol{R}\Gamma_{[X|Z]}a_X^! \omega_{pt} \to$$

by (2.3.2). Since $\boldsymbol{R}\Gamma_{[Z]}a_X^! \omega_{pt}=a_Z^! \omega_{pt}$ by (4.10.1), it is enough to show

(4.12.4) $$\boldsymbol{R}\Gamma_{[X|Z]}a_X^! \omega_{pt} \in D^b_{g,rh}(X, \mathcal{D})$$

by induction on dim X. By (3.10.2) we have a canonical isomorphism

(4.12.5) $$\boldsymbol{R}\Gamma_{[X|Z]}\pi_! a_{X'}^! \omega_{pt} = \pi_! \boldsymbol{R}\Gamma_{[X'|Z']}a_{X'}^! \omega_{pt}\,.$$

This implies a canonical morphism

(4.12.6) $$\pi_! \boldsymbol{R}\Gamma_{[X'|Z']}a_{X'}^! \omega_{pt} \to \boldsymbol{R}\Gamma_{[X|Z]}a_X^! \omega_{pt}$$

by (4.12.2). We show (4.12.6) is an isomorphism. Let M, M' be the mapping cone of (4.12.2), (4.12.6) so that $M'=\boldsymbol{R}\Gamma_{[X|Z]}M$. We have supp $\mathcal{H}^i M \subset Z$, because π is biholomorphic on U, and $M \in D^b_Z(X, \mathcal{D})$, cf. (2.1.4), because $M \in D^b_{rh}(X, \mathcal{D})$. This implies $\boldsymbol{R}\Gamma_{[Z]}M=M$ by (2.2.2), and $M'=\boldsymbol{R}\Gamma_{[X|Z]}M=0$. Therefore (4.12.4) is reduced to the case X smooth by 3.7 and [7][10], and follows from the triangle (4.12.3) and the inductive hypothesis, because (4.12.1) is clear in the smooth case.

4.13. Lemma. *With the notation of* (3.5.1), *let Z be a closed subspace of X. Then $\Gamma_{[Z]}M$ is induced by $\Gamma_{[Z]}L$, and $\mathcal{H}^i_{[Z]}M$ by $\mathcal{H}^i_{[Z]}L$. If L is a coherent $\mathcal{O}_X$-Module, $\mathcal{H}^i_{[Z]}L$ has an increasing exhaustive filtration G such that $\mathrm{Gr}^G_k\mathcal{H}^i_{[Z]}L$ are coherent $\mathcal{O}_Z$-Modules* (*i.e. annihilated by the Ideal of Z*) *and zero for $k \ll 0$.*

Proof. Let M' be the induced $\mathcal{D}$-Module by $\Gamma_{[Z]}L$. Then we have a canonical morphism $M' \to M$, and the first assertion is local, because it is enough to show that this morphism induces $M' \simeq \Gamma_{[Z]}M$ by taking $\Gamma_{[Z]}$. Then we may assume X smooth, and the assertion follows form the commutativity of local cohomology with inductive limit (using a resolution of $\mathcal{O}_X/\mathcal{J}$). The second assertion follows from the first by taking injective resolution of L, because a $\mathcal{D}$-Module induced by an injective $\mathcal{O}$-Module is $\Gamma_{[Z]}$-acyclic by the same reason as above. For the last assertion, we use the spectral sequence

(4.13.1) $$E_1^{i,j} = \mathcal{E}xt^{i+j}_{\mathcal{O}_X}(\mathrm{Gr}_G^{-i}\,\mathcal{O}_X, L) \Rightarrow \mathcal{H}^{i+j}_{[Z]}L$$

induced by the filtration G on $\mathcal{O}_X$ such that $G^k\mathcal{O}_X=\mathcal{J}_Z^k$, where $\mathcal{J}_Z$ is the Ideal of Z. In fact we replace L by an injective resolution so that $\mathcal{H}^i_{[Z]}L$ is the cohomology of the complex $\Gamma_{[Z]}L$. Then G induces an increasing filtration G of $\Gamma_{[Z]}L$ which gives (4.13.1), because $\mathrm{Gr}^G_k\,\Gamma_{[Z]}L=\mathcal{H}om_{\mathcal{O}_X}(\mathrm{Gr}^k_G\,\mathcal{O}_X, L)$ by the injectivity of L. Since $E_1^{i,j}$ are coherent $\mathcal{O}_Z$-Modules, the kernel of the projection $E_r^{i,j} \to E_\infty^{i,j}$ $(r > -i)$ is a union of coherent sub-Modules and is coherent by Noetherian property. So $\mathrm{Gr}_k^G \mathcal{H}_{[Z]}^i L$ are coherent $\mathcal{O}_Z$-Modules. This completes the proof of 4.13 and 4.11.

4.14. Theorem. *With the notation of* 4.11 *and the assumption of* 3.7, $D_{g,\mathrm{hol}}^b(X, \mathcal{D})$, $D_{g,rh}^b(X, \mathcal{D})$ *are stable by* f_*, *and* $D_{g,rh}^b(X, \mathcal{D}) = D_{rh}^b(X, \mathcal{D})$.

Proof. We first show the stability by f_*. It is enough to show the stability of holonomic and regular holonomic conditions by (3.7.1). Then the assertion is local on Y, and we may assume Y smooth. The assertion is well-known if X smooth by [7] [10], and the general case is reduced to this case using desingularization and induction on $\dim X$ as in the proof of 4.12. In fact, let $\pi\colon X' \to X$ and Z, Z' be as in the proof of 4.12. By induction and using the triangle (2.3.2), it is enough to show the assertion for $\boldsymbol{R}\Gamma_{[X|Z]}M$, where $M \in D_{g,\mathrm{hol}}^b(X, \mathcal{D})$ or $D_{g,rh}^b(X, \mathcal{D})$. We have a canonical isomorphism

$$\mathrm{Tr}\colon \pi_! \boldsymbol{R}\Gamma_{[X'|Z']}\pi^! M \xrightarrow{\sim} \boldsymbol{R}\Gamma_{[X|Z]}M \tag{4.14.1}$$

as in the proof of 4.12 using (3.10.2) (4.6.3), where we have $\pi_! \boldsymbol{R}\Gamma_{[X'|Z']}\pi^! M \in D_{\mathrm{hol}}^b(X, \mathcal{D})$ by [6], because the assertion is local on X. Since $\boldsymbol{R}\Gamma_{[X'|Z']}\pi^! M \in D_{g,\mathrm{hol}}^b(X', \mathcal{D})$ or $D_{g,rh}^b(X', \mathcal{D})$ by 4.11 (using the triangle (2.3.2)), we get the first assertion by (3.3.2). The proof of the last assertion is similar. If X is smooth, the assertion follows from the fact that a regular holonomic $\mathcal{D}$-Module has a globally good filtration [7] by Remark after 3.6. (Here the case of regular meromorphic connection is enough for our argument, if the resolution is taken appropriately.) Then we have the isomorphism (4.14.1) for $M \in D_{rh}^b(X, \mathcal{D})$, because $\pi^! M$ is regular holonomic by [7] [10], and belongs to $D_{g,rh}^b(X', \mathcal{D})$. So the last assertion follows from the first assertion (applied to $\boldsymbol{R}\Gamma_{[X'|Z']}\pi^! M$ and π) using the triangle (2.3.2) and induction.

4.15. *Remark.* Let $f\colon X \to Y$ be a morphism of complex analytic spaces. Assume f is bijective. Then f is topologically an isomorphism by Weierstrass preparation theorem, and we have an equivalence of categories

$$f_*\colon D^b(X, \mathcal{D}) \to D^b(Y, \mathcal{D}) \tag{4.15.1}$$

with a quasi-inverse $f^!\colon D^b(Y, \mathcal{D}) \to D^b(X, \mathcal{D})$. Note that the complexes with quasi-coherent (resp. holonomic, resp. regular holonomic) cohomologies are stable by these functors and (4.15.1) induces equivalences of categories for these complexes, but the stability of coherent complexes by $f^!$ is not clear. For the proof of (4.15.1) we have canonical morphisms

(4.15.2) $$M \to f^! f_* M\,, \quad f_* f^! N \to N$$

for $M \in D^b(X, \mathscr{D})$, $N \in D^b(Y, \mathscr{D})$ by 4.9, and it is enough to show that (4.15.2) are isomorphisms for $M \in M(X, \mathscr{D})$, $N \in M(Y, \mathscr{D})$. Here the assertion is stalkwise, because (4.15.2) is defined stalkwise by construction (in fact direct image by finite morphism and algebraic local cohomology are defined stalkwise, and (4.7.1) depends only on stalk of M.) Then we may assume the stalks of M, N are finite by using the commutativity of the functors $f^!$, f_* with inductive limit. So the assertion is reduced to the case M, N coherent (in fact, quasi-coherence is enough). Since the assertion is trivial on a Zariski open subset on which f is biholomorphic, we can proceed by induction on the dimension of X, Y by using the triangle (2.3.2) and the commutativity 3.10 as in the proof of 4.12 and 4.14, where the pull-back $f^!$ commutes with $\boldsymbol{R}\Gamma_{[Z]}$, $\boldsymbol{R}\Gamma_{[Y|Z]}$ by (4.1.6) and (2.3.2). Here we used the quasi-coherence to show that $\boldsymbol{R}\Gamma_{[X|Z]}$ or $\boldsymbol{R}\Gamma_{[Y|Z]}$ of the mapping cone of (4.15.2) is zero, if its restriction to the complement of Z is zero.

In the case X smooth, (4.15.1) was studied by [20] in the one dimensional case, and by [19] in a more general case (where X admits a stratification by closed smooth subspaces satisfying some condition, and the above argument can be used to complete some arguments in [loc. cit.]).

§5. Duality

5.1. Let X be a complex analytic space, and $\delta\colon X \to X \times X$ be the diagonal embedding. Let M, $N \in M(X, \mathscr{D})$, and $K \in M(X \times X, \mathscr{D})$ such that $\operatorname{supp} K \subset \operatorname{Im} \delta$. If X is smooth, we have a canonical isomorphism

(5.1.1) $$\operatorname{Hom}_{\mathscr{D}_X}(M, \mathscr{H}om_{\mathscr{D}_X}(N, \delta^{-1}K)) = \operatorname{Hom}_{\mathscr{D}_{X\times X}}(M \boxtimes N, K)\,,$$

cf. [16]. Using this we can construct $\mathscr{H}om_{\mathscr{D}}(N, \delta^{-1}K) \in M(X, \mathscr{D})$ with a morphism $\varepsilon\colon \mathscr{H}om_{\mathscr{D}}(N, \delta^{-1}K) \boxtimes N \to K$ which induces an isomorphism

(5.1.2) $$\operatorname{Hom}_{\mathscr{D}}(M, \mathscr{H}om_{\mathscr{D}}(N, \delta^{-1}K)) \xrightarrow{\sim} \operatorname{Hom}_{\mathscr{D}}(M \boxtimes N, K)$$

for any M by composition, where $\mathscr{D}$ means the morphisms in $M(X, \mathscr{D})$, etc. In fact, with the notation of 1.5, we define $\mathscr{H}om_{\mathscr{D}}(N, \delta^{-1}K)_W$ the representative of $\mathscr{H}om_{\mathscr{D}}(N, \delta^{-1}K)$ on $W = \{U \to V\} \in \mathcal{C}(X)$ by

(5.1.3) $$\mathscr{H}om_{\mathscr{D}}(N, \delta^{-1}K)_W = \mathscr{H}om_{\mathscr{D}_V}(N_W, \delta^{-1}K_{W\times W})$$

where $W \times W = \{U \times U \to V \times V\}$, and δ denotes the diagonal embedding for any spaces. By (5.1.1) we have a morphism

$$v_{W_2W_1}\colon f_!\,\mathcal{H}om_{\mathcal{D}}(N,\ \delta^{-1}K)_{W_1}\to\mathcal{H}om_{\mathcal{D}}(N,\ \delta^{-1}K)_{W_2} \tag{5.1.4}$$

using a commutative diagram

$$\begin{array}{ccc} f_!\,\mathcal{H}om_{\mathcal{D}}(N,\ \delta^{-1}K)_{W_1}\boxtimes f_!\,N_{W_1} & \longrightarrow & (f\times f)_!K_{W_1\times W_1} \\ \quad\downarrow{\scriptstyle id\boxtimes u_{W_2W_1}} & & \downarrow \\ f_!\,\mathcal{H}om_{\mathcal{D}}(N,\ \delta^{-1}K)_{W_1}\boxtimes N_{W_2} & \longrightarrow & K_{W_2\times W_2} \\ \quad\downarrow{\scriptstyle v_{W_2W_1}\boxtimes id} & \nearrow & \\ \mathcal{H}om_{\mathcal{D}}(N,\ \delta^{-1}K)_{W_2}\boxtimes N_{W_2}\,. & & \end{array} \tag{5.1.5}$$

Here we have the second horizontal morphism, because $u_{W_2W_1}$ is an isomorphism on $V_2\backslash(U_2\backslash U_1)$ and $f_!\,\mathcal{H}om_{\mathcal{D}}(N,\ \delta^{-1}K)_{W_1}$, $f_!N_{W_1}$ are zero on $U_2\backslash U_1$. (In fact we may assume $U_1=U_2$ by replacing V_2 with $V_2\backslash(U_2\backslash U_1)$.) We can check the compatibility condition (1.5.2) using (5.1.5), and this implies the isomorphism of $v_{W_2W_1}$ on $V_2\backslash(U_2\backslash U_1)$ using a section as in the proof of (1.3.3), because the assertion is clear in the closed embedding case. Then the morphism ε is well-defined by (5.1.5). We can check (5.1.2) using (5.1.1) and the morphism of $f_!M_{W_1}\boxtimes f_!N_{W_1}\to f_!M_{W_1}\boxtimes N_{W_2}\to M_{W_2}\boxtimes N_{W_2}$ to the left column of (5.1.5). Then we can check that $\mathcal{H}om_{\mathcal{D}}(N,\ \delta^{-1}K)$ is functorial in N and K using (5.1.2) with the diagrams

$$\begin{array}{ccc} \mathcal{H}om_{\mathcal{D}}(N_2,\ \delta^{-1}K)\boxtimes N_1 & \to & \mathcal{H}om_{\mathcal{D}}(N_2,\ \delta^{-1}K)\boxtimes N_2 \\ \downarrow & & \downarrow \\ \mathcal{H}om_{\mathcal{D}}(N_1,\ \delta^{-1}K)\boxtimes N_1 & \to & K \end{array} \tag{5.1.6}$$

$$\begin{array}{ccc} \mathcal{H}om_{\mathcal{D}}(N,\ \delta^{-1}K_1)\boxtimes N & \to & K_1 \\ \downarrow & & \downarrow \\ \mathcal{H}om_{\mathcal{D}}(N,\ \delta^{-1}K_2)\boxtimes N & \to & K_2 \end{array} \tag{5.1.7}$$

We have also

(5.1.8) $\mathcal{H}om_{\mathcal{D}}(N,\ \delta^{-1}K)$ is injective, if K is injective,

by (5.1.2), because $\boxtimes$ is exact in both factors, cf. 1.8.

Let $N\in D^b(X,\ \mathcal{D})$, $K\in D^b_X(X\times X,\ \mathcal{D})$, cf. (2.1.4), where X is identified with the image of δ. We define

$$\boldsymbol{R}\,\mathcal{H}om_{\mathcal{D}}(N,\ \delta^{-1}K)\in D^+(X,\ \mathcal{D}) \tag{5.1.9}$$

by taking injective resolution of K, where K is represented by a complex whose components are supported in $\mathrm{Im}\,\delta$ by 2.2, and its injective resolution has also support in Im δ by (1.2.6) (1.7.3). Then $\boldsymbol{R}\,\mathcal{H}om_{\mathcal{D}}(N, \delta^{-1}K)$ is acyclic if N or K is acyclic, and it is well-defined. In fact it is acyclic if K acyclic and injective, and it is independent of resolution of K. Then, if N is acyclic, the assertion is reduced to the smooth case in [16] by taking the representative on each W and using (1.2.6). We have a canonical morphism

$$\text{(5.1.10)} \qquad \varepsilon\colon \boldsymbol{R}\,\mathcal{H}om_{\mathcal{D}}(N, \delta^{-1}K)\boxtimes N \to K \quad \text{in} \quad D^{+}(X\times X, \mathcal{D}),$$

and this induces an isomorphism

$$\text{(5.1.11)} \quad \mathrm{Hom}_{D^b(X,\mathcal{D})}(M, \boldsymbol{R}\,\mathcal{H}om_{\mathcal{D}}(N, \delta^{-1}K)) \xrightarrow{\sim} \mathrm{Hom}_{D^b(X\times X,\mathcal{D})}(M\boxtimes N, K)$$

by (5.1.2) (5.1.6–8) (using an injective representative of K as above).

5.2. With the above notation, let $M\in D^b_{\mathrm{coh}}(X, \mathcal{D})$. We define $\boldsymbol{D}M\in D^b_{\mathrm{coh}}(X, \mathcal{D})$ by

$$\text{(5.2.1)} \qquad \boldsymbol{D}M = \boldsymbol{R}\,\mathcal{H}om_{\mathcal{D}}(M, \delta^{-1}\delta_{*}a_X^{!}\,\omega_{pt}).$$

Here $\boldsymbol{D}M\in D^b_{\mathrm{coh}}(X, \mathcal{D})$ follows from (2.1.11) and the canonical isomorphism

$$\text{(5.2.2)} \qquad (\boldsymbol{D}M)_W = \boldsymbol{D}(M_W) := \boldsymbol{R}\,\mathcal{H}om_{\mathcal{D}_V}(M_V, \omega_V[\dim V]\otimes_{\mathcal{O}_V}\mathcal{D}_V),$$

cf. [16], which follows from the definition (5.1.3). In fact, we have

$$\text{(5.2.3)} \quad \delta^{-1}\delta_{*}(a_X^{!}\,\omega_{pt})_W = (a_X^{!}\,\omega_{pt})_W\otimes_{\mathcal{O}_V}\mathcal{D}_V = \boldsymbol{R}\Gamma_{[U]}\omega_V[\dim V]\otimes_{\mathcal{O}_V}\mathcal{D}_V,$$

and $\boldsymbol{R}\,\mathcal{H}om_{\mathcal{D}_V}(M_W, \boldsymbol{R}\Gamma_{[U]}\omega_V[\dim V]\otimes_{\mathcal{O}_V}\mathcal{D}_V) \xrightarrow{\sim} \boldsymbol{R}\,\mathcal{H}om_{\mathcal{D}_V}(M_W, \omega_V[\dim V]\otimes_{\mathcal{O}_V}\mathcal{D}_V)$ by $M_W\in D^b(M_U(\mathcal{D}_V))$, cf. (1.2.5). This implies also

(5.2.4) $D^b_{\mathrm{hol}}(X, \mathcal{D})$ and $D^b_{\mathrm{rh}}(X, \mathcal{D})$ are stable by $\boldsymbol{D}$.

By definition we have a canonical pairing

$$\text{(5.2.5)} \qquad \boldsymbol{D}M\boxtimes M \to \delta_{*}a_X^{!}\,\omega_{pt} \quad \text{in} \quad D^b(X\times X, \mathcal{D})$$

inducing the isomorphism

$$\text{(5.2.6)} \quad \mathrm{Hom}_{D^b(X,\mathcal{D})}(N, \boldsymbol{D}M) \xrightarrow{\sim} \mathrm{Hom}_{D^b(X\times X,\mathcal{D})}(N\boxtimes M, \delta_{*}a_X^{!}\omega_{pt}).$$

by composition, cf. (5.1.10–11). We say that a morphism

$$\text{(5.2.7)} \qquad N\boxtimes M \to \delta_{*}a_X^{!}\,\omega_{pt} \quad \text{in} \quad D^b(X\times X, \mathcal{D})$$

is a *perfect pairing* of M, $N\in D^b_{\mathrm{coh}}(X, \mathcal{D})$, if the corresponding morphism by (5.2.6):

$$\text{(5.2.8)} \qquad N \to \boldsymbol{D}M \quad \text{in} \quad D^b(X, \mathcal{D}),$$

is an isomorphism. By (5.2.2) we have

(5.2.9) perfect pairings are invariant by closed embeddings,

i.e. for a closed embedding $i\colon X\to Y$, $N\boxtimes M\to\delta_*a_X^!\omega_{pt}$ is a perfect pairing if and only if so is the composition $i_*N\boxtimes i_*M\to\delta_*i_*a_X^!\omega_{pt}\to\delta_*a_Y^!\omega_{pt}$. Then we have a canonical isomorphism

$$\boldsymbol{D}\boldsymbol{D}M=M\,, \tag{5.2.10}$$

i.e. the transpose $M\boxtimes\boldsymbol{D}M\to\delta_*a_X^!\omega_{pt}$ of $\boldsymbol{D}M\boxtimes M\to\delta_*a_X^!\omega_{pt}$ is a perfect pairing, because the assertion is local, and reduced to the smooth case by (5.2.9).

Let $f\colon X\to Y$ be a morphism of complex analytic spaces. We define

$$f^*M=\boldsymbol{D}f^!\boldsymbol{D}M\qquad\text{for}\quad M\in D^b_{\text{hol}}(X,\mathcal{D}) \tag{5.2.11}$$

using (4.1.9) (5.2.4).

5.3. Let X be a complex analytic space, and $W=\{U\to V\}\in\mathcal{C}(X)$. We denote by $\tilde{K}_V$ the complex of currents on V shifted by 2 dim V to the left, and K_V its subcomplex consisting of forms of type (dim V, i) with $0\le i\le\dim V$, so that $\tilde{K}_V$ and K_V are quasi-isomorphic to $\boldsymbol{C}_V[2\dim V]$ and $\omega_V[\dim V]$ respectively. By [9] the stalks of the components of $\tilde{K}_V$ and K_V are injective $\mathcal{O}_{X,x}$-modules. We define

$$\tilde{K}_W=\Gamma_{[U]}\tilde{K}_V\,,\quad K_W=\mathcal{H}om_{\mathcal{O}_V}(\mathcal{O}_U,K_V)\,, \tag{5.3.1}$$

so that K_W is a subcomplex of $\tilde{K}_W$. Then, for a morphism $f\colon W_1\to W_2$ as in (1.5.1), we have canonical morphisms

$$f_!\tilde{K}_{W_1}\to\tilde{K}_{W_2}\,,\quad f_!K_{W_1}\to K_{W_2}\,, \tag{5.3.2}$$

induced by the push-down of currents $f_\sharp\colon f_!\tilde{K}_{V_1}\to\tilde{K}_{V_2}$, because $f_\sharp(mf^*g)=f_\sharp(m)g$ for $m\in f_!\tilde{K}_{V_1}$, $g\in\mathcal{O}_{V_2}$. Here $f_!$ denoted the sheaf theoretic direct image with proper supports, and the second morphism of (5.3.2) is a morphism of complexes of $\mathcal{O}$-Modules. Then they satisfy the condition (1.5.2) by the functoriality of push-down of currents. Since $\tilde{K}_W$ and K_W are differential complexes in the sense of [16], we denote by $\widetilde{\mathrm{DR}}^{-1}\tilde{K}_W$ and $\widetilde{\mathrm{DR}}^{-1}K_W$ the complexes of $\mathcal{D}$-Modules induced by $\tilde{K}_W$ and K_W respectively (i.e. $(\widetilde{\mathrm{DR}}^{-1}\tilde{K}_W)^i=\tilde{K}_W^i\otimes\mathcal{D}$, etc.), and they can be also viewed as complexes of $\mathcal{D}$-Modules on X by zero extension. Then, for f as above, we have a morphism of complexes of $\mathcal{D}$-Modules

$$f_!(\widetilde{\mathrm{DR}}^{-1}\tilde{K}_{W_1})=\widetilde{\mathrm{DR}}^{-1}(f_!\tilde{K}_{W_1})\to\widetilde{\mathrm{DR}}^{-1}\tilde{K}_{W_2} \tag{5.3.3}$$

$$f_!(\widetilde{\mathrm{DR}}^{-1}K_{W_1})=\widetilde{\mathrm{DR}}^{-1}(f_!K_{W_1})\to\widetilde{\mathrm{DR}}^{-1}K_{W_2} \tag{5.3.4}$$

induced by (5.3.2) and they satisfy (1.5.2), where the first $f_!$ in (5.3.3–4) denotes the direct image of $\mathcal{D}$-Modules with proper supports, and the first isomorphism is obtained by the same argument as in [16], because $f_!$ commutes with inductive limit. Here $f_!$ in (5.3.3–4) can be omitted if $\widetilde{\mathrm{DR}}^{-1}\tilde{K}_{W_i}$ and $\widetilde{\mathrm{DR}}^{-1}K_{W_i}$ are viewed as complexes of $\mathcal{D}$-Modules on X by zero extension as above, because $f_!$ is essentially the zero extension, cf. 1.4. Although the morphisms in (5.3.2–4) do not induce isomorphisms of complexes on $V_2\backslash(U_2\backslash U_1)$, they are quasi-isomorphisms on $V_2\backslash(U_2\backslash U_1)$, because it is clear in the closed embedding case, and we can use the section of f as in (1.3.3) in general.

With the notation of 1.6 (e.g. $\mathcal{W}=\{W_i\}_{i\in\Lambda}$ is a covering of X), we define complexes of induced $\mathcal{D}$-Modules $\widetilde{\mathrm{DR}}^{-1}\tilde{K}_W$ and $\widetilde{\mathrm{DR}}^{-1}K_W$ by the co-Čech double complexes whose (p, q)-components are

$$(5.3.5) \qquad \oplus_{1-|I|=p}(\widetilde{\mathrm{DR}}^{-1}\tilde{K}_{W_I})^q, \quad \oplus_{1-|I|=p}(\widetilde{\mathrm{DR}}^{-1}\tilde{K}_{W_I})^q$$

respectively, where the co-Čech morphism is induced by (3.5.3–4), and we use the zero extension of $\widetilde{\mathrm{DR}}^{-1}\tilde{K}_{W_I}$, etc. as above. Then we have a natural morphism

$$(5.3.6) \qquad \widetilde{\mathrm{DR}}^{-1}K_W \to \widetilde{\mathrm{DR}}^{-1}\tilde{K}_W .$$

We define similarly a complex of $\mathcal{O}_X$-Modules K_W by the double complex whose (p, q)-component is

$$(5.3.7) \qquad \oplus_{1-|I|=p} K^q_{W_I} .$$

In this case K_{W_I} are complexes of $\mathcal{O}_{U_I}$-Modules and the second $f_!$ in (5.3.2) is really the zero extension. Then $\widetilde{\mathrm{DR}}^{-1}K_W$ is the complex of $\mathcal{D}$-Modules induced by the complex of $\mathcal{O}_X$-Modules K_W, i.e. we have a canonical isomorphism of complexes

$$(5.3.8) \qquad \widetilde{\mathrm{DR}}^{-1}K_W = K_W \otimes \mathcal{D} .$$

In the derived category $D^b(X, \mathcal{D})$, $\widetilde{\mathrm{DR}}^{-1}K_W$ and $\widetilde{\mathrm{DR}}^{-1}\tilde{K}_W$ are independent of the choice of covering $\mathcal{W}$ (because (5.3.3–4) induces quasi-isomorphisms on the complement of $U_2\backslash U_1$) and we denote them by $\widetilde{\mathrm{DR}}^{-1}K_X$ and $\widetilde{\mathrm{DR}}^{-1}\tilde{K}_X$ respectively. (Similarly for K_W and $K_X \in D^b(\mathcal{O}_X)$.)

5.4. Let $f: X \to Y$ be a morphism of complex analytic spaces. By the functoriality of push-down of currents, we have the trace morphisms of $\mathcal{D}$-Modules

(5.4.1) $\quad \mathrm{Tr}_f : f_! \widetilde{\mathrm{DR}}^{-1} K_X \to \widetilde{\mathrm{DR}}^{-1} K_Y, \quad \mathrm{Tr}_f : f_! \widetilde{\mathrm{DR}}^{-1} \tilde{K}_X \to \widetilde{\mathrm{DR}}^{-1} \tilde{K}_Y$

compatible with the morphisms $\widetilde{\mathrm{DR}}^{-1} K_X \to \widetilde{\mathrm{DR}}^{-1} \tilde{K}_X$, $\widetilde{\mathrm{DR}}^{-1} K_Y \to \widetilde{\mathrm{DR}}^{-1} \tilde{K}_Y$ in (5.3.6), where we use $W \times W' \in \mathcal{C}(X \times Y)$ for $W \in \mathcal{C}(X)$, $W' \in \mathcal{C}(Y)$ as in (3.1.9). Here the first morphism in (5.4.1) is induced by the trace morphism of $\mathcal{O}$-Modules

(5.4.2) $$\mathrm{Tr}_f : f_! K_X \to K_Y$$

using the commutativity of $\widetilde{\mathrm{DR}}^{-1}$ with direct image. Note that we have a canonical isomorphism

(5.4.3) $$\widetilde{\mathrm{DR}}^{-1} \tilde{K}_X = a_X^! \omega_{pt}$$

compatible with the trace morphism for a_X: $X \to pt$ by construction of 4.2. We can also check that K_X is canonically isomorphic to the dualizing sheaf in [11] using Théorème 14 in [12], see Remark below, and in the f proper case, the trace morphism (5.4.2) induces the duality isomorphisms for the direct images of coherent complexes of $\mathcal{O}$-Modules by [12] [13].

Remark. Let X be a complex manifold of dimension d, and K' denote the dualizing complex of [11]. By definition K'^{-i} is the sheaf associated to the presheaf which assigns to U the inductive limit of the meromorphic sections of $\mathcal{H}^{d-i}_{[Z]} \Omega^d_U$ with Z running over closed subvarieties of dimension i in U, where the meromorphic sections are defined by the functor $\Gamma_{[Z|Z']}$ in (1.2.1), and the differential of K' is induced by a long exact sequence similar to (1.2.2). Here we may assume Z (and Z') are complete intersections in U, because we take inductive limit for U, Z (and Z'). By Théorème 14 of [12] we have a natural morphism $K' \to K_X$, where K_X is as in 5.3. This morphism is defined by using residues and principal values, combined with Koszul comlexes calculating the higher extensions (whose limit gives the algebraic local cohomology). This morphism is compatible with closed embedding of complex manifolds and also with the smooth projection as in (1.6.4) so that we get a quasi-isomorphism of the dualizing complex of [11] with K_X in 5.3 in the singular case.

5.5. Proposition. *Let M be a coherent induced $\mathcal{D}$-Module $L \otimes \mathcal{D}$, and $\boldsymbol{D}L = \boldsymbol{R}\,\mathcal{H}om_{\mathcal{O}}(L, K_X)$ the dual of L as $\mathcal{O}_X$-Module. We have a pairing*

(5.5.1) $$(\boldsymbol{D}L \otimes \mathcal{D}) \boxtimes (L \otimes \mathcal{D}) \to (\delta_* K_X) \otimes \mathcal{D} = \delta_*(K_X \otimes \mathcal{D}) \to \delta_* \widetilde{\mathrm{DR}}^{-1} \tilde{K}_X = \delta_* a_X^! \omega_{pt}$$

induced by (5 3.6)(5.4.3) *and the natural pairing $\boldsymbol{D}L \otimes L \to K_X$, and it is a perfect pairing, i.e. we get a canonical isomorphism in $D^b_{\mathrm{coh}}(X, \mathcal{D})$:*

(5.5.2) $$\boldsymbol{D}L\otimes\mathcal{D} = \boldsymbol{D}(L\otimes\mathcal{D})$$

Proof. The assertion is local, and we may assume X smooth by (5.2.9). Then the assertion is clear by definition, cf. [16].

5.6. Proposition. *The categories $D^b_{g,\mathrm{coh}}(X, \mathcal{D})$ and $D^b_{g,\mathrm{hol}}(X, \mathcal{D})$ are stable by $\boldsymbol{D}$.*

Proof. By (5.2.4) it is enough to show the stability of $D^b_{g,\mathrm{coh}}(X, \mathcal{D})$. We have a spectral sequence

(5.6.1) $$E_1^{i,j} = \mathcal{H}^{i+j}\boldsymbol{D}\mathrm{Gr}_i^G M \Rightarrow \mathcal{H}^{i+j}\boldsymbol{D}M$$

associated with a filtration G on M (by restricting X), and a long exact sequence

(5.6.2) $$\to \mathcal{H}^{i-1}\boldsymbol{D}M \to \mathcal{H}^{i-1}\boldsymbol{D}M' \to \mathcal{H}^{i}\boldsymbol{D}M'' \to \mathcal{H}^{i}\boldsymbol{D}M \to \mathcal{H}^{i}\boldsymbol{D}M' \to$$

associated with a short exact sequence as (3.7.7), i.e. M is coherent induced, M'' is quotient coherent induced, and M' is good coherent. Then $\mathcal{H}^i\boldsymbol{D}M$ are coherent induced by 5.5, and we can show $\mathcal{H}^i\boldsymbol{D}M', \mathcal{H}^i\boldsymbol{D}M'' \in D^b_{g,\mathrm{coh}}(X, \mathcal{D})$ by increasing induction on i using 3.6, because $D^b_{\mathrm{coh}}(X, \mathcal{D})$ is stable by $\boldsymbol{D}$.

5.7. Theorem. *Let $f\colon X\to Y$ be a proper morphism of complex analytic spaces. Then for $M\in D^b_{g,\mathrm{coh}}(X, \mathcal{D})$, we have a canonical and functorial isomorphism in $D^b_{g,\mathrm{coh}}(Y, \mathcal{D})$:*

(5.7.1) $$f_*\boldsymbol{D}M \xrightarrow{\sim} \boldsymbol{D}f_*M$$

induced by $\mathrm{Tr}_f\colon f_*a_X^!\omega_{pt} \to a_Y^!\omega_{pt}$ *in* (5.4.1)(5.4.3). *If f is a closed embedding,* (5.7.1) *holds for $M\in D^b_{\mathrm{coh}}(X, \mathcal{D})$.*

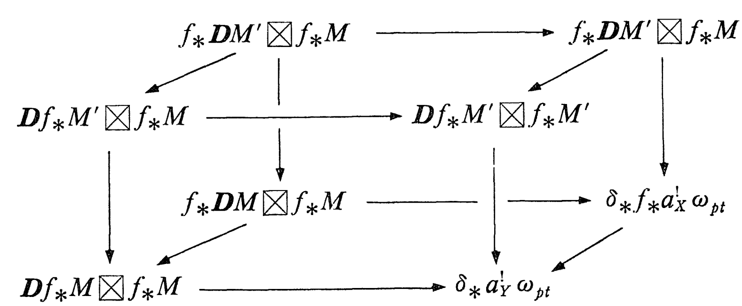
Proof. We have the perfect pairing

(5.7.2) $$\boldsymbol{D}M\boxtimes M \to \delta_* a_X^!\omega_{pt} \quad \text{in} \quad D^b(X\times X, \mathcal{D})$$

corresponding to the identity on $\boldsymbol{D}M$, cf. (5.2.5–6). Taking the direct image by $f\times f$ and composing it with the trace morphism, we get

(5.7.3) $$f_*\boldsymbol{D}M\boxtimes f_*M \to \delta_* f_* a_X^!\omega_{pt} \to \delta_* a_Y^!\omega_{pt} \quad \text{in} \quad D^b(Y\times Y, \mathcal{D}),$$

and (5.7.1) is obtained as the corresponding morphism by (5.2.6). Here $(f\times f)_*(\boldsymbol{D}M\boxtimes M) = f_*\boldsymbol{D}M\boxtimes f_*M$ follows from 3.9. So it is enough to show that (5.7.3) is a perfect pairing. Then the closed embedding case follows form (5.2.9). In general, we can check (5.7.1) is compatible with morphisms of M using (5.2.6) with the commutative diagram



for a morphism $M\to M'$. So we may assume M good coherent, cf. 3.5. If M is a coherent induced $\mathcal{D}$-Module $L\otimes\mathcal{D}$, and $\boldsymbol{D}L$ is as in 5.5, we have a commutative diagram

$$\begin{array}{ccccc} (f_*\boldsymbol{D}L\boxtimes f_*L)\otimes\mathcal{D} & \to & \delta_*f_*K_X\otimes\mathcal{D} & \to & \delta_*f_*\widetilde{\mathrm{DR}}^{-1}\tilde{K}_X \\ & & \downarrow & & \downarrow \\ & & \delta_*K_Y\otimes\mathcal{D} & \to & \delta_*\widetilde{\mathrm{DR}}^{-1}\tilde{K}_Y \end{array}$$

by (5.4.1), and the assertion follows from the duality for $\mathcal{O}$-Modules [12][13] by 5.5. Then we can show the injectivity and surjectivity of

$$\mathcal{H}^if_*\boldsymbol{D}M\to\mathcal{H}^i\boldsymbol{D}f_*M \tag{5.7.4}$$

for good coherent $\mathcal{D}$-Modules by increasing induction on i using the long exact sequences

$$\begin{array}{ccccccccccc} \to\mathcal{H}^{i-1}f_*\boldsymbol{D}M & \to & \mathcal{H}^{i-1}f_*\boldsymbol{D}M' & \to & \mathcal{H}^if_*\boldsymbol{D}M'' & \to & \mathcal{H}^if_*\boldsymbol{D}M & \to & \mathcal{H}^if_*\boldsymbol{D}M' & \to \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \to\mathcal{H}^{i-1}\boldsymbol{D}f_*M & \to & \mathcal{H}^{i-1}\boldsymbol{D}f_*M' & \to & \mathcal{H}^i\boldsymbol{D}f_*M'' & \to & \mathcal{H}^i\boldsymbol{D}f_*M & \to & \mathcal{H}^i\boldsymbol{D}f_*M' & \to \end{array} \tag{5.7.5}$$

associated with (3.7.7) and the long exact sequences associated with $0\to G_{i-1}M'\to G_iM'\to\mathrm{Gr}_i^G M'\to 0$, cf. 3.5 for G. For example, the surjectivity of (5.7.4) for i with M good coherent is reduced to the surjectivity for i with M quotient coherent induced and the injectivity for i with M good coherent, because the injectivity for $i+1$ with M quotient coherent induced follows from the injectivity for i with M good coherent.

5.8. Proposition. *Let $M\in D^b_{\mathrm{hol}}(X,\mathcal{D})$. The pairing*

$$\mathrm{DR}(\boldsymbol{D}M)\boxtimes\mathrm{DR}(M)\to\mathrm{DR}(\delta_*\,a_X^!\,\omega_{pt})=\delta_*\,a_X^!\boldsymbol{C} \tag{5.8.1}$$

induced by the canonical perfect pairing (5.2.5) *and the de Rham functor is a perfect pairing, i.e. we get a canonical isomorphism*

$$\mathrm{DR}(\boldsymbol{D}M)=\boldsymbol{D}\,\mathrm{DR}(M) \tag{5.8.2}$$

where $\boldsymbol{D}$ on the right is the dual functor in $D_c^b(\boldsymbol{C}_X)$.

Proof. The de Rham functor commutes with external product, because the assertion is local and well-known in the smooth case, cf. for example [1] (see also the remark after 4.4). So we get (5.8.1). Then the assertion is local, and reduced to the smooth case [5][7] (cf. also [16]), because (5.8.1) is compatible with the direct image by closed embeddings, cf. (5.7.3).

5.9. *Remark.* By construction the duality isomorphism of 5.7 in the good holonomic case is compatible with the duality isomorphism for constructible sheaves by the de Rham functor.

§6. Riemann-Hilbert correspondence

6.1. Let X be a complex analytic space. The de Rham functor $\mathrm{DR}=\mathrm{DR}_X$ in (3.1.7) induces

(6.1.1) $$\mathrm{DR}\colon D_{rh}^b(X, \mathcal{D}) \to D_c^b(\boldsymbol{C}_X)$$

by [5]. Let Z be a closed subspace, and $i\colon Z\to X$, $j\colon X\backslash Z\to X$ the natural inclusions. Then for $M\in D_{rh}^b(X, \mathcal{D})$, we have canonical isomorphisms

(6.1.2) $$\mathrm{DR}(\boldsymbol{R}\Gamma_{[Z]}M) = i_*i^!\mathrm{DR}(M)$$

(6.1.3) $$\mathrm{DR}(\boldsymbol{R}\Gamma_{[X|Z]}M) = \boldsymbol{R}j_*j^*\,\mathrm{DR}(M)$$

In fact the assertion is local by the functoriality of $i_*i^!$, $\boldsymbol{R}j_*j^*$ applied to the natural morphisms $\boldsymbol{R}\Gamma_{[Z]}M\to M\to\boldsymbol{R}\Gamma_{[X|Z]}M$ in (2.3.2). Then we may assume X smooth, and (6.1.2–3) are well-known in this case by [7][10]. Note that (6.1.2) implies

(6.1.4) $$\mathrm{DR}(i^!M) = i^!\mathrm{DR}(M)\,,\quad \mathrm{DR}(i^*M) = i^*\mathrm{DR}(M)$$

for $M\in D_{rh}^b(X, \mathcal{D})$, where the second follows from the first by definition (5.2.11) and duality.

6.2. Theorem. *The functor* (6.1.1) *is an equivalence of categories.*

Proof. Let $M, N\in D_{rh}^b(X, \mathcal{D})$. We have a canonical pairing $\boldsymbol{D}M\boxtimes M\to a_X^!\,\omega_{pt}$ in (5.2.5), and the induced pairing $\mathrm{DR}(\boldsymbol{D}M)\boxtimes\mathrm{DR}(M)\to\delta_*a_X^!\boldsymbol{C}$ in (5.8.1), and they induce a commutative diagram

(6.2.1) $$\begin{array}{ccc}\mathrm{Hom}_{D^b(X,\mathcal{D})}(N, \boldsymbol{D}M) & \xrightarrow{\sim} & \mathrm{Hom}_{D^b(X\times X,\mathcal{D})}(N\boxtimes M, \delta_*a_X^!\,\omega_{pt}) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{D^b(\boldsymbol{C}_X)}(\mathrm{DR}(N), \mathrm{DR}(\boldsymbol{D}M)) & \xrightarrow{\sim} & \mathrm{Hom}_{D^b(\boldsymbol{C}_{X\times X})}(\mathrm{DR}(N)\boxtimes\mathrm{DR}(M), \delta_*a_X^!\boldsymbol{C})\end{array}$$

because the composition is compatible with the de Rham functor. So we get

(6.2.2) DR: $D^b_{rh}(X, \mathcal{D}) \to D^b_c(\boldsymbol{C}_X)$ is fully faithful,

if the second vertical morphism of (6.2.1) is bijective. We have a canonical morphism $N \boxtimes M \to \delta_*\delta^*(N \boxtimes M)$ which induces an isomorphism

$$\text{(6.2.3)} \qquad \mathrm{Hom}_{D^b(X\times X, \mathcal{D})}(N \boxtimes M, \delta_* a^!_X \omega_{pt}) \xleftarrow{\sim} \mathrm{Hom}_{D^b(X\times X, \mathcal{D})}(\delta_*\delta^*(N \boxtimes M), \delta_* a^!_X \omega_{pt})$$

because it is isomorphic to

$$\mathrm{Hom}_{D^b(X\times X, \mathcal{D})}(\delta_* \boldsymbol{D} a^!_X \omega_{pt}, \boldsymbol{D}(N \boxtimes M)) \xleftarrow{\sim} \mathrm{Hom}_{D^b(X\times X, \mathcal{D})}(\delta_* \boldsymbol{D} a^!_X \omega_{pt}, \delta_*\delta^! \boldsymbol{D}(N \boxtimes M))$$

(cf. (2.1.12) (2.2.1)) by duality (5.2.10). Since δ_*, δ^* commute with DR by (3.3.4) (6.1.4), the isomorphism (6.2.3) is compatible with the de Rham functor (i.e. it gives a commutative diagram), and the assertion (6.2.2) is reduced to the isomorphism

$$\text{(6.2.4)} \qquad \mathrm{Hom}_{D^b(X, \mathcal{D})}(M, a^!_X \omega_{pt}) \xrightarrow{\sim} \mathrm{Hom}_{D^b(\boldsymbol{C}_X)}(\mathrm{DR}(M), a^!_X \boldsymbol{C}) \quad \text{for} \quad M \in D^b_{rh}(X, \mathcal{D})$$

induced by the de Rham functor, because the functor $\delta_*: D^b(X, \mathcal{D}) \to D^b(X \times X, \mathcal{D})$ is fully faithful by 2.2, and it is same for $\delta_*: D^b(\boldsymbol{C}_X) \to D^b(\boldsymbol{C}_{X\times X})$. Here M corresponds to the above $\delta^*(N \boxtimes M)$. We take an injective representative K of $a^!_X \omega_{pt}$ so that the left hand side is obtained by the hypercohomology of the sheaf complex $\mathcal{H}om_{\mathcal{D}}(M, K)$ on X, where $\mathcal{H}om_{\mathcal{D}}(M^i, K^j)$ is defined by the presheaf $U \to \mathrm{Hom}_{\mathcal{D}}(M^i|_U, K^j|_U)$, and is flasque by the injectivity of K. This construction is compatible with the de Rham functor, i.e. we have a natural morphism $\mathcal{H}om_{\mathcal{D}}(M, K) \to \mathcal{H}om_{\boldsymbol{C}}(\mathrm{DR}(M), K')$ which induces (6.2.4) taking hypercohomology, where K' is a resolution of $\mathrm{DR}(K)$. So the assertion is localized, and we may assume X smooth. In fact, for a closed embedding $i: X \to Y$, the morphism

$$\text{(6.2.5)} \qquad \mathrm{Hom}_{D^b(X, \mathcal{D})}(M, a^!_X \omega_{pt}) \to \mathrm{Hom}_{D^b(Y, \mathcal{D})}(i_*M, a^!_Y \omega_{pt})$$

induced by the trace morphism $i_* a^!_X \omega_{pt} \to a^!_Y \omega_{pt}$ is an isomorphism by (2.1.12), because $i^! a^!_Y \omega_{pt} = a^!_Y \omega_{pt}$, and moreover the isomorphism (6.2.5) is compatible with the de Rham functor (i.e. it gives a commutative diagram). Then (6.2.4) is well-known in this case [7][10] (see also [16]). So we get (6.2.2). Then by the standard argument [1][10] we can show

(6.2.6) DR: $D^b_{rh}(X, \mathscr{D}) \to D^b_c(\boldsymbol{C}_X)$ is essentially surjective,

i.e. for any $\mathscr{F} \in D^b_c(\boldsymbol{C}_X)$, there exists $M \in D^b_{rh}(X, \mathscr{D})$ such that $\mathrm{DR}(M) \simeq \mathscr{F}$. Here we may assume $\mathscr{F} = \boldsymbol{R}j_* L$ by using triangles and stratification of constructible sheaves, where L is a local system on a smooth Zariski open subset $j\colon U \to X$ (i.e. $X \backslash U$ is a closed analytic subset). Then we may assume X smooth and $X \backslash U$ a divisor with normal crossings, using Hironaka's resolution and (6.1.3) (3.10.2). The assertion follows from Deligne's theory of regular singularity [2] as is well-known.

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